GENERALIZED $d$-KOSZUL MODULES

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Abstract. Generalized $d$-Koszul modules are introduced to solve an open problem: the odd Ext-module $E^{odd}(M)$ of a $d$-Koszul module $M$ over a $d$-Koszul algebra $\Lambda$ is a Koszul module over the even Yoneda algebra $E^{ev}(\Lambda)$.

Introduction

For an integer $d \geq 2$, a $d$-Koszul algebra was introduced and studied by R. Berger [B1], and developed by E. L. Green et al. [GMMZ] to the nonlocal case. If $d = 2$ it is the usual Koszul algebra. This class of generalized Koszul structures turns out to be important for example in theory of the Artin-Shelter algebras, the Calabi-Yau algebras, and the Yang-Mills algebras (see e.g. [B1], [B2], [CD]).

Let $\Lambda$ be a $d$-Koszul algebra and $M$ a $d$-Koszul $\Lambda$-module. It was shown in Theorem 6.1 of [GMMZ] that the even Ext-algebra $E^{ev}(\Lambda)$ is a Koszul algebra and the even Ext-module $E^{ev}(M)$ is a Koszul $E^{ev}(\Lambda)$-module. This generalizes the corresponding result of J. Backelin and R. Fröberg [BF] on the local Koszul algebras. An open problem was raised by E. L. Green et al. [GMMZ], Section 6: Is the odd Ext-module $E^{odd}(M)$ also a Koszul module over $E^{ev}(\Lambda)$? E. N. Marcos and R. Martínez-Villa [MM] proved that this is the case if the orthogonal algebra $\Lambda^!$ is also a $d$-Koszul algebra. However, in general $\Lambda^!$ is not a $d$-Koszul algebra (see [B1]; also Example 2 in [MM]). So the problem remains to be open.

In this paper we introduce the so-called generalized $d$-Koszul modules. This is a natural class of graded modules. For example, the syzygies of a $d$-Koszul module are generalized $d$-Koszul modules up to shifts. Also for each $i$, $J^iM$ is a generalized $d$-Koszul module up to shift, where $M$ is a generalized $d$-Koszul module over a $d$-Koszul algebra, and $J$ is the graded Jacobson radical.

Our main result is as follows.

Main Theorem. Let $\Lambda$ be a $d$-Koszul algebra, and $M$ a generalized $d$-Koszul $\Lambda$-module. Then $E^{ev}(M)$ is a Koszul module over the Koszul algebra $E^{ev}(\Lambda)$.

As a consequence, we have

Corollary. Let $\Lambda$ be a $d$-Koszul algebra, and $M$ a $d$-Koszul $\Lambda$-module. Then $E^{odd}(M)$ is a Koszul module over the Koszul algebra $E^{ev}(\Lambda)$.

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This answers in the affirmative the open problem mentioned above.

1. Preliminaries

We fix the notations and recall some facts frequently used later. For the details we refer to [BGS], [GM], and [GMMZ].

1.1. Throughout $\Lambda$ is a standardly graded algebra over a field $k$ (see [GM], p.250), i.e., $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is a positively graded $k$-algebra satisfying the following three conditions:

(i) $\Lambda_0 = k^r$ for some integer $r \geq 1$,

(ii) $\dim_k \Lambda_i < \infty$, $\forall i \geq 0$,

(iii) $\Lambda_i \Lambda_j = \Lambda_{i+j}$, $\forall i, j \geq 0$.

A left graded $\Lambda$-module $M$ is a $\Lambda$-module together with a decomposition of $k$-spaces $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $\Lambda_i M_j \subseteq M_{i+j}$, $\forall i, j \in \mathbb{Z}$. Let $M$ and $N$ be graded $\Lambda$-modules. A $\Lambda$-homomorphism $f : M \rightarrow N$ is a graded homomorphism if $f(M_i) \subseteq N_i$, $\forall i \in \mathbb{Z}$. For $M \in \text{Gr}(\Lambda)$, let $M[n]$ denote the graded module with $M[n]_i = M_{i-n}$. Let $\Lambda$-Mod be the category of the left $\Lambda$-modules, $\text{Gr}(\Lambda)$ the category of the left graded $\Lambda$-modules and graded homomorphisms, and $\text{gr}(\Lambda)$ the full subcategory of $\text{Gr}(\Lambda)$ consisting of finitely generated $\Lambda$-modules. Then $\Lambda$-Mod and $\text{Gr}(\Lambda)$ are abelian categories; and $\text{gr}(\Lambda)$ is abelian if $\Lambda$ is noetherian. Let $\text{Hom}_{\text{Gr}(\Lambda)}$ and $\text{Ext}^i_{\text{Gr}(\Lambda)}$ denote the homomorphisms and extensions in $\text{Gr}(\Lambda)$, as opposed to the usual $\text{Hom}_\Lambda$ and $\text{Ext}^i_\Lambda$ in $\text{Gr}(\Lambda)$.

Let $I$ be a subset of $\mathbb{Z}$, and $M \in \text{Gr}(\Lambda)$. $M$ is generated in degrees in $I$, if $M = \Lambda(\bigoplus_{j \in I} M_j)$; $M$ is generated in degree $i$ if $M = \Lambda M_i$; $M$ is supported above degree $n$ if $M_j = 0$ for $j < n$; and $M$ is concentrated in degrees in $I$ if $M_i = 0$ for $i \notin I$.

Let $J$ be the ideal $\bigoplus_{i \geq 1} \Lambda_i$ of $\Lambda$. The trivial $\Lambda$-module $\Lambda_0$ is the lift of the $\Lambda_0$-module $\Lambda_0$ via the $k$-algebra homomorphism $\Lambda \rightarrow \Lambda/J = \Lambda_0$. It is a graded $\Lambda$-module concentrated in degree 0. We need the following well-known fact.

**Lemma 1.1.** Let $M \in \text{Gr}(\Lambda)$, and $I$ be a subset of $\mathbb{Z}$. If $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin I$, then $M$ is generated in degrees in $I$.

**Proof.** For the convenience of the reader we include a justification. Put $L := M/\Lambda(\bigoplus_{j \in I} M_j)$. If $L \neq 0$, then $L/JL \neq 0$. While $L/JL$ is a graded module over the semisimple algebra $\Lambda_0$, it follows that $\text{Hom}_{\text{Gr}(\Lambda)}(L/JL, \Lambda_0[j]) \neq 0$ for some $j \notin I$, and hence $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) \neq 0$ for some $j \notin I$, contrary to the assumption. \qed

1.2. Denote by $E(\Lambda)$ the Ext-algebra $\bigoplus_{i \geq 0} \text{Ext}_{\Lambda}^i(\Lambda_0, \Lambda_0)$, with the multiplication given by the Yoneda product. We also consider the even Ext-algebra

$$E^{ev}(\Lambda) := \bigoplus_{i \geq 0} \text{Ext}_{\Lambda}^{2i}(\Lambda_0, \Lambda_0),$$
which is a positively graded algebra with grading $E^\text{ev}(\Lambda)_n := \text{Ext}_\Lambda^{2n}(\Lambda_0, \Lambda_0)$. For a $\Lambda$-module $M$, let $E(M)$ be the graded $E(\Lambda)$-module $\bigoplus_{i \geq 0} \text{Ext}_\Lambda^i(M, \Lambda_0)$. We also consider the even Ext-module $E^\text{ev}(M) := \bigoplus_{n \geq 0} \text{Ext}_\Lambda^{2n}(M, \Lambda_0)$ over $E(\Lambda)$, and the odd Ext-module $E^\text{odd}(M) := \bigoplus_{n \geq 0} \text{Ext}_\Lambda^{2n+1}(M, \Lambda_0)$ over $E^\text{ev}(\Lambda)$: they are graded modules with gradings

$$E^\text{ev}(M)_n := \text{Ext}_\Lambda^{2n}(M, \Lambda_0), \quad \text{and} \quad E^\text{odd}(M)_n := \text{Ext}_\Lambda^{2n+1}(M, \Lambda_0), \quad \forall \ n \geq 0.$$ 

Every graded $\Lambda$-module $M$ has a graded projective resolution

$$(1) \quad Q^i : \cdots \to Q^i \to \cdots \to Q^1 \to Q^0 \to M \to 0.$$ 

If each $Q^i$ is finitely generated, then we say that $Q^\bullet$ is a finitely generated graded projective resolution of $M$. If $M \in \text{gr}(\Lambda)$, then $M$ admits a minimal graded projective resolution (1) in the sense that $\text{Im}(Q^i \to Q^{i-1}) \subseteq JQ^{i-1}$, $\forall \ i \geq 1$ (see Propositions 2.3 and 2.4 in [GM]).

If $M \in \text{gr}(\Lambda)$, then for each $N \in \text{Gr}(\Lambda)$, $\text{Hom}_\Lambda(M, N)$ is a graded $k$-space with the shift-grading: $\text{Hom}_\Lambda(M, N)_i = \text{Hom}_{\text{Gr}(\Lambda)}(M, N[i])$, i.e., $\text{Hom}_\Lambda(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Gr}(\Lambda)}(M, N[i])$.

If $M$ has a finitely generated graded projective resolution, then for each $N \in \text{Gr}(\Lambda)$ and each $n \geq 1$, $\text{Ext}_\Lambda^n(M, N)$ is a graded $k$-space with the shift grading: $\text{Ext}_\Lambda^n(M, N)_i = \text{Ext}_{\text{Gr}(\Lambda)}^n(M, N[i])$, i.e., $\text{Ext}_\Lambda^n(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}_{\text{Gr}(\Lambda)}^n(M, N[i])$.

Fix a minimal graded projective resolution of the trivial $\Lambda$-module $\Lambda_0$:

$$(2) \quad P^\bullet : \cdots \to P^n \to \cdots \to P^1 \to P^0 \to \Lambda_0 \to 0.$$ 

We need the following fact.

**Lemma 1.2.** ([GMMZ], Lemma 3.2) *Let $M$ be a graded module supported above degree 0 with a minimal graded projective resolution (1). For any integer $n \geq 1$, if $P^n$ in (2) is supported above degree $s$, then so is $Q^n$.*

**1.3.** For the theory of the Koszul algebras and the Koszul modules we refer to A. Beilinson, V. Ginzburg and W. Soergel [BGS], and E. L. Green and R. Martínez-Villa [GM].

**Definition 1.3** ([GMMZ], [MM]). *Let $d \geq 2$ be an integer. A graded $\Lambda$-module $M$ is a $d$-Koszul module if $M$ admits a finitely generated graded projective resolution (1) such that each $Q^i$ is generated in degree $\delta(i)$, where

$$\delta(i) := \begin{cases} \ nd, & \text{if } i = 2n, \\ \ nd + 1, & \text{if } i = 2n + 1. \end{cases}$$

If the trivial $\Lambda$-module $\Lambda_0$ is a $d$-Koszul module, then we call $\Lambda$ a $d$-Koszul algebra.*
Theorem 1.4. ([GMMZ], Theorem 6.1) Let $\Lambda$ be a $d$-Koszul algebra and $M$ a $d$-Koszul $\Lambda$-module. Then $E\text{ev}(\Lambda)$ is a Koszul algebra, and $E\text{ev}(M)$ is a Koszul $E\text{ev}(\Lambda)$-module.

2. Generalized $d$-Koszul modules

2.1. Let $d \geq 2$ be an integer. For each integer $i \geq 0$ we assign a subset $\Delta(i)$ of $\mathbb{N}_0$ as

$$\Delta(i) := \begin{cases} \{nd\}, & \text{if } i = 2n; \\ \{nd + 1, \ldots, nd + d - 1\}, & \text{if } i = 2n + 1. \end{cases}$$

Definition 2.1. A graded $\Lambda$-module $M$ is called a generalized $d$-Koszul module if $M$ admits a finitely generated graded projective resolution $Q^\bullet$ such that each $Q^i$ is generated in degrees in $\Delta(i)$, i.e., $Q^i = \Lambda(\bigoplus_{j \in \Delta(i)} Q^i_j)$, $i \geq 0$.

Remark 2.2. (i) As remarked by Beilinson-Ginzburg-Soergel [BGS] (p.476) in the Koszul situation, $Q^\bullet$ in Definition 2.1 is unique up to isomorphism. More precisely, if $L^\bullet$ is another graded projective resolution of $M$ such that each $L^i$ is also generated in degrees in $\Delta(i)$ (it is not assumed to be finitely generated), then $L^\bullet \cong Q^\bullet$ as complexes. In fact, $L^\bullet$ is homotopy equivalent to $Q^\bullet$; while any chain maps $Q^\bullet \to Q^\bullet$ and $L^\bullet \to L^\bullet$, which respect the grading on $Q^i$ and $L^i$ and are homotopic to zero must themselves be zero (since any element in $\Delta(i)$ is strictly smaller than any element in $\Delta(i+1)$, and $Q^i$ and $L^i$ are both generated in degrees in $\Delta(i)$). It follows that $L^\bullet \cong Q^\bullet$ as complexes.

(ii) We emphasize that, as in the $d$-Koszul situation, here $Q^i$ is also required to be finitely generated: it is for the application of the shift grading on $\text{Ext}^n_\Lambda(M, \cdot)$.

(iii) If $M$ is a generalized $d$-Koszul module, then such a graded projective resolution $Q^\bullet$ in the definition is minimal, and each syzygy $\Omega^i(M)$ is a graded $\Lambda$-module finitely generated in degrees in $\Delta(i)$. In particular, $M$ is finitely generated in degree 0.

(iv) A $d$-Koszul module is always generalized $d$-Koszul; and a generalized 2-Koszul module is a finitely generated Koszul module (if $\Lambda$ is noetherian, then a generalized 2-Koszul $\Lambda$-module is exactly a finitely generated Koszul $\Lambda$-module).

Example 2.3. Let $A$ be the algebra given by the quiver

$$\begin{array}{ccc}
\alpha & \to & 1 \\
\downarrow & & \downarrow \\
2 & \to & 3
\end{array}$$

with relations $\alpha^3$, $\gamma \beta \alpha$. Then the simple (left) module $S(1)$ has a minimal graded projective resolution

$$\cdots \to P(1)[4] \oplus P(2)[5] \to P(1)[3] \oplus P(3)[3] \to P(1)[1] \oplus P(2)[1] \to P(1) \to S(1) \to 0$$

where $\Omega^4 S(1) = (\Omega^2 S(1))[3]$. Thus $S(1)$ is a generalized 3-Koszul $A$-module. Since $Q^3 = P(1)[4] \oplus P(2)[5]$ is generated in degrees 4 and 5, but not generated in degree 4, it follows that $S(1)$ is not a 3-Koszul $A$-module (by an argument in Remark 2.2(i)).
2.2. We have the following characterization for a $d$-Koszul module and for a generalized $d$-Koszul module, which is the corresponding version of Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS] for the Koszul modules.

Lemma 2.4. Let $M$ be a graded $\Lambda$-module with a finitely generated graded projective resolution. Then

(i) $M$ is $d$-Koszul if and only if $\Ext^i_{\Lambda}(M, \Lambda_0)_j = \Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$, $\forall j \neq \delta(i)$.

(ii) $M$ is generalized $d$-Koszul if and only if $\Ext^i_{\Lambda}(M, \Lambda_0)$ is concentrated in degrees in $\Delta(i)$, with the shift grading, i.e., $\Ext^i_{\Lambda}(M, \Lambda_0)_j = \Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$, $\forall j \notin \Delta(i)$.

Proof. They can be similarly proved as Proposition 2.14.2 in [BGS]. For the convenience of the reader we include a justification of (ii).

Assume that $M$ is generalized $d$-Koszul. Then $M$ has a graded projective resolution $Q^\bullet$ such that each $Q^i$ is generated in degrees in $\Delta(i)$, and $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j])$ is the $i$-th cohomology group of the complex $\Hom_{\Gr(\Lambda)}(Q^\bullet, \Lambda_0[j])$. Since $Q^i$ is generated in degrees in $\Delta(i)$, and $\Lambda_0[j]$ is concentrated in degree $j$, it follows that $\Hom_{\Gr(\Lambda)}(Q^\bullet, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$, and hence $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$.

Conversely, assume that $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$. We construct inductively a graded projective resolution $L^\bullet$ of $M$ such that each $L^i$ is generated in degrees in $\Delta(i)$. Since $\Hom_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \neq 0$, by Lemma 1.1, $M$ is generated in degree 0, and hence we have a surjective graded $\Lambda$-homomorphism $L^0 \rightarrow M$ such that $L^0$ is generated in degree 0. Denote by $K^1$ its kernel. Then $\Hom_{\Gr(\Lambda)}(K^1, \Lambda_0[j]) = \Ext^1_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(1)$, and hence by Lemma 1.1, $K^1$ is generated in degrees in $\Delta(1)$. Thus we have a surjective graded $\Lambda$-homomorphism $L^1 \rightarrow K^1$ such that $L^1$ is generated in degrees in $\Delta(1)$. Repeating this process we are done.

By assumption we have already a finitely generated graded projective resolution $Q^\bullet$. By the argument in Remark 2.2(i), there are chain maps $f : L^\bullet \rightarrow Q^\bullet$ and $g : Q^\bullet \rightarrow L^\bullet$ such that $gf = \Id_{L^\bullet}$, which means that $L^\bullet$ is a direct summand of $Q^\bullet$. Thus $L^\bullet$ is also a finitely generated resolution. By definition $M$ is generalized $d$-Koszul.

2.3. For a $d$-Koszul module $M$, in general $\Omega^i M$ and $J^i M$ are not $d$-Koszul modules, up to shifts (see Proposition 5.2 in [GMMZ] for some special cases); however, they turn out to be generalized $d$-Koszul, after proper shifts. In the rest of this section we precisely state and prove these results, which will be important in the proof of Main Theorem and Corollary.

Lemma 2.5. Let $M$ be a $d$-Koszul $\Lambda$-module. Then

(i) $E^{\odd}(M) \cong E^{\even}(\Omega M)$ as graded $E(\Lambda)$-modules.

(ii) $(\Omega^i M)[−\delta(i)]$ is a generalized $d$-Koszul module for each $i \geq 0$.  

Proof. (i) By definition we have an isomorphism of graded $E(\Lambda)$-modules
\[
E^{\text{odd}}(M) = \bigoplus_{n \geq 0} \text{Ext}^{2n+1}_\Lambda(M, \Lambda_0) \cong \bigoplus_{n \geq 0} \text{Ext}^{2n}_\Lambda(\Omega^i M, \Lambda_0) = E^{\text{even}}(\Omega M).
\]

(ii) Taking a graded projective resolution $Q^* \otimes$ of $M$ such that each $Q^i$ is finitely generated in degree $\delta(i)$, we see that $(\Omega^i M)[\delta(i)]$ has a graded projective resolution
\[
L^*: 0 \to L^2 \to \cdots \to L^1 \to L^0 \to (\Omega^i M)[\delta(i)] \to 0
\]
where $L^j = Q^{i+j}[\delta(i)]$ is finitely generated in degree $\delta(i+j) - \delta(i)$ for $j \geq 0$.

If $i$ is even, then $L^j$ is generated in degree $\delta(j)$. This is, $(\Omega^i M)[\delta(i)]$ is a $d$-Koszul module, and hence a generalized $d$-Koszul module.

Assume that $i$ is odd. Then $L^j$ is generated in degree $nd$ if $j = 2n$, and $L^j$ is generated in degree $nd+d-1$ if $j = 2n+1$. By definition $(\Omega^i M)[\delta(i)]$ is generalized $d$-Koszul. □

Theorem 2.6. Let $\Lambda$ be a $d$-Koszul algebra and $M$ a generalized $d$-Koszul $\Lambda$-module. Then
\begin{enumerate}[(i)]
    \item $(J^i M)[-i]$ is generalized $d$-Koszul for each $i \geq 1$.
    \item For each $n \geq 1$ we have $k$-isomorphisms
    \[\text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(J^i M, \Lambda_0[nd]) \cong \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(J^{i+1} M, \Lambda_0[nd]) \cong \cdots \cong \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(J^{d-1} M, \Lambda_0[nd]).\]
\end{enumerate}

Proof. (i) It suffices to prove that $(J^i M)[-1]$ is generalized $d$-Koszul. Since $M = \bigoplus_{i \geq 0} M_i$ is finitely generated in degree $0$, $J^i M = \bigoplus_{i \geq 1} M_i$ is finitely generated in degree $1$.

We first prove the following claim: $JM[-1]$ admits a graded projective resolution $Q^*$ such that $Q^i$ is generated in degrees in $\Delta(i)$. By the proof of Lemma 2.4(ii), it suffices for each $n \geq 0$ to prove that $\text{Ext}^0_{\text{Gr}(\Lambda)}(JM[-1], \Lambda_0[j]) = \text{Ext}^0_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$, $\forall j \neq nd$, and that $\text{Ext}^1_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$, $\forall j \notin \Delta(2n+1) = \{nd+1, \cdots, nd+d-1\}$.

Applying $\text{Hom}_{\text{Gr}(\Lambda)}(-, \Lambda_0[j+1])$ to the graded exact sequence $0 \to JM \to M \to M/JM \to 0$ we get the following exact sequence of $k$-spaces
\[
\text{Ext}^0_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \text{Ext}^1_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \text{Ext}^2_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1])
\]
\[
\to \text{Ext}^0_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \text{Ext}^1_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \text{Ext}^2_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]).
\]

Since $\Lambda$ is a $d$-Koszul algebra, $Q^{2n}$ in (2) is supported above degrees $nd$, and hence by Lemma 1.2, $Q^{2n}$ is supported above degrees $nd$, where $Q^*$ is a minimal graded projective resolution of $JM[-1]$. Thus $\text{Ext}^0_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$ for $j < nd$.

Similarly, $\text{Ext}^0_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$ for $j < nd+d-1$. Since $M$ is generalized $d$-Koszul, by Lemma 2.4(ii), $\text{Ext}^{2n}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$ if $j \neq nd-1$, and $\text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$ if $j \notin \{nd, \cdots, nd+d-2\}$.

Note that $M/JM$ is a $\Lambda/J$-module and $\Lambda/J = \Lambda_0$ is a semisimple algebra. Thus $M/JM$ is a direct summand of a finite direct sum of copies of the trivial $\Lambda$-module.
$\Lambda_0$. In particular, $M/JM$ is a d-Koszul module. By Lemma 2.4(i),
$$\text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0 \text{ if } j \neq nd,$$
and $\text{Ext}^{2(n+1)}_{\text{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0 \text{ if } j \neq (n+1)d - 1$.

Now if $j \neq nd$, then by the exact sequence above we have the following exact sequence
$$\text{Ext}^{2n}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j+1]) \to \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0,$$
where if $j \neq nd - 1$ then $\text{Ext}^{2n}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$, and hence $\text{Ext}^{2n}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$; and if $j = nd - 1 < nd$, then we already know $\text{Ext}^{2n}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$.

Let $j \notin \Delta(2n+1) = \{nd + 1, \ldots, nd + d - 1\}$. Then by the exact sequence above we have the following exact sequence
$$\text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j+1]) \to \text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \text{Ext}^{2(n+1)}_{\text{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0,$$
where if $j \notin \{nd, \ldots, nd + d - 2\}$ then $\text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$, and hence $\text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$; and if $j \in \{nd, \ldots, nd + d - 2\}$, then $j = nd < nd + 1$, and in this case we already know $\text{Ext}^{2n+1}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$. This proves the claim.

Since $M/JM$ is a d-Koszul module, $M/JM$ has a finitely generated graded projective resolution, say $Q^*_2$, such that $Q^*_3$ is generated in degrees in $\delta(i)$. By the graded version of the Horseshoe Lemma, we get a graded projective resolution $Q^*_t$ of $M$, such that $Q^*_t = Q^*_i \oplus Q^*_t$ for each $i$. Thus $Q^*_t$ is also generated in degrees in $\Delta(i)$. Since $M$ is a generalized d-Koszul module, by Remark 2.2(i), we know that $Q^*_2$ is finitely generated, and hence $Q^*_i$ is finitely generated. By definition $JM[-1]$ is d-Koszul.

(iii) Let $d \geq 3$. Applying Hom$_{\Lambda}(\cdot, \Lambda_0)$ to the graded exact sequence $0 \to J^2M \to JM \to J^2M \to 0$, we get the following exact sequence
$$\text{Ext}^{2n-1}_{\Lambda}(JM/J^2M, \Lambda_0) \to \text{Ext}^{2n-1}_{\Lambda}(JM, \Lambda_0) \to \text{Ext}^{2n}_{\Lambda}(J^2M, \Lambda_0) \to \text{Ext}^{2n}_{\Lambda}(JM/J^2M, \Lambda_0).$$
Since $(JM/J^2M)[-1]$ is d-Koszul, by Lemma 2.4(i), $\text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JM/J^2M, \Lambda_0[j]) = 0$ if $j \neq nd - d + 2$, and $\text{Ext}^{2n}_{\text{Gr}(\Lambda)}(JM/J^2M, \Lambda_0[j]) = 0$ if $j \neq nd + 1$. Taking the $nd$-th homogeneous components of the exact sequence above, we obtain that $\text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JM, \Lambda_0[nd]) \cong \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(J^2M, \Lambda_0[nd])$. Repeating the process one gets (iii).

\section{3. Proofs of Main Theorem and Corollary}

3.1. We begin with a lemma, which seems to be of independent interest.

\textbf{Lemma 3.1.} Let $A$ be an arbitrary Koszul algebra and $C$ a full subcategory of $\text{Gr}(A)$. Suppose that for any $X \in C$, there exist exact sequences in $\text{Gr}(A)$

\begin{align*}
(3) & \quad 0 \to \Omega \to P^0 \to X \to 0, \\
(4) & \quad 0 \to X'' \to X' \to \Omega[-1] \to 0,
\end{align*}

such that $P^0$ is a graded projective $A$-module generated in degree 0 and $X', X'' \in C$. Then all modules in $C$ are Koszul $A$-modules.
follows from the exact sequence above that Ext
j
k
= 1. It suffices to prove that all the conditions in Lemma 3.1 are satisfied.

3.2. Proof of Main Theorem.

is supported above degree
−
1
. By Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS], it suffices to prove that for each
X
∈
C,
Ext
i
Gr(Λ)(X, A0[j]) = 0 unless
i
= 0. The sequence (4) implies that
Ω
is a graded
A
-module and is generated in degree 1, since
X′ ∈
C
is generated in degree 0. By Ext
1
Gr(Λ)(X, A0[j]) ≏ HomGr(Λ)(Ω, A0[j]), we see that Ext
1
Gr(Λ)(X, A0[j]) = 0 unless
j
= 1.

Let
n
≥ 1. Assume that for each
X
∈
C
and for each positive integer
i
with
i
≤

n
, Ext
i
Gr(Λ)(X, A0[j]) = 0 unless
i
= 0. The exact sequence (4) implies the following exact sequence for every integer
j

Ext
n−1
Gr(Λ)(X0, A0[j]) → Ext
n
Gr(Λ)(Ω[1], A0[j]) → Ext
n
Gr(Λ)(X′, A0[j]).

By the inductive hypothesis, we have Ext
n−1
Gr(Λ)(X0, A0[j]) = 0 unless
j
= 0, and Ext
n
Gr(Λ)(X′, A0[j]) = 0 unless
j
= 0. Let
Q
∗
be a minimal graded projective resolution of
Ω[1] (it exists since
Ω ⊆
P0
is supported above 0). By Lemma 1.2,
Q
n
is supported above degree
n
, which implies Ext
n
Gr(Λ)(Ω[1], A0[j]) = 0 for
j
<
n
. It follows from the exact sequence above that Ext
n
Gr(Λ)(Ω[1], A0[j]) = 0 unless
j
= 0. Thus Ext
n+1
Gr(Λ)(X, A0[j]) = Ext
n
Gr(Λ)(Ω, A0[j]) = Ext
n
Gr(Λ)(Ω[1], A0[j−1]) = 0 unless
j
= 0. This completes the proof. □

3.2. Proof of Main Theorem. By Theorem 1.4,
E
v
(Λ) is a Koszul algebra. Put

C := \{E
v
(N) ∈ Gr(E
v
(Λ)) | N is a generalized d-Koszul Λ-module\}.

It suffices to prove that all the conditions in Lemma 3.1 are satisfied.

The graded exact sequence
0 → JN → N → N/JN → 0
induces the following exact sequence of graded k-spaces for each
n
≥ 0

Ext
2n−1
Gr(Λ)(N, A0[n]) → Ext
2
Gr(Λ)(JN, A0[n]) → Ext
2
Gr(Λ)(N/JN, A0[n]) → Ext
2
Gr(Λ)(N, A0[n]) → Ext
2
Gr(Λ)(JN, A0[n]). (5)

Since
N
and
JN[−1]
are generalized d-Koszul, by Lemma 2.4(ii), we have Ext
2n−1
Gr(Λ)(N, A0[n]) = 0 = Ext
2
Gr(Λ)(JN, A0[n]). Taking the
n
-th homogeneous components of (5) we get the following exact sequence for each
n
≥ 0

0 → Ext
2
Gr(Λ)(JN, A0[n]) → Ext
2
Gr(Λ)(N/JN, A0[n]) → Ext
2
Gr(Λ)(N, A0[n]) → 0. (6)

Since
N
is generalized d-Koszul and
N/JN
is d-Koszul, by Lemma 2.4, we have

E
v
(N/JN) = \bigoplus
n
≥ 0
Ext
2
Gr(Λ)(N/JN, A0[n]), E
v
(N) = \bigoplus
n
≥ 0
Ext
2n
Gr(Λ)(N, A0[n]).

By taking direct sum of (6), we get the following short exact sequence in Gr(E
v
(Λ)):

0 → \Omega → E
v
(N/JN) → E
v
(N) → 0. (7)

where
Ω := \bigoplus
n
≥ 0
Ext
2n−1
Gr(Λ)(JN, A0[n]). In particular, \Omega
is a graded E
v
(Λ)-module with grading
Ω
:= Ext
2n−1
Gr(Λ)(JN, A0[n]). (One can also prove this directly as follows:}
since \( \Lambda \) is \( d \)-Koszul algebra, it follows from Lemma 2.4(i) that
\[
\Ext^2_{\Lambda} (\Lambda_0, \Lambda_0) \Ext^1_{\Gr(\Lambda)} (JN, \Lambda_0[nd]) = \Ext^2_{\Gr(\Lambda)} (\Lambda_0, \Lambda_0[nd]) \Ext^1_{\Gr(\Lambda)} (JN, \Lambda_0[n]) \\
\subseteq \Ext^1_{\Gr(\Lambda)} (JN, \Lambda_0[(n + m)d]).
\]

By Theorem 1.4, \( E^\ev (N/JN) \) is a Koszul \( E^\ev (\Lambda) \)-module, in particular it is generated in degree 0. Since \( N/JN \) is a direct summand of finite direct sum of copies of the trivial \( \Lambda \)-module \( \Lambda_0 \), \( E^\ev (N/JN) \) is a projective \( E^\ev (\Lambda) \)-module.

Similarly, the graded exact sequence \( 0 \to J^dN \to J^{d-1}N \to J^{d-1}N/J^dN \to 0 \) induces the following exact sequence of graded \( k \)-spaces for each \( n \geq 0 \)
\[
\Ext^2_{\Lambda} (J^{d-1}N, \Lambda_0) \to \Ext^2_{\Lambda} (JdN, \Lambda_0) \to \Ext^2_{\Lambda} (JdN, \Lambda_0) \\
\to \Ext^2_{\Lambda} (J^{d-1}N, \Lambda_0) \to \Ext^2_{\Lambda} (JdN, \Lambda_0).
\]
Note that by Theorem 2.6(ii), \( J^{d-1}N[-(d - 1)] \) and \( JdN[-d] \) are generalized \( d \)-Koszul \( \Lambda \)-modules, and that \( (JdN/J^dN)[-d] \) is a \( d \)-Koszul module. Taking the \((n + 1)d\)-th homogeneous components, and by the same arguments we get another exact sequence in \( \Gr(E^\ev (\Lambda)) \):
\[
0 \to \bigoplus_{n \geq 0} \Ext^2_{\Gr(\Lambda)} (J^dN, \Lambda_0[(n + 1)d]) \to \bigoplus_{n \geq 0} \Ext^2_{\Gr(\Lambda)} (J^{d-1}N/J^dN, \Lambda_0[(n + 1)d]) \\
\to \bigoplus_{n \geq 0} \Ext^2_{\Gr(\Lambda)} (J^{d-1}N, \Lambda_0[(n + 1)d]) \to 0,
\]

or equivalently,
\[
0 \to E^\ev (J^dN) \to E^\ev (J^{d-1}N/J^dN) \to \Omega[-1] \to 0,
\]
where
\[
\Omega[-1] = \bigoplus_{n \geq 0} \Ext^2_{\Gr(\Lambda)} (JN, \Lambda_0[(n + 1)d])[n] = \bigoplus_{n \geq 0} \Ext^2_{\Gr(\Lambda)} (JN, \Lambda_0)[(n + 1)d])
\]
\[
\cong \bigoplus_{n \geq 0} \Ext^2_{\Gr(\Lambda)} (J^{d-1}N, \Lambda_0[(n + 1)d]),
\]
where the last isomorphism follows from Theorem 2.6(ii).

Since \( (J^{d-1}N/J^dN)[-d] \) is \( d \)-Koszul, by Lemma 2.5(ii), \( \Omega(J^{d-1}N/J^dN)[-d] \) is generalized \( d \)-Koszul, and by Lemma 2.5(i), we have
\[
E^\ev (J^{d-1}N/J^dN) \cong E^\ev ((J^{d-1}N/J^dN)[-d]) \cong E^\ev (\Omega(J^{d-1}N/J^dN)[-d]),
\]
from which we see \( E^\ev (J^{d-1}N/J^dN) \in \mathcal{C} \).

Since \( N \) is generalized \( d \)-Koszul, by Theorem 2.6(i), \( (J^dN)[-d] \) is generalized \( d \)-Koszul. Thus \( E^\ev (J^dN) = E^\ev ((J^dN)[-d]) \in \mathcal{C} \).

Now (7) and (8) shows that all the conditions in Lemma 3.1 are satisfied. This completes the proof.

\[\square\]

3.3. **Proof of Corollary.** By Lemma 2.5(ii), \( (\Omega M)[-1] \) is a generalized \( d \)-Koszul module. It follows from Main Theorem that \( E^\ev ((\Omega M)[-1]) \) is a Koszul \( E^\ev (\Lambda) \)-module. Therefore by Lemma 2.5(i), \( E^\ev (M) \cong E^\ev (\Omega M) = E^\ev ((\Omega M)[-1]) \) is a Koszul \( E^\ev (\Lambda) \)-module.

\[\square\]
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