REMARKS ON $\mathbb{A}^1$-HOMOTOPY GROUPS
OF SMOOTH TORIC MODELS

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Abstract. We extend previous results on $\mathbb{A}^1$-homotopy groups of smooth proper toric varieties to the case of smooth proper toric models, i.e., smooth proper equivariant compactifications of possibly non-split tori, in characteristic 0.

1. Statement of results

Fix a field $k$ having characteristic 0, and let $Sm_k$ denote the category of schemes that are separated, smooth and have finite type over $k$. Suppose $X$ is a smooth proper $k$-scheme. Let $H(k)$ denote the $\mathbb{A}^1$-homotopy category of $k$-schemes as constructed in [8, §3.2]. Assume $X(k)$ is non-empty, and fix $x \in X(k)$. One can study the $\mathbb{A}^1$-homotopy sheaves of groups $\pi_{\mathbb{A}^1}^i(X, x)$ (the Nisnevich sheaves of groups on $Sm_k$ denoted $a_{\pi_{\mathbb{A}^1}^i}(X, x)$ on [8, p. 110]). Our aim in this short note is to show that the “geometric” decomposition of $\mathbb{A}^1$-homotopy sheaves of groups of smooth proper “split” toric varieties, i.e., equivariant compactifications of $\mathbb{G}_m \times \mathbb{A}^n$, studied in [1] and [9] extends to “non-split” toric varieties, i.e., equivariant compactifications of tori $T$ over $k$. We will refer to equivariant compactifications of tori $T$ over $k$ as toric $T$-models [6, §5].

Let $\overline{k}$ denote a fixed algebraic closure of $k$ and let $G_k$ denote the Galois group of $\overline{k}$ over $k$. For a $k$-scheme $Y$, let $Y_{\overline{k}}$ denote the variety obtained by extending scalars to $\overline{k}$. Suppose $X$ is a smooth proper toric $T$-model. One knows that $Pic(X_{\overline{k}})$ is a finitely generated $G_k$-module, and we denote the associated dual $k$-torus—the Neron-Severi torus—by $T_{NS}(X)$. With any toric $T$-model, one can associate a fan $\Sigma$ in $X^*(T_{\overline{k}})$ that is $G_k$-invariant. Cox’s construction [5] realizing any “split” smooth proper toric variety as a geometric quotient of an open subscheme of affine space by a free action of $T_{NS}(X)$ can be generalized to the non-split case: if $X$ is a smooth proper toric $T$-model, there are a $T_{NS}(X)$-torsor $f : U \to X$ and an open immersion $U \hookrightarrow \mathbb{A}^n_k$ ($n = \dim T + \dim T_{NS}(X))$ [6, Proposition 5.6]. Let $\mathcal{H}_{et}(T_{NS}(X))$ denote the Nisnevich sheafification of the presheaf (on $Sm_k$) $U \mapsto H_{et}^1(U, T_{NS(X)})$.

Theorem 1.1. Assume $k$ is a field having characteristic 0 and $T$ is a $k$-torus. Suppose $X$ is a smooth proper toric $T$-model, and let $x$ denote the $k$-rational point of $X$ corresponding to $1 \in T(k)$. The $T_{NS(X)}$-torsor $f : U \to X$ above is an $\mathbb{A}^1$-cover. In particular, if $\tilde{x}$ is any lift of $x$, there is a short exact sequence (of Nisnevich sheaves of groups)

$$1 \longrightarrow \pi_{\mathbb{A}^1}^1(U, \tilde{x}) \longrightarrow \pi_{\mathbb{A}^1}^1(X, x) \longrightarrow T_{NS(X)} \longrightarrow 1,$$

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and, for each integer $i > 1$, there are isomorphisms $\pi^A_1(U,x) \sim \pi^A_1(X,x)$. Finally, $f$ induces a morphism of sheaves $\pi^A_0(X) \to H^1_\text{ét}(T_{\text{Nis}(X)})$ that is an isomorphism on sections over finitely generated separable extensions $L/k$.

**Remark 1.2.** There are examples of $k$-tori $T$ and smooth proper toric $T$-models $X$ for which $\pi^A_0(X)(k)$ is non-trivial. Thus, over non separably closed fields, we have the interesting phenomenon that a smooth proper $\mathbb{A}^1$-disconnected space can have $\mathbb{A}^1$-connected covering spaces! For a manifestation of this phenomenon for non-proper smooth varieties, one can consider the morphism $\mathbb{A}^m \setminus 0 \to \mathbb{A}^m \setminus 0/\mu_n$ [1] Remark 3.13.

2. Torus torsors as $\mathbb{A}^1$-covering spaces

The word space, will mean “object of $\Delta^c \text{Sh}_{\text{Nis}}(\text{Sm}_k)$” (the category of simplicial Nisnevich sheaves on $\text{Sm}_k$); we use calligraphic letters (e.g., $\mathcal{X}, \mathcal{Y}$) to denote such objects. We set $[\mathcal{X}, \mathcal{Y}]_s := \text{hom}_{\mathcal{H}}(\text{Sm}_k_{\text{Nis}})(\mathcal{X}, \mathcal{Y})$, where $\mathcal{H}(\text{Sm}_k_{\text{Nis}})$ is as on [8] p. 49 and $[\mathcal{X}, \mathcal{Y}]_k := \text{hom}_{\mathcal{H}_k}(\mathcal{X}, \mathcal{Y})$. A morphism $f: \mathcal{X} \to \mathcal{Y}$ of $k$-spaces is an $\mathbb{A}^1$-cover (cf. [2] Section 4.1) if it has the unique right lifting property with respect to morphisms that are simultaneously $\mathbb{A}^1$-weak equivalences and monomorphisms of sheaves, i.e., $\mathbb{A}^1$-acyclic cofibrations.

**Proposition 2.1.** Let $T$ be a multiplicative group over a field $k$ having characteristic 0. If $X$ is a smooth scheme, and $\pi: U \to X$ is a $T$-torsor locally trivial in the étale topology, then $\pi$ is an $\mathbb{A}^1$-cover and, in particular, an $\mathbb{A}^1$-fibration.

Let $BT$ denote the simplicial classifying space of $T$ viewed as a Nisnevich sheaf of groups, and let $BT_{\text{ét}}$ denote the simplicial classifying space of $T$ viewed as an étale sheaf of groups. Let $\alpha: (\text{Sm}_k)_{\text{ét}} \to (\text{Sm}_k)_{\text{Nis}}$ be the morphism of sites induced by the identity functor. Set $B_{\text{ét}}T := R\alpha_1 BT_{\text{ét}}$; see [8] §4.1 for more details.

**Lemma 2.2** (cf. [2] Lemma 4.2.4). The space $B_{\text{ét}}T$ is $\mathbb{A}^1$-local.

**Proof.** By adjunction, one has canonical bijections

$$\text{hom}_{\mathcal{H}_k(\text{Sm}_k_{\text{Nis}})}(U, B_{\text{ét}}T) \sim \text{hom}_{\mathcal{H}_\text{ét}(\alpha)}(U, BT_{\text{ét}}).$$

Choosing a fibrant model for $BT_{\text{ét}}$, and using [8] §2 Proposition 3.19 and §4 Proposition 1.16], to check that $B_{\text{ét}}T$ is $\mathbb{A}^1$-local, it suffices to prove that the maps

$$H^i_{\text{ét}}(U, T) \to H^i_{\text{ét}}(U \times \mathbb{A}^1, T)$$

are bijections for $i = 0, 1$. For $i = 0$, this a consequence of étale descent: if $k'/k$ is a separable extension splitting $T$, then it suffices to observe that any morphism $U \times \mathbb{A}^1 \to \mathbb{G}_m^{\times n}$ factors through a morphism $U \to \mathbb{G}_m^{\times n}$. For $i = 1$ one could apply [2] Lemma 4.3.7 and Proposition 4.4.3. For a direct proof, observe that [4] Lemma 2.4, establishes the result for affine $X$ (Grothendieck showed that étale and flat cohomology coincide *Ibid.* p.159). We reduce the case of general $X$ to the affine case by comparing the exact sequences of low degree terms for the Leray spectral sequences associated with an open affine cover $u: U \to X$ and the corresponding cover $u \times id: U \times \mathbb{A}^1 \to X \times \mathbb{A}^1$. \qed
Proof of Proposition [2.7]. After Lemma [2.2] the proof is essentially [7, Lemma 4.5(2)]; here are the details. Start with an $\mathcal{A}^1$-acyclic cofibration $j : \mathcal{A} \to \mathcal{B}$ fitting into a diagram
\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{s_0} & U \\
\downarrow j & & \downarrow \pi \\
\mathcal{B} & \to & X.
\end{array}
\]

Now, since $B_{\text{et}}T$ is $\mathcal{A}^1$-local, the natural maps $[\mathcal{B}, B_{\text{et}}T]_s \to [\mathcal{A}, B_{\text{et}}T]_s$ and $[\mathcal{B}, T]_s \to [\mathcal{A}, T]_s$ are bijections. The pullback of $\pi$ to $\mathcal{A}$ admits a section and is therefore a trivial torsor. By the first bijection just mentioned, it follows that the pullback of $\pi$ to $\mathcal{B}$ is also trivial, and thus also admits a section, which we denote by $s$. The composite morphism $j \circ s$ need not be equal to $s_0$, but if it is not, then there is an element $t_0 \in [\mathcal{A}, T]_s$ such that $t_0 \cdot s = s_0$. By the second bijection mentioned at the beginning of this paragraph, the element $t_0$ determines a unique element $t$ of $[\mathcal{B}, T]_s$. The product $t^{-1} \cdot s$ is a new section of $\pi$ pulled back to $\mathcal{B}$. By construction this new section gives back $s_0$ upon restriction to $\mathcal{A}$ and thus provides the necessary (unique) lift. \hfill \Box

Proof of Theorem [7.7]. We return to the notation of the introduction: $X$ is a smooth proper toric $T$-model, $T_{NS(X)}$ is the associated Neron-Severi torus and $f : U \to X$ is the $T_{NS(X)}$-torsor constructed in [6, Proposition 5.6].

Since $X$ is proper, it follows from, e.g., [5, Lemma 1.4] that $U$ has complement of codimension $\geq 2$ in the affine space in which it sits since the same thing is true upon passing to a separable closure. Since $k$ has characteristic 0 and is thus infinite, it follows that $U$ is even connected by lines. (In fact, [1, Proposition 5.12] gives conditions guaranteeing that this complement has codimension $\geq d$, depending only on the fan of $X_{\text{et}}$.) In any case, we can choose a point $\tilde{x}$ lifting $x$.

By Proposition [2.1] $\pi$ is an $\mathcal{A}^1$-cover and thus an $\mathcal{A}^1$-fibration. Consider the long exact sequence in $\mathcal{A}^1$-homotopy groups of $\pi$, which exists by a formal argument in the theory of model categories (cf. [1, Remark 3.2]). The higher ($i > 1$) homotopy (sheaves of) groups of $B_{\text{et}}T_{NS(X)}$ are trivial, and $\pi^A_1(B_{\text{et}}T_{NS(X)}) = T_{NS(X)}$ (again, see [6, §4 Proposition 1.16]). We then have a long exact sequence of groups (and pointed sets)
\[
\begin{align*}
1 & \longrightarrow \pi^A_1(U, \tilde{x}) \longrightarrow \pi^A_1(X, x) \longrightarrow T_{NS(X)} \\
& \longrightarrow \pi^A_0(U) \longrightarrow \pi^A_0(X) \longrightarrow \pi^A_0(B_{\text{et}}T_{NS(X)}),
\end{align*}
\]

and for each $i > 1$, we have isomorphisms $\pi^A_i(U, \tilde{x}) \simeq \pi^A_i(X, x)$.

For the case $i = 0$, observe that the morphism $X \to B_{\text{et}}T_{NS(X)}$ classifying $f$ induces the morphism $\pi^A_0(X) \to \pi^A_0(B_{\text{et}}T_{NS(X)})$. Using the $\mathcal{A}^1$-weak equivalence $X \to \text{Sing}^A(X)$, there is an induced epimorphism $\pi^A_0(\text{Sing}^A(X)) \to \pi^A_0(X)$ by [6, §2 Corollary 3.22]. Again using the fact that $X$ is proper, we conclude $\pi^A_0(\text{Sing}^A(X))(L)$ is $X(L)/R$.

Since $B_{\text{et}}T_{NS(X)}$ is $\mathcal{A}^1$-local, $\pi^A_0(B_{\text{et}}T) = \mathcal{H}^1_{\text{et}}(T_{NS(X)})$. Taking sections over finitely generated separable extensions $L/k$ determines a morphism of functors (on
that coincides with the “obvious” such morphism gotten by restricting the $T_{NS}(X)$-torsor $f: U \to X$ to $L$-points of $X$. The torus $T_{NS}(X)$ is flasque (see, e.g., [3, Proposition 6]) so [3, §5 Corollaire 1] implies that the restriction map $X(L)/R \to H^1_\et(L, T_{NS}(X))$ is a bijection. It follows that $\pi^A_0(X) \to H^1_\et(T_{NS}(X))$ is an isomorphism on sections over separable finitely generated $L/k$. □

Remark 2.3. The statement in Theorem 1.1 involving $\pi^A_0$ provides an alternate proof of [2, Theorem 2.4.3] in the special case of smooth proper toric models. Furthermore, this statement can be strengthened slightly. Indeed, the multiplication morphism $T \times T \to T$ gives rise to a rational map $X \times X \to X$. Resolving indeterminacy, we get a morphism $X' \to X \times X$ (that is a composite of blow-ups). One can check that this induces a composition on $\pi^A_0(X)(L)$ for any $L/k$ (coinciding with the composition on $R$-equivalence classes). The map of the proposition is in fact a homomorphism of abelian groups. One would like to show that $\pi^A_0(X)$ can be equipped with the structure of a Nisnevich sheaf of abelian groups and that the map $\pi^A_0(X) \to H^1_\et(T_{NS}(X))$ is an isomorphism of sheaves.

References


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