Math. Res. Lett. **18** (2011), *no.* 04, 791–804

LOCAL GAGLIARDO–NIRENBERG ESTIMATES FOR ELLIPTIC SYSTEMS OF VECTOR FIELDS

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Abstract. We extend the global L^1 estimates proved by Bourgain and Brezis and Lanzani and Stein for the de Rham complex on $\mathbb{R}^N$ to the setup of local L^1 estimates for the differential complex associated to an involutive elliptic structure spanned by a family of linearly independent smooth complex vector fields.

1. Introduction

A standard a priori estimate in the classical Hodge theory is, for $1 < p < \infty$,

$$\|\nabla u\|_{L^p} \leq C_p \left(\|du\|_{L^p} + \|d^* u\|_{L^p}\right), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k \mathbb{R}^N), \tag{1.1}$$

where d is the exterior differential, d^* its dual operator and ∇u denotes the componentwise gradient of u. This estimate is known to be false when $p = 1, \infty$. By the Sobolev embedding theorem, (1.1) implies

$$\|u\|_{L^{p^*}} \leq C_p \left(\|du\|_{L^p} + \|d^* u\|_{L^p}\right), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k \mathbb{R}^N), \tag{1.2}$$

with $p^* = pN/(N-p)$. The limiting case as $p \searrow 1$ of (1.2) is

$$\|u\|_{L^{N/(N-1)}} \leq C \left(\|du\|_{L^1} + \|d^* u\|_{L^1}\right), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k \mathbb{R}^N), \tag{1.3}$$

which, however, cannot be obtained by combining the Gagliardo–Nirenberg inequality with (1.1), since $C_p \to \infty$ as $p \searrow 1$. Nevertheless, Bourgain and Brezis [2, 3] and Lanzani and Stein [9] have shown that (1.3) still holds for k-forms, as long as $k \neq 1$, $N-1$ (for $k = 1$ and $N-1$ appropriate substitute a priori estimates are also known). Estimates of this type have been extended in several directions, mainly within the framework of the de Rham complex and constant vector fields [10–12, 17–19], although related inequalities in the setup of nilpotent groups [4] and CR complexes [20] also have been considered quite recently.

Notice that if u is a zero-form, i.e., a function, (1.3) may be written as $\|u\|_{L^{N/(N-1)}} \leq C\|\nabla u\|_{L^1}$ so we may regard (1.3) as a generalization of the Sobolev–Gagliardo–Nirenberg inequality

$$\|u\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u \in C_c^\infty(\mathbb{R}^N), \tag{1.4}$$

where $L_j = \partial_{x_j}$ and $n = N$. In this paper, we address the following question. Suppose that $L_1, \dots, L_n$ is a system of linearly independent vector fields with smooth complex

Received by the editors February 4, 2011.
2010 *Mathematics Subject Classification.* Primary 35J46 46E35; Secondary 35F05 36N10.
Key words and phrases. de Rham complex, L^1 estimates, locally integrable systems.

coefficients defined on an open set $\Omega \subset \mathbb{R}^N$: for which systems estimates like (1.4) are valid, at least locally? The answer is given by

Theorem 1.1. *If the system of vector fields $L_1, \dots, L_n$, $n \geq 2$, is elliptic then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$*

$$\|u\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u \in C_c^\infty(U), \tag{1.5}$$

holds. Conversely, if (1.5) *holds, the system must be elliptic on U.*

We recall that the ellipticity of the system $\{L_1, \dots, L_n\}$ means that, for any *real* 1-form ω (i.e., any section of $T^*(\Omega)$): $\langle \omega, L_j \rangle = 0$, $j = 1, \dots, n$, $\implies$ $\omega = 0$.

When the system $\mathcal{L} = \{L_1, \dots, L_n\}$ is involutive, i.e., each commutator $[L_j, L_k]$, $1 \leq j, k \leq n$, is a linear combination of $L_1, \dots, L_n$, the subbundle of $\mathbb{C}T(\Omega)$ spanned by $\mathcal{L}$ is denoted by $(\Omega, \mathcal{L})$ and called an involutive (or formally integrable) structure. Examples of involutive structures include integrable distributions in the sense of Frobenius, complex structures and CR structures. If the system $\{L_1, \dots, L_n\}$ is, in addition, elliptic, an appropriate version of the Newlander–Nirenberg states that the structure $\mathcal{L}$ is locally integrable (on the subject of locally integrable structures we refer to [1, 16]). In particular, there is a natural complex of differential operators $d_{\mathcal{L}}$ (which is precisely the de Rham complex when $L_j = \partial_{x_j}$, $j = 1, \dots, N$) and it is natural to ask whether estimates analog to (1.3) hold with $d_{\mathcal{L}}$ and $d_{\mathcal{L}}^*$ in the place of d and d^*. Suppose $\mathcal{L} = \{L_1, \dots, L_n\}$ is a locally integrable system on Ω and denote by $E^k(\Omega)$ the space of k-forms, $0 \leq k \leq N$, with complex coefficients, i.e., the smooth sections of the vector bundle $\wedge^k \mathbb{C}T^*(\Omega)$ and let $\mathcal{L}^\perp(\Omega)$ be the subbundle of $\mathbb{C}T^*(\Omega)$ of all $\omega \in E^1(\Omega)$ such that $\langle \omega, L \rangle = 0$ for all sections of $\mathcal{L}$. Denote by $\mathcal{I}$ the ideal generated by $\mathcal{L}^\perp(\Omega)$ in $\bigotimes_{k=0}^N \wedge^k \mathbb{C}T^*(\Omega)$. Hence, if $\omega_1, \dots, \omega_m$, $n + m = N$, is a set of local generators of $\mathcal{L}^\perp(\Omega)$, then $\mathcal{I}^k(\Omega) \doteq \mathcal{I} \cap \wedge^k \mathbb{C}T^*(\Omega)$, $1 \leq k \leq n$ is spanned by the k-forms

$$\omega_j \wedge \omega', \quad j = 1 \dots, m, \quad \omega' \in E^{k-1}(\Omega).$$

Write $\mathfrak{N}^k(\Omega) = \wedge^k \mathbb{C}T^*(\Omega)/\mathcal{I}^k(\Omega)$, $0 \leq k \leq n$, and denote by $\tilde{E}^k(\Omega)$ the space of smooth sections of the vector bundle $\mathfrak{N}^k(\Omega)$. The de Rham complex [5] $d : E^k(\Omega) \to E^{k+1}(\Omega)$, given by the exterior derivative on complex-valued forms, gives rise to a new complex $d_{\mathcal{L}}$ associated to the structure $\mathcal{L}$,

$$d_{\mathcal{L},k} : \tilde{E}^k(\Omega) \longrightarrow \tilde{E}^{k+1}(\Omega), \quad 0 \leq k \leq n.$$

We have

Theorem 1.2. *Assume that the system of vector fields $L_1, \dots, L_n$, $n \geq 2$, is elliptic and involutive and that $0 \leq \ell \leq n$ is neither 1 nor $n - 1$. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$*

$$\|u\|_{L^{N/(N-1)}} \leq C\big(\|d_{\mathcal{L},\ell} u\|_{L^1} + \|d^*_{\mathcal{L},\ell-1} u\|_{L^1}\big), \quad u \in \tilde{E}^\ell_c(U).$$

For the special values $\ell = 1, n - 1$ we have estimates involving the norm of the localizable Hardy space $h^1(\mathbb{R}^N)$ of Goldberg [6].

Theorem 1.3. *Assume that the system of vector fields $L_1, \dots, L_n$, $n \geq 2$, is elliptic and involutive. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$*

$$\|u\|_{L^{N/(N-1)}} \leq C\big(\|d^*_{\mathcal{L},0}u\|_{h^1} + \|d_{\mathcal{L},1}u\|_{L^1}\big), \quad u \in \tilde{E}^1_{\mathrm{c}}(U),$$
$$\|u\|_{L^{N/(N-1)}} \leq C\big(\|d^*_{\mathcal{L},n-2}u\|_{L^1} + \|d_{\mathcal{L},n}u\|_{h^1}\big), \quad u \in \tilde{E}^{n-1}_{\mathrm{c}}(U).$$

The organization of the paper is as follows. In Section 2, we prove the converse part of Theorem 1.1 (Proposition 2.1) and a variation of a lemma of Van Schaftingen (Lemma 2.1) that will be instrumental in the proof of Theorem 2.1. The latter implies the other part of Theorem 1.1. The proof of Theorem 2.1 is given in Section 3 while Section 4 is devoted to the proof of Theorem 4.1, which implies Theorem 1.2. The special cases $\ell = 1$ and $n - 1$ are also dealt with in Theorem 4.2.

2. A Sobolev–Gagliardo–Nirenberg inequality

Consider n complex vector fields $L_1, \dots, L_n$, $n \geq 1$, with smooth coefficients defined on a neighborhood Ω of the origin $0 \in \mathbb{R}^N$, $N \geq 2$, that may be viewed as sections of the vector bundle $\mathbb{C}T(\Omega)$ as well as first-order differential operators. We will always assume that

(a) $L_1, \dots, L_n$ are everywhere linearly independent.

Most of the time, we will also assume that

(b) the system $\{L_1, \dots, L_n\}$ is *elliptic.*

This means that, for any *real* 1-form ω (i.e., any section of $T^*(\Omega)$)

$$\langle \omega, L_j \rangle = 0, \quad j = 1, \dots, n, \qquad \Longrightarrow \omega = 0.$$

This implies that the number n of vector fields must satisfy

$$\frac{N}{2} \leq n \leq N.$$

Alternatively, (b) is equivalent to saying that the second-order operator

$$L^{\mathrm{t}}_1 \bar{L}_1 + \cdots + L^{\mathrm{t}}_n \bar{L}_n$$

is elliptic. Here, $\bar{L}_j$, $j = 1, \dots, n$, denotes the vector field obtained from L_j by conjugating its coefficients and L^{t}_j is the formal transpose of L_j.

We are interested in the following question: characterize the systems of complex vector fields for which there exist local a priori estimates

$$\|u\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u \in C^\infty_{\mathrm{c}}(U), \tag{2.1}$$

for some neighborhood U of the origin and some $C > 0$.

Due to the local nature of the estimates, if (2.1) is valid, similar estimates replacing $N/(N-1)$ by any $p \in [1, N/(N-1)]$ will also hold true. Notice that in the case $n = N$, $L_k = \partial/\partial x_k$, $1 \leq k \leq N$, (2.1) is just the well known Sobolev–Gagliardo–Nirenberg inequality that holds without any restriction on the size of the support of the test functions.

Proposition 2.1. *If* (2.1) *holds the system* $\{L_1, \dots, L_n\}$ *is elliptic.*

Proof. Write $p = N/(N-1)$, $L_j = \sum_{k=1}^N a_{jk}(x)\partial_k$, $\partial_k = \partial/\partial x_k$, choose a bump function $\phi(x) \in C_c^\infty(U)$ and set $\phi_\varepsilon(x) = \phi(x/\varepsilon)$, so $\phi_\varepsilon \in C_c^\infty(U)$ for small $\varepsilon > 0$. Applying (2.1) to ϕ_ε we obtain

$$\|\phi\|_{L^p} \leq C \sum_{j=1}^n \|L_j^\varepsilon \phi\|_{L^1}$$

with $L_j^\varepsilon = \sum_{k=1}^N a_{jk}(\varepsilon x)\partial_k$. Letting $\varepsilon \to 0$ we see that

$$\|\phi\|_{L^p} \leq C \sum_{j=1}^n \|L_j^0 \phi\|_{L^1}. \tag{2.2}$$

If the system with constant coefficients $\{L_1^0, \dots, L_n^0\}$ is not elliptic, we may assume after a linear change of variables that $a_{j1}(0) = 0$, $j = 1, \dots, n$. Taking now $\phi_\varepsilon(x) = \phi_1(x_1/\varepsilon)\phi_2(x')$, $x' = (x_2, \dots, x_N)$ in (2.2) we get, with $\phi = \phi_1\phi_2$

$$\|\phi\|_{L^p} \leq C\varepsilon^{1-1/p} \sum_{j=1}^n \|L_j^0 \phi\|_{L^1} = C\varepsilon^{1/N} \sum_{j=1}^n \|L_j^0 \phi\|_{L^1},$$

which leads to a contradiction as $\varepsilon \searrow 0$. Hence, $\{L_1^0, \dots, L_n^0\}$ is elliptic and therefore the system $\{L_1, \dots, L_n\}$ is elliptic at $x = 0$. The same argument can be applied to a generic point of Ω. $\square$

Theorem 2.1. *Let $L_1, \dots, L_n$, $n \geq 2$, satisfy* (a) *and* (b). *There exists a neighborhood U of the origin and $C > 0$ such that*

$$\|u\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^n \|L_j u\|_{L^1}, \quad u \in C_c^\infty(U). \tag{2.3}$$

Note that the assumption $n \geq 2$ in Theorem 2.1 cannot be dropped. Indeed, (2.3) fails for the elliptic vector field $L = \partial_1 + \mathrm{i}\partial_2$ in any neighborhood of the origin $U \subseteq \mathbb{R}^2$ as can be seen by taking u of the form

$$\psi^\delta(x)(\phi_\varepsilon * E)(x), \quad E(x) = \frac{1}{\pi(x_1 + \mathrm{i}x_2)}, \quad \phi_\varepsilon(x) = \varepsilon^{-2}\phi(x/\varepsilon), \quad \psi^\delta(x) = \psi(x/\delta),$$

where $\phi(x)$ and $\psi(x)$ are bump functions. Recalling that E is a fundamental solution of the Cauchy–Riemann operator and letting $\varepsilon \searrow 0$, it is easy to blow up the estimate $\|u\|_{L^2} \leq C\|Lu\|_{L^1}$.

Since the vector fields

$$L_j = \sum_{k=1}^N a_{jk}(x)\frac{\partial}{\partial x_k}, \quad j = 1, \dots, n$$

are linearly independent, after shrinking Ω and relabeling of the indices we may assume that the matrix $(a_{jk})_{1\leq j,k\leq n}$ is invertible in a neighborhood of $\overline{\Omega}$. Let $(b_{jk})_{1\leq j,k\leq n}$ be the inverse matrix and set

$$L_j^\# = \sum_{k=1}^N b_{jk}(x)L_k, \quad j = 1, \dots, n.$$

As the coefficients b_{jk} are bounded in Ω it will be enough to prove (2.3) with the vectors fields L_j replaced by the vectors fields $L_j^{\#}$. Furthermore, we have

$$L_j^{\#} = \frac{\partial}{\partial x_j} + \sum_{k=1}^{m} c_{jk}(x)\frac{\partial}{\partial x_{n+k}}, \quad j = 1, \dots, n,$$

where the coefficients c_{jk} are smooth and $m = N - n$. Moreover, by modifying the functions $c_{jk}(x)$ outside a neighborhood of $\overline{\Omega}$ and extending them to $\mathbb{R}^N$ we may assume that $L_1^{\#}, \dots, L_n^{\#}$ are globally defined on $\mathbb{R}^N$ and that the functions $c_{jk}(x)$ have bounded derivatives of all orders. In other words, we may have assumed from the start that the vector fields of the system $\{L_1, \dots, L_n\}$ have the form

$$L_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^{m} c_{jk}(x)\frac{\partial}{\partial x_{n+k}}, \quad j = 1, \dots, n, \tag{a$'$}$$

with smooth coefficients globally defined on $\mathbb{R}^N$ that possess bounded derivatives of all orders. Similarly, hypothesis (b) could have been replaced by the uniform ellipticity of

$$\Delta_L \doteq L_1^{\mathrm{t}}\bar{L}_1 + \cdots + L_n^t\bar{L}_n,$$

where L_j^{t} is the formal transpose of L_j and $\bar{L}_j$ is its conjugate vector field, i.e., for some constant $c > 0$ and all $x, \xi \in \mathbb{R}^N$

$$\sum_{j=1}^{n}\left|\xi_j + \sum_{k=1}^{m} c_{jk}\xi_{n+k}\right|^2 \geq c|\xi|^2. \tag{b$'$}$$

From now on, we will work in this global setup. The following lemma, that does not require ellipticity, is one of the ingredients in the proof of Theorem 2.1.

Lemma 2.1. *Assume that $n \geq 2$ and the system $\{L_1, \dots, L_n\}$ satisfies* (a$'$). *For any $u \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ write $f_j = L_j u$. Then, there exist $C > 0$ such that for any $u, \phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ and $1 \leq j \leq n$*

$$\left|\int f_j(x)\phi(x)\,dx\right| \leq C\left(\sum_{j=1}^{n}\|f_j\|_{L^1} + \|u\|_{L^1}\right)\|\phi\|_{W^{1,N}}. \tag{2.4}$$

If, in addition, the vector fields $L_1, \dots, L_n$ have constant coefficients then

$$\left|\int f_j(x)\phi(x)\,dx\right| \leq C\left(\sum_{j=1}^{n}\|f_j\|_{L^1}\right)\|\nabla\phi\|_{L^N}. \tag{2.4$'$}$$

Note that taking the sup on the right-hand side (2.4) with ϕ in the unit ball of $W^{1,N}$ we obtain the equivalent estimate

$$\sum_{j=1}^{n}\|f_j\|_{W^{-1,N/(N-1)}} \leq C\left(\sum_{j=1}^{n}\|f_j\|_{L^1} + \|u\|_{L^1}\right). \tag{2.5}$$

The proof of Lemma 2.1 adapts the arguments of Van Schaftingen in [17], in particular, we recall a useful decomposition of a test function $\phi(x) \in C_{\mathrm{c}}^\infty(\mathbb{R}^\mu)$, depending on a parameter $\varepsilon > 0$. Choose a bump function $\eta(x) \in C_{\mathrm{c}}^\infty(\mathbb{R}^\mu)$ with $\int \eta(x)\,dx = 1$, write $\eta_\varepsilon(x) = \varepsilon^{-\mu}\eta(x/\varepsilon)$ and set $\phi_1 = \phi - \eta_\varepsilon * \phi$, $\phi_2 = \eta_\varepsilon * \phi$. Then, for any $p > \mu$ there exists a positive constant $C = C(\mu, p)$ such that the decomposition $\phi = \phi_1 + \phi_2$ satisfies for any $\varepsilon > 0$ and $\gamma = 1 - \mu/p$

$$\|\phi_1\|_{L^\infty} \leq C\varepsilon^\gamma \|\nabla\phi\|_{L^p}, \tag{2.6}$$

$$\|\phi_2\|_{L^\infty} \leq C\varepsilon^{\gamma-1} \|\phi\|_{L^p}, \tag{2.7}$$

$$\|\nabla\phi_2\|_{L^\infty} \leq C\varepsilon^{\gamma-1} \|\nabla\phi\|_{L^p}\,. \tag{2.8}$$

We may now prove Lemma 2.1.

Proof. For $j = 2$, the left-hand side of (2.4) may be written as

$$\int_{\mathbb{R}} J(x_1)\, dx_1, \quad J(x_1) = \int_{\mathbb{R}^{N-1}} f_2^{x_1}(x')\phi^{x_1}(x')\, dx',$$

with $x' = (x_2, \ldots, x_N)$, $f_2^{x_1}(x') = f_2(x_1, x')$ and $\phi^{x_1}(x') = \phi(x_1, x')$. Using the above decomposition for $\phi^{x_1} \in C_c^\infty(R^{N-1})$ we write $J(x_1) = J_1(x_1) + J_2(x_1)$ with

$$J_k(x_1) = \int_{\mathbb{R}^{N-1}} f_2^{x_1}(x')\phi_k^{x_1}(x')\, dx', \quad k = 1, 2.$$

Then

$$|J_1(x_1)| \leq \|f_2^{x_1}\|_{L^1(\mathbb{R}^{N-1})} \|\phi_1^{x_1}\|_{L^\infty} \tag{2.9}$$

and

$$J_2(x_1) = \int_{-\infty}^{x_1} \int_{\mathbb{R}^{N-1}} \frac{\partial f_2}{\partial s}(s, x')\phi_2^{x_1}(x')\, dx' ds. \tag{2.10}$$

Next we write $L_1 f_2 = L_2 f_1 + [L_1, L_2]u$ as

$$\frac{\partial f_2}{\partial x_1} = -\sum_{k=n+1}^{N} a_{1k} \frac{\partial f_2}{\partial x_k} + [L_1, L_2]u + L_2 f_1. \tag{2.11}$$

Note that, (a$'$) shows that no term on the right-hand side of (2.11) involves derivatives with respect to x_1. Plugging (2.11) in the integrand of (2.10) and integrating by parts in the integral over $\mathbb{R}^{N-1}$ to switch the derivatives of f_1, f_2 and u over $\phi_2^{x_1}$ we obtain

$$|J_2(x_1)| \leq C(\|f\|_{L^1} + \|u\|_{L^1})(\|\nabla\Phi_2^{x_1}\|_{L^\infty} + \|\Phi_2^{x_1}\|_{L^\infty}) \tag{2.12}$$

with the notation $\|f\|_{L^1} = \sum_{j=1}^{n} \|f_j\|_{L^1}$. To estimate $\|\nabla\Phi_2^{x_1}\|_{L^\infty} + \|\Phi_2^{x_1}\|_{L^\infty}$ we will apply estimates (2.7) and (2.8) with $p = N$, $\mu = N - 1$, $\gamma = 1/N$, with an appropriate choice of $\varepsilon = \varepsilon(x_1)$. If $J(x_1) \neq 0$ note that $\|f_2^{x_1}\|_{L^1} > 0$. In this case, we set

$$\varepsilon(x_1) = \frac{b}{a(x_1)}, \quad a(x_1) = \|f_2^{x_1}\|_{L^1}, \quad b = \|f\|_{L^1} + \|u\|_{L^1}.$$

If follows from (2.6)–(2.9) and (2.12) that

$$\begin{aligned}
|J(x_1)| &\leq C(a(x_1)\|\phi_1^{x_1}\|_{L^\infty} + b\, \|\nabla\Phi_2^{x_1}\|_{L^\infty} + b\, \|\Phi_2^{x_1}\|_{L^\infty}) \\
&\leq C\big(a\, \varepsilon^{1/N} \|\phi^{x_1}\|_{L^N} + b\, \varepsilon^{(1-N)/N} (\|\phi^{x_1}\|_{L^N} + \|\nabla\phi^{x_1}\|_{L^N})\big) \\
&\leq C\, (\|f\|_{L^1} + \|u\|_{L^1})^{1/N} \|f_2^{x_1}\|_{L^1}^{(N-1)/N} \big(\|\phi^{x_1}\|_{L^N} + \|\nabla\phi^{x_1}\|_{L^N}\big).
\end{aligned}$$

If $J(x_1) = 0$ the inequality is trivially true. Integrating on $\mathbb{R}$ the latter estimate and using Hölder inequality with exponents N and $N/(N-1)$ we obtain

$$\int_{\mathbb{R}} |J(x_1)| \leq C\, b^{1/N} \|f_2\|_{L^1}^{1-1/N} \|\phi\|_{W^{1,N}} \leq C(2\|f\|_{L^1} + \|u\|_{L^1})\|\phi\|_{W^{1,N}}.$$

This proves (2.4) for $j = 2$ and a similar argument can be given for any $1 \leq j \leq n$.

When the L_j's have constant coefficients we have $[L_j, L_k] = 0$ and $L_j^t = -L_j$, so the same calculations give the better estimate (2.4′). □

3. Proof of Theorem 2.1

Proof. To estimate the right-hand side of (2.3) we must look at

$$\langle u, \phi\rangle = \int u(x)\phi(x)\, dx$$

with $\phi \in C_c^\infty(U)$ and $\|\phi\|_{L^N} \leq 1$. Here U is a ball $B(0, \rho)$ centered at the origin with small radius $\rho < 1/2$ to be chosen later. We assume that the vector fields $L_1, \ldots, L_n$ satisfy (a′) and (b′), in particular, the second-order partial differential operator

$$\Delta_L(x, D) = L_1^{\mathrm{t}} \bar{L}_1 + \cdots + L_n^{\mathrm{t}} \bar{L}_n,$$

may be regarded as an elliptic pseudo-differential operator with symbol in the Hörmander class $S^2 = S^2_{1,0}(\mathbb{R}^N)$ (since we will only work with symbols of type (1,0), the type will be omitted in the notation; on the subject of pseudo-differential operators we refer, for instance, to [8, Chapter 3, 15]). Then, there exist symbols $q(x, \xi) \in S^{-2}$ and $r(x, \xi) \in S^{-\infty}$, such that

$$\phi = \Delta_L(x, D)q(x, D)\phi + r(x, D)\phi, \quad \phi \in C_c^\infty(R^N).$$

Then

$$\begin{aligned}
\langle u, \phi\rangle &= \langle u, \Delta_L(x, D)q(x, D)\phi\rangle + \langle u, r(x, D)\phi\rangle \\
&= \sum_{j=1}^{N} \langle L_j u, \bar{L}_j(x, D)q(x, D)\phi\rangle + \langle u, r(x, D)\phi\rangle \\
&= \sum_{j=1}^{N} \langle L_j u, \bar{L}_j(x, D)\chi(x)q(x, D)\phi\rangle + \langle u, r(x, D)\phi\rangle, \qquad (3.1)
\end{aligned}$$

where $\chi(x) \in C_c^\infty(U^*)$ is identically equal to 1 on a neighborhood of $\bar{U}$ and $U^* = B(0, 1)$. Set $\psi_j = \bar{L}_j(x, D)\chi(x)q(x, D)\phi$, $f_j = L_j u$ and apply (2.4) to obtain

$$\begin{aligned}
|\langle f_j, \psi_j\rangle| &\leq C\left(\sum_{j=1}^{n} \|f_j\|_{L^1} + \|u\|_{L^1}\right)\left(\|\nabla\psi_j\|_{L^N} + \|\psi_j\|_{L^N}\right) \\
&\leq C\left(\sum_{j=1}^{n} \|f_j\|_{L^1} + \|u\|_{L^1}\right)\|\nabla\psi_j\|_{L^N} \\
&\leq C\left(\sum_{j=1}^{n} \|f_j\|_{L^1} + \|u\|_{L^1}\right)\|\phi\|_{L^N}. \qquad (3.2)
\end{aligned}$$

Here we have used that $\psi_j \in C_c^\infty(U^*)$ and the fact that the peudodifferential operator $\nabla \bar{L}_j(x,D)\chi(x)q(x,D)$ of order zero is bounded in $L^N(\mathbb{R}^N)$. Furthermore, since $r(x,\xi) \in S^{-\infty}$ we may write $r(x,D)\phi = \int k(x,y)\phi(y)\,dy$ with a continuous kernel $k(x,y)$ that decreases rapidly as $y \to \infty$ uniformly for $x \in U$. Hence, for $x \in U$,

$$|r(x,D)\phi| \leq \sup_{x\in U} \left(\int |k(x,y)|^{N/(N-1)}\,dy \right)^{(N-1)/N} \|\phi\|_{L^N} \leq C\|\phi\|_{L^N}$$

and it follows that

$$|\langle u, r(x,D)\phi\rangle| \leq C\|u\|_{L^1}\|\phi\|_{L^N}. \tag{3.3}$$

Thus, (3.1)–(3.3) give

$$|\langle u,\phi\rangle| \leq C\left(\sum_{j=1}^n \|f_j\|_{L^1} + \|u\|_{L^1}\right)\|\phi\|_{L^N}, \quad \phi \in C_c^\infty(\mathbb{R}^N),$$

which implies

$$\|u\|_{L^{N/(N-1)}} \leq C\sum_{j=1}^n \|f_j\|_{L^1} + C\|u\|_{L^1}, \quad u \in C_c^\infty(U).$$

Since $\|u\|_{L^1} \leq C\rho^{1/N}\|u\|_{L^{N/(N-1)}}$, the term $C\|u\|_{L^1}$ may be absorbed for small ρ and this proves (2.3). □

Since $L_j = -L_j^t + c_j(x)$, $c_j \in C^\infty(\mathbb{R}^N)$, $j = 1,\dots,n$, we easily get from (2.3) after shrinking U if necessary

$$\|u\|_{L^{N/(N-1)}} \leq C\sum_{j=1}^n \|L_j^t u\|_{L^1}, \quad u \in C_c^\infty(U). \tag{3.4}$$

A standard duality consequence of estimate (3.4) is the following local solvability result

Corollary 3.1. *Let $L_1,\dots,L_n$, $n \geq 2$ satisfy* (a) *and* (b). *There exists a neighborhood U of the origin such that for every $f \in L^N(U)$, the underdetermined equation*

$$L_1u_1 + \cdots + L_nu_n = f$$

can be solved in U with $(u_1,\dots,u_n) \in L^\infty(U)^N$.

Note that the result is false in general when $n = 1$.

4. Involutive systems

Let $L_1,\dots,L_n$ be defined on a neighborhood Ω of the origin $0 \in \mathbb{R}^N$, $N \geq 2$, and assume they are linearly independent. The system $\{L_1,\dots,L_n\}$ is said to be involutive if the vector fields L_j, $j = 1,\dots,n$, satisfy the Frobenius condition

(c) $[L_j, L_k] = \sum_{\ell=1}^n c_{jk\ell}(x)L_\ell$, $1 \leq j,k, \leq n$ for some complex–valued functions $c_{jk\ell}(x) \in C^\infty(\Omega)$.

Note that this is a property of the subbundle $\mathcal{L} \subset \mathbb{C}T(\Omega)$ generated by $L_1, \ldots, L_n$, i.e., every other set of generators of $\mathcal{L}$ will satisfy (c). After a local change of generators and extension to $\mathbb{R}^N$ preserving involutivity, we may assume that the vector fields L_j are of the form

$$(\mathrm{a}') \qquad L_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^{m} c_{jk}(x) \frac{\partial}{\partial x_{n+k}}, \quad j = 1, \ldots, n,$$

with globally defined smooth coefficients possessing bounded derivatives of all orders. The special form of $L_1, \ldots, L_n$ shows that (c) reduces to

$$(\mathrm{c}') \qquad [L_j, L_k] = 0, \quad 1 \le j, k, \le n.$$

Now the proof of Lemma 2.1 gives a strengthened form of (2.4), namely

$$(4.1) \qquad \left| \int L_j u(x) \phi(x)\, dx \right| \le C \|\phi\|_{W^{1,N}} \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u, \phi \in C_c^\infty(\mathbb{R}^N),$$

since the term $\|u\|_{L^1}$ in (2.4) was due to the presence of the commutators $[L_j, L_k]$. We then have

$$(4.2) \qquad \sum_{j=1}^{n} \|L_j u\|_{W^{-1,N/(N-1)}} \le C \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u \in C_c^\infty(\mathbb{R}^N).$$

We now consider elliptic involutive systems.

4.1. Elliptic involutive systems. Let $L_1, \ldots, L_n$ be a set of linearly independent smooth complex vector fields defined on a neighborhood $\Omega \subset \mathbb{R}^N$ of the origin and assume that the system $\{L_1, \ldots, L_n\}$ is elliptic and involutive, so they are generators of an involutive subbundle $\mathcal{L}$ of $\mathbb{C}T(\Omega)$. Denote by $E^k(\Omega)$ the space of k-forms, $0 \le k \le N$, with complex coefficients, i.e., the smooth sections of the vector bundle $\wedge^k \mathbb{C}T^*(\Omega)$ and let $\mathcal{L}^\perp(\Omega)$ be the subbundle of $\mathbb{C}T^*(\Omega)$ of all $\omega \in E^1(\Omega)$ such that $\langle \omega, L \rangle = 0$ for all sections of $\mathcal{L}$. Denote by $\mathcal{I}$ the ideal generated by $\mathcal{L}^\perp(\Omega)$ in $\bigotimes_{k=0}^N \wedge^k \mathbb{C}T^*(\Omega)$. Hence, if $\omega_1, \ldots, \omega_m$, $n + m = N$, is a set of local generators of $\mathcal{L}^\perp(\Omega)$, then $\mathcal{I}^k(\Omega) \doteq \mathcal{I} \cap \wedge^k \mathbb{C}T^*(\Omega)$, $1 \le k \le n$ is spanned by the k-forms

$$\omega_j \wedge \omega', \quad j = 1 \ldots, m, \quad \omega' \in E^{k-1}(\Omega).$$

Write $\mathfrak{N}^k(\Omega) = \wedge^k \mathbb{C}T^*(\Omega)/\mathcal{I}^k(\Omega)$, $0 \le k \le n$, and denote by $\tilde{E}^k(\Omega)$ the space of smooth sections of the vector bundle $\mathfrak{N}^k(\Omega)$. The de Rham complex [5] $d : E^k(\Omega) \to E^{k+1}(\Omega)$, given by the exterior derivative on complex-valued forms, gives rise to a new complex $d_{\mathcal{L}}$ associated to the structure $\mathcal{L}$,

$$d_{\mathcal{L},k} : \tilde{E}^k(\Omega) \longrightarrow \tilde{E}^{k+1}(\Omega), \quad 0 \le k \le n,$$

defined as follows: if $u \in E^k(\Omega)$ then $d_{\mathcal{L},k}(u + \mathcal{I}^k) \doteq d_k u + \mathcal{I}^{k+1}$. This is well defined because the involutivity of $\mathcal{L}$ implies that $d_k \mathcal{I}^k \subset \mathcal{I}^{k+1}$. In particular, we have the basic complex property $d_{\mathcal{L},k+1} d_{\mathcal{L},k} = 0$.

By an elegant combination due to Treves of the Newlander–Nirenberg theorem [13] and the Frobenius theorem on real integrable distributions (see, e.g., [1, Theorem I.12.1; [16]]), there exist local coordinates

$$x_1, \ldots, x_r, y_1, \ldots y_r, t_1 \ldots, t_s, \quad r + s = n,$$

defined on a neighborhood Ω' of the origin such that $\mathcal{L}$ is generated over Ω' by the n vector fields

$$D_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, r,$$
$$T_k = \frac{\partial}{\partial t_k}, \quad k = 1, \dots, s.$$

In this case, $\mathcal{L}^\perp$ is generated over Ω' by $d\bar{z}_j \doteq dx_j + \mathrm{i}dy_j$, $1 \le j \le r$, and dt_k, $1 \le k \le s$. Then $\tilde{E}^k(\Omega')$ may be identified with the subspace of $E^k(\Omega')$ generated by the monomials

$$dt_{k_1} \wedge \cdots \wedge dt_{k_\ell} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_\mu}, \quad k_p \in \{1, \dots, s\},\ j_q \in \{1, \dots, r\},\ \ell + \mu = k$$

and for $u \in \tilde{E}^1(\Omega') = C^\infty(\Omega')$ we have

$$d_{\mathcal{L},0} u = \sum_{k=1}^{s} T_k u\, dt_k + \sum_{j=1}^{r} D_j u\, d\bar{z}_j.$$

Note that when $r = 0$, the complex $d_{\mathcal{L}}$ is just the de Rham complex while when $s = 0$ it is the Dolbeault complex [7]; this justifies the notation $d_{\mathcal{L}} = d + \bar{\partial}$ in the general case. In other words, in appropriate local coordinates, the elliptic complex $d_{\mathcal{L}}$ has constant coefficients and

$$\Delta_L = \sum_{k=1}^{s} T_k^2 + \sum_{j=1}^{r} \bar{D}_j D_j$$

is a slight variation of the Laplace operator in $\mathbb{R}^N = \mathbb{R}^{s+2r}$. Applying (2.4′) to the system $\{T_1, \dots, T_s, D_1, \dots, D_r\}$ gives, for $N = s + 2r \ge 2$ and setting $f_k = T_k u$, $g_j = D_j u$

$$\left|\int f_k(x)\phi(x)\,dx\right| \le C\left(\sum_{k=1}^{s} \|f_j\|_{L^1} + \sum_{j=1}^{r} \|g_j\|_{L^1}\right)\|\nabla\phi\|_{L^N}, \tag{4.3}$$

$$\left|\int g_j(x)\phi(x)\,dx\right| \le C\left(\sum_{k=1}^{s} \|f_k\|_{L^1} + \sum_{j=1}^{r} \|g_j\|_{L^1}\right)\|\nabla\phi\|_{L^N} \tag{4.3a}$$

for any $u, \phi \in C_c^\infty(\mathbb{R}^N)$, $1 \le k \le s$, $1 \le j \le r$. Consider the norm

$$\|u\|_{\overset{\circ}{W}{}^{-1,N/(N-1)}} \doteq \sup_{\|\nabla\phi\|_{L^N} \le 1} |\langle u, \phi\rangle| \simeq \|Iu\|_{L^{N/(N-1)}},\ u \in C_c^\infty(\mathbb{R}^N),$$

where I denotes the Riesz operator

$$Iu(x) = \int \frac{u(y)}{|x-y|^{n-1}}\,dy.$$

Then (4.3) and (4.3a) could be written as

$$\|d_{\mathcal{L},0}u\|_{\overset{\circ}{W}{}^{-1,N/(N-1)}} \le C\|d_{\mathcal{L},0}u\|_{L^1}, \quad u \in C_c^\infty(\mathbb{R}^N). \tag{4.4}$$

More generally, for k-forms we have

$$\|d_{\mathcal{L},k}u\|_{\overset{\circ}{W}{}^{-1,N/(N-1)}} \le C\|d_{\mathcal{L},k}u\|_{L^1}, \quad u \in \tilde{E}_c^k(\mathbb{R}^N),\ 0 \le k \le n-2, \tag{4.5}$$

where $\tilde{E}^k_{\mathrm{c}}(\mathbb{R}^N)$ denotes the compactly supported elements of $\tilde{E}^k(\mathbb{R}^N)$. Similarly, if $d^*_{\mathcal{L}}$ denotes the dual complex

$$d^*_{\mathcal{L},k} : \tilde{E}^{k+1}(\mathbb{R}^N) \longrightarrow \tilde{E}^k(\mathbb{R}^N), \quad 0 \le k \le n-1,$$

determined by

$$\int d_{\mathcal{L},k} u \cdot \bar{v}\, dx = \int u \cdot \overline{d^*_{\mathcal{L},k} v}\, dx, \quad u \in \tilde{E}^k_{\mathrm{c}}(\mathbb{R}^N),\ v \in \tilde{E}^{k+1}_{\mathrm{c}}(\mathbb{R}^N),$$

where the dot indicates the standard pairing on forms of the same degree, we have

$$\|d^*_{\mathcal{L},k} u\|_{\overset{\circ}{W}{}^{-1,N/(N-1)}} \le C \|d^*_{\mathcal{L},k} u\|_{L^1},\ u \in \tilde{E}^{k+1}_{\mathrm{c}}(\mathbb{R}^N),\ 1 \le k \le n-1. \tag{4.6}$$

Estimates (4.5) and (4.6) follow from an application of the next lemma which extends estimate (2.4′) (the proof is similar and will be omitted).

Lemma 4.1. *Let $L_1, \dots, L_n$, $n \ge 2$, be linearly independent vector fields with complex constant coefficients defined on $\mathbb{R}^N$. Consider test functions $f_1, \dots, f_\nu \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ and suppose that for any $1 \le k \le \nu$ there exists $1 \le j \le n$ such that*

$$L_j f_k = \sum_{j' \ne j} \sum_{\ell=1}^{\nu} c_{kj'\ell} L_{j'} f_\ell,$$

with $c_{kj'\ell} \in \mathbb{C}$. Then

$$\sum_{k=1}^{\nu} \|f_k\|_{\overset{\circ}{W}{}^{-1,N/(N-1)}} \le C \sum_{k=1}^{\nu} \|f_k\|_{L^1}. \tag{4.7}$$

Observe that

$$\Delta_k \doteq d^*_{\mathcal{L},k} d_{\mathcal{L},k} + d_{\mathcal{L},k-1} d^*_{\mathcal{L},k-1}, \quad 0 \le k \le n$$

is just a tiny variation of the standard Laplace–Beltrami operator on k-forms defined on $\mathbb{R}^N$. Note also that for $k = 0$ and $k = n$ the operators $d_{\mathcal{L},n}$, $d^*_{\mathcal{L},-1}$ should be understood as zero. If $\phi \in \tilde{E}^k_c(\mathbb{R}^N)$ we have $\phi = \Delta_k G_k \phi$, where G_k is the canonic fundamental solution of Δ_k (cf. [1, p. 339] for explicit formulas). Assume that $2 \le \ell \le n-2$. Thus, the arguments of Theorem 2.1 with (3.1) simplified to

$$\int u \cdot \overline{\phi} = \int u \cdot \overline{\Delta_\ell G_\ell \phi} = \int d_{\mathcal{L},\ell} u \cdot \overline{d_{\mathcal{L},\ell} G_\ell \phi} + \int d^*_{\mathcal{L},\ell-1} u \cdot \overline{d^*_{\mathcal{L},\ell-1} G_\ell \phi}$$

yield, taking account of (4.5)–(4.7) and invoking the Calderón–Zygmund theory,

$$\begin{aligned} \left| \int u \cdot \overline{\phi} \right| &\le \|d_{\mathcal{L},\ell} u\|_{L^1} \|\nabla d_{\mathcal{L},\ell} G_\ell \phi\|_{L^N} + \|d^*_{\mathcal{L},\ell-1} u\|_{L^1} \|\nabla d^*_{\mathcal{L},\ell-1} G_\ell \phi\|_{L^N} \\ &\le C\big(\|d_{\mathcal{L},\ell} u\|_{L^1} + \|d^*_{\mathcal{L},\ell-1} u\|_{L^1}\big) \|\phi\|_{L^N}. \end{aligned}$$

This implies by duality that

$$\|u\|_{L^{N/(N-1)}} \le C\big(\|d_{\mathcal{L},\ell} u\|_{L^1} + \|d^*_{\mathcal{L},\ell-1} u\|_{L^1}\big), \quad u \in \tilde{E}^\ell_{\mathrm{c}}(\mathbb{R}^N). \tag{4.8}$$

Note that the argument breaks down for $\ell = 1, n-1$ because (4.6) fails for $k = 1$ and (4.5) fails for $k = n-1$. Nevertheless, (4.8) holds for $\ell = 0$ and for $\ell = n$ by an application of (2.4′). This proves

Theorem 4.1. *Let $u \in \tilde{E}^\ell_{\mathrm{c}}(\mathbb{R}^N)$ and assume that $0 \le \ell \le n$ is neither 1 nor $n-1$. Then*

$$\|u\|_{L^{N/(N-1)}} \le C\big(\|(d+\bar\partial)u\|_{L^1} + \|(d+\bar\partial)^*u\|_{L^1}\big), \quad u \in \tilde{E}^\ell_{\mathrm{c}}(\mathbb{R}^N). \tag{4.9}$$

For $\ell = 1$ and $n-1$ the a priori estimates (4.9) fail. There are two alternatives, either we replace the L^1 norm on the right-hand side by a stronger norm or we may impose an additional differential condition on u. Suppose that $\ell = 1$, $u \in \tilde{E}^1_{\mathrm{c}}(\mathbb{R}^N)$ and

$$(d+\bar\partial)^*u = 0.$$

Then, for any $\phi \in \tilde{E}^1_{\mathrm{c}}(\mathbb{R}^N)$,

$$\int u \cdot \overline{\phi} = \int u \cdot \overline{\Delta_1 G_1 \phi} = \int (d+\bar\partial)u \cdot \overline{(d+\bar\partial)G_1\phi}$$

so

$$\left|\int u \cdot \overline{\phi}\right| \le \|(d+\bar\partial)u\|_{L^1}\|\nabla(d+\bar\partial)G_1\phi\|_{L^N} \le C\|(d+\bar\partial)u\|_{L^1}\|\phi\|_{L^N},$$

which implies for

$$\|u\|_{L^{N/(N-1)}} \le C\|(d+\bar\partial)u\|_{L^1} \quad \text{if } (d+\bar\partial)^*u = 0,\ u \in \tilde{E}^1_{\mathrm{c}}(\mathbb{R}^N). \tag{4.9$'$}$$

Similarly,

$$\|u\|_{L^{N/(N-1)}} \le C\|(d+\bar\partial)^*u\|_{L^1} \quad \text{if } (d+\bar\partial)u = 0,\ u \in \tilde{E}^{n-1}_{\mathrm{c}}(\mathbb{R}^N). \tag{4.9$''$}$$

Next we discuss the other possible strategy, namely, to replace at the appropriate place the L^1 norm by a stronger one, rather that imposing additional conditions on u. In the case of the de Rahm complex, Lanzani and Stein showed that when $\ell = 1$, an estimate analogous to (1.3) holds as soon as $\|d^*u\|_{H^1}$ in replaced by $\|d^*u\|_{L^1}$. Similarly, for $\ell = N-1$, one has to replace $\|du\|_{L^1}$ by $\|du\|_{H^1}$. Here, $H^1(\mathbb{R}^N)$ is the Hardy real space (on the subject of Hardy spaces we refer to [14]). Since we are dealing with local estimates, and we would like them to be invariant under coordinate changes, the natural replacement is Goldberg's localizable Hardy space $h^1(\mathbb{R}^N)$ introduced in [6]. We have

Theorem 4.2. *If the diameter of Ω is sufficiently small, the following local estimates hold:*

$$\|u\|_{L^{N/(N-1)}} \le C\big(\|d^*_{\mathcal{L},0}u\|_{h^1} + \|d_{\mathcal{L},1}u\|_{L^1}\big), \quad u \in \tilde{E}^1_{\mathrm{c}}(\Omega), \tag{4.10}$$

$$\|u\|_{L^{N/(N-1)}} \le C\big(\|d^*_{\mathcal{L},n-2}u\|_{L^1} + \|d_{\mathcal{L},n}u\|_{h^1}\big), \quad u \in \tilde{E}^{n-1}_{\mathrm{c}}(\Omega). \tag{4.11}$$

Proof. Here we proceed as in Theorem 2.1. Let $\ell = 1$ and consider the identity

$$\phi = \Delta_1 q(x,D)\phi + r(x,D)\phi, \quad \phi \in \tilde{E}^1_{\mathrm{c}}(\Omega),$$

for $q(x,\xi) \in S^{-2}$ and $r(x,\xi) \in S^{-\infty}$. Then

$$\begin{aligned}\langle u, \phi\rangle &= \langle u, \Delta_1(x,D)q(x,D)\phi\rangle + \langle u, r(x,D)\phi\rangle \\ &= \langle d^*_{\mathcal{L},0}u, d^*_{\mathcal{L},0}q(x,D)\phi\rangle + \langle d_{\mathcal{L},1}u, d_{\mathcal{L},1}q(x,D)\phi\rangle + \langle u, r(x,D)\phi\rangle.\end{aligned}$$

The second and third terms of the right-hand side may be majorized by $\big(\|d_{\mathcal{L},1}u\|_{L^1} + \|u\|_{L^1}\big)\|\phi\|_{L^N}$ as in the proof of estimates (3.2) and (3.3). This approach cannot be used to handle the first term so we write instead

$$\begin{aligned}(d^*_{\mathcal{L},0}u, d^*_{\mathcal{L},0}q(x,D)\phi) &= (q(x,D)d^*_{\mathcal{L},0}u, d^*_{\mathcal{L},0}\phi)\\ &= (j(x,D)q(x,D)d^*_{\mathcal{L},0}u, j^{-1}(x,D)d^*_{\mathcal{L},0}\phi)\end{aligned}$$

with $j(x,\xi) = (1+|\xi|^2)^{1/2}$, $j^{-1}(x,\xi) = (1+|\xi|^2)^{-1/2}$. Thus

$$\begin{aligned}&\big|(d^*_{\mathcal{L},0}u, d^*_{\mathcal{L},0}q(x,D)\phi)\big|\\ &\qquad \le C\big\|j(x,D)q(x,D)d^*_{\mathcal{L},0}u\big\|_{L^{\frac{N}{N-1}}}\big\|j^{-1}(x,D)d^*_{\mathcal{L},0}\phi\big\|_{L^N}.\end{aligned}$$

Since $j^{-1}(x,D)d^*_{\mathcal{L},0} \in S^0$ we have $\big\|j^{-1}(x,D)d^*_{\mathcal{L}}\phi\big\|_{L^N} \le C\,\|\phi\|_{L^N}$ and by Gagliardo–Nirenberg's inequality

$$\big\|j(x,D)q(x,D)d^*_{\mathcal{L},0}u\big\|_{L^{\frac{N}{N-1}}} \le C\,\big\|\nabla j(x,D)q(x,D)d^*_{\mathcal{L},0}u\big\|_{L^1}.$$

Since $\nabla j(x,D)q(x,D) \in S^0$, it takes continuously $L^1 \subset h^1$ into h^1 [6] so

$$\big\|\nabla j(x,D)q(x,D)d^*_{\mathcal{L},0}u\big\|_{L^1} \le C\,\big\|d^*_{\mathcal{L},0}u\big\|_{h^1}.$$

Hence,

$$\big|(d^*_{\mathcal{L},0}u, d^*_{\mathcal{L},0}q(x,D)\phi)\big| \le \big\|d^*_{\mathcal{L},0}u\big\|_{h^1}\,\|\phi\|_{L^N}.$$

This shows that $|\langle u,\phi\rangle| \le C\big(\|d^*_{\mathcal{L},0}u\|_{h^1} + \|d_{\mathcal{L},1}u\|_{L^1} + \|u\|_{L^1}\big)\|\phi\|_{L^N}$ and proves (4.10) after taking the sup on $\{\|\phi\|_{L^N} \le 1\}$ and shrinking Ω if necessary. The proof of (4.11) is similar. $\square$

Acknowledgment

This work was supported in part by CAPES, CNPq and FAPESP (Brazil).

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