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HARMONIC SPINORS AND LOCAL DEFORMATIONS OF THE METRIC

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Abstract. Let (M,g) be a compact Riemannian spin manifold. The Atiyah–Singer index theorem yields a lower bound for the dimension of the kernel of the Dirac operator. We prove that this bound can be attained by changing the Riemannian metric g on an arbitrarily small open set.

1. Introduction and statement of results

Let M be a spin manifold, we assume that all spin manifolds come equipped with a choice of orientation and spin structure. The Dirac operator D^g of (M,g) is a first order differential operator acting on sections of the spinor bundle associated to the spin structure on M. This is an elliptic, formally self-adjoint operator. If M is compact, then the spectrum of D^g is real, discrete, and the eigenvalues tend to plus and minus infinity. In this case, the operator D^g is invertible if and only if 0 is not an eigenvalue, which is the same as vanishing of the kernel.

The Atiyah–Singer index theorem states that the index of the Dirac operator is equal to a topological invariant of the manifold,

$$\mathrm{ind}(D^g) = \alpha(M),$$

see, for example, [11, Theorem 16.6, p. 276]. Depending on the dimension n of M this formula has slightly different interpretations. To explain this interpretation, it is important to remark that we will always consider the spinor bundle as a complex vector bundle, similar results with different dimensions would also hold for the real spinor bundle or the $\mathrm{C}\ell_n$-linear spinor bundle. If n is even there is a $\pm$-grading of the spinor bundle and the Dirac operator D^g has a part $(D^g)^+$, which maps from positive to negative spinors. If $n \equiv 0,4 \bmod 8$ the index is integer-valued and computed as the dimension of the kernel minus the dimension of the cokernel of $(D^g)^+$. If $n \equiv 1,2 \bmod 8$ the index is $\mathbb{Z}/2\mathbb{Z}$-valued and given by the dimension modulo 2 of the kernel of D^g (if $n \equiv 1 \bmod 8$) resp. $(D^g)^+$ (if $n \equiv 2 \bmod 8$). In other dimensions the index is zero. In all dimensions, $\alpha(M)$ is a topological invariant depending only on the spin bordism class of M. In particular, $\alpha(M)$ does not depend on the metric, but it depends on the spin structure in dimension $n \equiv 1,2 \bmod 8$. For further details see [11, Chapter II, §7].

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The index theorem implies a lower bound on the dimension of the kernel of D^g which we can write succinctly as

$$\dim\ker D^g \geq a(M), \tag{1}$$

where

$$a(M) := \begin{cases} |\widehat{A}(M)| & \text{if } n \equiv 0 \bmod 4; \\ 1 & \text{if } n \equiv 1 \bmod 8 \text{ and } \alpha(M) \neq 0; \\ 2 & \text{if } n \equiv 2 \bmod 8 \text{ and } \alpha(M) \neq 0; \\ 0 & \text{otherwise.} \end{cases}.$$

If M is not connected, then this lower bound can be improved by studying each connected component of M. For this reason we restrict to connected manifolds from now on.

Metrics g for which equality holds in (1) are called D-minimal, see [3, Section 3]. The existence of D-minimal metrics on all connected compact spin manifolds was established in [1] following previous work in [12, 3]. In this note we will strengthen this existence result by showing that one can find a D-minimal metric coinciding with a given metric outside a small open set. For a Riemannian manifold (M, g) we denote by $U_p(r)$ the set of points for which the distance to the point p is strictly less than r. We will prove the following theorem.

Theorem 1.1. *Let (M, g) be a compact connected Riemannian spin manifold of dimension $n \geq 2$. Let $p \in M$ and $r > 0$. Then there is a D-minimal metric $\widetilde{g}$ on M with $\widetilde{g} = g$ on $M \setminus U_p(r)$.*

The new ingredient in the proof of this theorem is the use of the "invertible double" construction which gives a D-minimal metric on any spin manifold of the type $(-M)\#M$ where # denotes connected sum and where $-M$ denotes M equipped with the opposite orientation. For dimension $n \geq 5$, we can then use the surgery method from [3] with surgeries of codimension ≥ 3. For $n = 3, 4$, we need the stronger surgery result of [1] preserving D-minimality under surgeries of codimension ≥ 2. The case $n = 2$ follows from [1] and classical facts about Riemann surfaces.

If a manifold has one D-minimal metric, then generic metrics are D-minimal, to formulate this precisely we introduce some notation. We denote by $\mathcal{R}(M, U_p(r), g)$ the set of all smooth Riemannian metrics on M which coincide with the metric g outside $U_p(r)$ and by $\mathcal{R}_{\min}(M, U_p(r), g)$ the subset of D-minimal metrics. From Theorem 1.1 it follows that a generic metric from $\mathcal{R}(M, U_p(r), g)$ is actually in $\mathcal{R}_{\min}(M, U_p(r), g)$, as made precise in the following corollary.

Corollary 1.1. *Let (M, g) be a compact connected Riemannian spin manifold of dimension ≥ 3. Let $p \in M$ and $r > 0$. Then $\mathcal{R}_{\min}(M, U_p(r), g)$ is open in the C^1-topology on $\mathcal{R}(M, U_p(r), g)$ and it is dense in all C^k-topologies, $k \geq 1$.*

The proof follows ideas described in [2, Theorem 1.2] or [12, Proposition 3.1]. The first observation of the argument is that the eigenvalues of D^g are continuous functions of g in the C^1-topology, from which the property of being open follows. The second observation is that spectral data of D^{g_t} for a linear family of metrics $g_t = (1-t)g_0 + tg_1$ depends real analytically on the parameter t. If $g_0 \in \mathcal{R}_{\min}(M, U_p(r), g)$ it follows that metrics g_t with t arbitrarily close to 1 are also in this set, from which we conclude the property of being dense.

2. Preliminaries

2.1. Spin manifolds and spin structure preserving maps. An orientation on an n-dimensional manifold M can be viewed as a refinement of the frame bundle $\mathrm{GL}(M)$ for the tangent bundle TM to a sub-bundle $\mathrm{GL}_+(M)$ with structure group $\mathrm{GL}_+(n,\mathbb{R})$. Such a refinement exists if and only if the first Stiefel–Whitney class $w_1(TM)$ vanishes. Here the group $\mathrm{GL}_+(n,\mathbb{R})$ consists of all invertible $n\times n$-matrices with positive determinant and has fundamental group $\mathbb{Z}$ if $n=2$ and $\mathbb{Z}/2\mathbb{Z}$ if $n\geq 3$. Let $\widetilde{\mathrm{GL}_+}(n,\mathbb{R})$ be the unique connected double cover of $\mathrm{GL}_+(n,\mathbb{R})$.

A (topological) spin structure on an oriented manifold M is a $(\widetilde{\mathrm{GL}_+}(n,\mathbb{R})\to \mathrm{GL}(n,\mathbb{R}))$-equivariant lift of $\mathrm{GL}_+(M)$ to a bundle with the structure group $\widetilde{\mathrm{GL}_+}(n,\mathbb{R})$. Such a lift exists if and only if the second Stiefel–Whitney class $W_2(TM))$ vanishes.

If these structures exist they are in general not unique, the orientation can be chosen independently on each connected component of M, or equivalently the space of orientations on M is an affine space for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^0(M,\mathbb{Z}/2\mathbb{Z})$. Similarly, the space of spin structures is an affine space for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^1(M,\mathbb{Z}/2\mathbb{Z})$.

As already mentioned we use the term "spin manifold" for a manifold together with the choice of an orientation and a spin structure.

If $f: M_1\to M_2$ is a diffeomorphism between two manifolds, any orientation and spin structure on M_2 pulls back to an orientation and spin structure on M_1. A diffeomorphism f between two spin manifolds M_1 and M_2 is called a spin structure preserving diffeomorphism if the orientation and spin structure on M_1 coincide with the pullbacks from M_2.

If the manifold M is further equipped with a Riemannian metric the above topological spin structure reduces to a geometrical spin structure which is a $(\mathrm{Spin}(n)\to \mathrm{SO}(n))$-equivariant lift $\mathrm{Spin}(M)$ of the bundle $\mathrm{SO}(M)$ of oriented orthonormal frames of the tangent bundle. The spinor bundle ΣM on M is a vector bundle associated to $\mathrm{Spin}(M)$, it has a natural first order elliptic operator $D:\Gamma(\Sigma M)\to\Gamma(\Sigma M)$, see, for example, [8] for details. Any spin structure preserving diffeomorphism $f: M_1\to M_2$ which is also an isometry induces an isomorphism between the spinor bundles $f_*:\Sigma M_1\to\Sigma M_2$ which is compatible with the Dirac operators in the sense that all sections φ of ΣM_1 satisfy $D^{M_2}(f_*\circ\varphi\circ f^{-1}) = f_*\circ(D^{M_1}\varphi)\circ f^{-1}$.

If W is a manifold with boundary $\partial W = M$, then an orientation and spin structure on W induce an orientation and a spin structure on M. Conversely, if an orientation and a spin structure on M are given, then there is a unique orientation and spin structure on $W = M\times[0,1]$ such that the restricted structures on $M\cong M\times\{1\}$ coincide with the given ones. The boundary component $M\times\{0\}$ is obviously diffeomorphic to M as well, but the restriction of the orientation of $M\times[0,1]$ is the opposite of the orientation of M. We write $-M := M\times\{0\}$ for the spin manifold with this opposite orientation and the spin structure obtained from $M\times[0,1]$.

2.2. The invertible double. Let N be a compact connected spin manifold with boundary. The double of N is formed by gluing N and $-N$ along the common boundary ∂N and is denoted by $(-N)\cup_{\partial N} N$. If N is equipped with a Riemannian metric which has product structure near the boundary, then this metric naturally gives a metric on $(-N)\cup_{\partial N} N$. The spin structures can be glued together to obtain a spin structure on $(-N)\cup_{\partial N} N$. The spinor bundle of $(-N)\cup_{\partial N} N$ is obtained by gluing the spinor bundle of N with the spinor bundle of $-N$ along their common boundary ∂N. It is straightforward to check that the appropriate gluing map is the map used in [6, Chapter 9].

The Dirac operator on $(-N) \cup_{\partial N} N$ is invertible due to the following argument. Assume that a spinor field φ is in the kernel of the Dirac operator on $(-N) \cup_{\partial N} N$. The restriction $\varphi|_{-N}$ can be "reflected along ∂N" to a spinor field $\tilde{\varphi}$ on N as indicated in the appendix. On the boundary ∂N one has $\tilde{\varphi}|_N = \nu \cdot \varphi|_N$ and thus $\nu \cdot \tilde{\varphi}|_N = -\varphi|_N$ for the exterior unit normal field ν on ∂N, see Lemma A.2. Green's formula for the Dirac operator yields

$$0 = \int_N \langle D\tilde{\varphi}, \varphi \rangle - \int_N \langle \tilde{\varphi}, D\varphi \rangle = \int_{\partial N} \langle \nu \cdot \tilde{\varphi}, \varphi \rangle = -\|\varphi|_{\partial N}\|^2_{L^2(\partial N)}.$$

Thus $\varphi|_{\partial N} = 0$, and by the weak unique continuation property of the Dirac operator it follows that $\varphi = 0$. For more details on this argument see [6, Chapter 9] and [5, Proposition 1.4]. In the appendix we also show that the doubling construction of [6, Chapter 9] coincides with the spinor bundle and Dirac operator on the doubled manifold.

Proposition 2.1. *Let (M, g) be a compact connected Riemannian spin manifold. Let $p \in M$ and $r > 0$. Let $(-M)\#M$ be the connected sum formed at the points $p \in M$ and $p \in -M$. Then there is a metric on $(-M)\#M$ with invertible Dirac operator which coincides with g outside $U_p(r)$*

This proposition is proved by applying the double construction to the manifold with boundary $N = M \setminus U_p(r/2)$, where N is equipped with a metric we get by deforming the metric g on $U_p(r) \setminus U_p(r/2)$ to become a product near the boundary.

Metrics with invertible Dirac operator are obviously D-minimal, so the metric provided by Proposition 2.1 is D-minimal.

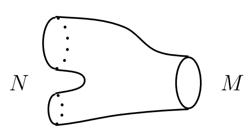
3. Proof of Theorem 1.1

Let M and N be compact spin manifolds of dimension n. Recall that a spin bordism from M to N is a manifold with boundary W of dimension $n+1$ together with a spin structure preserving diffeomorphism from $N \amalg (-M)$ to the boundary of W. The manifolds M and N are said to be spin bordant if such a bordism exists.

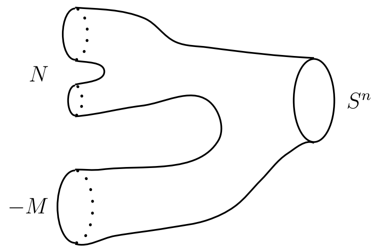
For the proof of Theorem 1.1 we have to distinguish several cases.

3.1. Proof of Theorem 1.1 in dimension $n \geq 5$.

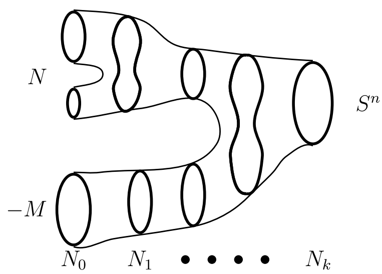
Proof. To prove the Gromov–Lawson conjecture, Stolz [13] showed that any compact spin manifold with vanishing index is spin bordant to a manifold of positive scalar curvature. Using this, we see that M is spin bordant to a manifold N which has a D-minimal metric h, where the manifold N is not necessarily connected. For details see [3, Proposition 3.9].



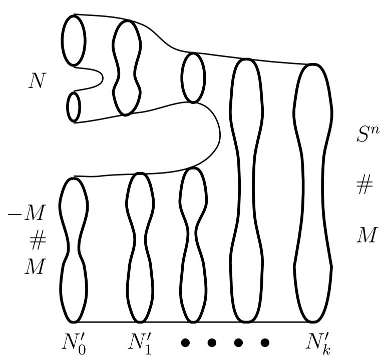
By removing an open ball from the interior of a spin bordism from M to N we get that $N \amalg (-M)$ is spin bordant to the sphere S^n.



Since S^n is simply connected and $n \geq 5$ it follows from [11, Proof of Theorem 4.4, p. 300] that S^n can be obtained from $N \amalg (-M)$ by a sequence of surgeries of codimension at least 3. By making r smaller and possibly move the surgery spheres slightly we may assume that no surgery hits $U_p(r) \subset M$. We obtain a sequence of manifolds $N_0, N_1, \ldots, N_k$, where $N_0 = N \amalg (-M)$, $N_k = S^n$, and N_{i+1} is obtained from N_i by a surgery of codimension at least 3.



Since the surgeries do not hit $U_p(r) \subset M \subset N \amalg (-M) = N_0$ we can consider $U_p(r)$ as a subset of every N_i. We define the sequence of manifolds $N'_0, N'_1, \ldots, N'_k$ by forming the connected sum $N'_i = M \# N_i$ at the points p. Then $N'_0 = N \amalg (-M) \# M$, $N'_k = S^n \# M = M$, and N'_{i+1} is obtained from N'_i by a surgery of codimension at least 3 which does not hit $M \setminus U_p(r)$.



We now equip N'_0 with a Riemannian metric. On N we choose a D-minimal metric. The manifold $(-M) \# M$ has vanishing index, so a D-minimal metric is a metric with an invertible Dirac operator. From Proposition 2.1, we know that there exists such a metric on $(-M) \# M$ which coincides with g outside $U_p(r)$. Note that here we use the assumption that M is connected. Together we get a D-minimal metric g'_0 on N'_0.

From [3, Proposition 3.6] we know that the property of being D-minimal is preserved under surgery of codimension at least 3. We apply the surgery procedure to g_0' to produce a sequence of D-minimal metrics g_i' on N_i'. Since the surgery procedure of [3, Theorem 1.2] does not affect the Riemannian metrics outside arbitrarily small neighborhoods of the surgery spheres we may assume that all g_i' coincide with g on $M \setminus U_p(r)$. The Theorem is proved by choosing $\tilde{g} = g_k'$ on $N_k' = M$. □

3.2. Proof of Theorem 1.1 in dimensions $n = 3$ and $n = 4$.

Proof. In these cases the argument works almost the same, except that we can only conclude that S^n is obtained from $N \amalg (-M)$ by surgeries of codimension at least 2, see [9, VII, Theorem 3] for $n = 3$ and [10, VIII, Proposition 3.1] for $n = 4$. To take care of surgeries of codimension 2 we use [1, Theorem 1.2]. Since this surgery construction affects the Riemannian metric only in a small neighborhood of the surgery sphere we can finish the proof as described in the case $n \geq 5$. □

Alternatively, it is straightforward to adapt the perturbation proof by Maier [12] to prove Theorem 1.1 in dimensions 3 and 4.

3.3. Proof of Theorem 1.1 in dimension $n = 2$.

Proof. The argument in the case $n = 2$ is different. Assume that a metric g on a compact surface with chosen spin structure is given. In [1, Theorem 1.1] it is shown that for any $\varepsilon > 0$ there is a D-minimal metric $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$. Using the following Lemma 3.1, we see that for $\varepsilon > 0$ sufficiently small, there is a spin structure preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to g on $M \setminus U_p(r)$. As the dimension of the kernel of the Dirac operator is preserved under spin structure preserving conformal diffeomorphisms, $\tilde{g}$ is D-minimal as well. □

Lemma 3.1. *Let M be a compact surface with a Riemannian metric g and a spin structure. Then for any $r > 0$ there is an $\varepsilon > 0$ with the following property: For any $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$ there is a spin structure preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to g on $M \setminus U_p(r)$.*

To prove the lemma one has to show that a certain differential is surjective. This proof can be carried out in different mathematical languages. One alternative is to use Teichmüller theory formulated in terms of quadratic differentials, we will use a presentation in terms of Riemannian metrics following [14].

Sketch of Proof of Lemma 3.1. If g_1 and g_2 are metrics on M, then we say that g_1 is Teichmüller equivalent to g_2 if there is a diffeomorphism $\psi : M \to M$ such that ψ is homotopic to the identity and $\psi^* g_2$ is conformal to g_1. This is an equivalence relation on the set of metrics on M, and the equivalence class of g_1 is denoted by $\Phi(g_1)$. Let $\mathcal{T}$ be the set of equivalence classes, this is the Teichmüller space which has a natural structure of a smooth finite-dimensional manifold. Note that any diffeomorphism $\psi : M \to M$ homotopic to the identity is also isotopic to the identity, i.e. the homotopy can be chosen as a path in the diffeomorphism group, see, e.g., [7]. As along this path, the spin structure is preserved, ψ perserves the spin structure.

Showing the lemma is thus equivalent to showing that $\Phi(\mathcal{R}(M, U_p(r), g))$ is a neighborhood of $\Phi(g)$ in $\mathcal{T}$.

Variations of metrics are given by symmetric (2,0)-tensors, that is, by sections of S^2T^*M. The tangent space of $\mathcal{T}$ can be identified with the space of transverse (= divergence free) traceless sections,

$$S^{TT} := \{h \in \Gamma(S^2T^*M) \mid \mathrm{div}^g h = 0, \mathrm{tr}^g h = 0\},$$

see, for example, [4, Lemma 4.57] and [14].

The two-dimensional manifold M has a complex structure which is denoted by J. The map $H : T^*M \to S^2T^*M$ defined by $H(\alpha) := \alpha\otimes\alpha - \alpha\circ J\otimes\alpha\circ J$ is quadratic, it is 2-to-1 outside the zero section, and its image are the trace free symmetric tensors. Furthermore $H(\alpha\circ J) = -H(\alpha)$. Hence by polarization we obtain an isomorphism of real vector bundles from $T^*M \otimes_{\mathbb{C}} T^*M$ to the trace free part of S^2T^*M. Here the complex tensor product is used when T^*M is considered as a complex line bundle using J. A trace-free section of S^2T^*M is divergence free if and only if the corresponding section $T^*M \otimes_{\mathbb{C}} T^*M$ is holomorphic, see [14, p. 45–46]. We get that S^{TT} is finite dimensional, and it follows that $\mathcal{T}$ is finite dimensional.

In order to show that $\Phi(\mathcal{R}(M, U_p(r), g))$ is a neighborhood of $\Phi(g)$ in $\mathcal{T}$ we show that the differential $d\Phi : T\mathcal{R}(M, U_p(r), g) \to T\mathcal{T}$ is surjective at g. Using the above identification $T\mathcal{T} = S^{TT}$, $d\Phi$ is just orthogonal projection from $\Gamma(S^2T^*M)$ to S^{TT}.

Assume that $h_0 \in S^{TT}$ is orthogonal to $d\Phi(T\mathcal{R}(M, U_p(r), g))$. Then h_0 is L^2-orthogonal to $T\mathcal{R}(M, U_p(r), g)$. As $T\mathcal{R}(M, U_p(r), g))$ consists of all sections of S^2T^*M with support in $U_p(r)$ we conclude that h_0 vanishes on $U_p(r)$. Since h_0 can be identified with a holomorphic section of $T^*M \otimes_{\mathbb{C}} T^*M$ we see that h_0 vanishes everywhere on M. The surjectivity of $d\Phi$ and the lemma follow. □

Appendix A. Notes about reflections at hypersurfaces and the doubling construction

Let M be a connected Riemannian spin manifold, with a reflection φ at a hyperplane N. That is φ is an isometry with fixed point set N, orientation reversing, and N separates M into two components. Let $-M$ be the manifold M with the opposite orientation, i.e., $\varphi : M \to -M$ is orientation preserving. It is also required that φ preserves the spin structure. The reflection φ lifts to the frame bundle by mapping the frame $\mathcal{E} = (e_1, \ldots, e_n)$ to $\varphi_*\mathcal{E} := (-d\varphi(e_1), d\varphi(e_2), \ldots, d\varphi(e_n))$, so $\varphi_* : \mathrm{SO}(M) \to \mathrm{SO}(M)$. This map φ_* is not $\mathrm{SO}(n)$ equivariant, but if we define $J = \mathrm{diag}(-1, 1, 1, 1, \ldots 1)$, then

$$\varphi_*(\mathcal{E}A) = \varphi_*(\mathcal{E})JAJ.$$

If $\mathcal{E}$ is a frame over N whose first vector is normal to N, then $\varphi_*(\mathcal{E}) = \mathcal{E}$.

The above-mentioned compatibility with the spin structure is the fact that the pullback of the double covering $\vartheta : \mathrm{Spin}(M) \to \mathrm{SO}(M)$ via φ_* is again the covering $\mathrm{Spin}(M) \to \mathrm{SO}(M)$. In other words, a lift $\widetilde{\varphi}_* : \mathrm{Spin}(M) \to \mathrm{Spin}(M)$ can be chosen such that $\vartheta \circ \widetilde{\varphi}_* = \varphi_* \circ \vartheta$. This implies that $(\widetilde{\varphi}_*)^2 = \pm\,\mathrm{Id}$. Choose $\widetilde{\mathcal{E}} \in \mathrm{Spin}(M)$ over N, such that the first vector of $\vartheta(\widetilde{\mathcal{E}})$ is normal to N. Then $\widetilde{\varphi}_*(\widetilde{\mathcal{E}}) = \pm\widetilde{\mathcal{E}}$, thus $(\widetilde{\varphi}_*)^2(\widetilde{\mathcal{E}}) = \widetilde{\mathcal{E}}$. It follows that $(\widetilde{\varphi}_*)^2 = \mathrm{Id}$.

The conjugation with J is an automorphism of $\mathrm{SO}(n)$ and lifts to $\mathrm{Spin}(n) \subset \mathrm{Cl}_n$, as a conjugation with $E_1 := (1, 0..., 0)$ in the Clifford algebra sense. We therefore have

$$\widetilde{\varphi}_*(\widetilde{\mathcal{E}}B) = \widetilde{\varphi}_*(\widetilde{\mathcal{E}})(-E_1 B E_1).$$

Let $\sigma : \mathrm{Cl}_n \to \mathrm{End}(\Sigma_n)$ be an irreducible representation of the Clifford algebra. We set $\Sigma M := \mathrm{Spin}(M) \times_\sigma \Sigma_n$.

Lemma A.1 (Lift to the spinor bundle)**.** *The map*

$$\mathrm{Spin}(M) \times \Sigma_n \ni (\widetilde{\mathcal{E}}, \rho) \mapsto (\widetilde{\varphi}_* \widetilde{\mathcal{E}}, \sigma(E_1)\rho) \in \mathrm{Spin}(M) \times \Sigma_n$$

is compatible with the equivalence relation given by σ. Thus it descends to a map

$$\varphi_\# : \Sigma M = \mathrm{Spin}(M) \times_\sigma \Sigma_n \to \Sigma M = \mathrm{Spin}(M) \times_\sigma \Sigma_n.$$

Proof. $(\widetilde{\mathcal{E}}B, \sigma^{-1}(B)\rho)$ is mapped to

$$(\widetilde{\varphi}_*(\widetilde{\mathcal{E}}B), \sigma(E_1)\sigma^{-1}(B)\rho) = \widetilde{\varphi}_*(\widetilde{\mathcal{E}})(-E_1 B E_1), \sigma((-E_1 B E_1)^{-1})\sigma(E_1)\rho). \qquad \square$$

Obviously $(\varphi_\#)^2 = -\mathrm{Id}$, and $\varphi_\# : \Sigma_p M \to \Sigma_{\varphi(p)} M$. In even dimensions $\varphi_\#$ maps positive spinors to negative ones and vice versa.

Lemma A.2 (On the fixed point set N)**.** *Assume that $\psi \in \Sigma M|_N$. Then $\varphi_\#(\psi) = \pm\nu \cdot \psi$ for a unit normal vector ν of N in M. The sign depends on the choice of ν and the choice of the lift $\tilde{\varphi}_*$.*

Proof. Choose $\widetilde{\mathcal{E}} \in \mathrm{Spin}(M)$ over the base point of ψ, such that ν is the first vector of $\vartheta(\tilde{\mathcal{E}})$. Determine $\rho \in \Sigma_n$ such that $(\widetilde{\mathcal{E}}, \rho)$ represents ψ. Then $\varphi_\#(\psi)$ is represented by $(\pm\widetilde{\mathcal{E}}, \nu \cdot \rho)$. $\square$

Lemma A.3 (Compatibility with the Clifford action)**.**

$$d\varphi(X) \cdot \varphi_\#(\psi) = -\varphi_\#(X \cdot \psi)$$

for $X \in T_p M$, $\psi \in \Sigma_p M$.

Proof. We view $T_p M$ as an associated bundle to $\mathrm{Spin}(M)$. Then $d\varphi([\widetilde{\mathcal{E}}, v]) = [\widetilde{\varphi}_*(\mathcal{E}), Jv]$. Thus

$$\begin{aligned} d\varphi([\widetilde{\mathcal{E}}, v]) \cdot \varphi_\#([\widetilde{\mathcal{E}}, \rho]) &= [\widetilde{\varphi}_*(\widetilde{\mathcal{E}}), \sigma(Jv)\sigma(E_1)\rho] \\ &= [\widetilde{\varphi}_*(\widetilde{\mathcal{E}}), -\sigma(E_1)\sigma(v)\rho] \\ &= -\varphi_\#([\widetilde{\mathcal{E}}, v] \cdot [\widetilde{\mathcal{E}}, \rho]). \end{aligned}$$

Here we used that $Jv = E_1 \cdot v \cdot E_1$ in Cl_n. $\square$

Lemma A.4. *Let $X \in T_p M$, $\psi \in \Gamma(\Sigma M)$. Then*

$$\nabla_{d\varphi(X)} \varphi_\#(\psi) = \varphi_\#(\nabla_X \psi).$$

Proof. The differential of $\varphi_* : \mathrm{SO}(M) \to \mathrm{SO}(M)$ maps $T\,\mathrm{SO}(M)$ to $T\,\mathrm{SO}(M)$. The connection 1-form $\omega : \mathrm{SO}(M) \to \mathfrak{so}(n)$ then pulls back according to

$$\omega((d(\varphi_*))(Y)) = J\omega(Y)J$$

for $Y \in T_{\mathcal{E}}\,\mathrm{SO}(M)$, a lift of $X \in T_M$ under the projection $\mathrm{SO}(M) \to M$. We lift this to a connection 1-form $\tilde{\omega} : \mathrm{Spin}(M) \to \mathrm{Cl}_n$, which thus transforms as

$$\tilde{\omega}((d(\tilde{\varphi}_*))(\tilde{Y})) = -E_1 \omega(\tilde{Y}) E_1$$

where $\tilde{Y} \in T\,\mathrm{Spin}(M)$ is a lift of Y. And this induces the relation

$$\nabla_{d\varphi(X)}\varphi_\#(\psi) = \varphi_\#(\nabla_X\psi). \qquad \square$$

We obtain

$$\begin{aligned}\varphi_\#(D\psi) &= \sum_i \varphi_\#(e_i \cdot \nabla_{e_i}\psi)\\ &= -\sum_i d\varphi(e_i)\cdot\varphi_\#(\nabla_{e_i}\psi)\\ &= -\sum_i d\varphi(e_i)\cdot\nabla_{d\varphi(e_i)}\varphi_\#\psi\\ &= -D\varphi_\#\psi\end{aligned}$$

This formula can also be read as

$$D\psi = \varphi_\# D\varphi_\#\psi \tag{2}$$

As a conclusion we obtain the following proposition.

Proposition A.1. *If one constructs the double for a manifold with the classical spinor bundle and Dirac operator as in* [6, Theorem 9.3], *then we obtain the classical spinor bundle and the classical Dirac operator on the double.*

To prove the proposition one has to compare the definitions in [6] with ours. The map $\varphi_\# : \Sigma_p^+ M \to \Sigma_{\varphi(p)}^- M$ corresponds to the map G in [6]. It follows that G^{-1} corresponds to $-\varphi_\# : \Sigma_p^- M \to \Sigma_{\varphi(p)}^+ M$. In [6], the map G is used to identify $\Sigma_p^+ M$ with $\Sigma_{\varphi(p)}^- M$. Pay attention that with respect to this identification, the map $\varphi_\# : \Sigma_p^+ M \to \Sigma_{\varphi(p)}^- M$ is the identity, whereas $\varphi_\# : \Sigma_p^- M \to \Sigma_{\varphi(p)}^+ M$ is $-\mathrm{Id}$. Equation (2) says that this identification is compatible with the Dirac operator, and corresponds to (9.10) in [6].

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