HARMONIC SPINORS AND LOCAL DEFORMATIONS OF THE METRIC

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Abstract. Let \((M, g)\) be a compact Riemannian spin manifold. The Atiyah–Singer index theorem yields a lower bound for the dimension of the kernel of the Dirac operator. We prove that this bound can be attained by changing the Riemannian metric \(g\) on an arbitrarily small open set.

1. Introduction and statement of results

Let \(M\) be a spin manifold, we assume that all spin manifolds come equipped with a choice of orientation and spin structure. The Dirac operator \(D^g\) of \((M, g)\) is a first order differential operator acting on sections of the spinor bundle associated to the spin structure on \(M\). This is an elliptic, formally self-adjoint operator. If \(M\) is compact, then the spectrum of \(D^g\) is real, discrete, and the eigenvalues tend to plus and minus infinity. In this case, the operator \(D^g\) is invertible if and only if 0 is not an eigenvalue, which is the same as vanishing of the kernel.

The Atiyah–Singer index theorem states that the index of the Dirac operator is equal to a topological invariant of the manifold,

\[
\text{ind}(D^g) = \alpha(M),
\]

see, for example, [11, Theorem 16.6, p. 276]. Depending on the dimension \(n\) of \(M\) this formula has slightly different interpretations. To explain this interpretation, it is important to remark that we will always consider the spinor bundle as a complex vector bundle, similar results with different dimensions would also hold for the real spinor bundle or the \(C\ell_n\)-linear spinor bundle. If \(n\) is even there is a \(\pm\)-grading of the spinor bundle and the Dirac operator \(D^g\) has a part \((D^g)^+\), which maps from positive to negative spinors. If \(n \equiv 0, 4 \mod 8\) the index is integer-valued and computed as the dimension of the kernel minus the dimension of the cokernel of \((D^g)^+\). If \(n \equiv 1, 2 \mod 8\) the index is \(\mathbb{Z}/2\mathbb{Z}\)-valued and given by the dimension modulo 2 of the kernel of \(D^g\) (if \(n \equiv 1 \mod 8\)) resp. \((D^g)^+\) (if \(n \equiv 2 \mod 8\)). In other dimensions the index is zero. In all dimensions, \(\alpha(M)\) is a topological invariant depending only on the spin bordism class of \(M\). In particular, \(\alpha(M)\) does not depend on the metric, but it depends on the spin structure in dimension \(n \equiv 1, 2 \mod 8\). For further details see [11, Chapter II, §7].

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The index theorem implies a lower bound on the dimension of the kernel of $D^g$ which we can write succinctly as
\[(1) \quad \dim \ker D^g \geq a(M),\]
where
\[a(M) := \begin{cases} 
|\hat{A}(M)| & \text{if } n \equiv 0 \mod 4; \\
1 & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\
2 & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\
0 & \text{otherwise}.
\end{cases}\]

If $M$ is not connected, then this lower bound can be improved by studying each connected component of $M$. For this reason we restrict to connected manifolds from now on.

Metrics $g$ for which equality holds in (1) are called $D$-minimal, see [3, Section 3]. The existence of $D$-minimal metrics on all connected compact spin manifolds was established in [1] following previous work in [12, 3]. In this note we will strengthen this existence result by showing that one can find a $D$-minimal metric coinciding with a given metric outside a small open set. For a Riemannian manifold $(M, g)$ we denote by $U_p(r)$ the set of points for which the distance to the point $p$ is strictly less than $r$.

We will prove the following theorem.

**Theorem 1.1.** Let $(M, g)$ be a compact connected Riemannian spin manifold of dimension $n \geq 2$. Let $p \in M$ and $r > 0$. Then there is a $D$-minimal metric $\tilde{g}$ on $M$ with $\tilde{g} = g$ on $M \setminus U_p(r)$.

The new ingredient in the proof of this theorem is the use of the "invertible double" construction which gives a $D$-minimal metric on any spin manifold of the type $(-M)\# M$ where $\#$ denotes connected sum and where $-M$ denotes $M$ equipped with the opposite orientation. For dimension $n \geq 5$, we can then use the surgery method from [3] with surgeries of codimension $\geq 3$. For $n = 3, 4$, we need the stronger surgery result of [1] preserving $D$-minimality under surgeries of codimension $\geq 2$. The case $n = 2$ follows from [1] and classical facts about Riemann surfaces.

If a manifold has one $D$-minimal metric, then generic metrics are $D$-minimal, to formulate this precisely we introduce some notation. We denote by $\mathcal{R}(M, U_p(r), g)$ the set of all smooth Riemannian metrics on $M$ which coincide with the metric $g$ outside $U_p(r)$ and by $\mathcal{R}_{\text{min}}(M, U_p(r), g)$ the subset of $D$-minimal metrics. From Theorem 1.1 it follows that a generic metric from $\mathcal{R}(M, U_p(r), g)$ is actually in $\mathcal{R}_{\text{min}}(M, U_p(r), g)$, as made precise in the following corollary.

**Corollary 1.1.** Let $(M, g)$ be a compact connected Riemannian spin manifold of dimension $\geq 3$. Let $p \in M$ and $r > 0$. Then $\mathcal{R}_{\text{min}}(M, U_p(r), g)$ is open in the $C^1$-topology on $\mathcal{R}(M, U_p(r), g)$ and it is dense in all $C^k$-topologies, $k \geq 1$.

The proof follows ideas described in [2, Theorem 1.2] or [12, Proposition 3.1]. The first observation of the argument is that the eigenvalues of $D^g$ are continuous functions of $g$ in the $C^1$-topology, from which the property of being open follows. The second observation is that spectral data of $D^g$ for a linear family of metrics $g_t = (1-t)g_0 + tg_1$ depends real analytically on the parameter $t$. If $g_0 \in \mathcal{R}_{\text{min}}(M, U_p(r), g)$ it follows that metrics $g_t$ with $t$ arbitrarily close to 1 are also in this set, from which we conclude the property of being dense.
2. Preliminaries

2.1. Spin manifolds and spin structure preserving maps. An orientation on an $n$-dimensional manifold $M$ can be viewed as a refinement of the frame bundle $GL(M)$ for the tangent bundle $TM$ to a sub-bundle $GL_+(M)$ with structure group $GL_+(n,\mathbb{R})$. Such a refinement exists if and only if the first Stiefel–Whitney class $w_1(TM)$ vanishes. Here the group $GL_+(n,\mathbb{R})$ consists of all invertible $n \times n$-matrices with positive determinant and has fundamental group $\mathbb{Z}$ if $n = 2$ and $\mathbb{Z}/2\mathbb{Z}$ if $n \geq 3$.

Let $GL_+(n,\mathbb{R})$ be the unique connected double cover of $GL_+(n,\mathbb{R})$.

A (topological) spin structure on an oriented manifold $M$ is a $(GL_+(n,\mathbb{R}) \to GL(n,\mathbb{R}))$-equivariant lift of $GL_+(M)$ to a bundle with the structure group $GL_+(n,\mathbb{R})$. Such a lift exists if and only if the second Stiefel–Whitney class $w_2(TM)$ vanishes.

If these structures exist they are in general not unique, the orientation can be chosen independently on each connected component of $M$, or equivalently the space of orientations on $M$ is an affine space for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^0(M,\mathbb{Z}/2\mathbb{Z})$. Similarly, the space of spin structures is an affine space for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^1(M,\mathbb{Z}/2\mathbb{Z})$.

As already mentioned we use the term “spin manifold” for a manifold together with the choice of an orientation and a spin structure.

If $f : M_1 \to M_2$ is a diffeomorphism between two manifolds, any orientation and spin structure on $M_2$ pulls back to an orientation and spin structure on $M_1$. A diffeomorphism $f$ between two spin manifolds $M_1$ and $M_2$ is called a spin structure preserving diffeomorphism if the orientation and spin structure on $M_1$ coincide with the pullbacks from $M_2$.

If the manifold $M$ is further equipped with a Riemannian metric the above topological spin structure reduces to a geometrical spin structure which is a $(Spin(n) \to SO(n))$-equivariant lift $Spin(M)$ of the bundle $SO(M)$ of oriented orthonormal frames of the tangent bundle. The spinor bundle $\Sigma M$ on $M$ is a vector bundle associated to $Spin(M)$, it has a natural first order elliptic operator $D : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$, see, for example, [8] for details. Any spin structure preserving diffeomorphism $f : M_1 \to M_2$ which is also an isometry induces an isomorphism between the spinor bundles $f_* : \Sigma M_1 \to \Sigma M_2$ which is compatible with the Dirac operators in the sense that all sections $\varphi$ of $\Sigma M_1$ satisfy $D^{M_2}(f_* \circ \varphi \circ f^{-1}) = f_* \circ (D^{M_1}(\varphi) \circ f^{-1})$.

If $W$ is a manifold with boundary $\partial W = M$, then an orientation and spin structure on $W$ induce an orientation and a spin structure on $M$. Conversely, if an orientation and a spin structure on $M$ are given, then there is a unique orientation and spin structure on $W = M \times [0,1]$ such that the restricted structures on $M \cong M \times \{1\}$ coincide with the given ones. The boundary component $M \times \{0\}$ is obviously diffeomorphic to $M$ as well, but the restriction of the orientation of $M \times [0,1]$ is the opposite of the orientation of $M$. We write $-M := M \times \{0\}$ for the spin manifold with this opposite orientation and the spin structure obtained from $M \times [0,1]$.

2.2. The invertible double. Let $N$ be a compact connected spin manifold with boundary. The double of $N$ is formed by gluing $N$ and $-N$ along the common boundary $\partial N$ and is denoted by $(N) \cup_{\partial N} N$. If $N$ is equipped with a Riemannian metric which has product structure near the boundary, then this metric naturally gives a metric on $(N) \cup_{\partial N} N$. The spin structures can be glued together to obtain a spin structure on $(N) \cup_{\partial N} N$. The spinor bundle of $(N) \cup_{\partial N} N$ is obtained by
gluing the spinor bundle of $N$ with the spinor bundle of $-N$ along their common boundary $\partial N$. It is straightforward to check that the appropriate gluing map is the map used in [6, Chapter 9].

The Dirac operator on $(-N) \cup_{\partial N} N$ is invertible due to the following argument. Assume that a spinor field $\varphi$ is in the kernel of the Dirac operator on $(-N) \cup_{\partial N} N$. The restriction $\varphi|_{-N}$ can be “reflected along $\partial N$” to a spinor field $\tilde{\varphi}$ on $N$ as indicated in the appendix. On the boundary $\partial N$ one has $\tilde{\varphi}|_N = \nu \cdot \varphi|_N$ and thus $\nu \cdot \tilde{\varphi}|_N = -\varphi|_N$ for the exterior unit normal field $\nu$ on $\partial N$, see Lemma A.2. Green’s formula for the Dirac operator yields

$$0 = \int_N \langle D\tilde{\varphi}, \varphi \rangle - \int_N \langle \tilde{\varphi}, D\varphi \rangle = \int_{\partial N} \langle \nu \cdot \tilde{\varphi}, \varphi \rangle = -\|\varphi|_{\partial N}\|^2_{L^2(\partial N)}.$$ 

Thus $\varphi|_{\partial N} = 0$, and by the weak unique continuation property of the Dirac operator it follows that $\varphi = 0$. For more details on this argument see [6, Chapter 9] and [5, Proposition 1.4]. In the appendix we also show that the doubling construction of [6, Chapter 9] coincides with the spinor bundle and Dirac operator on the doubled manifold.

**Proposition 2.1.** Let $(M, g)$ be a compact connected Riemannian spin manifold. Let $p \in M$ and $r > 0$. Let $(-M)\#M$ be the connected sum formed at the points $p \in M$ and $p \in -M$. Then there is a metric on $(-M)\#M$ with invertible Dirac operator which coincides with $g$ outside $U_p(r)$

This proposition is proved by applying the double construction to the manifold with boundary $N = M \setminus U_p(r/2)$, where $N$ is equipped with a metric we get by deforming the metric $g$ on $U_p(r) \setminus U_p(r/2)$ to become a product near the boundary.

Metrics with invertible Dirac operator are obviously $D$-minimal, so the metric provided by Proposition 2.1 is $D$-minimal.

### 3. Proof of Theorem 1.1

Let $M$ and $N$ be compact spin manifolds of dimension $n$. Recall that a spin bordism from $M$ to $N$ is a manifold with boundary $W$ of dimension $n + 1$ together with a spin structure preserving diffeomorphism from $N \amalg (-M)$ to the boundary of $W$. The manifolds $M$ and $N$ are said to be spin bordant if such a bordism exists.

For the proof of Theorem 1.1 we have to distinguish several cases.

#### 3.1. Proof of Theorem 1.1 in dimension $n \geq 5$.

**Proof.** To prove the Gromov–Lawson conjecture, Stolz [13] showed that any compact spin manifold with vanishing index is spin bordant to a manifold of positive scalar curvature. Using this, we see that $M$ is spin bordant to a manifold $N$ which has a $D$-minimal metric $h$, where the manifold $N$ is not necessarily connected. For details see [3, Proposition 3.9].
By removing an open ball from the interior of a spin bordism from $M$ to $N$ we get that $N \amalg (-M)$ is spin bordant to the sphere $S^n$.

Since $S^n$ is simply connected and $n \geq 5$ it follows from [11, Proof of Theorem 4.4, p. 300] that $S^n$ can be obtained from $N \amalg (-M)$ by a sequence of surgeries of codimension at least 3. By making $r$ smaller and possibly move the surgery spheres slightly we may assume that no surgery hits $U_p(r) \subset M$. We obtain a sequence of manifolds $N_0, N_1, \ldots, N_k$, where $N_0 = N \amalg (-M)$, $N_k = S^n$, and $N_{i+1}$ is obtained from $N_i$ by a surgery of codimension at least 3.

Since the surgeries do not hit $U_p(r) \subset M \subset N \amalg (-M) = N_0$ we can consider $U_p(r)$ as a subset of every $N_i$. We define the sequence of manifolds $N'_0, N'_1, \ldots, N'_k$ by forming the connected sum $N'_i = M \# N_i$ at the points $p$. Then $N'_0 = N \amalg (-M) \# M$, $N'_k = S^n \# M = M$, and $N'_{i+1}$ is obtained from $N'_i$ by a surgery of codimension at least 3 which does not hit $M \setminus U_p(r)$.

We now equip $N'_0$ with a Riemannian metric. On $N$ we choose a $D$-minimal metric. The manifold $(-M) \# M$ has vanishing index, so a $D$-minimal metric is a metric with an invertible Dirac operator. From Proposition 2.1, we know that there exists such a metric on $(-M) \# M$ which coincides with $g$ outside $U_p(r)$. Note that here we use the assumption that $M$ is connected. Together we get a $D$-minimal metric $g'_0$ on $N'_0$. 
From [3, Proposition 3.6] we know that the property of being $D$-minimal is preserved under surgery of codimension at least 3. We apply the surgery procedure to $g'_0$ to produce a sequence of $D$-minimal metrics $g'_i$ on $N'_i$. Since the surgery procedure of [3, Theorem 1.2] does not affect the Riemannian metrics outside arbitrarily small neighborhoods of the surgery spheres we may assume that all $g'_i$ coincide with $g$ on $M \setminus U_p(r)$. The Theorem is proved by choosing $\tilde{g} = g'_k$ on $N'_k = M$. □

3.2. Proof of Theorem 1.1 in dimensions $n = 3$ and $n = 4$.

Proof. In these cases the argument works almost the same, except that we can only conclude that $S^n$ is obtained from $\pi_1(\mathbb{R})$ by surgeries of codimension at least 2, see [9, VII, Theorem 3] for $n = 3$ and [10, VIII, Proposition 3.1] for $n = 4$. To take care of surgeries of codimension 2 we use [1, Theorem 1.2]. Since this surgery construction affects the Riemannian metric only in a small neighborhood of the surgery sphere we can finish the proof as described in the case $n \geq 5$. □

Alternatively, it is straightforward to adapt the perturbation proof by Maier [12] to prove Theorem 1.1 in dimensions 3 and 4.

3.3. Proof of Theorem 1.1 in dimension $n = 2$.

Proof. The argument in the case $n = 2$ is different. Assume that a metric $g$ on a compact surface with chosen spin structure is given. In [1, Theorem 1.1] it is shown that for any $\varepsilon > 0$ there is a $D$-minimal metric $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$. Using the following Lemma 3.1, we see that for $\varepsilon > 0$ sufficiently small, there is a spin structure preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to $g$ on $M \setminus U_p(r)$. As the dimension of the kernel of the Dirac operator is preserved under spin structure preserving conformal diffeomorphisms, $\tilde{g}$ is $D$-minimal as well. □

Lemma 3.1. Let $M$ be a compact surface with a Riemannian metric $g$ and a spin structure. Then for any $r > 0$ there is an $\varepsilon > 0$ with the following property: For any $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$ there is a spin structure preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to $g$ on $M \setminus U_p(r)$.

To prove the lemma one has to show that a certain differential is surjective. This proof can be carried out in different mathematical languages. One alternative is to use Teichmüller theory formulated in terms of quadratic differentials, we will use a presentation in terms of Riemannian metrics following [14].

Sketch of Proof of Lemma 3.1. If $g_1$ and $g_2$ are metrics on $M$, then we say that $g_1$ is Teichmüller equivalent to $g_2$ if there is a diffeomorphism $\psi : M \to M$ such that $\psi$ is homotopic to the identity and $\psi^* g_2$ is conformal to $g_1$. This is an equivalence relation on the set of metrics on $M$, and the equivalence class of $g_1$ is denoted by $\Phi(g_1)$. Let $T$ be the set of equivalence classes, this is the Teichmüller space which has a natural structure of a smooth finite-dimensional manifold. Note that any diffeomorphism $\psi : M \to M$ homotopic to the identity is also isotopic to the identity, i.e. the homotopy can be chosen as a path in the diffeomorphism group, see, e.g., [7]. As along this path, the spin structure is preserved, $\psi$ perserves the spin structure.

Showing the lemma is thus equivalent to showing that $\Phi(R(M, U_p(r), g))$ is a neighborhood of $\Phi(g)$ in $T$. 
Variations of metrics are given by symmetric $(2,0)$-tensors, that is, by sections of $S^2T^*M$. The tangent space of $\mathcal{T}$ can be identified with the space of transverse (= divergence free) traceless sections,
\[ S^{TT} := \{ h \in \Gamma(S^2T^*M) \mid \text{div}^g h = 0, \text{tr}^g h = 0 \}, \]
see, for example, [4, Lemma 4.57] and [14].

The two-dimensional manifold $M$ has a complex structure which is denoted by $J$. The map $H : T^*M \to S^2T^*M$ defined by $H(\alpha) := \alpha \otimes \alpha - \alpha \circ J \otimes \alpha \circ J$ is quadratic, it is 2-to-1 outside the zero section, and its image are the trace free symmetric tensors. Furthermore $H(\alpha \circ J) = -H(\alpha)$. Hence by polarization we obtain an isomorphism of real vector bundles from $T^*M \otimes \mathbb{C} T^*M$ to the trace free part of $S^2T^*M$. Here the complex tensor product is used when $T^*M$ is considered as a complex line bundle using $J$. A trace-free section of $S^2T^*M$ is divergence free if and only if the corresponding section $T^*M \otimes \mathbb{C} T^*M$ is holomorphic, see [14, p. 45–46]. We get that $S^{TT}$ is finite dimensional, and it follows that $\mathcal{T}$ is finite dimensional.

In order to show that $\Phi(\mathcal{R}(M, U_p(r), g))$ is a neighborhood of $\Phi(g)$ in $\mathcal{T}$ we show that the differential $d\Phi : T\mathcal{R}(M, U_p(r), g) \to T\mathcal{T}$ is surjective at $g$. Using the above identification $TT = S^{TT}$, $d\Phi$ is just orthogonal projection from $\Gamma(S^2T^*M)$ to $S^{TT}$.

Assume that $h_0 \in S^{TT}$ is orthogonal to $d\Phi(T\mathcal{R}(M, U_p(r), g))$. Then $h_0$ is $L^2$-orthogonal to $T\mathcal{R}(M, U_p(r), g)$. As $T\mathcal{R}(M, U_p(r), g))$ consists of all sections of $S^2T^*M$ with support in $U_p(r)$ we conclude that $h_0$ vanishes on $U_p(r)$. Since $h_0$ can be identified with a holomorphic section of $T^*M \otimes \mathbb{C} T^*M$ we see that $h_0$ vanishes everywhere on $M$. The surjectivity of $d\Phi$ and the lemma follow. \hfill \Box

Appendix A. Notes about reflections at hypersurfaces and the doubling construction

Let $M$ be a connected Riemannian spin manifold, with a reflection $\varphi$ at a hyperplane $N$. That is $\varphi$ is an isometry with fixed point set $N$, orientation reversing, and $N$ separates $M$ into two components. Let $-M$ be the manifold $M$ with the opposite orientation, i.e., $\varphi : M \to -M$ is orientation preserving. It is also required that $\varphi$ preserves the spin structure. The reflection $\varphi$ lifts to the frame bundle by mapping the frame $\mathcal{E} = (e_1, \ldots, e_n)$ to $\varphi_*\mathcal{E} := (-d\varphi(e_1), d\varphi(e_2), \ldots, d\varphi(e_n))$, so $\varphi_* : \text{SO}(M) \to \text{SO}(M)$. This map $\varphi_*$ is not $\text{SO}(n)$ equivariant, but if we define $J = \text{diag}(-1, 1, 1, 1, \ldots, 1)$, then
\[ \varphi_*(\mathcal{E}A) = \varphi_*(\mathcal{E})JAJ. \]

If $\mathcal{E}$ is a frame over $N$ whose first vector is normal to $N$, then $\varphi_*(\mathcal{E}) = \mathcal{E}$.

The above-mentioned compatibility with the spin structure is the fact that the pullback of the double covering $\vartheta : \text{Spin}(M) \to \text{SO}(M)$ via $\varphi_*$ is again the covering $\text{Spin}(M) \to \text{SO}(M)$. In other words, a lift $\tilde{\varphi}_* : \text{Spin}(M) \to \text{Spin}(M)$ can be chosen such that $\vartheta \circ \tilde{\varphi}_* = \varphi_* \circ \vartheta$. This implies that $(\tilde{\varphi}_*)^2 = \pm \text{Id}$. Choose $\tilde{\mathcal{E}} \in \text{Spin}(M)$ over $N$, such that the first vector of $\vartheta(\tilde{\mathcal{E}})$ is normal to $N$. Then $\tilde{\varphi}_*(\tilde{\mathcal{E}}) = \pm \tilde{\mathcal{E}}$, thus $(\tilde{\varphi}_*)^2(\tilde{\mathcal{E}}) = \tilde{\mathcal{E}}$. It follows that $(\tilde{\varphi}_*)^2 = \text{Id}.

The conjugation with $J$ is an automorphism of $\text{SO}(n)$ and lifts to $\text{Spin}(n) \subset \text{Cl}_n$, as a conjugation with $E_1 := (1, 0, \ldots, 0)$ in the Clifford algebra sense. We therefore
Lemma A.3

(Compatibility with the Clifford action)

Here we used that $Jv$ to a connection 1-form $\tilde{\varphi}$ by $(\tilde{\varphi}, \tilde{\varepsilon}, \sigma(E_1)\rho) \in \text{Spin}(M) \times \Sigma_n$.

Proof. The differential of $\varphi : \text{Spin}(M) \times \Sigma_n \ni (\tilde{\varphi}, \tilde{\varepsilon}) \mapsto (\varphi_\#(\tilde{\varphi}_*(\tilde{\varepsilon})), \sigma(\tilde{\varepsilon})\sigma(E_1)\rho) \in \text{Spin}(M) \times \Sigma_n$ is compatible with the equivalence relation given by $\sigma$. Thus it descends to a map $\varphi_\# : \Sigma M = \text{Spin}(M) \times \Sigma_n \rightarrow \Sigma M = \text{Spin}(M) \times \Sigma_n$.

Proof. $(\tilde{\varphi}_*(\tilde{\varepsilon})), \sigma(E_1)\sigma^{-1}(B)\rho)$ is mapped to

$$(\varphi_\#(\tilde{\varphi}_*(\tilde{\varepsilon})), \sigma(E_1)\sigma^{-1}(B)\rho) = (\varphi_*(\tilde{\varepsilon})), \sigma((-E_1BE_1)^{-1})\sigma(E_1)\rho).$$

Obviously $(\varphi_\#)^2 = -\text{Id}$, and $\varphi_\# : \Sigma \rho \rightarrow \Sigma_{\varphi(\rho)}M$. In even dimensions $\varphi_\#$ maps positive spinors to negative ones and vice versa.

Lemma A.2 (On the fixed point set $N$). Assume that $\psi \in \Sigma M|_N$. Then $\varphi_\#(\psi) = \pm \nu \cdot \psi$ for a unit normal vector $\nu$ of $N$ in $M$. The sign depends on the choice of $\nu$ and the choice of the lift $\tilde{\varphi}$.

Proof. Choose $\tilde{\varepsilon} \in \text{Spin}(M)$ over the base point of $\psi$, such that $\nu$ is the first vector of $\nu(\tilde{\varepsilon})$. Determine $\rho \in \Sigma_n$ such that $\nu(\tilde{\varepsilon}), \rho$ represents $\psi$. Then $\varphi_\#(\psi)$ is represented by $(\pm \tilde{\varepsilon}, \nu \cdot \rho)$.

Lemma A.3 (Compatibility with the Clifford action).

$$d\varphi(X) \cdot \varphi_\#(\psi) = -\varphi_\#(X \cdot \psi)$$

for $X \in T_pM$, $\psi \in \Sigma_pM$.

Proof. We view $T_pM$ as an associated bundle to $\text{Spin}(M)$. Then $d\varphi([\tilde{\varepsilon}, v]) = [\tilde{\varphi}_*(\tilde{\varepsilon}), Jv]$ Thus

$$d\varphi([\tilde{\varepsilon}, v]) \cdot \varphi_\#([\tilde{\varepsilon}, \rho]) = [\tilde{\varphi}_*(\tilde{\varepsilon}), \sigma(Jv)\sigma(E_1)\rho]$$

$$= [\tilde{\varphi}_*(\tilde{\varepsilon}), -\sigma(E_1)\sigma(v)\rho]$$

$$= -\varphi_\#([\tilde{\varepsilon}, v] \cdot [\tilde{\varepsilon}, \rho]).$$

Here we used that $Jv = E_1 \cdot v \cdot E_1$ in $\text{Cl}_{n}$.

Lemma A.4. Let $X \in T_pM$, $\psi \in \Gamma(\Sigma M)$. Then

$$\nabla_{d\varphi(X)}\varphi_\#(\psi) = \varphi_\#(\nabla_X\psi).$$

Proof. The differential of $\varphi_* : \text{SO}(M) \rightarrow \text{SO}(M)$ maps $T\text{SO}(M)$ to $T\text{SO}(M)$. The connection 1-form $\omega : \text{SO}(M) \rightarrow \mathfrak{so}(n)$ then pulls back according to

$$\omega((d(\varphi_*))(Y)) = J\omega(Y)J$$

for $Y \in T\text{SO}(M)$, a lift of $X \in T_M$ under the projection $\text{SO}(M) \rightarrow M$. We lift this to a connection 1-form $\tilde{\omega} : \text{Spin}(M) \rightarrow \text{Cl}_{n}$, which thus transforms as

$$\tilde{\omega}((d(\varphi_*))(\tilde{Y})) = -E_1\omega(\tilde{Y})E_1.$$
where $\tilde{Y} \in T\text{Spin}(M)$ is a lift of $Y$. And this induces the relation
\[
\nabla_{d\varphi(X)} \varphi_\#(\psi) = \varphi_\#(\nabla_X \psi).
\]
\[\square\]
We obtain
\[
\varphi_\#(D\psi) = \sum_i \varphi_\#(e_i \cdot \nabla_{e_i} \psi)
\]
\[
= -\sum_i d\varphi(e_i) \cdot \varphi_\#(\nabla_{e_i} \psi)
\]
\[
= -\sum_i d\varphi(e_i) \cdot \nabla_{d\varphi(e_i)} \varphi_\# \psi
\]
\[
= -D\varphi_\# \psi
\]

This formula can also be read as
\[
(2) \quad D\psi = \varphi_\# D\varphi_\# \psi
\]
As a conclusion we obtain the following proposition.

**Proposition A.1.** If one constructs the double for a manifold with the classical spinor bundle and Dirac operator as in [6, Theorem 9.3], then we obtain the classical spinor bundle and the classical Dirac operator on the double.

To prove the proposition one has to compare the definitions in [6] with ours. The map $\varphi_\# : \Sigma_p^+ M \to \Sigma_{\varphi(p)}^- M$ corresponds to the map $G$ in [6]. It follows that $G^{-1}$ corresponds to $-\varphi_\# : \Sigma_p^- M \to \Sigma_{\varphi(p)}^+ M$. In [6], the map $G$ is used to identify $\Sigma_p^+ M$ with $\Sigma_{\varphi(p)}^- M$. Pay attention that with respect to this identification, the map $\varphi_\# : \Sigma_p^+ M \to \Sigma_{\varphi(p)}^- M$ is the identity, whereas $\varphi_\# : \Sigma_p^- M \to \Sigma_{\varphi(p)}^+ M$ is $-\text{Id}$. Equation (2) says that this identification is compatible with the Dirac operator, and corresponds to (9.10) in [6].

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