ON CABLED KNOTS, DEHN SURGERY, AND LEFT-ORDERABLE FUNDAMENTAL GROUPS

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Abstract. Previous work of the authors establishes a criterion on the fundamental group of a knot complement that determines when Dehn surgery on the knot will have a fundamental group that is not left-orderable [6]. We provide a refinement of this criterion by introducing the notion of a decayed knot; it is shown that Dehn surgery on decayed knots produces surgery manifolds that have non-left-orderable fundamental group for all sufficiently positive surgeries. As an application, we prove that sufficiently positive cables of decayed knots are always decayed knots. These results mirror properties of L-space surgeries in the context of Heegaard Floer homology.

1. Introduction

Definition 1.1. A group $G$ is left-orderable if there exists a partition of the group elements

$$G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$$

satisfying $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P} \neq \emptyset$. The subset $\mathcal{P}$ is called a positive cone.

This is equivalent to $G$ admitting a left-invariant strict total ordering. For background on left-orderable groups relevant to this paper see [2, 6]; a standard reference for the theory of left-orderable groups is [12]. As established by Boyer et al. [2] (compare [11]), the fundamental group $\pi_1(K)$ of the complement of a knot $K$ in $S^3$ is always left-orderable. Indeed, this follows from the fact that any compact, connected, irreducible, orientable three-manifold with positive first Betti number has left-orderable fundamental group [2, Theorem 1.1]. However, the question of left-orderability for fundamental groups of rational homology three-spheres is considerably more subtle (see [2, 6]) and seems closely tied to certain codimension one structures on the three-manifold (see [2, 3, 17]). Continuing along the lines of [6] this paper focuses on Dehn surgery, an operation on knots that produces rational homology three-spheres. We recall this construction in order to fix notation and conventions.

For any knot $K$ in $S^3$ there is a preferred generating set for the peripheral subgroup $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(K)$ provided by the knot meridian $\mu$ and the Seifert longitude $\lambda$. The latter is uniquely determined (up to orientation) by the existence of a Seifert surface for $K$. We orient $\mu$ so that it links positively with $K$, and orient $\lambda$ so that $\mu \cdot \lambda = 1$. For any rational number $r$ with reduced form $p/q$ we denote the peripheral element $\mu^p \lambda^q$ by $\alpha_r$. At the level of the fundamental group, the result of Dehn surgery along $\alpha_r$ is summarized by the short exact sequence

$$1 \to \langle \langle \alpha_r \rangle \rangle \to \pi_1(K) \to \pi_1(S^3_r(K)) \to 1.$$
Here $\langle \alpha_r \rangle$ denotes the normal closure of $\alpha_r$, and $S^3_r(K)$ is the three-manifold obtained by attaching a solid torus to the boundary of $S^3 \setminus \nu(K)$, sending the meridian of the torus to a simple closed curve representing the class $[\alpha_r] \in H_1(\partial(S^3 \setminus \nu(K)); \mathbb{Z})/\{\pm 1\}$.

We will blur the distinction between $\alpha_r$ as an element of the fundamental group or as a primitive class in the (projective) first homology of the boundary, and refer to these peripheral elements as slopes.

While many examples of rational homology three-spheres have left-orderable fundamental group [2], there exist infinite families of knots for which sufficiently positive Dehn surgery (that is, along a slope parameterized by a suitable large rational number) yields a manifold with non-left-orderable fundamental group [6]. To make this precise, consider the set of slopes

$$S_r = \{ \alpha_r | r' \geq r \}$$

for some fixed rational $r$.

**Definition 1.2.** A non-trivial knot $K$ in $S^3$ is called $r$-decayed if, for any positive cone $P$ in $\pi_1(K)$, either $P \cap S_r = S_r$ or $P \cap S_r = \emptyset$.

The existence of decayed knots is established in [6]. For example, the torus knot $T_{p,q}$ is $(pq - 1)$-decayed (for $p, q > 0$), and the $(-2, 3, q)$-pretzel knot is $(10 + q)$-decayed for odd $q \geq 5$ (see Theorem 2.1). Our interest in this property stems from the following:

**Theorem 1.1.** If $K$ is $r$-decayed then $\pi_1(S^3_r(K))$ is not left-orderable for all $r' \geq r$.

As a result, it is not restrictive to assume that $r$ is a positive rational number since $\pi_1(S^3_0(K))$ is always left-orderable [2]. Notice however that it is not immediately clear how Theorem 1.1 might be applied in practice, as there is no obvious method for checking when a knot is $r$-decayed. For this reason, in Section 2 we describe an equivalent formulation of $r$-decay whose statement is more technical, but easier to verify, than the definition. Together with the proof of Theorem 1.1, the results of Section 2 provide a useful refinement of the ideas from [6].

Results connecting left-orderability and Dehn surgery may be expected to mirror similar results relating to L-spaces, since there is no known example of an L-space with left-orderable fundamental group, while many L-spaces have fundamental group that is not left-orderable, (see [1, 5, 6, 16, 21]). Recall that an L-space is a rational homology sphere with Heegaard Floer homology that is as simple as possible, in the sense that $rkHF(Y) = |H_1(Y; \mathbb{Z})|$ (see [15]). Theorem 1.1 mirrors a fundamental property of knots admitting L-space surgeries: if $S^3_n(K)$ is an L-space, then $S^3_r(K)$ is an L-space as well for any $r \geq n$.

In the interest of further investigating left-orderability of fundamental groups of 3-manifolds along the lines of [6], we consider the behaviour of Dehn surgery on cables of $r$-decayed knots (for necessary background, see Section 3). Denoting the $(p, q)$-cable of the knot $K$ as $C_{p,q}(K)$, the main theorem of this article is:

**Theorem 1.2.** If $K$ is $r$-decayed then $C_{p,q}(K)$ is $pq$-decayed whenever $\frac{q}{p} > r$. 
The proof of Theorem 1.2 is contained in Section 3. Notice that combining Theorem 2.1 and Theorem 1.2 provides a rather large class of knots for which sufficiently positive surgery yields a non-left-orderable fundamental group.

Dehn surgery on cabled knots and non-left-orderability of the resulting fundamental groups may again be viewed in the context of Heegaard Floer homology. Referring to knots admitting L-space surgeries as L-space knots, Hedden proves:

**Theorem 1.3.** [9, Theorem 1.10] If \( K \) is an L-space knot then \( C_{p,q}(K) \) is an L-space knot whenever \( \frac{q}{p} \geq 2g(K) - 1 \).

Here, the quantity \( g(K) \) is the Seifert genus of \( K \). Note that the converse of this statement has been recently established by Hom [10].

In order to assess the strength of Theorem 1.2, it is natural to ask when Dehn surgery on a cable knot yields a manifold that has left-orderable fundamental group. It turns out that, in the case that \( K \) is \( r \)-decayed, Theorem 1.2 is close to describing all possible non-left-orderable surgeries on a cable knot \( C_{p,q}(K) \), in the following sense:

**Theorem 1.4.** Suppose that \( C \) is the \((p,q)\)-cable of some knot. If \( r \in \mathbb{Q} \) satisfies \( r < pq - p - q \), then \( \pi_1(S^3_+(C)) \) is left-orderable.

This result is a special case of a more general observation pertaining to satellite knots that is discussed in Section 4. Notice that Theorem 6 makes no reference to the original knot being \( r \)-decayed. However, restricted to \( r \)-decayed knots, Theorem 1.2 and Theorem 1.4 combine to produce an interval of surgery coefficients for which the left-orderability of the associated quotient is not determined. More precisely:

**Question 1.1.** If \( K \) is \( r \)-decayed and \( C \) is a \((p,q)\)-cable of \( K \) with \( \frac{q}{p} > r \), can Theorem 1.2 and Theorem 1.4 be sharpened to determine when \( \pi_1(S^3_+(C)) \) is left-orderable for \( r' \) satisfying \( pq - p - q < r' < pq \)?

2. A practical reformulation of Theorem 1.1

We begin with a reformulation of \( r \)-decay that will be essential in connecting this work with the results of [6]. This will require the following lemma:

**Lemma 2.1.** Let \( G \) be a left-orderable group containing elements \( g, h \). If \( g \in \mathcal{P} \) implies \( h \in \mathcal{P} \) for every positive cone \( \mathcal{P} \), then \( g \in \mathcal{P} \) if and only if \( h \in \mathcal{P} \).

**Proof.** We need only show the converse, namely \( h \in \mathcal{P} \) implies \( g \in \mathcal{P} \) for every positive cone \( \mathcal{P} \subset G \). For a contradiction, suppose this is not the case, so there exists a positive cone such that \( h \in \mathcal{P} \) and \( g \notin \mathcal{P} \). Consider the positive cone \( \mathcal{Q} = \mathcal{P}^{-1} \), defining the reverse ordering of \( G \). This gives \( g \in \mathcal{Q} \) and \( h \notin \mathcal{Q} \), contradicting our assumption. \( \square \)

**Proposition 2.1.** A knot \( K \) is \( r \)-decayed if and only if for every positive cone \( \mathcal{P} \subset \pi_1(K) \) there exists a strictly increasing sequence of positive rational numbers \( \{r_i\} \) with \( r_i \to \infty \) satisfying

\[
\begin{align*}
(1) \quad & r = r_0, \text{ and} \\
(2) \quad & \alpha_r \in \mathcal{P} \text{ implies } \alpha_{r_i} \in \mathcal{P} \text{ for all } i.
\end{align*}
\]
Proof. Suppose that $K$ is $r$-decayed, and let $\mathcal{P}$ be any positive cone. Choose a strictly increasing sequence of rational numbers $\{r_i\}$ with $r_0 = r$ and $r_i \to \infty$. Whenever $\alpha_r = \alpha_{r_0} \in \mathcal{P}$ we have $\mathcal{S}_r \cap \mathcal{P} \neq \emptyset$, so that $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$ since $K$ is $r$-decayed. It follows that $\alpha_{r_i} \in \mathcal{S}_r \subset \mathcal{P}$ for all $i$.

To prove the converse, let $\mathcal{P}$ be a positive cone for $\pi_1(K)$. Fix a strictly increasing sequence $\{r_i\}$ of rational numbers limiting to infinity and satisfying (1) and (2). Suppose that $\alpha_r \in \mathcal{P}$, then by assumption $\alpha_{r_i} \in \mathcal{P}$ for all $i > 0$.

Now suppose that $\mu^m\lambda^n$ is an element of $\mathcal{S}_r$. Choose $r_i, r_{i+1}$ with corresponding reduced forms $\frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}$ such that $r_i < \frac{m}{n} < r_{i+1}$. By solving

\begin{align*}
q_ia + q_{i+1}b &= cn, \\
p_ia + p_{i+1}b &= cm,
\end{align*}

we can find positive integers $a, b$ and $c$ such that $(\mu^{p_i}\lambda^{q_i})a(\mu^{p_{i+1}}\lambda^{q_{i+1}})^b = (\mu^m\lambda^n)^c$. Explicitly, Cramer’s rule gives

\begin{align*}
\begin{vmatrix}
q_{i+1} & n \\
p_{i+1} & m
\end{vmatrix}, \quad
\begin{vmatrix}
q_i & n \\
p_i & m
\end{vmatrix}, \quad
\begin{vmatrix}
q_i & q_{i+1} \\
p_i & p_{i+1}
\end{vmatrix};
\end{align*}

note that all these quantities are positive because of our restriction $r_i < \frac{m}{n} < r_{i+1}$ (compare [6, Lemma 17]). This shows that $\mu^m\lambda^n$ is positive, since its $c$th power is expressed as a product of positive elements. Hence $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$.

This establishes the implication $\alpha_r \in \mathcal{P} \Rightarrow \mathcal{S}_r \subset \mathcal{P}$ for every positive cone $\mathcal{P}$. By Lemma 2.1, this is equivalent to $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$ or $\mathcal{S}_r \cap \mathcal{P} = \emptyset$ for every positive cone $\mathcal{P}$, so that $K$ is $r$-decayed. \hfill \Box

Remark 2.1. In practice, it is often more natural to establish $\alpha_r \in \mathcal{P}$ implies $\alpha_{r_i}^{w_i} \in \mathcal{P}$ for all $i$, where $w_i \in \mathbb{N}$ (see in particular the proofs of Lemmas 3.2 and 3.3). This situation arises when one constructs (for a given positive cone $\mathcal{P}$) a sequence of unreduced rationals $\{r_i\} = \left\{ \frac{p_i}{q_i} \right\}$ for which $\gcd(p_i, q_i) = w_i \geq 1$, and $\mu^{p_0}\lambda^{q_0} \in \mathcal{P}$ implies $\mu^p\lambda^q \in \mathcal{P}$ for all $i$. Notice that the implication $\alpha_r \in \mathcal{P}$ implies $\alpha_{r_i}^{w_i} \in \mathcal{P}$ still allows us to apply Proposition 2.1, since $\alpha_{r_i}^{w_i} \in \mathcal{P}$ if and only if $\alpha_{r_i} \in \mathcal{P}$ (this simple observation holds in any left-orderable group). Ultimately, this results in more flexibility in selecting the sequence $\{r_i\}$.

The equivalence established in Proposition 2.1 shows that all examples considered in [6] are $r$-decayed for certain $r$, as [6, Corollary 11] is a special case of Proposition 2.1.

Theorem 2.1. [6, Theorems 24, 28 and 30]

1. The $(p, q)$-torus knot is $(pq - 1)$-decayed for all positive, relatively prime pairs of integers $p, q$.
2. The $(-2, 3, q)$-pretzel knot is $(10 + q)$-decayed for all odd $q \geq 5$.
3. The $(3, q)$-torus knot with one positive full twist added along two strands is $(3q + 2)$-decayed, for all positive $q$ congruent to 2 modulo 3.

Proof. We consider the case of $K_q$, the $(-2, 3, q)$-pretzel knot with $q \geq 5$ odd, the other cases are similar. Set $r = 10 + q$, and $r_i = r + i$. It is shown in [6] that for every positive cone $\mathcal{P}$ in $\pi_1(K_q)$, the implication $\alpha_r \in \mathcal{P} \Rightarrow \alpha_{r_i} \in \mathcal{P}$ holds for all $i \geq 0$. This means that for every left-ordering of $\pi_1(K_q)$, the integer sequence $\{r_i\}$ satisfies the properties required by Proposition 2.1, and we conclude that $K_q$ is $r$-decayed. \hfill \Box
Note that the above proof illustrates some particularly special behaviour, as the rational sequences \( \{r_i\} \) required by Proposition 2.1 (which a priori may be different for each left-ordering) are replaced by a single integer sequence sufficient for every left-ordering. Thus, Proposition 2.1 provides a more workable method (than used previously) for checking when a knot has surgeries that yield a non-left-orderable fundamental group. Combined with the material established in [6, Section 2], we provide a short proof of Theorem 1.1.

**Proof of Theorem 1.1.** For contradiction, assume that \( \pi_1(S^3_{r_0}(K)) \) is left-orderable for some \( r' \geq r \), and consider the short exact sequence

\[
1 \to \langle \langle \alpha_{r'} \rangle \rangle \overset{i}{\to} \pi_1(K) \overset{f}{\to} \pi_1(S^3_{r'}(K)) \to 1,
\]

as defined in the introduction. Let \( \mu, \lambda \in \pi_1(K) \) denote the meridian and longitude. Since \( \pi_1(S^3_{r'}(K)) \) is left-orderable, \( \langle \langle \alpha_{r'} \rangle \rangle \cap (\mu, \lambda) = \langle \alpha_{r'} \rangle \) (see [6, Proof of Proposition 20]). In particular, if we fix an arbitrary rational number \( s_0 > r' \), then \( f(\alpha_{s_0}) \neq 1 \). Thus, we may choose a positive cone \( Q \) in \( \pi_1(S^3_{r'}(K)) \) that contains \( f(\alpha_{s_0}) \). Next, choose a positive cone \( Q' \subset \langle \langle \alpha_{r'} \rangle \rangle \) not containing \( \alpha_{r'} \), and define a positive cone \( P \subset \pi_1(K) \) by

\[
P = i(Q') \cup f^{-1}(Q).
\]

Note that \( \alpha_{r'} \notin P \), and \( \alpha_{s_0} \in P \).

This is a standard construction for creating a left-ordering of a group using a short exact sequence, here the result is a left-ordering of \( \pi_1(K) \) with positive cone \( P \), relative to which the subgroup \( \langle \langle \alpha_{r'} \rangle \rangle \) is convex. Because \( \langle \langle \alpha_{r'} \rangle \rangle \cap (\mu, \lambda) = \langle \alpha_{r'} \rangle \) is convex in the restriction ordering of \( (\mu, \lambda) \). Therefore, [6, Proposition 18] shows that all slopes \( \alpha_s \) with \( s > r' \) must have the same sign. In particular, since \( \alpha_{s_0} \) is positive it follows that all slopes \( \alpha_s \) with \( s > r' \) are positive, so that

\[
Q \cap S_r = \{ \alpha_s | s > r' \}.
\]

Therefore, \( K \) is not \( r \)-decayed. \( \square \)

We remark that there is a more geometric argument establishing Theorem 1.1, that relies upon an understanding of the topology of the space of left-orderings of \( \mathbb{Z} \oplus \mathbb{Z} \) (see [18, Section 3] and [4, Chapter 6]). Roughly, every left-ordering of the knot group \( \pi_1(K) \) restricts to a left-ordering of the peripheral subgroup that defines a line in \( \mathbb{Z} \oplus \mathbb{Z} \), with all positive elements of \( \mathbb{Z} \oplus \mathbb{Z} \) on one side of the line, and all the negative elements on the other side. As a result, given two rationals \( r_1 < r_2 \) corresponding to slopes \( \alpha_{r_1} \) and \( \alpha_{r_2} \) that have the same sign in every left-ordering, no left-ordering can restrict to an ordering of the peripheral subgroup with corresponding slope \( s \) between \( r_1 \) and \( r_2 \). The proof of Theorem 1.1 then follows from checking that whenever \( \pi_1(S^3_{r'}(K)) \) is left-orderable, we can define a left-ordering of \( \pi_1(K) \) that restricts to yield a line of slope \( r' \) in the peripheral subgroup (compare [6, Proof of Theorem 9]).

### 3. The proof of Theorem 1.2

We recall the construction of a cabled knot in order to fix notation. Consider the \((p, q)\)-torus knot \( T_{p,q} \), where \( p, q > 0 \) are relatively prime. As the closure of a \( p \)-strand braid, this knot may be naturally viewed in a solid torus \( T \) by removing a tubular
neighbourhood of the braid axis. The complement of \(T_{p,q}\) in \(T\) is referred to as a \((p,q)\)-cable space. Now given any knot \(K\) in \(S^3\), the cable knot \(C_{p,q}(K)\) is obtained by identifying the boundary of \(T\) with the boundary of \(S^3 \setminus \nu(K)\), identifying the longitude of \(T\) with the longitude \(\lambda\) of \(K\). We will denote this cable knot by \(C\) whenever this simplified notation does not cause confusion.

The knot group \(\pi_1(C)\) may be calculate via the Seifert–Van Kampen Theorem, by viewing the complement \(S^3 \setminus \nu(C)\) as the identification of the boundaries of \(S^3 \setminus \nu(K)\) and a solid torus \(D^2 \times S^1\) along an essential annulus with core curve given by the slope \(\mu^q\lambda^p\). If \(\pi_1(D^2 \times S^1) = \langle t \rangle\) then this gives rise to a natural amalgamated product

\[
\pi_1(C) \cong \pi_1(K) \ast_{\mu^q\lambda^p=tv} \mathbb{Z}.
\]

Consulting [6, Section 3], the meridian and longitude for \(C\) may be calculated as

\[
\mu_C = \mu^ut^v\lambda^{-v} \quad \text{and} \quad \lambda_C = \mu_C^{-pq}t^p,
\]

where \(u\) and \(v\) are positive integers satisfying \(pu - qv = 1\) (compare [20, Proof of Theorem 3.1]).

Suppose that the knot \(K\) is \(r\)-decayed, and choose cabling coefficients \(p\) and \(q\) so that \(q/p > r\). To begin, we choose a positive cone \(\mathcal{P} \subset \pi_1(C)\) and assume, without loss of generality, that \(\mu_C^{pq}\lambda = t^p\) is positive. This means that \(t^p = \mu^q\lambda^p \in \mathcal{P}\), so every element \(\mu^m\lambda^n\) is positive whenever \(m/n > r\), since \(K\) is \(r\)-decayed. To see this, note that the inclusion \(\pi_1(K) \subset \pi_1(C)\) means that every left-ordering of \(\pi_1(C)\) induces a left-ordering of \(\pi_1(K)\) by restriction. In particular, if two elements of the subgroup \(\langle \mu, \lambda \rangle \subset \pi_1(C)\) have opposite signs in a left-ordering of \(\pi_1(C)\), then they will also have opposite signs in the induced left-ordering of \(\pi_1(K)\). As a result we can use \(r\)-decay of the subgroup \(\pi_1(K)\) as a property of left-orderings of the larger group \(\pi_1(C)\).

Our method of proof will be to check that the cable is \(pq\)-decayed by using the equivalence from Proposition 2.1. In particular, we will show that for the given positive cone \(\mathcal{P} \subset \pi_1(C)\) there exists an unbounded sequence of increasing rationals \(\{r_i\}\) with \(r_0 = pq\), such that our assumption \(\alpha_{pq} = \mu_C^{pq}\lambda_C \in \mathcal{P}\) implies \(\alpha_{r_i} \in \mathcal{P}\) for all \(i > 0\).

First consider the case when \(\mu_C\) is positive in the left-ordering defined by \(\mathcal{P}\). Here, \(\mu_C^{pq+N}\lambda_C\) is positive for \(N \geq 0\), as it is a product of positive elements. Therefore in this case it suffices to choose \(r_i = pq + i\) for all \(i \geq 0\).

For the remainder of the proof, we assume that \(\mu_C\) is negative. For repeated use below, we also observe the crucial identity

\[
(t^{-v})^p(\mu^u\lambda^v)^p = (t^p)^{-v}\mu^{up}\lambda^{vp} = \mu^{-qv}\lambda^{-pv}\mu^{up}\lambda^{vp} = \mu^{pv-qv} = \mu,
\]

and recall that \(t^p\) commutes with \(\mu, \lambda, \mu_C, \) and \(\lambda_C\). Therefore, we also have

\[
(t^{-v})^p = (t^{-v})^p(\mu^u\lambda^v)^p = \mu.
\]

Let \(k\) be an arbitrary non-negative integer, and consider the element

\[
\mu^{-k}(t^{-v}\mu^u\lambda^v)^k.
\]

If this element is positive for some \(k\), then the required sequence is provided by Lemma 3.2 (proved below). Therefore, we may assume that

\[
(3.1) \quad \mu^{-k}(t^{-v}\mu^u\lambda^v)^k \notin \mathcal{P},
\]

for all \(k\).
Similarly, for \( k \) a non-negative integer, we consider
\[
(\mu^{-k} t^{-v} \mu^k)^{-1}(\mu^u \lambda^v)^{-1}.
\]
If this element is positive for some non-negative \( k \), then we can create the required sequence using Lemma 3.3 (proved below). Therefore, we may assume that
\[
(\mu^{-k} t^{-v} \mu^k)^{-1}(\mu^u \lambda^v)^{-1} \notin \mathcal{P},
\]
for all \( k \geq 0 \).

Observe that
\[
(\mu^{-k} t^{-v} \mu^k)^{-1}(\mu^u \lambda^v)^{-1} = (\mu^{-k} t^{-v} \mu^k)(\mu^{-k} t^{-v} \mu^k)(\mu^{up-u} \lambda^{vp-v}),
\]
which, recalling that \( t^p \) commutes with the elements \( \mu \) and \( \lambda \), simplifies to give
\[
(\mu^{-k} t^{-v} \mu^k)(\mu^{-u} \lambda^{-v})t^{-vp} \lambda^{vp} = (\mu^{-k} t^{-v} \mu^k)(\mu^{-u} \lambda^{-v}) \mu = \mu^{-k} t^{-v} \lambda^{-v} \mu - u \mu^{k+1} \notin \mathcal{P}
\]
for all \( k \geq 0 \). Taking inverses yields
\[
\mu^{-k-1} u \lambda^{-v} t^{-v} \mu^k = \mu^{-k-1} \mu_C \lambda^k \in \mathcal{P}.
\]

For the following lemmas, let \( > \) denote the left-ordering defined by the positive cone \( \mathcal{P} \), so that \( h > g \) whenever \( g^{-1} h \in \mathcal{P} \). We can then calculate:

**Lemma 3.1.** If \( \mu^{-k-1} \mu_C \lambda^k \in \mathcal{P} \) holds for all \( k \geq 0 \), then \( \mu^{N+q} \lambda^p \) must be positive for all \( N \geq 0 \).

**Proof.** Since \( \mu^{-k-1} \mu_C \lambda^k > 1 \), left-multiplying by \( \mu^{k+1} \) gives \( \mu_C \lambda^k > \mu^{k+1} \) for all \( k \geq 0 \). Setting \( k = 0 \) we obtain \( \mu_C > \mu \), so that left-multiplying by \( \mu_C \) gives rise to
\[
\mu_C^2 > \mu_C \mu.
\]

By setting \( k = 1 \), we get
\[
\mu_C \mu > \mu^2,
\]
which combines with the previous expression to give \( \mu_C^2 > \mu^2 \). Continuing in this manner, we obtain \( \mu_C^N > \mu^N \) for all \( N \geq 0 \). Left-multiplying by \( t^p \), it follows that
\[
\mu_C^{N+pq} \lambda_C = \mu_C^N t^p > \mu^{N+t^p} = \mu^{N+q} \lambda^p > 1,
\]
where the final inequality follows from the fact that \( (N+q)/p > q/p > r \) and \( K \) is \( r \)-decayed.

As a result, when (3.1) and (3.2) hold we may choose the sequence of rationals \( r_i = pq + i \) for all \( i \geq 0 \), and the requirements of Proposition 2.1 are met.

To conclude the proof, we establish Lemmas 3.2 and 3.3.

**Lemma 3.2.** If \( \mu^{-k} (t^{-v} \mu^u \lambda^v) \mu^k \in \mathcal{P} \) for some \( k \geq 0 \), then there exists a sequence of rationals \( \{r_i\} \) with \( r_0 = pq \) such that \( \alpha_{r_i} > 1 \) for all \( i \).

**Proof.** For \( N \geq 0 \), we rewrite \( \mu_C^N \) as
\[
\mu_C^N = \mu^u \lambda^v \mu^k (\mu^{-k} (t^{-v} \mu^u \lambda^v)^N \mu^k) \mu^{-k} \lambda^{-v}.
\]
Fix a positive integer \( s \) that is large enough so that \( (sq - u - k)/{(sp - v)} > r \), this is possible because \( q/p > r \). Next, the product \( \mu_C^{N+pq} \lambda^s \) becomes \( \mu_C^{N+pq} \lambda^s \mu^{sp} \), which is equal to
\[
\mu^{u+k} \lambda^v (\mu^{-k} (t^{-v} \mu^u \lambda^v)^N \mu^k) (\mu^{qs-u-k} \lambda^{ps-v}).
\]
This is a product of positive elements, because:

(a) $\mu^{u+k}\lambda^v > 1$ because $(u+k)/v > u/v > q/p > r$ (recalling that $pu - qv = 1$), and

(b) $\mu^{qs-u-k}\lambda^{ps-v} > 1$ because $(sq - u - k)/(sp - v) > r$,

while the quantity $\mu^{-k}(t^{-v}\mu^u\lambda^v)^N\mu^k$ is positive by assumption. Therefore, in this case we choose our sequence of rationals to be

$$r_i = \frac{pq + i}{s}$$

for $i \geq 0$, this guarantees that the associated slopes $\alpha_{r_i}$ are positive in the given left-ordering.

\[\square\]

**Lemma 3.3.** If $\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k \notin \mathcal{P}$ for all $k \geq 0$, and $(\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1} \in \mathcal{P}$ for some $k \geq 0$, then there exists a sequence of rationals $\{r_i\}$ with $r_0 = pq$ such that $\alpha_{r_i} > 1$ for all $i$.

**Proof.** Fix $k \geq 0$ such that $\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k < 1$ and $(\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1} > 1$, and let $n$ be the smallest positive integer such that

$$(\mu^{-k}t^{-v}\mu^k)^n(\mu^u\lambda^v)^n > 1,$$

and

$$(\mu^{-k}t^{-v}\mu^k)^{n-1}(\mu^u\lambda^v)^{n-1} < 1,$$

note that $1 < n \leq p - 1$ (If we take $n = p$, then the equality $(t^{-v})^p(\mu^u\lambda^v)^p = \mu$ reduces the first expression to the identity, so $n = p$ is not possible). Note that we may rearrange these two expressions, so that

$$\mu^{-k}t^{-v}(\mu^u\lambda^v)^n\mu^k > 1,$$

and

$$\mu^{-k}(\mu^u\lambda^v)^{1-n}t^{-v(1-n)}\mu^k > 1.$$

Then, for $N \geq 1$, we can rewrite the expression for $\mu_C^N$ as follows:

$$\mu^{u+k}\lambda^v(\mu^{-k}t^{-v(1-n)}\mu^k)((\mu^{-k}t^{-v}(\mu^u\lambda^v)^n\mu^k)\mu^{-k}(\mu^u\lambda^v)^{1-n}t^{-v(1-n)}\mu^k)^N \mu^{-k}t^{-v}.$$

In the above expression, the quantity inside the square brackets is a product of positive elements. Denote this quantity by $P$. Choose an integer $s$ such that $(qs - k)/ps > r$.

Then considering the slope $\mu_C^{N+pq(v+s)}\lambda^{v+s} = \mu_C^{Nt(p(v+s))}$, we find

$$\mu_C^{Nt(p(v+s))} = \mu^{u+k}\lambda^v(\mu^{-k}t^{-v(1-n)}\mu^k)P^{N-1}\mu^{qs-k}\lambda^{ps}\lambda^{t(pv-vn)}.$$

This is a product of positive elements, because:

(a) $\mu^{u+k}\lambda^v > 1$, since $(u+k)/v > q/p > r$ (as before).

(b) $\mu^{-k}t^{-v(1-n)}\mu^k > 1$, because if we consider its $p$th power, we can use the fact that $t^p$ commutes with all peripheral elements so that

$$(\mu^{-k}t^{-v(1-n)}\mu^k)^p = t^{-pv(1-n)} > 1.$$

The final inequality follows from $-pv(1-n) > 0$.

(c) $\mu^{qs-k}\lambda^{ps} > 1$, because $s$ is chosen so that $(qs - k)/ps > r$.

(d) $t^{pv-vn} > 1$, because $pv - vn > 0$. 
Therefore, in this case we may choose our sequence of rationals to be

\[ r_i = \frac{i + pq(v + s)}{v + s} \]

for \( i \geq 0 \), as the corresponding elements \( \mu_C^{i+pq(v+s)}\lambda_C^{v+s} \) are positive in the left-ordering for \( i \geq 0 \).

\[ \square \]

4. Surgery on satellites

Let \( T \) denote the solid torus containing a knot \( K^P \), we require that \( K^P \) is not contained in any three-ball inside \( T \). The knot \( K^P \) will be called the pattern knot. Let \( K^C \) denote a knot in \( S^3 \), \( K^C \) will be called the companion knot. We construct the satellite knot \( K \) with pattern \( K^P \) and companion \( K^C \) as follows.

Let \( h : \partial T \to \partial(S^3 \setminus \nu(K^C)) \) denote a diffeomorphism from the boundary of \( T \) to the boundary of the complement of \( \nu(K^C) \), which carries the longitude of \( \partial T \) onto the longitude of the knot \( K^C \). The knot \( K \) is then realized as the image of the knot \( K^P \) in the manifold

\[ S^3 \setminus \nu(K^C) \sqcup_h T = S^3. \]

**Lemma 4.1.** [19, Proposition 3.4] There exists a homomorphism \( \phi : \pi_1(K) \to \pi_1(K^P) \) that preserves peripheral structure.

**Proof.** We can compute the fundamental group \( \pi_1(K) \) by using the Seifert-Van Kampen theorem. Since

\[ \pi_1(K) = \pi_1(K^C) \ast \pi_1(T \setminus \nu(K^P)), \]

the group \( \pi_1(K) \) is the free product \( \pi_1(K^C) \ast \pi_1(T \setminus \nu(K^P)) \), with amalgamation as follows: the meridian of \( K^C \) is identified with the meridian of \( T \), and the longitude of \( K^C \) is identified with the longitude of \( T \).

Let \( N \) denote the normal closure in \( \pi_1(K) \) of the commutator subgroup of \( \pi_1(K^C) \). The quotient \( \pi_1(K)/N \) can be considered as the result of killing the longitude of \( T \). Topologically we can think of this quotient as gluing a second solid torus \( T' \) to the torus \( T \) containing \( K^P \), in such a way that the meridian of \( T' \) is glued to the longitude of \( T \). The result is that \( \pi_1(K^C) \) collapses to a single infinite cyclic subgroup, and the group \( \pi_1(K)/N \) is isomorphic to \( \pi_1(K^P) \). The desired homomorphism \( \phi \) is the quotient map \( \pi_1(K) \to \pi_1(K)/N \).

\[ \square \]

**Proposition 4.1.** Suppose that \( K \) is a satellite knot with pattern knot \( K^P \), and \( r \in \mathbb{Q} \)

is any rational number. If \( \pi_1(S^3_r(K^P)) \) is left-orderable and \( S^3_r(K) \) is irreducible, then \( \pi_1(S^3_r(K)) \) is left-orderable.

**Proof.** By Lemma 4.1, there exists a homomorphism \( \phi : \pi_1(K) \to \pi_1(K^P) \) that preserves peripheral structure, so there exists an induced map

\[ \phi_r : \pi_1(S^3_r(K)) \to \pi_1(S^3_r(K^P)), \]

for every \( r \in \mathbb{Q} \). Whenever \( \pi_1(S^3_r(K^P)) \) is left-orderable (and hence non-trivial, since our definition of left-orderable does not allow \( P = \emptyset \)) the image of \( \phi_r \) is non-trivial and \( \pi_1(S^3_r(K)) \) is left-orderable [2, Theorem 1.1].

\[ \square \]
Proof of Theorem 1.4. By [7], \(pq\)-surgery on a \((p, q)\)-cable knot yields a reducible manifold. Since the minimal geometric intersection number between reducible slopes is \(\pm 1\) [8], \(r\)-surgery on a \((p, q)\)-cable yields an irreducible manifold whenever \(r < pq - p - q\). Moreover, a \((p, q)\)-cable knot can be described as a satellite knot with pattern knot \(T_{p,q}\), the \((p, q)\)-torus knot. Therefore, for \(r < pq - p - q\) we can apply Proposition 4.1 to conclude that \(\pi_1(S_r^3(K))\) will be left-orderable whenever \(\pi_1(S_r^3(T_{p,q}))\) is left-orderable.

We may now combine known results for surgery on torus knots in this setting. On the one hand, \(\pi_1(S_r^3(T_{p,q}))\) is an L-space whenever \(r \geq 2g - 1\) [14, Proposition 9.5] (see in particular [9, Lemma 2.13]), where \(g = g(T_{p,q})\) is the Seifert genus given by \(g(T_{p,q}) = \frac{1}{2}(p-1)(q-1)\). On the other, since \(S_r^3(T_{p,q})\) is Seifert fibred or a connect sum of lens spaces for every \(r\) [13], \(S_r^3(T_{p,q})\) is an L-space if and only if \(\pi_1(S_r^3(T_{p,q}))\) is not left-orderable [1] (see also [16, 21]). In particular, \(\pi_1(S_r^3(T_{p,q}))\) is left-orderable whenever \(r\) is less than \(2g(T_{p,q}) - 1\) and the result follows.

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References


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