COHOMOLOGY OF LINE BUNDLES ON COMPACTIFIED JACOBIANS

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Abstract. Let $C$ be an integral projective curve with planar singularities. For the compactified Jacobian $J$ of $C$, we prove that topologically trivial line bundles on $J$ are in one-to-one correspondence with line bundles on $C$ (the autoduality conjecture), and compute the cohomology of $J$ with coefficients in these line bundles. We also show that the natural Fourier–Mukai functor from the derived category of quasi-coherent sheaves on $J$ (where $J$ is the Jacobian of $X$) to that of quasi-coherent sheaves on $J$ is fully faithful.

0. Introduction

Let $C$ be a smooth irreducible projective curve over a field $k$, and let $J$ be the Jacobian of $C$. As an abelian variety, $J$ is self-dual. More precisely, $J \times J$ carries a natural line bundle (the Poincaré bundle) $P$ that is universal as a family of topologically trivial line bundles on $J$.

The Poincaré bundle defines the Fourier–Mukai functor

$$
\mathfrak{F} : D^b(J) \to D^b(J) : \mathcal{F} \mapsto Rp_{1,2}^*(p_1^*(\mathcal{F}) \otimes P).
$$

Here $D^b(J)$ is the derived category of quasi-coherent sheaves on $J$ and $p_{1,2} : J \times J \to J$ are the projections. Mukai [22] proved that $\mathfrak{F}$ is an equivalence of categories; the proof uses the formula

$$
Rp_{1,*}P \simeq O_\zeta[-g],
$$

where $O_\zeta$ is the structure sheaf of the zero element $\zeta \in J$ and $g$ is the genus of $C$. Formula (0.1) goes back to Mumford (see the proof of the theorem in [23, Section III.13]).

Now suppose that $C$ is a singular curve, which we assume to be projective and integral. The Jacobian $J$ is no longer projective, but it admits a natural compactification $\overline{J} \supset J$. By definition, $\overline{J}$ is the moduli space of torsion-free sheaves $F$ on $C$ such that $F$ has generic rank one and $\chi(F) = \chi(O_C)$; $J$ is identified with the open subset of locally free sheaves. It is natural to ask whether $\overline{J}$ is in some sense self-dual. For instance, one can look for a Poincaré sheaf (or complex of sheaves) $\overline{P}$ on $\overline{J} \times \overline{J}$. One can then ask whether $\overline{P}$ is, in some sense, a universal family of sheaves on $\overline{J}$ and whether the corresponding Fourier–Mukai functor $\mathfrak{F} : D^b(\overline{J}) \to D^b(\overline{J})$ is an equivalence.

In the case when the singularities of $C$ are nodes or cusps, such Poincaré sheaf $\overline{P}$ is constructed by E. Esteves and S. Kleiman in [12]; they also prove the universality of

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In addition, if $C$ is a singular plane cubic, $\mathbb{F}$ is known to be an equivalence ([8, 9], also formulated as Theorem 5.2 in [6]).

If the singularities of $C$ are more general, constructing the Poincaré sheaf $\mathbb{P}$ on $\overline{J} \times \overline{J}$ is much harder (see Remark (i) at the end of the introduction). However, it is easy to construct a Poincaré bundle $P$ on $J \times \overline{J}$. It can then be used to define a Fourier–Mukai transform

$$\mathbb{F} : D^b(J) \to D^b(\overline{J}) : \mathcal{F} \mapsto R_{p_2, *} (p_1^*(\mathcal{F}) \otimes P).$$

In this paper, we assume that $C$ is an integral projective curve with planar singularities; the main result is that formula (0.1) still holds in this case. This implies that (0.2) is fully faithful. As a simple corollary, we prove the following autoduality result: $P$ is the universal family of topologically trivial line bundles on $\overline{J}$, so that $J$ is identified with the connected component of the trivial bundle in the moduli space of line bundles on $\overline{J}$. This generalizes the Autoduality Theorem of [11] (see the remark after Theorem 1.3).

**Remarks.** (i) Suppose that there exists an extension of $P$ to a sheaf $\mathbb{P}$ on $\overline{J} \times \overline{J}$ such that the corresponding Fourier–Mukai transform $\mathbb{F} : D^b(\overline{J}) \to D^b(\overline{J})$ is an equivalence. After the first version of this paper was completed, such an extension was constructed in [3]. Then (0.2) is a composition of $\mathbb{F}$ and the direct image $j_* : D^b(J) \to D^b(\overline{J})$ for the open embedding $j : J \hookrightarrow \overline{J}$. Since $j_*$ is fully faithful, so is (0.2). Thus our result is natural assuming existence of $\mathbb{F}$.

(ii) In this paper, we work with bounded derived categories of quasi-coherent sheaves. However, formula (0.2) also defines a functor between unbounded derived categories of quasi-coherent sheaves, and our results remain true in these settings: this extended functor $\mathbb{F}$ is still fully faithful. On the other hand, $\mathbb{F}$ does not preserve coherence; therefore, it is important to consider the derived categories of quasi-coherent (rather than coherent) sheaves.

(iii) Compactified Jacobians appear as (singular) fibers of the Hitchin fibration for the group $GL(n)$; therefore, our results can be interpreted as a kind of autoduality of the Hitchin fibration. Conversely, some of our results can be derived from a theorem of E. Frenkel and C. Teleman [15] (see Theorem 7.1). We explore this relation in more details in Section 7.

(iv) Recall that the curve $C$ is assumed to be integral with planar singularities. We assume integrality of $C$ to avoid working with stability conditions for sheaves on $C$. It is likely that our argument works without this assumption if one fixes an ample line bundle on $C$ and defines the compactified Jacobian $\overline{J}$ to be the moduli space of semi-stable torsion-free sheaves of degree zero. Such generalization is natural in view of the previous remark, because some fibers of the Hitchin fibration are compactified Jacobians of non-integral curves.

On the other hand, the assumption that $C$ has planar singularities is more important. There are two reasons why the assumption is natural. First of all, for an integral curve $C$, $\overline{J}$ is irreducible if and only if the singularities of $C$ are planar ([20]); so if one drops this assumption, $J$ is no longer dense in $\overline{J}$. Secondly, only compactified Jacobians of curves with planar singularities appear in the Hitchin fibration.
1. Main results

Fix a ground field $\mathbb{k}$. For convenience, let us assume that $\mathbb{k}$ is algebraically closed. Let $C$ be an integral projective curve over $\mathbb{k}$. Denote by $J$ its Jacobian, that is, $J$ is the moduli space of line bundles on $C$ of degree zero. Denote by $\overline{J}$ the compactified Jacobian; in other words, $\overline{J}$ is the moduli space of torsion-free sheaves on $C$ of generic rank one and degree zero. (For a sheaf $F$ of generic rank one, the degree is $\deg(F) = \chi(F) - \chi(O_C).$)

Let $P$ be the Poincaré bundle; it is a line bundle on $J \times \overline{J}$. Its fiber over $(L, F) \in (J \times \overline{J})$ equals

\[
P_{(L,F)} = \det R\Gamma(L \otimes F) \otimes \det R\Gamma(O_C) \otimes \det R\Gamma(L)^{-1} \otimes \det R\Gamma(F)^{-1}.
\]

More explicitly, we can write $L \simeq O(\sum a_i x_i)$ for a divisor $\sum a_i x_i$ supported by the smooth locus of $C$, and then

\[
P_{(L,F)} = \bigotimes (F_{x_i})^{\otimes a_i}.
\]

From now on, we assume that $C$ has planar singularities; that is, the tangent space to $C$ at any point is at most two-dimensional. Our main result is the computation of the direct image of $P$:

**Theorem 1.1.**

\[
Rp_{1,*} P = \det(H^1(C, O_C)) \otimes O_{\mathbb{C}}[-g].
\]

Here $p_1 : J \times \overline{J} \to J$ is the projection, and $O_{\mathbb{C}}$ is the structure sheaf of the neutral element $\zeta = [O_C] \in J$ (so $O_{\mathbb{C}}$ is a sky-scraper sheaf at $\zeta$).

Let us now view $P$ as a family of line bundles on $\overline{J}$ parameterized by $J$. For fixed $L \in J$, denote the corresponding line bundle on $\overline{J}$ by $P_L$. In other words, $P_L$ is the restriction of $P$ to $\{L\} \times \overline{J}$. Applying base change, we can use Theorem 1.1 to compute cohomology of $P_L$:

**Theorem 1.2.**

(i) If $L \not\simeq O_C$, then $H^i(\overline{J}, P_L) = 0$ for any $i$;

(ii) If $L = O_C$, then $P_L = O_{\overline{J}}$ and $H^i(\overline{J}, O_{\overline{J}}) = \bigwedge^i H^1(C, O_C)$. (The identification is described more explicitly in Proposition 6.1.) \qedsymbol

Let Pic($\overline{J}$) be the moduli space of line bundles on $\overline{J}$. The correspondence $L \mapsto P_L$ can be viewed as a morphism $\rho : J \to \text{Pic}(\overline{J})$. Denote by Pic$^0(\overline{J}) \subset \text{Pic}(\overline{J})$ the connected component of the identity $[O_{\overline{J}}] \in \text{Pic}(\overline{J})$. In Section 6, we derive the following statement.

**Theorem 1.3.** $\rho$ gives an isomorphism $J \cong \text{Pic}^0(\overline{J})$.

**Remark.** Theorem 1.3 answers the question raised in [11]. Following [17], set

\[
\begin{align*}
\text{Pic}^+(\overline{J}) &= \{L \in \text{Pic}(\overline{J}) : L \otimes n \in \text{Pic}^0(\overline{J}) \text{ for some } n > 0\}, \\
\text{Pic}^\sigma(\overline{J}) &= \{L \in \text{Pic}(\overline{J}) : L \otimes n \in \text{Pic}^0(\overline{J}) \text{ for some } n \text{ coprime to char } \mathbb{k}\}
\end{align*}
\]

(if $\text{char} \mathbb{k} = 0$, $\text{Pic}^\sigma(\overline{J}) = \text{Pic}^+(\overline{J})$ by definition). The main result of [11] is the Autoduality Theorem, which claims that if all singularities of $C$ are double points, then $\rho : J \cong \text{Pic}^0(\overline{J})$ and $\text{Pic}^0(\overline{J}) = \text{Pic}^+(\overline{J})$. Theorem 1.3 generalizes the first statement to curves with planar singularities; as for the second statement, we show in
Proposition 6.2 that $\text{Pic}^0(\mathcal{J}) = \text{Pic}^\sigma(\mathcal{J})$. We do not know whether $\text{Pic}^\tau(\mathcal{J})$ and $\text{Pic}^\sigma(\mathcal{J})$ coincide when $\text{char}(k) > 0$ and $C$ has planar singularities.

Theorem 1.1 can be reformulated in terms of the Fourier functor

$$\mathcal{F} : D^b(J) \to D^b(\mathcal{J}) : \mathcal{F} \mapsto R\pi_2^*(\pi_1^*(\mathcal{F}) \otimes P)$$

given by $P$. Recall that $D^b(J)$ stands for the (bounded) derived category of quasi-coherent sheaves on $J$. The functor $\mathcal{F}$ admits a left adjoint given by

$$\mathcal{F}^\vee : D^b(J) \to D^b(\mathcal{J}) : \mathcal{F} \mapsto R\pi_1^*(\pi_2^*(\mathcal{F}) \otimes P^{-1}) \otimes \det(H^1(C, \mathcal{O}_C))^{-1}[g].$$

This formula relies on the computation of the dualizing sheaf on $J$: see Corollary 4.1.

Theorem 1.4.

(i) The composition $\mathcal{F}^\vee \circ \mathcal{F}$ is isomorphic to the identity functor.

(ii) $\mathcal{F}$ is fully faithful.

Proof. The first statement follows from Theorem 1.1 by base change. (This is completely analogous to the original argument of [22, Theorem 2.2].) This implies the second statement, because the functors $\mathcal{F}^\vee$ and $\mathcal{F}$ are adjoint.

Remark. For simplicity, we considered a single curve $C$ in this section. However, all our results hold for families of curves. Actually, we prove Theorem 1.1 for the universal family of curves (Theorem 5.1); base change then implies that the statement holds for any family, and, in particular, for any single curve.

2. Line bundles on a compactified Jacobian

Proposition 2.1. Suppose $H^i(\mathcal{J}, P_L) \neq 0$ for some $i$. Then $(P_L)|_J \simeq O_J$.

Proof. Let $T \to J$ be the $\mathbb{G}_m$-torsor corresponding to $(P_L)|_J$. One easily sees that $T$ is naturally an abelian group that is an extension of $J$ by $\mathbb{G}_m$. The action of $J$ on $\mathcal{J}$ lifts to an action of $T$ on $P_L$, therefore, $T$ also acts on $H^i(\mathcal{J}, P_L)$. Note that $\mathbb{G}_m \subset T$ acts via the tautological character.

Let $V \subset H^i(\mathcal{J}, P_L)$ be an irreducible $T$-submodule. Since $T$ is commutative, $\dim(V) = 1$. The action of $T$ on $V$ is given by a character $\chi : T \to \mathbb{G}_m$. Since $\chi|_{\mathbb{G}_m} = id$, we see that $\chi$ gives a splitting $T \simeq \mathbb{G}_m \times J$. This implies the statement.

Remark. If $C$ is smooth, Proposition 2.1 is equivalent to observation (vii) in [23, Section II.8]; however, our proof uses a slightly different idea, which is better adapted to the singular case.

Let $C^0 \subset C$ be the smooth locus of $C$.

Corollary 2.1. Suppose $H^i(\mathcal{J}, P_L) \neq 0$ for some $i$. Then $L|_{C^0} \simeq O_{C^0}$.

Proof. Fix a degree minus one line bundle $\ell$ on $C$. It defines an Abel–Jacobi map

$$\alpha : C \to \mathcal{J} : c \mapsto \ell(c).$$

Here $\ell(c)$ is defined as the sheaf of homomorphisms from the ideal sheaf of $c \in C$ to $\ell$. Notice that $\alpha^*(P_L) \simeq L$ and $\alpha(C^0) \subset J$. Now Proposition 2.1 completes the proof.
Set
\[ N = \{ L \in J : H^i(J, P_L) \neq 0 \text{ for some } i \} \subset J. \]
Clearly, \( N \subset J \) is closed (by the Semicontinuity Theorem), and \( N = \text{supp}(Rp_{1*}P) \), where \( p_1 : J \times J \to J \) is the projection (by base change).

**Corollary 2.2.** Let \( g \) be the (arithmetic) genus of \( C \) and \( \tilde{g} \) be its geometric genus, that is, the genus of its normalization. Then \( \dim(N) \leq (g - \tilde{g}) \).

**Proof.** Let \( \nu : \tilde{C} \to C \) be the normalization, and let \( \tilde{J} \) be the Jacobian of \( \tilde{C} \). The map \( \nu^* : J \to \tilde{J} \) is smooth and surjective; its fibers have dimension \( (g - \tilde{g}) \).

Denote by \( \tilde{N} \subset \tilde{J} \) the set of line bundles on \( \tilde{C} \) that are trivial on \( \nu^{-1}(C^0) \subset \tilde{C} \). By Corollary 2.1, \( \nu^*(N) \subset \tilde{N} \). Now it suffices to note that \( \tilde{N} \) is a countable set. \( \square \)

### 3. Moduli of curves

Let \( M = M_g \) be the moduli stack of integral projective curves \( C \) of genus \( g \) with planar singularities. The following properties of \( M \) are well known:

**Proposition 3.1.** \( M \) is a smooth algebraic stack of finite type; \( \dim(M) = 3g - 3. \) \( \square \)

**Remark.** Denote by \( C \) the universal curve over \( M \); that is, \( C \) is the moduli stack of pairs \( (C, c) \in M \). One easily checks that \( C \) is a smooth stack of dimension \( 3g - 2 \). This is similar to the statement (ii’) after Theorem 4.1.

Consider the normalization \( \tilde{C} \) of a curve \( C \in M \), and let \( \tilde{g} \) be the genus of \( \tilde{C} \) (that is, the geometric genus of \( C \)). We need some results on the stratification of \( M \) by geometric genus due to Teissier [26], Diaz and Harris [10], and Laumon [21]. Since our settings are somewhat different, we provide the proofs.

Denote by \( M^{(\tilde{g})} \subset M \) the locus of curves \( C \in M \) of geometric genus \( \tilde{g} \). Note that we view \( M^{(\tilde{g})} \) simply as a subset of the set of points of \( M \), rather than a substack.

**Proposition 3.2.** \( M^{(\tilde{g})} \) is a stratification of \( M \):
\[ \overline{(M^{(\tilde{g})})} \subset \bigcup_{\gamma \leq \tilde{g}} M^{(\gamma)}. \]
In particular, \( M^{(\tilde{g})} \subset M \) is locally closed.

**Proof.** Let \( S \) be the stack of birational morphisms \( (\nu : \tilde{C} \to C) \), where \( C \in M \), and \( \tilde{C} \) is an integral projective curve of genus \( \tilde{g} \) (with arbitrary singularity). Consider the forgetful map
\[ \pi : S \to M : (\nu : \tilde{C} \to C) \mapsto C. \]
Clearly,
\[ \pi(S) \subset \bigcup_{\gamma \leq \tilde{g}} M^{(\gamma)}. \]
Therefore, it suffices to show that \( \pi \) is projective.

Let \( S'' \) be the stack of collections \( (C, F, s) \), where \( C \in M \), \( F \) is a torsion-free sheaf on \( C \) of generic rank one and degree \( g - \tilde{g} \), \( s \in H^0(C, F) - \{0\} \). Also, let \( S' \) be the stack of collections \( (C, F, s, \mu) \), where \( (C, F, s) \in S'' \) and \( \mu : F \otimes F \to F \) is such that \( \mu(s \otimes s) = s \). Consider
\[ S \to S' : (\nu : \tilde{C} \to C) \mapsto (C, \nu_*(O_{\tilde{C}}), 1, \mu), \]
where $\mu$ is the product on the sheaf of algebras $\nu_*(O_{\tilde{C}})$. This identifies $\mathcal{S}$ and $\mathcal{S}'$. The forgetful map 

$$\mathcal{S}' \to \mathcal{S}'' : (C, F, s, \mu) \mapsto (C, F, s)$$

is a closed embedding (essentially because $\mu$ is uniquely determined by $\mu(s \otimes s) = s$). Finally, the map 

$$\mathcal{S}'' \to \mathcal{M} : (C, F, s) \mapsto C$$

is projective.

Proposition 3.3. \text{codim}(\mathcal{M}(\overline{g})) \geq (g - \tilde{g}).

Proof. Let $\mathcal{S}$ be as in the proof of Proposition 3.2. Denote by $\mathcal{S}^0$ the substack of morphisms $(\nu : \tilde{C} \to C) \in \mathcal{S}$ with smooth $\tilde{C}$; clearly, $\mathcal{M}(\overline{g}) = \pi(\mathcal{S}^0)$. Therefore, we need to show that \text{dim}(\mathcal{S}^0) \leq 2g + \tilde{g} - 3.

Consider the morphism 

$$\tilde{\pi} : \mathcal{S}^0 \to \mathcal{M}_{\tilde{g}} : (\nu : \tilde{C} \to C) \mapsto \tilde{C}.$$ 

It suffices to show \text{dim}(\tilde{\pi}^{-1}(\tilde{C})) \leq 2(g - \tilde{g}) for any $\tilde{C} \in \mathcal{M}_{\tilde{g}}$. Fix $(\nu : \tilde{C} \to C) \in \mathcal{S}^0$.

Let us prove that the dimension of the tangent space $T_{\nu}\tilde{\pi}^{-1}(\tilde{C})$ at this point is at most $2(g - \tilde{g})$.

$T_{\nu}\tilde{\pi}^{-1}(\tilde{C})$ is isomorphic to the space of first-order deformations of $O_C$ viewed as a sheaf of subalgebras of $\nu_*O_{\tilde{C}}$. This yields an isomorphism

$$T_{\nu}\tilde{\pi}^{-1}(\tilde{C}) = \{\text{derivations } O_C \to \nu_*O_{\tilde{C}}/O_C\} = \text{Hom}_{O_C}(\Omega_C, \nu_*O_{\tilde{C}}/O_C).$$

Now it suffices to notice that the fibers of the cotangent sheaf $\Omega_C$ are at most two-dimensional, and that the length of the sky-scraper sheaf $\nu_*O_{\tilde{C}}/O_C$ equals $g - \tilde{g}$. \hfill \Box

Remark. By looking at nodal curves, one sees that \text{codim}(\mathcal{M}(\overline{g})) = g - \tilde{g}.

4. Universal Jacobian

Let $\overline{\mathcal{J}}$ (resp. $\mathcal{J} \subset \overline{\mathcal{J}}$) be the relative compactified Jacobian (resp. relative Jacobian) of $\mathcal{C}$ over $\mathcal{M}$. Here is the precise definition:

Definition 4.1. For a scheme $S$, let $\mathcal{J}_S$ be the following groupoid:

- Objects of $\mathcal{J}_S$ are pairs $(C, F)$, where $C \to S$ is a flat family of integral projective curves with planar singularities (that is, $C \in \mathcal{M}_S$), and $F$ is an $S$-flat coherent sheaf on $C$ whose restriction to the fibers of $C \to S$ is torsion free of generic rank one and degree zero;
- Morphisms $(C_1, F_1) \to (C_2, F_2)$ are collections

$$(\phi : C_1 \to C_2, \ell, \Phi : F_1 \to \phi^*(F_2) \otimes_{O_S} \ell),$$

where $\phi$ is a morphism of $S$-schemes, and $\ell$ is an invertible sheaf on $S$.

As $S$ varies, groupoids $\mathcal{J}_S$ form a pre-stack; let $\overline{\mathcal{J}}$ be the stack associated to it. Also, consider pairs $(C, F)$ where $C \in \mathcal{M}_S$ and $F$ is a line bundle on $C$ (of degree zero along the fibers of $C \to S$); such pairs form a sub-prestack of $\mathcal{J}$; let $\mathcal{J} \subset \overline{\mathcal{J}}$ be the associated stack.

Clearly, $\mathcal{J} \subset \overline{\mathcal{J}}$ is an open substack. The main properties of these stacks are summarized in the following theorem ([1]):
Theorem 4.1 (Altman, Iarrobino, Kleiman).

(i) $\overline{p} : \overline{J} \to \overline{\mathcal{M}}$ is a projective morphism with irreducible fibers of dimension $g$;
(ii) $\overline{p}$ is locally a complete intersection;
(iii) The restriction $p : \overline{J} \to \mathcal{M}$ is smooth.

Remark. By [14, Corollary B.2], (ii) can be strengthened:

(ii') $\overline{J}$ is smooth.

Clearly, (ii') together with (i) imply (ii).

Remark. The key step in the proof of (i) is Iarrobino’s calculation (see [19]):

\begin{equation}
\dim(\text{Hilb}_k(k[[x,y]])) = k - 1,
\end{equation}

where $\text{Hilb}_k(k[[x,y]])$ is the Hilbert scheme of codimension $k$ ideals in $k[[x,y]]$. For other proofs of (4.1), see [7], [24, Theorem 1.13] and [5]. Also, J. Rego gives an alternative inductive proof of (i) in [25].

Denote by $j$ the rank $g$ vector bundle on $\mathcal{M}$ whose fiber over $C \in \mathcal{M}$ is $H^1(C,O_C)$. Alternatively, $j$ can be viewed as the bundle of (commutative) Lie algebras corresponding to the group scheme $p : \overline{J} \to \mathcal{M}$. The relative dualizing sheaf for $p$ then equals $\Omega^g_{\overline{J}/\mathcal{M}} = \overline{p}^*(\det(j)^{-1})$. It is easy to find the dualizing sheaf for $\overline{p} : \overline{J} \to \mathcal{M}$:

Corollary 4.1. The relative dualizing sheaf $\omega_{\overline{p}^*}$ of $\overline{p}$ equals $\overline{p}^*(\det(j)^{-1})$.

Proof. By Theorem 4.1(ii), $\overline{p}$ is Gorenstein, so $\omega_{\overline{p}^*}$ is a line bundle. Since $\omega_{\overline{p}^*}|\overline{J} = \Omega^g_{\overline{J}/\mathcal{M}}$, it suffices to check that $\text{codim}(\overline{J} - \overline{J}) \geq 2$. But this is clear because a generic curve $C \in \mathcal{M}$ is smooth (see Proposition 3.3).

\section{5. Proof of Theorem 1.1}

Consider the Poincaré bundle on $\overline{\mathcal{J}} \times_{\mathcal{M}} \overline{\mathcal{J}}$. We still denote it by $P$.

Theorem 5.1. Let $p_1 : \overline{\mathcal{J}} \times_{\mathcal{M}} \overline{\mathcal{J}} \to \overline{\mathcal{J}}$ be the projection. Then

\[ R_{p_1*}P = (\Omega^g_{\overline{J}/\mathcal{M}})^{-1} \otimes \zeta_* O_{\mathcal{M}}[-g] = \zeta_* \det(j)[-g], \]

where $\zeta : \mathcal{M} \to \overline{\mathcal{J}}$ is the zero section.

Proof. Consider the dual $P^{-1} = \mathcal{H}om(P,O)$ of $P$. (Actually $P^{-1} = (\nu \times \text{id})^* P$, where $\nu : \overline{J} \to \overline{J}$ is the involution $L \mapsto L^{-1}$.) By Corollary 4.1, the dualizing sheaf of $p_1$ is isomorphic to $p_1^* \Omega^g_{\overline{J}/\mathcal{M}}$. Therefore,

\begin{equation}
R \mathcal{H}om(R_{p_1*}P,O_{\overline{J}}) = (R_{p_1*}P^{-1}) \otimes \Omega^g_{\overline{J}/\mathcal{M}}[g]
\end{equation}

by Serre’s duality.

Combining Corollary 2.2 and Proposition 3.3, we see that

\[ \text{codim}(\text{supp}(R_{p_1*}P)) \geq g. \]

By (5.1), we see that both $R_{p_1*}P$ and $R \mathcal{H}om(R_{p_1*}P,O_{\overline{J}})[-g]$ are concentrated in cohomological degrees from zero to $g$. It is now easy to see that $R_{p_1*}P$ is concentrated in cohomological degree $g$. Indeed, $\overline{J}$ is smooth, so given any coherent sheaf $G$ on $\overline{J}$, we have

\[ \mathcal{E}xt^i(G,O_{\overline{J}}) = 0 \quad \text{for} \quad i < \text{codim(\text{supp} G)}. \]
Equivalently, $R\mathcal{H}om(G, O_{\mathcal{J}})$ is concentrated in cohomological degree $\text{codim}(\text{supp} \, G)$ and above. Since $R\mathcal{H}om(Rp_1_*P, O_{\mathcal{J}})[-g]$ is concentrated in cohomological degrees from zero to $g$, its dual

$$R\mathcal{H}om(R\mathcal{H}om(Rp_1_*P, O_{\mathcal{J}})[-g], O_{\mathcal{J}}) = Rp_1_*P[g]$$

is concentrated in non-negative cohomological degrees. Thus, $Rp_1_*P[g]$ is a sheaf.

By the same argument, $R\mathcal{H}om(Rp_1_*P, O_{\mathcal{J}})$ is a sheaf. Equivalently,

$$\mathcal{E}xt^i(R^gP_1_*P, O_{\mathcal{J}}) = 0, \quad \text{for all } i \neq g.$$ 

That is, $R^gP_1_*P$ is a coherent Cohen–Macaulay sheaf of codimension $g$.

Next, notice that the restriction of $P$ to $\zeta(\mathcal{M}) \times_{\mathcal{M}} \mathcal{J}$ is trivial. This provides a map

$$\zeta^*(R^gP_1_*P) \to R^gP_\ast(O_{\mathcal{J}}).$$

By Serre’s duality, $R^gP_\ast O_{\mathcal{J}} = \det(j)$. Now by adjunction, we obtain a morphism

$$(5.2) \quad R^gP_1_*P \to \zeta_* \det j.$$

It remains to verify that (5.2) is an isomorphism. Since (5.2) is an isomorphism over $\zeta(\mathcal{M})$ by construction, it is enough to verify that $\text{supp}(R^gP_1_*P) = \zeta(\mathcal{M}) \subset \mathcal{J}$.

Let us check that $\text{supp}(R^gP_1_*P)$ equals $\zeta(\mathcal{M})$ as a set. As a set, $\text{supp}(R^gP_1_*P)$ consists of pairs $(C, L) \in \mathcal{J}$ such that the line bundle $L$ on $C$ satisfies $H^0(\mathcal{J}, P_L^-) \neq 0$. In this case, $H^0(\mathcal{J}, P_L^-) \neq 0$ by Serre’s duality. Since $\mathcal{J}$ is irreducible, we see that the line bundle $P_L^- \neq 0$ has a subsheaf isomorphic to $O_{\mathcal{J}}$. On the other hand, the line bundles $P_L^- = P_L^{-1}$ and $O_{\mathcal{J}} = O_C$ are algebraically equivalent, and therefore their Hilbert polynomials coincide. Hence $P_L \simeq O_{\mathcal{J}}$. Finally, we can restrict $P_L$ to the image of the Abel–Jacobi map (see the proof of Corollary 2.1) to obtain $L \simeq O_C$.

To complete the proof, let us verify that $\text{supp}(R^gP_1_*P) = \zeta(\mathcal{M})$ as a scheme. Since $R^gP_1_*P$ is Cohen–Macaulay of codimension $g$, it suffices to check the claim generically on $\zeta(\mathcal{M})$. We can thus restrict ourselves to the open substack of smooth curves in $\mathcal{J}$, and the claim reduces to (0.1). \qed

**Remark.** The proof is similar to an argument of S. Lysenko (see proof of Theorem 4 in [4]), see also D. Mumford’s proof of the theorem in [23, Section III.13].

Theorem 1.1 follows from Theorem 5.1 and base change.

### 6. Autoduality

Recall that the morphism $\rho : J \to \text{Pic} \mathcal{J}$ is given by $L \mapsto P_L$. Since the tangent space to $J$ at $[O_C]$ (resp. to $\text{Pic} \mathcal{J}$ at $[O_{\mathcal{J}}]$) equals $H^1(C, O_C)$ (resp. $H^1(\mathcal{J}, O_{\mathcal{J}})$), the differential of $\rho$ at $[O_C] \in J$ becomes a linear operator

$$d\rho : H^1(C, O_C) \to H^1(\mathcal{J}, O_{\mathcal{J}}).$$

Let us give a more precise form of Theorem 1.2(ii):

**Proposition 6.1.** $d\rho$ is an isomorphism, and the (super-commutative) cohomology algebra $H^\bullet(\mathcal{J}, O_{\mathcal{J}})$ is freely generated by $H^1(\mathcal{J}, O_{\mathcal{J}})$. 

Proof. Let $O_\zeta$ be the structure sheaf of the zero $[O_C] \in J$ viewed as a coherent sheaf on $J$ (it is a sky-scraper sheaf of length one). Note that $\mathfrak{F}(O_\zeta) = O_\mathcal{J}$, where $\mathfrak{F} : D^b(J) \to D^b(\mathcal{J})$ is the Fourier transform of Theorem 1.4. Since $\mathfrak{F}$ is fully faithful, it induces an isomorphism

$$\text{Ext}^\bullet(O_\zeta, O_\zeta) \simeq \text{Ext}^\bullet(O_\mathcal{J}, O_\mathcal{J}) = H^\bullet(\mathcal{J}, O_\mathcal{J}).$$

Finally, $J$ is smooth; therefore, $\text{Ext}^\bullet(O_\zeta, O_\zeta) = \bigwedge^\bullet H^1(C, O_C)$.

Let us fix a line bundle $\ell$ on $C$ of degree minus one. It defines an Abel-Jacobi map $\alpha : C \to \mathcal{J}$, as in the proof of Corollary 2.1. We then obtain a morphism

$$\alpha^* : \text{Pic}(\mathcal{J}) \to \text{Pic}(C) : L \mapsto \alpha^* L.$$

By construction, $\alpha^*$ is a left inverse of $\rho$ (cf. [11, Proposition 2.2]).

Remark. Injectivity of $d\rho$ follows from the existence of the left inverse. Once injectivity is known, bijectivity follows from the equality

$$\dim H^1(\mathcal{J}, O_\mathcal{J}) = \dim H^1(C, O_C) = g.$$

Proof of Theorem 1.3. $\text{Pic}(\mathcal{J})$ is a group scheme of locally finite type (see [16, Theorem 3.1], [13, Theorem 9.4.8], or [2, Corollary (6.4)]). Set

$$\text{Pic}'(\mathcal{J}) = (\alpha^*)^{-1}(J) = \{L \in \text{Pic}(\mathcal{J}) : \deg(\alpha^* L) = 0\}$$

$$K = \ker(\alpha^*) = \{L \in \text{Pic}(\mathcal{J}) : \alpha^* L \simeq O_C\}.$$

Clearly, $K \subset \text{Pic}'(\mathcal{J})$ is closed, and $\text{Pic}'(\mathcal{J}) \subset \text{Pic}(\mathcal{J})$ is both open and closed. The map

$$J \times K \to \text{Pic}'(\mathcal{J}) : (L_1, L_2) \mapsto \rho(L_1) \cdot L_2$$

is an isomorphism. Bijectivity of $d\rho$ (Proposition 6.1) implies that $K$ is a disjoint union of points. Therefore, the connected component of identity of $\text{Pic}(\mathcal{J})$ is contained in $\rho(J)$. Now it remains to notice that $J$ is connected. $\square$

Proposition 6.2. $\text{Pic}^\sigma(\mathcal{J}) = \text{Pic}^0(\mathcal{J})$ (where $\text{Pic}^\sigma$ is defined in (1.2)).

Proof. Consider $\overline{p} : \mathcal{J} \to \mathcal{M}$. It is a projective flat morphism with integral fibers (Theorem 4.1); we can therefore construct the corresponding family of Picard schemes $\text{Pic}(\mathcal{J}/\mathcal{M}) \to \mathcal{M}$ (see the references in the proof of Theorem 1.3). The family is separated and its fiber over $C \in \mathcal{M}$ is $\text{Pic}(\mathcal{J}_C)$.

Let us work in the smooth topology of $\mathcal{M}$. Locally, we can choose a degree minus one line bundle $\ell$ on the universal curve $C \to \mathcal{M}$. As in the proof of Theorem 1.3, we then introduce a map

$$\alpha^* : \text{Pic}(\mathcal{J}/\mathcal{M}) \to \text{Pic}(C/\mathcal{M})$$

and substacks $\text{Pic}'(\mathcal{J}/\mathcal{M}) = (\alpha^*)^{-1}(J)$ and $K = \ker(\alpha^*)$ such that

$$\text{Pic}'(\mathcal{J}/\mathcal{M}) = J \times_\mathcal{M} K.$$

Let $\text{Pic}^\sigma(\mathcal{J}/\mathcal{M}) \subset \text{Pic}(\mathcal{J}/\mathcal{M})$ be the substack whose fiber over $C \in \mathcal{M}$ is $\text{Pic}^\sigma(\mathcal{J}_C)$. We have

$$\text{Pic}^\sigma(\mathcal{J}/\mathcal{M}) = J \times_\mathcal{M} K^\sigma,$$

where

$$K^\sigma = \{L \in K : L^\otimes n \simeq O \text{ for some } n \text{ coprime to char } k\}.$$
By [17, Theorem 2.5], the map

$$\text{Pic} (\mathcal{J}/\mathcal{M}) \to \text{Pic} (\mathcal{J}/\mathcal{M}) : L \mapsto L \otimes n$$

is étale for all $n$ coprime to char $k$. Therefore, $\mathcal{K}^\sigma$ is étale over $\mathcal{M}$.

Finally, the morphism $\mathcal{K}^\sigma \to \mathcal{M}$ is separated, and over the locus of smooth curves $C \in \mathcal{M}$, we have $\text{Pic}^0 (\mathcal{J}_C) = \text{Pic}^\sigma (\mathcal{J}_C)$ by [23, Corollary IV.19.2]. Therefore, $\mathcal{K}^\sigma$ is the zero group scheme, and $\text{Pic}^\sigma (\mathcal{J}/\mathcal{M}) = \mathcal{J}$, as required. □

7. Fibers of the Hitchin fibration

Recall the construction of the Hitchin fibration [18] (for $GL(n)$). Fix a smooth curve $X$ and an integer $n$.

**Definition 7.1.** A Higgs bundle is a rank $n$ vector bundle $E$ on $X$ together with a Higgs field $A : E \to E \otimes \Omega_X$.

Given a Higgs bundle $(E, A)$, consider the characteristic polynomial of $A$:

$$\det (\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n; \quad a_i \in H^0 (X, \Omega_X^{\otimes i}).$$

The zero locus of (7.1) is a curve $C \subset T^*X$: the spectral curve of $A$. Higgs bundle $(E, A)$ gives rise to a coherent sheaf $F$ on $C$; informally, $F$ is the ‘sheaf of co-eigenspaces’: its fiber over a point $(x, \mu) \in T^*X$ is the co-eigenspace

$$\text{coker} (A(x) - \mu : E_x \to E_x \otimes \Omega_{X,x}).$$

Here $x \in X$, $\mu \in \Omega_{X,x}$.

**Proposition 7.1.**

(i) $F$ is a torsion-free sheaf on $C$ whose stalk at any generic point of $C$ has length equal to the multiplicity of the corresponding component of $C$. In particular, if $C$ is reduced, $F$ is a torsion-free sheaf of generic rank one.

(ii) Fix a spectral curve $C$ (that is, fix a polynomial (7.1)). Then $(E, A) \mapsto F$ is a one-to-one correspondence between Higgs bundles with spectral curve $C$ and sheaves $F$ as in (i). □

Given $F$, $E$ is reconstructed as the push-forward of $F$ with respect to $C \to X$. Therefore, $F$ and $E$ have equal Euler characteristics. We have $\chi (O_C) = n^2 \chi (O_X) = n^2 (1 - g)$, where $g$ is the genus of $X$. Hence $\text{deg}(F) = 0$ if and only if $\text{deg}(E) = n(n - 1)(1 - g)$. (Recall that $\text{deg}(F) = \chi (F) - \chi (O_C)$.) Also, note that $(E, A)$ is (semi)stable if and only if $F$ is (semi)stable. If $C$ is integral, $F$ has generic rank one and stability is automatic.

Let $\mathcal{Higgs}$ be the moduli space of semi-stable Higgs bundles $(E, A)$ with $\text{rk} (E) = n$ and $\text{deg} (E) = n(n - 1)(1 - g)$. Also, let $\mathcal{SCurves}$ be the space of spectral curves $C \subset T^*X$; explicitly, $\mathcal{SCurves}$ is the space of coefficients $(a_1, \ldots, a_n)$ of (7.1):

$$\mathcal{SCurves} = \prod_{i=1}^n H^0 (X, \Omega_X^{\otimes i}).$$

Finally, let $\mathcal{SCurves}' \subset \mathcal{SCurves}$ be the locus of integral spectral curves $C \subset T^*X$.

The correspondence $(E, A) \mapsto C$ gives a map $h : \mathcal{Higgs} \to \mathcal{SCurves}$ (the Hitchin fibration). For $C \in \mathcal{SCurves}$, the fiber $h^{-1}(C)$ is the space of Higgs bundles with
spectral curve $C$; Proposition 7.1 identifies $h^{-1}(C)$ with the moduli space of semi-stable coherent sheaves $F$ on $C$ that satisfy Proposition 7.1(i) and have degree zero. In other words, the fiber is the compactified Jacobian of $C$.

The results of this paper can be applied to integral spectral curves $C \in SCurves'$. For instance, Theorem 1.2(ii) implies that

$$H^i(h^{-1}(C), O) = \bigwedge^i H^1(C, O_C).$$

Actually, applying the relative version of Theorem 1.2(ii) to the universal family of spectral curves, we obtain an isomorphism

$$(7.2) \quad (R^i h_* O_{Higgs})|_{SCurves'} = \Omega^i_{SCurves'},$$

where we used the symplectic form on $T^*X$ to identify $H^1(C, O_C)$ with the cotangent space to $C \in SCurves'$. Recently, E. Frenkel and C. Teleman proved that the isomorphism (7.2) can be extended to the space of all spectral curves:

**Theorem 7.1.** There is an isomorphism

$$R^i h_* O_{Higgs} = \Omega^i_{SCurves}. \quad \square$$

When $i = 0, 1$, Theorem 7.1 is proved by N. Hitchin ([18, Theorems 6.2 and 6.5]); the general case is announced in [15].

**Remarks.** (i) In [18], N. Hitchin works with the Hitchin fibration for the group $SL(2)$, but his argument can be used to compute $R^i h_* O_{Higgs}$ for arbitrary $n$ (still assuming $i = 0, 1$). Actually, essentially the same argument computes $R^i \overline{p}_* O_{\overline{J}}$ for $i = 0, 1$. (Recall that $\overline{p} : \overline{J} \to M$ is the universal compactified Jacobian over the moduli stack of curves $M$.)

(ii) In [15], Theorem 7.1 is stated for the Hitchin fibration for arbitrary group, not just $GL(n)$.

(iii) One can derive some of our results from Theorem 7.1, at least for integral curves $C$ that appear as spectral curves of the Hitchin fibration. Indeed, for such $C \in SCurves'$, Theorem 7.1 implies Theorem 1.2(ii). In turn, this implies Theorem 4.1. Also, one can easily derive from Theorem 1.2(ii) that the isomorphism of Theorem 1.1 exists on some neighborhood $U$ of $\zeta \in J$, so Theorem 1.2(i) holds for $L \in U$. Similarly, we see that $P$ defines a fully faithful Fourier–Mukai transform from $D^b(U)$ to $D^b(J)$.

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