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COHOMOLOGY OF LINE BUNDLES ON COMPACTIFIED JACOBIANS

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Abstract. Let C be an integral projective curve with planar singularities. For the compactified Jacobian $\overline{J}$ of C, we prove that topologically trivial line bundles on $\overline{J}$ are in one-to-one correspondence with line bundles on C (the autoduality conjecture), and compute the cohomology of $\overline{J}$ with coefficients in these line bundles. We also show that the natural Fourier–Mukai functor from the derived category of quasi-coherent sheaves on J (where J is the Jacobian of X) to that of quasi-coherent sheaves on $\overline{J}$ is fully faithful.

0. Introduction

Let C be a smooth irreducible projective curve over a field $\Bbbk$, and let J be the Jacobian of C. As an abelian variety, J is self-dual. More precisely, $J \times J$ carries a natural line bundle (the Poincaré bundle) P that is universal as a family of topologically trivial line bundles on J.

The Poincaré bundle defines the Fourier–Mukai functor

$$\mathfrak{F} : D^b(J) \to D^b(J) : \mathcal{F} \mapsto Rp_{2,*}(p_1^*(\mathcal{F}) \otimes P).$$

Here $D^b(J)$ is the derived category of quasi-coherent sheaves on J and $p_{1,2} : J \times J \to J$ are the projections. Mukai [22] proved that $\mathfrak{F}$ is an equivalence of categories; the proof uses the formula

$$Rp_{1,*}P \simeq O_\zeta[-g], \tag{0.1}$$

where O_ζ is the structure sheaf of the zero element $\zeta \in J$ and g is the genus of C. Formula (0.1) goes back to Mumford (see the proof of the theorem in [23, Section III.13]).

Now suppose that C is a singular curve, which we assume to be projective and integral. The Jacobian J is no longer projective, but it admits a natural compactification $\overline{J} \supset J$. By definition, $\overline{J}$ is the moduli space of torsion-free sheaves F on C such that F has generic rank one and $\chi(F) = \chi(O_C)$; J is identified with the open subset of locally free sheaves. It is natural to ask whether $\overline{J}$ is in some sense self-dual. For instance, one can look for a Poincaré sheaf (or complex of sheaves) $\overline{P}$ on $\overline{J} \times \overline{J}$. One can then ask whether $\overline{P}$ is, in some sense, a universal family of sheaves on $\overline{J}$ and whether the corresponding Fourier–Mukai functor $\overline{\mathfrak{F}} : D^b(\overline{J}) \to D^b(\overline{J})$ is an equivalence.

In the case when the singularities of C are nodes or cusps, such Poincaré sheaf $\overline{P}$ is constructed by E. Esteves and S. Kleiman in [12]; they also prove the universality of

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$\overline{P}$. In addition, if C is a singular plane cubic, $\overline{\mathfrak{F}}$ is known to be an equivalence ([8, 9], also formulated as Theorem 5.2 in [6]).

If the singularities of C are more general, constructing the Poincaré sheaf $\overline{P}$ on $\overline{J} \times \overline{J}$ is much harder (see Remark (i) at the end of the introduction). However, it is easy to construct a Poincaré bundle P on $J \times \overline{J}$. It can then be used to define a Fourier–Mukai transform

$$\mathfrak{F} : D^b(J) \to D^b(\overline{J}) : \mathcal{F} \mapsto Rp_{2,*}(p_1^*(\mathcal{F}) \otimes P). \tag{0.2}$$

In this paper, we assume that C is an integral projective curve with planar singularities; the main result is that formula (0.1) still holds in this case. This implies that (0.2) is fully faithful. As a simple corollary, we prove the following autoduality result: P is the universal family of topologically trivial line bundles on $\overline{J}$, so that J is identified with the connected component of the trivial bundle in the moduli space of line bundles on $\overline{J}$. This generalizes the Autoduality Theorem of [11] (see the remark after Theorem 1.3).

Remarks. (i) Suppose that there exists an extension of P to a sheaf $\overline{P}$ on $\overline{J} \times \overline{J}$ such that the corresponding Fourier–Mukai transform $\overline{\mathfrak{F}} : D^b(\overline{J}) \to D^b(\overline{J})$ is an equivalence. After the first version of this paper was completed, such an extension was constructed in [3]. Then (0.2) is a composition of $\overline{\mathfrak{F}}$ and the direct image $j_* : D^b(J) \to D^b(\overline{J})$ for the open embedding $j : J \hookrightarrow \overline{J}$. Since j_* is fully faithful, so is (0.2). Thus our result is natural assuming existence of $\overline{\mathfrak{F}}$.

(ii) In this paper, we work with bounded derived categories of quasi-coherent sheaves. However, formula (0.2) also defines a functor between unbounded derived categories of quasi-coherent sheaves, and our results remain true in these settings: this extended functor $\mathfrak{F}$ is still fully faithful. On the other hand, $\mathfrak{F}$ does not preserve coherence; therefore, it is important to consider the derived categories of quasi-coherent (rather than coherent) sheaves.

(iii) Compactified Jacobians appear as (singular) fibers of the Hitchin fibration for the group $GL(n)$; therefore, our results can be interpreted as a kind of autoduality of the Hitchin fibration. Conversely, some of our results can be derived from a theorem of E. Frenkel and C. Teleman [15] (see Theorem 7.1). We explore this relation in more details in Section 7.

(iv) Recall that the curve C is assumed to be integral with planar singularities. We assume integrality of C to avoid working with stability conditions for sheaves on C. It is likely that our argument works without this assumption if one fixes an ample line bundle on C and defines the compactified Jacobian $\overline{J}$ to be the moduli space of semi-stable torsion-free sheaves of degree zero. Such generalization is natural in view of the previous remark, because some fibers of the Hitchin fibration are compactified Jacobians of non-integral curves.

On the other hand, the assumption that C has planar singularities is more important. There are two reasons why the assumption is natural. First of all, for an integral curve C, $\overline{J}$ is irreducible if and only if the singularities of C are planar ([20]); so if one drops this assumption, J is no longer dense in $\overline{J}$. Secondly, only compactified Jacobians of curves with planar singularities appear in the Hitchin fibration.

1. Main results

Fix a ground field $\Bbbk$. For convenience, let us assume that $\Bbbk$ is algebraically closed. Let C be an integral projective curve over $\Bbbk$. Denote by J its Jacobian, that is, J is the moduli space of line bundles on C of degree zero. Denote by $\overline{J}$ the compactified Jacobian; in other words, $\overline{J}$ is the moduli space of torsion-free sheaves on C of generic rank one and degree zero. (For a sheaf F of generic rank one, the degree is $\deg(F) = \chi(F) - \chi(O_C)$.)

Let P be the Poincaré bundle; it is a line bundle on $J \times \overline{J}$. Its fiber over $(L, F) \in (J \times \overline{J})$ equals

$$P_{(L,F)} = \det \mathrm{R}\Gamma(L \otimes F) \otimes \det \mathrm{R}\Gamma(O_C) \otimes \det \mathrm{R}\Gamma(L)^{-1} \otimes \det \mathrm{R}\Gamma(F)^{-1}. \tag{1.1}$$

More explicitly, we can write $L \simeq O(\sum a_i x_i)$ for a divisor $\sum a_i x_i$ supported by the smooth locus of C, and then

$$P_{(L,F)} = \bigotimes (F_{x_i})^{\otimes a_i}.$$

From now on, we assume that C has planar singularities; that is, the tangent space to C at any point is at most two-dimensional. Our main result is the computation of the direct image of P:

Theorem 1.1.

$$Rp_{1,*}P = \det(H^1(C, O_C)) \otimes O_\zeta[-g].$$

Here $p_1 : J \times \overline{J} \to J$ is the projection, and O_ζ is the structure sheaf of the neutral element $\zeta = [O_C] \in J$ (so O_ζ is a sky-scraper sheaf at ζ).

Let us now view P as a family of line bundles on $\overline{J}$ parameterized by J. For fixed $L \in J$, denote the corresponding line bundle on $\overline{J}$ by P_L. In other words, P_L is the restriction of P to $\{L\} \times \overline{J}$. Applying base change, we can use Theorem 1.1 to compute cohomology of P_L:

Theorem 1.2. (i) *If $L \not\simeq O_C$, then $H^i(\overline{J}, P_L) = 0$ for any i;*

(ii) *If $L = O_C$, then $P_L = O_{\overline{J}}$ and $H^i(\overline{J}, O_{\overline{J}}) = \bigwedge^i H^1(C, O_C)$. (The identification is described more explicitly in Proposition 6.1.)* □

Let $\mathrm{Pic}(\overline{J})$ be the moduli space of line bundles on $\overline{J}$. The correspondence $L \mapsto P_L$ can be viewed as a morphism $\rho : J \to \mathrm{Pic}(\overline{J})$. Denote by $\mathrm{Pic}^0(\overline{J}) \subset \mathrm{Pic}(\overline{J})$ the connected component of the identity $[O_{\overline{J}}] \in \mathrm{Pic}(\overline{J})$. In Section 6, we derive the following statement.

Theorem 1.3. *ρ gives an isomorphism $J \widetilde{\to} \mathrm{Pic}^0(\overline{J})$.*

Remark. Theorem 1.3 answers the question raised in [11]. Following [17], set

$$\begin{aligned} &\mathrm{Pic}^\tau(\overline{J}) = \{L \in \mathrm{Pic}(\overline{J}) : L^{\otimes n} \in \mathrm{Pic}^0(\overline{J}) \text{ for some } n > 0\}, \\ &\mathrm{Pic}^\sigma(\overline{J}) = \{L \in \mathrm{Pic}(\overline{J}) : L^{\otimes n} \in \mathrm{Pic}^0(\overline{J}) \text{ for some } n \text{ coprime to } \operatorname{char}\Bbbk\} \end{aligned} \tag{1.2}$$

(if $\operatorname{char}\Bbbk = 0$, $\mathrm{Pic}^\sigma(\overline{J}) = \mathrm{Pic}^\tau(\overline{J})$ by definition). The main result of [11] is the Autoduality Theorem, which claims that if all singularities of C are double points, then $\rho : J \widetilde{\to} \mathrm{Pic}^0(\overline{J})$ and $\mathrm{Pic}^0(\overline{J}) = \mathrm{Pic}^\tau(\overline{J})$. Theorem 1.3 generalizes the first statement to curves with planar singularities; as for the second statement, we show in

Proposition 6.2 that $\mathrm{Pic}^0(\overline{J}) = \mathrm{Pic}^\sigma(\overline{J})$. We do not know whether $\mathrm{Pic}^\tau(\overline{J})$ and $\mathrm{Pic}^\sigma(\overline{J})$ coincide when $\mathrm{char}(\Bbbk) > 0$ and C has planar singularities.

Theorem 1.1 can be reformulated in terms of the Fourier functor

$$\mathfrak{F} : D^b(J) \to D^b(\overline{J}) : \mathcal{F} \mapsto Rp_{2,*}(p_1^*(\mathcal{F}) \otimes P)$$

given by P. Recall that $D^b(J)$ stands for the (bounded) derived category of quasi-coherent sheaves on J. The functor $\mathfrak{F}$ admits a left adjoint given by

$$\mathfrak{F}^\vee : D^b(\overline{J}) \to D^b(J) : \mathcal{F} \mapsto Rp_{1,*}(p_2^*(\mathcal{F}) \otimes P^{-1}) \otimes \det(H^1(C, O_C))^{-1}[g].$$

This formula relies on the computation of the dualizing sheaf on $\overline{J}$: see Corollary 4.1.

Theorem 1.4. (i) *The composition* $\mathfrak{F}^\vee \circ \mathfrak{F}$ *is isomorphic to the identity functor.*
(ii) $\mathfrak{F}$ *is fully faithful.*

Proof. The first statement follows from Theorem 1.1 by base change. (This is completely analogous to the original argument of [22, Theorem 2.2].) This implies the second statement, because the functors $\mathfrak{F}^\vee$ and $\mathfrak{F}$ are adjoint. □

Remark. For simplicity, we considered a single curve C in this section. However, all our results hold for families of curves. Actually, we prove Theorem 1.1 for the universal family of curves (Theorem 5.1); base change then implies that the statement holds for any family, and, in particular, for any single curve.

2. Line bundles on a compactified Jacobian

Proposition 2.1. *Suppose* $H^i(\overline{J}, P_L) \neq 0$ *for some* i. *Then* $(P_L)|_J \simeq O_J$.

Proof. Let $T \to J$ be the $\mathbf{G_m}$-torsor corresponding to $(P_L)|_J$. One easily sees that T is naturally an abelian group that is an extension of J by $\mathbf{G_m}$. The action of J on $\overline{J}$ lifts to an action of T on P_L, therefore, T also acts on $H^i(\overline{J}, P_L)$. Note that $\mathbf{G_m} \subset T$ acts via the tautological character.

Let $V \subset H^i(\overline{J}, P_L)$ be an irreducible T-submodule. Since T is commutative, $\dim(V) = 1$. The action of T on V is given by a character $\chi : T \to \mathbf{G_m}$. Since $\chi|_{\mathbf{G_m}} = id$, we see that χ gives a splitting $T \simeq \mathbf{G_m} \times J$. This implies the statement. □

Remark. If C is smooth, Proposition 2.1 is equivalent to observation (vii) in [23, Section II.8]; however, our proof uses a slightly different idea, which is better adapted to the singular case.

Let $C^0 \subset C$ be the smooth locus of C.

Corollary 2.1. *Suppose* $H^i(\overline{J}, P_L) \neq 0$ *for some* i. *Then* $L|_{C^0} \simeq O_{C^0}$.

Proof. Fix a degree minus one line bundle ℓ on C. It defines an Abel–Jacobi map

$$\alpha : C \to \overline{J} : c \mapsto \ell(c).$$

Here $\ell(c)$ is defined as the sheaf of homomorphisms from the ideal sheaf of $c \in C$ to ℓ. Notice that $\alpha^*(P_L) \simeq L$ and $\alpha(C^0) \subset J$. Now Proposition 2.1 completes the proof. □

Set

$$N = \{L \in J : H^i(\overline{J}, P_L) \neq 0 \text{ for some } i\} \subset J.$$

Clearly, $N \subset J$ is closed (by the Semicontinuity Theorem), and $N = \operatorname{supp}(Rp_{1,*}P)$, where $p_1 : J \times \overline{J} \to J$ is the projection (by base change).

Corollary 2.2. *Let g be the (arithmetic) genus of C and $\tilde{g}$ be its geometric genus, that is, the genus of its normalization. Then* $\dim(N) \leq (g - \tilde{g})$.

Proof. Let $\nu : \tilde{C} \to C$ be the normalization, and let $\tilde{J}$ be the Jacobian of $\tilde{C}$. The map $\nu^* : J \to \tilde{J}$ is smooth and surjective; its fibers have dimension $(g - \tilde{g})$.

Denote by $\tilde{N} \subset \tilde{J}$ the set of line bundles on $\tilde{C}$ that are trivial on $\nu^{-1}(C^0) \subset \tilde{C}$. By Corollary 2.1, $\nu^*(N) \subset \tilde{N}$. Now it suffices to note that $\tilde{N}$ is a countable set. □

3. Moduli of curves

Let $\mathcal{M} = \mathcal{M}_g$ be the moduli stack of integral projective curves C of genus g with planar singularities. The following properties of $\mathcal{M}$ are well known:

Proposition 3.1. *$\mathcal{M}$ is a smooth algebraic stack of finite type;* $\dim(\mathcal{M}) = 3g - 3$. □

Remark. Denote by $\mathcal{C}$ the universal curve over $\mathcal{M}$; that is, $\mathcal{C}$ is the moduli stack of pairs $(C \in \mathcal{M}, c \in C)$. One easily checks that $\mathcal{C}$ is a smooth stack of dimension $3g - 2$. This is similar to the statement (ii') after Theorem 4.1.

Consider the normalization $\tilde{C}$ of a curve $C \in \mathcal{M}$, and let $\tilde{g}$ be the genus of $\tilde{C}$ (that is, the geometric genus of C). We need some results on the stratification of $\mathcal{M}$ by geometric genus due to Teissier [26], Diaz and Harris [10], and Laumon [21]. Since our settings are somewhat different, we provide the proofs.

Denote by $\mathcal{M}^{(\tilde{g})} \subset \mathcal{M}$ the locus of curves $C \in \mathcal{M}$ of geometric genus $\tilde{g}$. Note that we view $\mathcal{M}^{(\tilde{g})}$ simply as a subset of the set of points of $\mathcal{M}$, rather than a substack.

Proposition 3.2. *$\mathcal{M}^{(\tilde{g})}$ is a stratification of $\mathcal{M}$:*

$$\overline{(\mathcal{M}^{(\tilde{g})})} \subset \bigcup_{\gamma \leq \tilde{g}} \mathcal{M}^{(\gamma)}.$$

In particular, $\mathcal{M}^{(\tilde{g})} \subset \mathcal{M}$ is locally closed.

Proof. Let $\mathcal{S}$ be the stack of birational morphisms $(\nu : \tilde{C} \to C)$, where $C \in \mathcal{M}$, and $\tilde{C}$ is an integral projective curve of genus $\tilde{g}$ (with arbitrary singularity). Consider the forgetful map

$$\pi : \mathcal{S} \to \mathcal{M} : (\nu : \tilde{C} \to C) \mapsto C.$$

Clearly,

$$\pi(\mathcal{S}) \subset \bigcup_{\gamma \leq \tilde{g}} \mathcal{M}^{(\gamma)}.$$

Therefore, it suffices to show that π is projective.

Let $\mathcal{S}''$ be the stack of collections (C, F, s), where $C \in \mathcal{M}$, F is a torsion-free sheaf on C of generic rank one and degree $g - \tilde{g}$, $s \in H^0(C, F) - \{0\}$. Also, let $\mathcal{S}'$ be the stack of collections (C, F, s, μ), where $(C, F, s) \in \mathcal{S}''$ and $\mu : F \otimes F \to F$ is such that $\mu(s \otimes s) = s$. Consider

$$\mathcal{S} \to \mathcal{S}' : (\nu : \tilde{C} \to C) \mapsto (C, \nu_*(O_{\tilde{C}}), 1, \mu),$$

where μ is the product on the sheaf of algebras $\nu_*(O_{\tilde{C}})$. This identifies $\mathcal{S}$ and $\mathcal{S}'$. The forgetful map

$$\mathcal{S}' \to \mathcal{S}'' : (C, F, s, \mu) \mapsto (C, F, s)$$

is a closed embedding (essentially because μ is uniquely determined by $\mu(s \otimes s) = s$). Finally, the map

$$\mathcal{S}'' \to \mathcal{M} : (C, F, s) \mapsto C$$

is projective. □

Proposition 3.3. $\operatorname{codim}(\mathcal{M}^{(\tilde{g})}) \geq (g - \tilde{g})$.

Proof. Let $\mathcal{S}$ be as in the proof of Proposition 3.2. Denote by $\mathcal{S}^0$ the substack of morphisms $(\nu : \tilde{C} \to C) \in \mathcal{S}$ with smooth $\tilde{C}$; clearly, $\mathcal{M}^{(\tilde{g})} = \pi(\mathcal{S}^0)$. Therefore, we need to show that $\dim(\mathcal{S}^0) \leq 2g + \tilde{g} - 3$.

Consider the morphism

$$\tilde{\pi} : \mathcal{S}^0 \to \mathcal{M}_{\tilde{g}} : (\nu : \tilde{C} \to C) \mapsto \tilde{C}.$$

It suffices to show $\dim(\tilde{\pi}^{-1}(\tilde{C})) \leq 2(g - \tilde{g})$ for any $\tilde{C} \in \mathcal{M}_{\tilde{g}}$. Fix $(\nu : \tilde{C} \to C) \in \mathcal{S}^0$. Let us prove that the dimension of the tangent space $T_\nu \tilde{\pi}^{-1}(\tilde{C})$ to $\tilde{\pi}^{-1}(\tilde{C})$ at this point is at most $2(g - \tilde{g})$.

$T_\nu \tilde{\pi}^{-1}(\tilde{C})$ is isomorphic to the space of first-order deformations of O_C viewed as a sheaf of subalgebras of $\nu_* O_{\tilde{C}}$. This yields an isomorphism

$$T_\nu \tilde{\pi}^{-1}(\tilde{C}) = \{\text{derivations } O_C \to \nu_* O_{\tilde{C}}/O_C\} = \operatorname{Hom}_{O_C}(\Omega_C, \nu_* O_{\tilde{C}}/O_C).$$

Now it suffices to notice that the fibers of the cotangent sheaf Ω_C are at most two-dimensional, and that the length of the sky-scraper sheaf $\nu_* O_{\tilde{C}}/O_C$ equals $g - \tilde{g}$. □

Remark. By looking at nodal curves, one sees that $\operatorname{codim}(\mathcal{M}^{(\tilde{g})}) = g - \tilde{g}$.

4. Universal Jacobian

Let $\overline{\mathcal{J}}$ (resp. $\mathcal{J} \subset \overline{\mathcal{J}}$) be the relative compactified Jacobian (resp. relative Jacobian) of $\mathcal{C}$ over $\mathcal{M}$. Here is the precise definition:

Definition 4.1. For a scheme S, let $\hat{\mathcal{J}}_S$ be the following groupoid:

- Objects of $\hat{\mathcal{J}}_S$ are pairs (C, F), where $C \to S$ is a flat family of integral projective curves with planar singularities (that is, $C \in \mathcal{M}_S$), and F is a S-flat coherent sheaf on C whose restriction to the fibers of $C \to S$ is torsion free of generic rank one and degree zero;
- Morphisms $(C_1, F_1) \to (C_2, F_2)$ are collections

$$(\phi : C_1 \tilde{\to} C_2, \ell, \Phi : F_1 \tilde{\to} \phi^*(F_2) \otimes_{O_S} \ell),$$

where ϕ is a morphism of S-schemes, and ℓ is an invertible sheaf on S.

As S varies, groupoids $\hat{\mathcal{J}}_S$ form a pre-stack; let $\overline{\mathcal{J}}$ be the stack associated to it. Also, consider pairs (C, F) where $C \in \mathcal{M}_S$ and F is a line bundle on C (of degree zero along the fibers of $C \to S$); such pairs form a sub-prestack of $\hat{\mathcal{J}}$; let $\mathcal{J} \subset \overline{\mathcal{J}}$ be the associated stack.

Clearly, $\mathcal{J} \subset \overline{\mathcal{J}}$ is an open substack. The main properties of these stacks are summarized in the following theorem ([1]):

Theorem 4.1 (Altman, Iarrobino, Kleiman)**.**

(i) *$\overline{p} : \overline{\mathcal{J}} \to \mathcal{M}$ is a projective morphism with irreducible fibers of dimension g;*
(ii) *$\overline{p}$ is locally a complete intersection;*
(iii) *The restriction $p : \mathcal{J} \to \mathcal{M}$ is smooth.*

Remark. By [14, Corollary B.2], (ii) can be strengthened:

(ii') $\overline{\mathcal{J}}$ is smooth.

Clearly, (ii') together with (i) imply (ii).

Remark. The key step in the proof of (i) is Iarrobino's calculation (see [19]):

$$\dim(\mathrm{Hilb}_k(\Bbbk[[x,y]])) = k-1, \tag{4.1}$$

where $\mathrm{Hilb}_k(\Bbbk[[x,y]])$ is the Hilbert scheme of codimension k ideals in $\Bbbk[[x,y]]$. For other proofs of (4.1), see [7], [24, Theorem 1.13] and [5]. Also, J. Rego gives an alternative inductive proof of (i) in [25].

Denote by $\mathfrak{j}$ the rank g vector bundle on $\mathcal{M}$ whose fiber over $C \in \mathcal{M}$ is $H^1(C, O_C)$. Alternatively, $\mathfrak{j}$ can be viewed as the bundle of (commutative) Lie algebras corresponding to the group scheme $p : \mathcal{J} \to \mathcal{M}$. The relative dualizing sheaf for p then equals $\Omega^g_{\mathcal{J}/\mathcal{M}} = p^*(\det(\mathfrak{j})^{-1})$. It is easy to find the dualizing sheaf for $\overline{p} : \overline{\mathcal{J}} \to \mathcal{M}$:

Corollary 4.1. *The relative dualizing sheaf $\omega_{\overline{p}}$ of $\overline{p}$ equals $\overline{p}^*(\det(\mathfrak{j})^{-1})$.*

Proof. By Theorem 4.1(ii), $\overline{p}$ is Gorenstein, so $\omega_{\overline{p}}$ is a line bundle. Since $\omega_{\overline{p}}|_{\mathcal{J}} = \Omega^g_{\mathcal{J}/\mathcal{M}}$, it suffices to check that $\mathrm{codim}(\overline{\mathcal{J}} - \mathcal{J}) \geq 2$. But this is clear because a generic curve $C \in \mathcal{M}$ is smooth (see Proposition 3.3). □

5. Proof of Theorem 1.1

Consider the Poincaré bundle on $\mathcal{J} \times_{\mathcal{M}} \overline{\mathcal{J}}$. We still denote it by P.

Theorem 5.1. *Let $p_1 : \mathcal{J} \times_{\mathcal{M}} \overline{\mathcal{J}} \to \mathcal{J}$ be the projection. Then*

$$Rp_{1,*}P = (\Omega^g_{\mathcal{J}/\mathcal{M}})^{-1} \otimes \zeta_* O_{\mathcal{M}}[-g] = \zeta_* \det(\mathfrak{j})[-g],$$

where $\zeta : \mathcal{M} \to \mathcal{J}$ is the zero section.

Proof. Consider the dual $P^{-1} = \mathcal{H}om(P, O)$ of P. (Actually $P^{-1} = (\nu \times id)^* P$, where $\nu : \mathcal{J} \to \mathcal{J}$ is the involution $L \mapsto L^{-1}$.) By Corollary 4.1, the dualizing sheaf of p_1 is isomorphic to $p_1^* \Omega^g_{\mathcal{J}/\mathcal{M}}$. Therefore,

$$R\mathcal{H}om(Rp_{1,*}P, O_{\mathcal{J}}) = (Rp_{1,*}P^{-1}) \otimes \Omega^g_{\mathcal{J}/\mathcal{M}}[g] \tag{5.1}$$

by Serre's duality.

Combining Corollary 2.2 and Proposition 3.3, we see that

$$\mathrm{codim}(\mathrm{supp}(Rp_{1,*}P)) \geq g.$$

By (5.1), we see that both $Rp_{1,*}P$ and $R\mathcal{H}om(Rp_{1,*}P, O_{\mathcal{J}})[-g]$ are concentrated in cohomological degrees from zero to g. It is now easy to see that $Rp_{1,*}P$ is concentrated in cohomological degree g. Indeed, $\mathcal{J}$ is smooth, so given any coherent sheaf G on $\mathcal{J}$, we have

$$\mathcal{E}xt^i(G, O_{\mathcal{J}}) = 0 \quad \text{for} \quad i < \mathrm{codim}(\mathrm{supp}\, G).$$

Equivalently, $R\mathcal{H}om(G, O_{\mathcal{J}})$ is concentrated in cohomological degree $\operatorname{codim}(\operatorname{supp} G)$ and above. Since $R\operatorname{Hom}(Rp_{1,*}P, O_{\mathcal{J}})[-g]$ is concentrated in cohomological degrees from zero to g, its dual

$$R\mathcal{H}om(R\mathcal{H}om(Rp_{1,*}P, O_{\mathcal{J}})[-g], O_{\mathcal{J}}) = Rp_{1,*}P[g]$$

is concentrated in non-negative cohomological degrees. Thus, $Rp_{1,*}P[g]$ is a sheaf.

By the same argument, $R\mathcal{H}om(Rp_{1,*}P, O_{\mathcal{J}})$ is a sheaf. Equivalently,

$$\mathcal{E}xt^i(R^g p_{1,*}P, O_{\mathcal{J}}) = 0, \quad \text{for all} \quad i \neq g.$$

That is, $R^g p_{1,*}P$ is a coherent Cohen–Macaulay sheaf of codimension g.

Next, notice that the restriction of P to $\zeta(\mathcal{M}) \times_{\mathcal{M}} \overline{\mathcal{J}}$ is trivial. This provides a map

$$\zeta^*(R^g p_{1,*}P) \to R^g \overline{p}_*(O_{\overline{\mathcal{J}}}).$$

By Serre's duality, $R^g \overline{p}_* O_{\overline{\mathcal{J}}} = \det(\mathfrak{j})$. Now by adjunction, we obtain a morphism

$$R^g p_{1,*}P \to \zeta_* \det \mathfrak{j}. \tag{5.2}$$

It remains to verify that (5.2) is an isomorphism. Since (5.2) is an isomorphism over $\zeta(\mathcal{M})$ by construction, it is enough to verify that $\operatorname{supp}(R^g p_{1,*}P) = \zeta(\mathcal{M}) \subset \mathcal{J}$.

Let us check that $\operatorname{supp}(R^g p_{1,*}P)$ equals $\zeta(\mathcal{M})$ as a set. As a set, $\operatorname{supp}(R^g p_{1,*}P)$ consists of pairs $(C, L) \in \mathcal{J}$ such that the line bundle L on C satisfies $H^g(\overline{J}, P_L) \neq 0$. In this case, $H^0(\overline{J}, P_L^{-1}) \neq 0$ by Serre's duality. Since $\overline{J}$ is irreducible, we see that the line bundle P_L^{-1} has a subsheaf isomorphic to $O_{\overline{J}}$. On the other hand, the line bundles $P_L^{-1} = P_{L^{-1}}$ and $O_{\overline{J}} = P_O$ are algebraically equivalent, and therefore their Hilbert polynomials coincide. Hence $P_L \simeq O_{\overline{J}}$. Finally, we can restrict P_L to the image of the Abel–Jacobi map (see the proof of Corollary 2.1) to obtain $L \simeq O_C$.

To complete the proof, let us verify that $\operatorname{supp}(R^g_{1,*}P) = \zeta(\mathcal{M})$ as a scheme. Since $R^g_{1,*}P$ is Cohen–Macaulay of codimension g, it suffices to check the claim generically on $\zeta(\mathcal{M})$. We can thus restrict ourselves to the open substack of smooth curves in $\mathcal{M}$, and the claim reduces to (0.1). □

Remark. The proof is similar to an argument of S. Lysenko (see proof of Theorem 4 in [4]), see also D. Mumford's proof of the theorem in [23, Section III.13].

Theorem 1.1 follows from Theorem 5.1 and base change.

6. Autoduality

Recall that the morphism $\rho : J \to \operatorname{Pic}_{\overline{J}}$ is given by $L \mapsto P_L$. Since the tangent space to J at $[O_C]$ (resp. to $\operatorname{Pic}(\overline{J})$ at $[O_{\overline{J}}]$) equals $H^1(C, O_C)$ (resp. $H^1(\overline{J}, O_{\overline{J}})$), the differential of ρ at $[O_C] \in J$ becomes a linear operator

$$d\rho : H^1(C, O_C) \to H^1(\overline{J}, O_{\overline{J}}).$$

Let us give a more precise form of Theorem 1.2(ii):

Proposition 6.1. *$d\rho$ is an isomorphism, and the (super-commutative) cohomology algebra $H^\bullet(\overline{J}, O_{\overline{J}})$ is freely generated by $H^1(\overline{J}, O_{\overline{J}})$.*

Proof. Let O_ζ be the structure sheaf of the zero $[O_C] \in J$ viewed as a coherent sheaf on J (it is a sky-scraper sheaf of length one). Note that $\mathfrak{F}(O_\zeta) = O_{\overline{J}}$, where $\mathfrak{F} : D^b(J) \to D^b(\overline{J})$ is the Fourier transform of Theorem 1.4. Since $\mathfrak{F}$ is fully faithful, it induces an isomorphism

$$\mathrm{Ext}^\bullet(O_\zeta, O_\zeta) \simeq \mathrm{Ext}^\bullet(O_{\overline{J}}, O_{\overline{J}}) = H^\bullet(\overline{J}, O_{\overline{J}}).$$

Finally, J is smooth; therefore, $\mathrm{Ext}^\bullet(O_\zeta, O_\zeta) = \bigwedge^\bullet H^1(C, O_C)$. □

Let us fix a line bundle ℓ on C of degree minus one. It defines an Abel-Jacobi map $\alpha : C \to \overline{J}$, as in the proof of Corollary 2.1. We then obtain a morphism

$$\alpha^* : \mathrm{Pic}(\overline{J}) \to \mathrm{Pic}(C) : L \mapsto \alpha^* L.$$

By construction, α^* is a left inverse of ρ (cf. [11, Proposition 2.2]).

Remark. Injectivity of $d\rho$ follows from the existence of the left inverse. Once injectivity is known, bijectivity follows from the equality

$$\dim H^1(\overline{J}, O_{\overline{J}}) = \dim H^1(C, O_C) = g.$$

Proof of Theorem 1.3. $\mathrm{Pic}(\overline{J})$ is a group scheme of locally finite type (see [16, Theorem 3.1], [13, Theorem 9.4.8], or [2, Corollary (6.4)]). Set

$$\mathrm{Pic}'(\overline{J}) = (\alpha^*)^{-1}(J) = \{L \in \mathrm{Pic}(\overline{J}) : \deg(\alpha^* L) = 0\}$$
$$K = \ker(\alpha^*) = \{L \in \mathrm{Pic}(\overline{J}) : \alpha^* L \simeq O_C\}.$$

Clearly, $K \subset \mathrm{Pic}'(\overline{J})$ is closed, and $\mathrm{Pic}'(\overline{J}) \subset \mathrm{Pic}(\overline{J})$ is both open and closed. The map

$$J \times K \to \mathrm{Pic}'(\overline{J}) : (L_1, L_2) \mapsto \rho(L_1) \cdot L_2$$

is an isomorphism. Bijectivity of $d\rho$ (Proposition 6.1) implies that K is a disjoint union of points. Therefore, the connected component of identity of $\mathrm{Pic}(\overline{J})$ is contained in $\rho(J)$. Now it remains to notice that J is connected. □

Proposition 6.2. $\mathrm{Pic}^\sigma(\overline{J}) = \mathrm{Pic}^0(\overline{J})$ *(where* Pic^σ *is defined in* (1.2)*).*

Proof. Consider $\overline{p} : \overline{\mathcal{J}} \to \mathcal{M}$. It is a projective flat morphism with integral fibers (Theorem 4.1); we can therefore construct the corresponding family of Picard schemes $\mathrm{Pic}(\overline{\mathcal{J}}/\mathcal{M}) \to \mathcal{M}$ (see the references in the proof of Theorem 1.3). The family is separated and its fiber over $C \in \mathcal{M}$ is $\mathrm{Pic}(\overline{J}_C)$.

Let us work in the smooth topology of $\mathcal{M}$. Locally, we can choose a degree minus one line bundle ℓ on the universal curve $\mathcal{C} \to \mathcal{M}$. As in the proof of Theorem 1.3, we then introduce a map

$$\alpha^* : \mathrm{Pic}(\overline{\mathcal{J}}/\mathcal{M}) \to \mathrm{Pic}(\mathcal{C}/\mathcal{M})$$

and substacks $\mathrm{Pic}'(\overline{\mathcal{J}}/\mathcal{M}) = (\alpha^*)^{-1}(\mathcal{J})$ and $\mathcal{K} = \ker(\alpha^*)$ such that

$$\mathrm{Pic}'(\overline{\mathcal{J}}/\mathcal{M}) = \mathcal{J} \times_{\mathcal{M}} \mathcal{K}.$$

Let $\mathrm{Pic}^\sigma(\overline{\mathcal{J}}/\mathcal{M}) \subset \mathrm{Pic}(\overline{\mathcal{J}}/\mathcal{M})$ be the substack whose fiber over $C \in \mathcal{M}$ is $\mathrm{Pic}^\sigma(\overline{J}_C)$. We have

$$\mathrm{Pic}^\sigma(\overline{\mathcal{J}}/\mathcal{M}) = \mathcal{J} \times_{\mathcal{M}} \mathcal{K}^\sigma,$$

where

$$\mathcal{K}^\sigma = \{L \in \mathcal{K} : L^{\otimes n} \simeq O \text{ for some } n \text{ coprime to } \operatorname{char} \Bbbk\}.$$

By [17, Theorem 2.5], the map

$$\mathrm{Pic}(\overline{\mathcal{J}}/\mathcal{M}) \to \mathrm{Pic}(\overline{\mathcal{J}}/\mathcal{M}) : L \mapsto L^{\otimes n}$$

is étale for all n coprime to $\mathrm{char}\,\Bbbk$. Therefore, $\mathcal{K}^\sigma$ is étale over $\mathcal{M}$.

Finally, the morphism $\mathcal{K}^\sigma \to \mathcal{M}$ is separated, and over the locus of smooth curves $C \in \mathcal{M}$, we have $\mathrm{Pic}^0(\overline{J}_C) = \mathrm{Pic}^\sigma(\overline{J}_C)$ by [23, Corollary IV.19.2]. Therefore, $\mathcal{K}^\sigma$ is the zero group scheme, and $\mathrm{Pic}^\sigma(\overline{\mathcal{J}}/\mathcal{M}) = \mathcal{J}$, as required. □

7. Fibers of the Hitchin fibration

Recall the construction of the Hitchin fibration [18] (for $GL(n)$). Fix a smooth curve X and an integer n.

Definition 7.1. A *Higgs bundle* is a rank n vector bundle E on X together with a *Higgs field* $A : E \to E \otimes \Omega_X$.

Given a Higgs bundle (E, A), consider the characteristic polynomial of A:

$$\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n; \qquad a_i \in H^0(X, \Omega_X^{\otimes i}). \tag{7.1}$$

The zero locus of (7.1) is a curve $C \subset T^*X$: the *spectral curve* of A. Higgs bundle (E, A) gives rise to a coherent sheaf F on C; informally, F is the 'sheaf of co-eigenspaces': its fiber over a point $(x, \mu) \in T^*X$ is the co-eigenspace

$$\mathrm{coker}(A(x) - \mu : E_x \to E_x \otimes \Omega_{X,x}).$$

Here $x \in X$, $\mu \in \Omega_{X,x}$.

Proposition 7.1. (i) *F is a torsion-free sheaf on C whose stalk at any generic point of C has length equal to the multiplicity of the corresponding component of C. In particular, if C is reduced, F is a torsion-free sheaf of generic rank one.*

(ii) *Fix a spectral curve C (that is, fix a polynomial* (7.1)*). Then $(E, A) \mapsto F$ is a one-to-one correspondence between Higgs bundles with spectral curve C and sheaves F as in* (i). □

Given F, E is reconstructed as the push-forward of F with respect to $C \to X$. Therefore, F and E have equal Euler characteristics. We have $\chi(O_C) = n^2\chi(O_X) = n^2(1 - g)$, where g is the genus of X. Hence $\deg(F) = 0$ if and only if $\deg(E) = n(n-1)(1-g)$. (Recall that $\deg(F) = \chi(F) - \chi(O_C)$.) Also, note that (E, A) is (semi)stable if and only if F is (semi)stable. If C is integral, F has generic rank one and stability is automatic.

Let $\mathcal{H}iggs$ be the moduli space of semi-stable Higgs bundles (E, A) with $\mathrm{rk}(E) = n$ and $\deg(E) = n(n-1)(1-g)$. Also, let $\mathcal{SC}urves$ be the space of spectral curves $C \subset T^*X$; explicitly, $\mathcal{SC}urves$ is the space of coefficients $(a_1, \ldots, a_n)$ of (7.1):

$$\mathcal{SC}urves = \prod_{i=1}^{n} H^0(X, \Omega_X^{\otimes i}).$$

Finally, let $\mathcal{SC}urves' \subset \mathcal{SC}urves$ be the locus of integral spectral curves $C \subset T^*X$.

The correspondence $(E, A) \mapsto C$ gives a map $h : \mathcal{H}iggs \to \mathcal{SC}urves$ (the *Hitchin fibration*). For $C \in \mathcal{SC}urves$, the fiber $h^{-1}(C)$ is the space of Higgs bundles with spectral curve C; Proposition 7.1 identifies $h^{-1}(C)$ with the moduli space of semistable coherent sheaves F on C that satisfy Proposition 7.1(i) and have degree zero. In other words, the fiber is the compactified Jacobian of C.

The results of this paper can be applied to integral spectral curves $C \in \mathcal{SC}urves'$. For instance, Theorem 1.2(ii) implies that

$$H^i(h^{-1}(C), O) = \bigwedge^i H^1(C, O_C).$$

Actually, applying the relative version of Theorem 1.2(ii) to the universal family of spectral curves, we obtain an isomorphism

$$(R^i h_* O_{\mathcal{H}iggs})|_{\mathcal{SC}urves'} = \Omega^i_{\mathcal{SC}urves'}, \tag{7.2}$$

where we used the symplectic form on T^*X to identify $H^1(C, O_C)$ with the cotangent space to $C \in \mathcal{SC}urves'$. Recently, E. Frenkel and C. Teleman proved that the isomorphism (7.2) can be extended to the space of all spectral curves:

Theorem 7.1. *There is an isomorphism*

$$R^i h_* O_{\mathcal{H}iggs} = \Omega^i_{\mathcal{SC}urves}.$$

□

When $i = 0, 1$, Theorem 7.1 is proved by N. Hitchin ([18, Theorems 6.2 and 6.5]); the general case is announced in [15].

Remarks. (i) In [18], N. Hitchin works with the Hitchin fibration for the group $SL(2)$, but his argument can be used to compute $R^i h_* O_{\mathcal{H}iggs}$ for arbitrary n (still assuming $i = 0, 1$). Actually, essentially the same argument computes $R^i \overline{p}_* O_{\overline{\mathcal{J}}}$ for $i = 0, 1$. (Recall that $\overline{p} : \overline{\mathcal{J}} \to \mathcal{M}$ is the universal compactified Jacobian over the moduli stack of curves $\mathcal{M}$.)

(ii) In [15], Theorem 7.1 is stated for the Hitchin fibration for arbitrary group, not just $GL(n)$.

(iii) One can derive some of our results from Theorem 7.1, at least for integral curves C that appear as spectral curves of the Hitchin fibration. Indeed, for such $C \in \mathcal{SC}urves'$, Theorem 7.1 implies Theorem 1.2(ii). In turn, this implies Theorem 4.1. Also, one can easily derive from Theorem 1.2(ii) that the isomorphism of Theorem 1.1 exists on some neighborhood U of $\zeta \in J$, so Theorem 1.2(i) holds for $L \in U$. Similarly, we see that P defines a fully faithful Fourier–Mukai transform from $D^b(U)$ to $D^b(\overline{J})$.

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