MODULAR INVARIANT D-MODULES

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ABSTRACT. We study D-modules over the modular curve of level 1 defined as an orbifold, and show that if such D-modules are of rank 1, these monodromy representations map the S-action to the identity. This means that in the orbifold case, there are linear representations of the fundamental groups which do not come from D-modules.

1. Introduction

It is well known that any linear representation of the fundamental group of a complex manifold is obtained as the monodromy of a certain D-module over this manifold. The aim of this paper is to show that the modular curve of level 1 defined as an orbifold does not satisfy this property. Our method is motivated by a result of Nakamura and Schneps (cf. [N, Section 4] and [NS, Section 7]) which concerns the Galois actions on the algebraic fundamental groups of the modular curves $X_i$ of level $i = 1, 2$. We study D-modules over $X_1$, called modular invariant D-modules, using the natural covering map

$$X_2 \cong \mathbb{P}^1 - \{0, 1, \infty\} \to X_1,$$

and show that if a D-module over $X_1$ has rank 1, then this monodromy representation maps the modular transformation by $S : \tau \mapsto -1/\tau$ to the identity. This implies that there are linear representations of $\pi_1(X_1)$ which are not obtained as the monodromy representations of modular invariant D-modules.

2. D-modules over modular curves

2.1. D-modules over orbifolds. A complex orbifold has an open covering $\{[U_\lambda/G_\lambda]\}_\lambda$, where $G_\lambda$ is a finite group acting on a complex manifold $U_\lambda$. Denote by $p_\lambda : U_\lambda \to U_\lambda/G_\lambda$ the natural projection to the geometric quotient of $U_\lambda$ by the action of $G_\lambda$. Then $M$ is called a D-module (of finite rank) over a complex orbifold $X$ if there exists an open covering $\{[U_\lambda/G_\lambda]\}_\lambda$ of $X$ such that $M$ is a compatible system $(F_\lambda, \nabla_\lambda)$ of vector bundles with meromorphic connection over $U_\lambda/G_\lambda$ such that $p_\lambda^*(F_\lambda, \nabla_\lambda)$ is isomorphic to a vector bundle with holomorphic connection over $U_\lambda$. For each D-module over $X$, one can associate naturally its monodromy which is a linear representation of $\pi_1(X)$.

2.2. Modular curves. Let $\zeta_n = \exp(2\pi i/n)$ be an $n$th root of 1, and for $a, \tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$, put $q^a = \exp(2\pi i a \tau)$. Let $H = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ denote the Poincaré upper-half plane with natural action of $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm 1\}$. The principal congruence subgroup of $\text{SL}_2(\mathbb{Z})$ of level 2 is $\Gamma(2) = \text{Ker}(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/2\mathbb{Z}))$, and $\text{PGL}(2)$ is defined as $\Gamma(2)/\{\pm 1\}$. Then $X_1 = H/\text{PSL}_2(\mathbb{Z})$ and $X_2 = H/\text{PGL}(2)$ are

Received by the editors April 12, 2010.
Let \( \pi \) be a trivial bundle over \( \mathbb{P}^1 \) with meromorphic connection of the form \( (A_0/z + A_1/(z - 1)) \, dz \), where \( z \) is the natural coordinate of \( \mathbb{P}^1 \). Then this connection matrix \( \Phi(A_0, A_1) \) is defined as \( G_1(z)^{-1} \cdot G_0(z) \), where \( G_i(z) \) (for \( i = 0, 1 \)) be the solutions of

\[
G'(z) = \left( \frac{A_0}{z} + \frac{A_1}{z - 1} \right) \cdot G(z),
\]

called the modular curves of level 1 and 2, respectively. We consider \( X_1 \) as a complex orbifold with fundamental group

\[
\pi_1(X_1) = \pi_1(X_1; \overline{01}) \cong PSL_2(\mathbb{Z}),
\]

(\( \overline{01} \) : the tangential base point for the \( q \)-coordinate) which has generators

\[
T = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

with relations \( S^2 = (TS)^3 = 1 \). Then the generators \( S, TS \) of \( PSL_2(\mathbb{Z}) \) stabilize \( \zeta_4 = i, \zeta_6 \in H \) respectively, and \( X_1 \) is expressed locally as orbifolds:

\[
\begin{cases}
[U/\{\pm1\}], & \text{around the image of } i, \\
[U/\langle\zeta_3\rangle], & \text{around the image of } \zeta_6, \\
[U] = U, & \text{otherwise},
\end{cases}
\]

where \( U = \{z \in \mathbb{C} \mid |z| < 1\} \). Let \( \tilde{X}_1 = X_1 \cup \{i\infty\} \) and \( \tilde{X}_2 = X_2 \cup \{0, 1, i\infty\} \) be the completion of \( X_1 \) and \( X_2 \) obtained by adding their cusps.

We describe natural models of \( \tilde{X}_1 \) which are examples of the canonical models in Shimura’s theory when these are considered as defined over \( \mathbb{Q} \). First, the Legendre \( \lambda \)-function gives an isomorphism \( \lambda : X_2 \cong \mathbb{P}^1 - \{0, 1, \infty\} \), and this extends to an isomorphism \( \tilde{\lambda} : \tilde{X}_2 \cong \mathbb{P}^1 \) mapping the cusps \( i\infty, 0 \) to \( 0, 1, \infty \), respectively. Then \( \lambda(\tau) = 16q^{1/2} + \cdots \) at \( \tau = i\infty \), and one can see that

\[
\lambda(T(\tau)) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}, \quad \lambda(S(\tau)) = 1 - \lambda(\tau)
\]

by seeing the changes of the \( \tilde{\lambda} \)-values of the cusps under \( \tau \mapsto T(\tau), S(\tau) \). Second, the \( j \)-function gives a surjective holomorphic map \( j : X_1 \to \mathbb{C} \), and this extends to \( \tilde{j} : \tilde{X}_1 \to \mathbb{P}^1 \) mapping \( \zeta_6, i \in H \) and the cusp \( i\infty \) to \( 0, 1 \) and \( \infty \), respectively. Since

\[
j = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},
\]

\( j = 1 \) has double roots at

\[
\lambda \in R_2 = \{\lambda(i) = 1/2, \lambda(T(i)) = 2, \lambda(S(T(i))) = -1\},
\]

and \( j = 0 \) has triple roots at

\[
\lambda \in R_3 = \{\lambda(\zeta_6) = \zeta_6, \lambda(\zeta_3) = \zeta_6^{-1}\}.
\]

Let \( \pi : X_2 \to X_1 \) and \( \tilde{\pi} : \tilde{X}_2 \to \tilde{X}_1 \) be the natural projections of degree 6. Then \( \tilde{j} \circ \tilde{\pi} \circ (\tilde{\lambda})^{-1} : \mathbb{P}^1 \to \mathbb{P}^1 \) is ramified in \( \{0, 1, \infty\} \cup R_2 \) with ramification index 2, and is ramified in \( R_3 \) with ramification index 3.

### 2.3. Connection matrices

We recall the definition of connection matrices. Let \( F \) be a trivial bundle over \( \mathbb{P}^1 \) with meromorphic connection of the form \( (A_0/z + A_1/(z - 1)) \, dz \), where \( z \) is the natural coordinate of \( \mathbb{P}^1 \). Then this connection matrix \( \Phi(A_0, A_1) \) is defined as \( G_1(z)^{-1} \cdot G_0(z) \), where \( G_i(z) \) (for \( i = 0, 1 \)) be the solutions of

\[
G'(z) = \left( \frac{A_0}{z} + \frac{A_1}{z - 1} \right) \cdot G(z),
\]
such that \( \lim_{z \to 0} G_0(z)/z^{A_0} = \lim_{z \to 1} G_1(z)/(1-z)^{A_1} = 1 \), where \( z \) runs in \((0,1)\) and \( \varepsilon^A = \exp(\log(\varepsilon) \cdot A) \) for \( \varepsilon > 0 \).

**Theorem 2.1.** Let \( M \) be a \( D \)-module over \( X_1 \) of finite rank \( r \).

1. Assume that there exist an extension \( \bar{M} \) of \( M \) to \( \bar{X}_1 \) as a vector bundle with meromorphic connection having logarithmic pole at \( q = 0 \), and a trivial bundle \( F \) over \( \mathbb{P}^1 \) with meromorphic connection \( \nabla \) such that \( \bar{\lambda}^*(F, \nabla) \cong \pi^*(\bar{M}) \) and that \( \nabla \) is holomorphic except \( 0, 1, \infty \) at which \( \nabla \) has logarithmic poles. Denote by

\[
\omega = \left( \frac{A_0}{z} + \frac{A_1}{z - 1} \right) dz
\]

the connection form of \( \nabla \). Then the monodromy of \( M \) maps \( S \) to the connection matrix \( \Phi(A_0, A_1) \).

2. Assume that \( r = 1 \). Then the monodromy maps \( S \) to 1, and \( T \) to a cubic root of 1.

**Proof.** First, we prove (1). Put \( V = \mathbb{C}^r \), and identify \( V \times I \) \((I = [0,1])\) with the pull back by \( \pi \circ (\lambda)^{-1} |_{\bar{X}_1} \) of the trivial bundle \( F \). By that \( \lambda(-1/\tau) = 1 - \lambda(\tau) \) and that the connection form of \( \nabla \) has the residues \( A_p \) at \( p = 0, 1 \), the transformation by \( S \) along the line \( i \mathbb{R} \subset H \) from \( i \infty \) to 0 gives an element of End_{\mathbb{C}}(V) represented as

\[
\lim_{\varepsilon \to 0} (\varepsilon^{-A_1} S \varepsilon^{A_0}),
\]

where \( S : V \cong V \times \{\varepsilon\} \to V \times \{1 - \varepsilon\} \cong V \). Therefore, using iterated integrals of \( \omega \),

\[
\lim_{\varepsilon \to 0} (\varepsilon^{-A_1} S \varepsilon^{A_0}) = \lim_{\varepsilon \to 0} \left\{ \varepsilon^{-A_1} \left( \sum_{n=0}^{\infty} \int_{\varepsilon}^{1-\varepsilon} \omega \cdots \omega \right) \varepsilon^{A_0} \right\},
\]

which is \( \Phi(A_0, A_1) \).

Second, we prove (2). By a theorem of Frobenius, there is an extension \( \bar{M} \) of \( M \) by gluing the trivial line bundle around \( q = 0 \) with meromorphic connection of the form \( Adq/q \), where exp(2\pi i A) is the monodromy of \( M \) around \( q = 0 \). Since \( (\pi \circ \lambda^{-1})*M \) is isomorphic to a \( D \)-module over \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \), there are a line bundle \( F' \) over \( \mathbb{P}^1 \) and a meromorphic connection \( \nabla' \) on \( F' \) holomorphic except \( 0, 1, \infty \) at which \( \nabla' \) has logarithmic poles such that \( (\pi \circ (\lambda)^{-1})*M \) is isomorphic to \( (F', \nabla') \). Represent the fiber of \( F' \) around \( \infty \) as \( W \cong \mathbb{C} \). Then changing the trivialization of \( F' \) around \( \infty \) by \( a \mapsto a \cdot z^n \) \((a \in W)\) for certain \( n \in \mathbb{Z} \), \((F', \nabla')\) becomes a trivial line bundle with meromorphic connection \( (F, \nabla) \) over \( \mathbb{P}^1 \) satisfying the desired property. Since \( M \) is of rank 1, \( A_0 \) and \( A_1 \) given in (1) are commutative, and hence \( G_0(z) = z^{A_0}(1-z)^{A_1} = G_1(z) \) which implies that \( S \) is mapped to \( \Phi(A_0, A_1) = 1 \). Therefore, by the relation \((T S)^3 = 1\), \( T \) is mapped to a cubic root of 1. \( \square \)

**Corollary 2.1.** There is a representation \( \pi_1(X_1) \to \mathbb{C}^\times \) which is not obtained as the monodromy of any \( D \)-module over \( X_1 \).

**Proof.** Let \( \rho : \pi_1(X_1) = \langle T, S \rangle \to \mathbb{C}^\times \) be the representation which maps \( T, S \) to \( e^{2\pi i/6}, -1 \) respectively. Then by Theorem 2.1 (2), \( \rho \) cannot be obtained as the monodromy of any \( D \)-module over \( X_1 \). \( \square \)
2.4. The cubic root of $j$. By results of Kac and Peterson [KP] and of Tsuchiya et al. [TUY], the conformal field theory for the family of elliptic curves gives rise to examples of $D$-modules over $X_1$ (satisfying the assumption of Theorem 2.1 (1)) whose sections are described by the characters for affine Lie algebras. For example, the cubic root

$$j^{1/3}(\tau) = q^{-1/3} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n} (q \cdot j(\tau) - 1)^n \right)$$

$$= q^{-1/3} (1 + 248q + 4124q^2 + \cdots)$$

of $j(\tau)$ satisfies the differential equation

$$\frac{d}{d\lambda} j^{1/3} = \left( -\frac{2}{3} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} \right) + \sum_{\gamma \in R_3} \frac{1}{\lambda - \gamma} \right) j^{1/3}$$

associated with a $D$-module over $X_1$ of rank 1, and satisfies the functional equations

$$j^{1/3}(T(\tau)) = e^{-2\pi i/3} \cdot j^{1/3}(\tau), \quad j^{1/3}(S(\tau)) = j^{1/3}(\tau).$$

By a result of Kac [K], $j^{1/3}$ becomes the character for the affine Lie algebra of type $E_8$.

References


