STATISTICS OF THE JACOBIANS OF HYPERELLIPTIC CURVES
OVER FINITE FIELDS

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Abstract. Let \( C \) be a smooth projective curve of genus \( g \geq 1 \) over a finite field \( \mathbb{F}_q \) of cardinality \( q \). Denote by \( \# \mathcal{J}_C \) the size of the Jacobian of \( C \) over \( \mathbb{F}_q \). We first obtain an estimate on \( \# \mathcal{J}_C \) when \( \mathbb{F}_q(C)/\mathbb{F}_q(X) \) is a geometric Galois extension, which improves a general result of Shparlinski [19]. Then we study the behavior of the quantity \( \# \mathcal{J}_C \) as \( C \) varies over a large family of hyperelliptic curves of genus \( g \). When \( g \) is fixed and \( q \to \infty \), its limiting distribution is given by the powerful theorem of Katz and Sarnak in terms of the trace of a random matrix. When \( q \) is fixed and the genus \( g \to \infty \), we also find explicitly the limiting distribution and show that the result is consistent with that of Katz and Sarnak when both \( q, g \to \infty \).

1. Introduction

Let \( C \) be a smooth projective curve of genus \( g \geq 1 \) over a finite field \( \mathbb{F}_q \) of cardinality \( q \). The Jacobian \( \text{Jac}(C) \) is a \( g \)-dimensional abelian variety. The set of the \( \mathbb{F}_q \)-rational points on \( \text{Jac}(C) \), denoted by \( \mathcal{J}_C = \text{Jac}(C)(\mathbb{F}_q) \), is a finite abelian group. The group \( \mathcal{J}_C \) has been studied extensively, partly because of its importance in the theory of algebraic curves and its surprising applications in public-key cryptography and computational number theory. For example, such groups are extremely useful in primality testing [3] and integer factorization [12, 13]. Statistics of group structures of \( \mathcal{J}_C \), for instance the analog of the Cohen–Lenstra conjecture over function fields remains an inspiring problem in number theory and provides insight for number fields case. Interested readers may refer to [1, 2, 22] for details and current development. The main purpose of this paper is to study \( \# \mathcal{J}_C \), the size of the Jacobian over \( \mathbb{F}_q \). This quantity is also the class number of the function field \( \mathbb{F}_q(C) \) [17, Theorem 5.9], a subject of study with a rich history.

The zeta function of \( C/\mathbb{F}_q \) is a rational function of the form
\[
Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)},
\]
where \( P_C(u) \in \mathbb{Z}[u] \) is a polynomial of degree \( 2g \) with \( P_C(0) = 1 \), satisfying a functional equation and having all its zeros on the circle \( |u| = 1/\sqrt{q} \) (the Riemann hypothesis for curves [23]). Moreover, there is a unitary symplectic matrix \( \Theta_C \in \text{USp}(2g) \), defined up to conjugacy, so that
\[
P_C(u) = \det(I - u\sqrt{q} \Theta_C).
\]
The eigenvalues of \( \Theta_C \) are of the form \( e(\theta_{C,j}) \), \( j = 1, \ldots, 2g \), where \( e(\theta) = e^{2\pi i \theta} \)
It is known that $\# J_C = P_C(1)$ (see [14, Corollary VIII.6.3]). From this we immediately derive that
\[(q^{1/2} - 1)^{2g} \leq \# J_C \leq (q^{1/2} + 1)^{2g},\]
which is tight in the case $g = 1$ due to the classical result of Deuring [6]. Many improvements of this bound have been obtained in [15,16,19–21]. In particular in an interesting paper [19], Shparlinski proves that if $C$ is a smooth absolutely irreducible curve of genus $g$ over $\mathbb{F}_q$ with gonality $d$, then
\[(1.1) \quad \log \# J_C = g \log q + O \left( g \log^{-1}(g/d) \right)\]
as $g \to \infty$, where the implied constant may depend on $q$. (The gonality of a curve $C$ is the smallest integer $d$ such that $C$ admits a non-constant map of degree $d$ to the projective line over the ground field $\mathbb{F}_q$. For example, a hyperelliptic curve is a curve given by an affine model $Y^2 = F(X)$ for some $F \in \mathbb{F}_q[X]$, so the gonality is $d = 2$.) This generalizes and improves similar results of Tsfasman [21].

In this paper, we first prove that if the function field $\mathbb{F}_q(C)$ is a geometric Galois extension of $\mathbb{F}_q(X)$, a sharper estimate can be obtained. Here “geometric” means that the constant field of $\mathbb{F}_q(C)$ is still $\mathbb{F}_q$.

**Theorem 1.1.** Let $C$ be a smooth projective curve of genus $g \geq 1$ over $\mathbb{F}_q$. Assume that the function field $\mathbb{F}_q(C)$ is a geometric Galois extension of the rational function field $\mathbb{F}_q(X)$ with $N = \# \text{Gal} (\mathbb{F}_q(C)/\mathbb{F}_q(X))$. Then
\[(1.2) \quad |\log \# J_C - g \log q| \leq (N - 1) \left( \log \max \left\{ 1, \frac{\log(7g/(N - 1))}{\log q} \right\} + 3 \right).\]

We remark that under the condition of Theorem 1.1, the gonality of the curve clearly satisfies $d \leq N$, and the quantity $|\log \# J_C - g \log q|$ is essentially bounded by $O(\log \log g)$, which is significantly smaller than $O(g/\log g)$ implied from (1.1).

Next we will study how the value $(\log \# J_C - g \log q)$ fluctuates when $C$ varies inside a family. More precisely, assume that $q$ is odd. For each positive integer $d \geq 3$, denote by $\mathcal{H}_{d,q}$ the family of hyperelliptic curves having an affine equation of the form $Y^2 = F(X)$, with $F \in \mathbb{F}_q[X]$ a monic square-free polynomials of degree $d$. The genus of a curve $C \in \mathcal{H}_{d,q}$ is given by
\[g = g(C) = \left\lfloor \frac{d - 1}{2} \right\rfloor,\]
where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$. For any $C \in \mathcal{H}_{d,q}$, since $\mathbb{F}_q(C)/\mathbb{F}_q(X)$ is a geometric Galois extension with Galois group $\mathbb{Z}/2\mathbb{Z}$, Theorem 1.1 implies that
\[|\log \# J_C - g \log q| \leq \log \max \left\{ 1, \frac{\log(7g)}{\log q} \right\} + 3.\]

We study how the value $(\log \# J_C - g \log q)$ is distributed as $C$ varies over the family $\mathcal{H}_{d,q}$. The measure on $\mathcal{H}_{d,q}$ is simply the uniform probability measure on the set of such polynomials.
Writing

\[ P_C(u) = \prod_{i=1}^{2g} (1 - \sqrt{q} e(\theta_{C,i}) u), \]

then

\[ \log \# \mathcal{J}_C - g \log q = \sum_{i=1}^{2g} \log \left( 1 - q^{-1/2} e(\theta_{C,i}) \right). \]

Katz and Sarnak [10] showed that for fixed genus \( g \), the conjugacy classes \( \{ \Theta_C : C \in \mathcal{H}_{d,q} \} \) become uniformly distributed in USp(2g) in the limit \( q \to \infty \). In particular, since

\[ \lim_{q \to \infty} \sqrt{q} (\log \# \mathcal{J}_C - g \log q) = -\sum_{i=1}^{2g} e(\theta_{C,i}), \]

it implies that

(i) When \( q \) is fixed and \( g \to \infty \), the value \( -\sqrt{q} (\log \# \mathcal{J}_C - g \log q) \) for \( C \in \mathcal{H}_{d,q} \) is distributed asymptotically as the trace of a random matrix in USp(2g).

Furthermore, since the limiting distribution of traces of a random matrix in USp(2g), as \( g \to \infty \), is a standard Gaussian by a theorem of Diaconis and Shahshahani [7], it also implies that

(ii) If \( q \to \infty \) and then \( g \to \infty \), the value \( \sqrt{q} (\log \# \mathcal{J}_C - g \log q) \) is distributed as a standard Gaussian.

Katz and Sarnak’s powerful theorem [10] provides an almost complete answer, except that in their argument, it is crucial to take the limit that \( q \to \infty \). What happens if \( g \to \infty \) instead? Complementary to (i) and (ii) above, we prove the following.

**Theorem 1.2.** (1) If \( q \) is fixed and \( g \to \infty \), then for \( C \in \mathcal{H}_{d,q} \), the quantity

\[ \log \# \mathcal{J}_C - g \log q + \delta_{d/2} \log (1 - q^{-1}) \]

converges weakly to a random variable \( X \), whose characteristic function \( \phi(t) = \mathbb{E}(e^{itX}) \) is given by

\[ \phi(t) = 1 + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{P_1, \ldots, P_r \text{ distinct}} \prod_{j=1}^{r} \frac{(1 - |P_j|^{-1})^{-it} + (1 + |P_j|^{-1})^{-it} - 2}{2(1 + |P_j|^{-1})}, \quad \forall t \in \mathbb{R}, \]

where we denote

\[ \delta_\gamma = \begin{cases} 1, & \gamma \in \mathbb{Z}, \\ 0, & \gamma \notin \mathbb{Z}, \end{cases} \]

and the sum on the right is over all distinct monic irreducible polynomials \( P_1, \ldots, P_r \in \mathbb{F}_q[X] \) and \( |P_j| = q^{\deg P_j} \).

(2) If both \( q, g \to \infty \), then for \( C \in \mathcal{H}_{d,q} \), \( \sqrt{q} (\log \# \mathcal{J}_C - g \log q) \) is distributed as a standard Gaussian, that is, for any \( \gamma \in \mathbb{R} \), we have

\[ \lim_{q \to \infty} \frac{1}{\# \mathcal{H}_{d,q}} \# \{ C \in \mathcal{H}_{d,q} : \sqrt{q} (\log \# \mathcal{J}_C - g \log q) \leq \gamma \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma} e^{-t^2} dt. \]

**Remark 1.1.** (1) Kurlberg and Rudnick [11] and Faifman and Rudnick [8] initiated the investigation of such problems under the limit that \( q \) is fixed and \( g \to \infty \). Bucur et al. [4, 5] made further important development. Theorem 1.2 is similar to their work. Theorem 1.2 can also be considered as a function field
analog of the distribution of $L(1, \chi_d)$ (over $\mathbb{Q}$) investigated by Granville and Soundararajan in [9]. The proof of Theorem 1.2 borrows techniques developed by Rudnick [18] and Faifman and Rudnick [8].

(2) Statement (2) of Theorem 1.2 is more general than statement (ii) which could be derived from the theorem of Katz and Sarnak because there is no requirement that $q \to \infty$ first.

(3) Instead of averaging over $\mathcal{H}_{d,q}$, the proof can be easily adapted to the moduli space of hyperelliptic curves of a fixed genus. Interested readers may refer to [4,5] for terminology and treatment.

(4) The authors are grateful to Alina Bucur for suggesting the following insightful heuristics: First notice $#J_C = P_C(1)$ and by the functional equation

$$P_C(1) = q^g P_C(1/q) = q^g \frac{Z_C(1/q)}{Z_{P^1}(1/q)}.$$  

The Euler product expansion of $Z_C(u)/Z_{P^1}(u)$ converges absolutely at $u = 1/q$, so we can write $#J_C$ as $q^g$ times a product over Euler factors corresponding to monic irreducible polynomials evaluated at $1/q$. Explicitly, for $P$ a monic irreducible polynomial, the corresponding Euler factor evaluated at $1/q$ will be

\[
\begin{cases}
(1 - |P|^{-1})^{-1} & \text{if } C \text{ splits at } P, \\
(1 + |P|^{-1})^{-1} & \text{if } C \text{ is inert at } P, \\
1 & \text{if } C \text{ ramifies at } P.
\end{cases}
\]

This suggests that the difference $\log #J_C - g \log q$ should be modeled by a sum of i.i.d. random variables, one for each monic irreducible polynomials. In this model, the probability that $C$ ramifies the above some polynomial $P$ is computed in the usual way: the residue field at $P$ has $r = |P|$ elements, so that probability of ramification is $(r - 1)/(r^2 - 1) = 1/(r + 1) = (1 + |P|)^{-1}$. This is counting the reductions modulo $P^2$ that are not zero, but are divisible by $P$ of the defining polynomial of the curve. The split and inert cases occur with equal probability, namely $\frac{|P|}{2(1 + |P|)}$. Thus the random variable corresponding to $P$ has characteristic function

$$\phi_P(t) = \frac{1}{1 + |P|} + (1 - |P|^{-1})^{-it} \frac{|P|}{2(1 + |P|)} + (1 + |P|^{-1})^{-it} \frac{|P|}{2(1 + |P|)},$$

One can check that

$$\phi(t) = \prod_P \phi_P(t),$$

which confirms statement (1) of Theorem 1.2.

### 2. Preliminaries

In this section we collect several results which will be used later. Interested readers can refer to [17] for more details.

#### 2.1. Zeta functions of function fields

Let $K = \mathbb{F}_q(X)$ be the rational function field over the finite field $\mathbb{F}_q$ and let $L/K$ be a finite geometric Galois extension. Here “geometric” means that the constant field of $L$ is still $\mathbb{F}_q$. We list several facts about such extensions $L/K$ as follows (see [17, Chapter 9] for more details).
First, the zeta function $\zeta_L(s)$ of $L$ is defined by

$$
\zeta_L(s) = \prod_{P \in \mathcal{S}_L} (1 - |P|^{-s})^{-1},
$$

where the product is over $\mathcal{S}_L$, the set of all primes of $L$, and for each $P \in \mathcal{S}_L$, $|P|$ is the cardinality of the residue field of $L$ at $P$. For the rational function field $K$, the zeta function $\zeta_K(s)$ turns out to be

$$
\zeta_K(s) = (1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}.
$$

If $C$ is a smooth projective curve of genus $g \geq 1$ over $\mathbb{F}_q$ with function field $\mathbb{F}_q(C) = L$, then $Z_C(q^{-s}) = \zeta_L(s)$, i.e., the zeta function of the curve $C$ coincides with the zeta function of the function field $\mathbb{F}_q(C)$ (see [17, p. 57, Chapter 5] for details).

Let $G = \text{Gal}(L/K)$ be the Galois group of $L/K$ and $\rho : G \rightarrow \text{Aut}_C(V)$ a representation of $G$, where $V$ is a finite-dimensional vector space over the complex numbers $\mathbb{C}$ of dimension $m$. One defines the Artin L-series associated to the representation $\rho$ as follows.

If $P$ is a prime of $K$ which is unramified in $L$ and $\mathcal{B}$ is a prime of $L$ lying above $P$, one defines the local factor $L_P(s, \rho)$ as

$$
L_P(s, \rho) = \det \left( I - \rho((\mathcal{B}, L/K)|P|^{-s})^{-1},
$$

where $I$ is the identity automorphism on $V$ and $(\mathcal{B}, L/K) \in G$ is the Frobenius automorphism at $\mathcal{B}$. Since $L/K$ is Galois, this definition does not depend on the choice of $\mathcal{B}$ over $P$.

Let $\{\alpha_1(P), \alpha_2(P), \ldots, \alpha_m(P)\}$ be the eigenvalues of $\rho((\mathcal{B}, L/K))$. In terms of these eigenvalues, we get another useful expression for $L_P(s, \rho)$:

$$
L_P(s, \rho)^{-1} = (1 - \alpha_1(P)|P|^{-s})(1 - \alpha_2(P)|P|^{-s}) \cdots (1 - \alpha_m(P)|P|^{-s}).
$$

We note that these eigenvalues $\alpha_i(P)$ are all roots of unity because $(\mathcal{B}, L/K)$ has finite order.

At a prime $P$ of $K$ which is ramified in $L$, the local factor $L_P(s, \rho)$ can also be defined. The definition is similar to (2.1), except that the action $\rho((\mathcal{B}, L/K))$ is restricted to a subspace of $V$ which is fixed by the inertia group $I(\mathcal{B}/P)$. We are contended with the fact that there are only finitely many primes $P$ which are ramified in $L$ and in either case we can write $L_P(s, \rho)$ as

$$
L_P(s, \rho)^{-1} = (1 - \alpha_1(P)|P|^{-s})(1 - \alpha_2(P)|P|^{-s}) \cdots (1 - \alpha_m(P)|P|^{-s}),
$$

where the values $\alpha_i(P)$'s are either roots of unity or zero. The Artin L-series $L(s, \rho)$ is defined by the infinite product

$$
L(s, \rho) = \prod_{P \in \mathcal{S}_K} L_P(s, \rho),
$$

where $\mathcal{S}_K$ is the set of all primes in $K = \mathbb{F}_q(X)$.

It is known that if $\rho = \rho_0$, the trivial representation, then $L(s, \rho_0) = \zeta_K(s)$, and if $\rho = \rho_{\text{reg}}$, the regular representation, then $L(s, \rho_{\text{reg}}) = \zeta_L(s)$. It is also known that $L(s, \rho)$ depends only on the character $\chi$ of $\rho$, so we can write it as $L(s, \chi)$. 
Finally, let \( L/K \) be a finite, geometric and Galois extension with Galois group \( G = \text{Gal}(L/K) \). Let \( \{\chi_1, \chi_2, \ldots, \chi_h\} \) be the set of irreducible characters of \( G \). We set \( \chi_1 = \chi_0 \), the trivial character. Denote by \( d_i \) the degree of \( \chi_i \), i.e., \( d_i = \chi_i(e) \) is the dimension of the representation space corresponding to \( \chi_i \). Then using results about group characters and formal properties of Artin L-series, one derives that

\[
(2.2) \quad \zeta_L(s) = \zeta_K(s) \prod_{i=2}^h L(s, \chi_i)^{d_i}.
\]

2.2. Averaging over \( H_{d,q} \). Let \( H_{d,q} \subset \mathbb{F}_q[X] \) be the set of all monic square-free polynomials of degree \( d \geq 3 \).

Lemma 2.1. For any Dirichlet character \( \chi : \mathbb{F}_q[X] \to \mathbb{C} \) modulo \( f \in \mathbb{F}_q[X] \), we have

\[
\frac{1}{\#H_{d,q}} \sum_{F \in H_{d,q}} \chi(F) \leq \frac{2^{\deg f - 1}}{(1 - q^{-1}) q^{d/2}}.
\]

Proof. This is [8, Lemma 3.1], which proves the case when \( \chi = \left( \frac{f}{.} \right) \) is a quadratic character. For the general case, the proof follows exactly the same line of argument, so we omit the details here. \( \square \)

Lemma 2.2. Let \( h \in \mathbb{F}_q[X] \) be a monic square-free polynomial. Then

\[
\frac{1}{\#H_{d,q}} \sum_{F \in H_{d,q}} \sum_{P \mid h, \gcd(F, h) = 1} (1 + |P|)^{-1} = 1 \prod_{P \mid h} (1 + |P|^{-1})^{-1} + O\left( q^{-d/2} \sigma(h) \right),
\]

where \( \sigma(h) = \sum_{D \mid h} 1 \).

Proof. This is essentially [18, Lemma 5], which treats the case that \( h = P \) is a monic irreducible polynomial. In fact in this case Rudnick [18, Lemma 5] yields a much stronger error term \( O(q^{-d}) \). The extra saving is obtained by carefully analyzing the functional equation of the zeta function. To obtain the error term \( O(q^{-d/2} \sigma(h)) \), the proof follows a standard procedure which is included [18, Lemma 5]. We also omit details here. \( \square \)

3. Proof of Theorem 1.1

Let \( C \) be a smooth projective curve of genus \( g \geq 1 \) over \( \mathbb{F}_q \). The zeta function \( Z_C(u) \) is of the form

\[
Z_C(u) = \frac{P_C(u)}{(1 - u)(1 - qu)},
\]

where \( P_C(u) \in \mathbb{Z}[u] \) is a polynomial of degree \( 2g \) with \( P_C(0) = 1 \), satisfying the functional equation

\[
P_C(u) = (qu^2)^g P_C\left( \frac{1}{qu} \right),
\]

and having all its zeros on the circle \( |u| = 1/\sqrt{q} \). We may write \( P_C(u) \) as

\[
P_C(u) = \prod_{i=1}^{2g} (1 - \sqrt{q} e(\theta_i) u),
\]
where these $\theta_i \in [0, 1]$ and $e(\alpha)$ stand for $e^{2\pi i \alpha}$ for any $\alpha \in \mathbb{R}$.

Since $\# \mathcal{J}_C = P_C(1)$, we have

$$\# \mathcal{J}_C = \prod_{i=1}^{2g} (1 - \sqrt{q} e(\theta_i)) = q^g \prod_{i=1}^{2g} \left(1 - q^{-1/2} e(\theta_i)\right).$$

Taking logarithms on both sides and using the expansion

$$- \log(1 - z) = \sum_{n \geq 1} \frac{z^n}{n}, \quad |z| < 1,$$

we obtain the equation

$$\log \# \mathcal{J}_C - g \log q = \sum_{n \geq 1} q^{-n/2} n^{-1} \sum_{i=1}^{2g} -e(n \theta_i).$$

Denote $L = \mathbb{F}_q(C)$ and $K = \mathbb{F}_q(X)$. The zeta functions of $L$ and $K$ can be written as

$$\zeta_L(s) = (1 - q^{-s})^{-1} (1 - q^{1-s})^{-1} \prod_{i=1}^{2g} (1 - \sqrt{q} e(\theta_i) q^{-s}),$$

and

$$\zeta_K(s) = (1 - q^{-s})^{-1} (1 - q^{1-s})^{-1}.$$

Since $L/K$ is a geometric Galois extension with $G = \text{Gal}(L/K)$ and $\#G = N$, let \{\chi_1, \chi_2, \ldots, \chi_h\} be the set of irreducible characters of $G$ with $\chi_1 = \chi_0$, the trivial character and denote by $d_i$ the degree of $\chi_i$. From (2.2) we find that

$$\prod_{i=2}^{h} L(s, \chi_i)^{d_i} = \prod_{i=1}^{2g} (1 - \sqrt{q} e(\theta_i) q^{-s}),$$

where for each $i$ with $2 \leq i \leq h$, the Artin L-series associated to $\chi_i$ can be written as

$$L(s, \chi_i)^{-1} = \prod_P (1 - \alpha_{i,1}(P) |P|^{-s}) (1 - \alpha_{i,2}(P) |P|^{-s}) \cdots (1 - \alpha_{i,d_i}(P) |P|^{-s}).$$

Here the product is over all monic irreducible polynomials $P \in \mathbb{F}_q(X)$ and $P = \infty$ with $|P| = q^{\deg P}$ (deg $\infty = 1$ hence $|\infty| = q$) and these $\alpha_{i,j}(P)$’s are either roots of unity or zero.

Taking logarithms on both sides of (3.3), using the expansion (3.1) again and equating the coefficients, we obtain for any positive integer $n$ the identity

$$q^{n/2} \sum_{j=1}^{2g} -e(n \theta_i) = \sum_{\deg f = n} \Lambda(f) \sum_{i=2}^{h} d_i \sum_{j=1}^{d_i} \alpha_{i,j}(f),$$

where the sum on the right side over $\deg f = n$ is over all monic polynomials $f \in \mathbb{F}_q[X]$ with $\deg f = n$, $\Lambda(f) = \deg P$ if $f = P^k$ is a prime power, and $\Lambda(f) = 0$ otherwise.

Let $Z$ be a positive integer which will be chosen later. Denote

$$\epsilon_{1,Z} = \sum_{n \leq Z} q^{-n/2} n^{-1} \sum_{i=1}^{2g} -e(n \theta_i)$$
and
\[ \epsilon_{2,Z} = \sum_{n > Z} q^{-n/2} n^{-1} \sum_{i=1}^{2g} -e(n\theta_i). \]

From (3.2) we can write
\[ \log \# \mathcal{J}_C - g \log q = \epsilon_{1,Z} + \epsilon_{2,Z}. \]

If \( Z \geq 2 \) we have
\[ |\epsilon_{2,Z}| \leq \sum_{n \geq Z + 1} q^{-n/2} n^{-1} 2g \leq \frac{2g}{Z + 1} q^{-(Z+1)/2} \left( 1 - q^{-1/2} \right)^{-1}, \]
and if \( Z = 1 \) we have
\[ |\epsilon_{2,Z}| \leq 2g \left( -\log \left( 1 - q^{-1/2} \right) - q^{-1/2} \right) \leq \frac{2g}{q - \sqrt{q}}. \]

For \( \epsilon_{1,Z} \), we use the identity (3.4). Since \( |\alpha_{i,j}| \leq 1 \) for all \( i, j \), we obtain the inequality
\[ |\epsilon_{1,Z}| \leq \sum_{n \leq Z} q^{-n} n^{-1} \sum_{\deg f = n} \Lambda(f) \sum_{i=2}^{h} d_i^2. \]

It is known that
\[ 1 + \sum_{i=2}^{h} d_i^2 = N = \#G \]
and
\[ \sum_{\deg f = n} \Lambda(f) = q^n + 1. \]

Here the extra “1” on the right side in the above equation accounts for \( f = \infty^n \). Hence
\[ |\epsilon_{1,Z}| \leq (N - 1) \left( \sum_{n \leq Z} \frac{1}{n} + \sum_{n \leq Z} \frac{1}{nq^n} \right). \]

If \( Z = 1 \), this is
\[ |\epsilon_{1,Z}| \leq (N - 1) \left( 1 + q^{-1} \right), \]
and if \( Z \geq 2 \), we use
\[ \sum_{n \leq Z} \frac{1}{n} \leq 1.5 + \log Z - \log 2 \]
and
\[ \sum_{n \leq Z} \frac{1}{nq^n} \leq -\log (1 - q^{-1}) \leq \frac{1}{q - 1} \]
to obtain
\[ |\epsilon_{1,Z}| \leq (N - 1) \left( 1.5 - \log 2 + \frac{1}{q - 1} + \log Z \right), \quad Z \geq 2. \]
Case 1: If \(2 \left(1 - q^{-1/2}\right)^{-1} g \geq (N - 1)q\), we choose

\[
Z = \left[ 2 \log \frac{2(1-q^{-1/2})^{-1}g}{N-1} \log q \right] \geq 2.
\]

We find from (3.8) that

\[
|\epsilon_{1,Z}| \leq (N - 1) \left\{ 1.5 + \frac{1}{q - 1} + \log \left( \frac{\log \frac{2(1-q^{-1/2})^{-1}g}{N-1} \log q}{\log q} \right) \right\}
\]

and from (3.5) that

\[
|\epsilon_{2,Z}| \leq \frac{N - 1}{2}.
\]

In this case noticing that \(q \geq 2\), we obtain

\[
|\log \#\mathcal{J}_C - g \log q| \leq (N - 1) \left( \log \left( \frac{\log 7g}{\log q} \right) + 3 \right).
\]

Case 2: If \(2 \left(1 - q^{-1/2}\right)^{-1} g < (N - 1)q\), we choose \(Z = 1\), and from (3.7) and (3.6) we obtain that

\[
|\log \#\mathcal{J}_C - g \log q| \leq (N - 1) (2 + q^{-1}) < 3(N - 1).
\]

In either case we conclude that

\[
|\log \#\mathcal{J}_C - g \log q| \leq (N - 1) \left( \log \max \left\{ 1, \frac{\log(7g/(N-1))}{\log q} \right\} + 3 \right).
\]

This completes the proof of Theorem 1.1. \(\square\)

4. Proof of Theorem 1.2

4.1. Preparation. Let \(\mathbb{F}_q\) be a finite field of cardinality \(q\) with \(q\) odd. Denote

\[
\mathcal{H}_{d,q} = \{ F \in \mathbb{F}_q[X] : F \text{ is monic, square-free and } \deg F = d \}.
\]

For any \(F \in \mathcal{H}_{d,q}\), the hyperelliptic curve \(C_F\) is given by the affine model

\[
C_F : Y^2 = F(X).
\]

It has genus

\[
g = g_F = \left\lceil \frac{d - 1}{2} \right\rceil.
\]

Suppose that the zeta function \(Z_{C_F}(u)\) is of the form

\[
Z_{C_F}(u) = \prod_{i=1}^{2g} \frac{1 - \sqrt{q} e(\theta_{i,F}) u}{(1 - u)(1 - qu)},
\]

where the \(\theta_{i,F}\)'s are real numbers. Then

\[
\#\mathcal{J}_C = \prod_{i=1}^{2g} (1 - \sqrt{q} e(\theta_{i,F})) = q^g \prod_{i=1}^{2g} (1 - q^{-1/2} e(\theta_{i,F})).
\]
Taking logarithms on both sides we obtain the equation

$$
\log \# \mathcal{J}_{C_F} - g \log q = \sum_{n \geq 1} q^{-n/2} n^{-1} \sum_{i=1}^{2g} -e(n\theta_{i,F}).
$$

As \( d \to \infty \) or \( d, q \to \infty \), the genus \( g = \left\lceil \frac{d-1}{2} \right\rceil \to \infty \). Choose

(4.1) \[ Z = \left\lceil \frac{d}{(\log d)^2} \right\rceil. \]

We write

(4.2) \[ \log \# \mathcal{J}_{C_F} - g \log q = \sum_{n \leq Z} q^{-n/2} n^{-1} \sum_{i=1}^{2g} -e(n\theta_{i,F}) + \epsilon_{1,Z}(F), \]

where

$$
\epsilon_{1,Z}(F) = \sum_{n > Z} q^{-n/2} n^{-1} \sum_{i=1}^{2g} -e(n\theta_{i,F}).
$$

It is easy to see that

$$
|\epsilon_{1,Z}(F)| \leq \sum_{n > Z} q^{-n/2} n^{-1} 2g \leq \frac{9g}{Z} q^{-Z/2}.
$$

Denote \( L = \mathbb{F}_q(C_F) \) and \( K = \mathbb{F}_q(X) \). Since \( L/K \) is a geometric quadratic extension and the Legendre symbol \( \chi := \left( \frac{\cdot}{F} \right) \) generates the Galois group \( \text{Gal}(L/K) \), from (2.2) we have

(4.3) \[ L(s, \chi) = \prod_{i=1}^{2g} \left( 1 - \sqrt{q} e(\theta_{i,F}) q^{-s} \right), \]

and by definition

(4.4) \[ L(s, \chi) = \prod_P \left( 1 - \left( \frac{F}{P} \right) |P|^{-s} \right)^{-1}. \]

Here the product is over all monic irreducible polynomials \( P \in \mathbb{F}_q(X) \) and \( P = \infty \) with \( |P| = q^\deg P \) (\( \deg \infty = 1 \) hence \( |\infty| = q \)).

Computing \( \frac{d}{ds} L(s, \chi) \) in two different ways using (4.3) and (4.4) and equating the coefficients we obtain for each positive integer \( n \) the identity

(4.5) \[ \sum_{i=1}^{2g} -e(n\theta_{i,F}) = q^{-n/2} \sum_{\deg f = n} \Lambda(f) \left( \frac{F}{f} \right) + q^{-n/2} \delta_{d/2}, \]

where the sum over \( \deg f = n \) on the right side is over all monic polynomials \( f \in \mathbb{F}_q[X] \) with \( \deg f = n \), and for any \( \gamma \in \mathbb{R}, \delta_\gamma = 1 \) if \( \gamma \in \mathbb{Z} \), and \( \delta_\gamma = 0 \) if \( \gamma \notin \mathbb{Z} \). The extra term \( q^{-n/2} \delta_{n/2} \) comes from \( f = \infty^n \), noting the fact that \( F \in \mathcal{H}_{d,q} \) is monic and

$$
\left( \frac{F}{\infty} \right) = \begin{cases} 
1, & \text{deg } F \equiv 0 \pmod{2}, \\
0, & \text{deg } F \equiv 1 \pmod{2}.
\end{cases}
$$
Using the identity (4.5) in (4.2) and denoting

\[ N_F = \log \# J_{C_F} - g \log q + \delta_{d/2} \log \left(1 - q^{-1}\right), \]

we find that

\[ N_F = \Delta_Z(F) + \epsilon_Z(F), \]

where

\[ \Delta_Z(F) = \sum_{n \leq Z} q^{-n} n^{-1} \sum_{\deg f = n} \Lambda(f) \left(\frac{F}{f}\right) \]

and

\[ |\epsilon_Z(F)| \leq \frac{10g}{Z} q^{-Z/2}. \]

An upper bound for \( \Delta_Z(F) \) is given by

\[ |\Delta_Z(F)| \leq \sum_{n \leq Z} q^{-n} n^{-1} \sum_{\deg f = n} \Lambda(f) \leq 1 + \log Z. \]

4.2. The \( r \)-th moment \( \Delta_Z \). For any function \( \chi : \mathcal{H}_d \to \mathbb{C} \), we denote by \( \langle \chi \rangle \) the mean value of \( \chi \) on \( \mathcal{H}_{d,q} \), that is,

\[ \langle \chi \rangle := \frac{1}{\# \mathcal{H}_{d,q}} \sum_{F \in \mathcal{H}_{d,q}} \chi(F). \]

For any positive integer \( r \), we find

\[ \Delta_Z(F)^r = \sum_{n_1, \ldots, n_r \leq Z} \prod_{i=1}^r q^{-n_i} n_i^{-1} \sum_{\deg f_i = n_i} \Lambda(f_1) \cdots \Lambda(f_r) \left(\frac{F}{f_1 \cdots f_r}\right), \]

hence

\[ \langle (\Delta_Z)^r \rangle = \sum_{n_1, \ldots, n_r \leq Z} \prod_{i=1}^r q^{-n_i} n_i^{-1} \sum_{\deg f_i = n_i} \Lambda(f_1) \cdots \Lambda(f_r) \left\langle \left(\frac{\cdot}{f_1 \cdots f_r}\right) \right\rangle. \]

If \( f_1 \cdots f_r \) is not a square in \( \mathbb{F}_q[X] \), then \( \left(\frac{\cdot}{f_1 \cdots f_r}\right) : \mathbb{F}_q[X] \to \mathbb{C} \) is a non-trivial Dirichlet character modulo \( h \) with \( \deg h \leq \sum_{i=1}^r \deg f_i \), by Lemma 2.1 we find that

\[ \left\langle \left(\frac{\cdot}{f_1 \cdots f_r}\right) \right\rangle \leq \frac{2^{n_1+\cdots+n_r-1}}{(1 - q^{-1})^{q^{d/2}}}. \]

The total contribution to \( \langle (\Delta_Z)^r \rangle \) from this case is bounded by

\[ T_1 \leq \sum_{n_1, \ldots, n_r \leq Z} \prod_{i=1}^r q^{-n_i} n_i^{-1} \sum_{\deg f_i = n_i} \Lambda(f_1) \cdots \Lambda(f_r) \frac{2^{n_1+\cdots+n_r-1}}{(1 - q^{-1})^{q^{d/2}}}. \]

This can be estimated as

\[ T_1 \leq \frac{q^{-d/2}2^{(Z+1)r}}{2(1 - q^{-1})} \leq q^{-d/2}2^{(Z+1)r} \ll q^{-d/3}. \]
If \( f_1 \cdots f_r \) is a square in \( \mathbb{F}_q[X] \), denote \( f_1 \cdots f_r = h^2 \) and \( \tilde{h} = \prod_{P|h} P \), then \( (\tilde{h}^2) \) is a trivial character, by Lemma 2.2 we find that
\[
\left\langle \left( \frac{1}{h^2} \right) \right\rangle = \frac{1}{\# \mathcal{H}_{d,q}} \sum_{F \in \mathcal{H}_{d,q}, \gcd(F, h) = 1} 1 = \prod_{P|h} \left( 1 + |P|^{-1} \right)^{-1} + O \left( q^{-d/2} \sigma(\tilde{h}) \right).
\]

Since \( f_i \)'s are always prime powers, \( \sigma(\tilde{h}) \leq 2^r \). The total contribution to \( \langle (\triangle Z)^r \rangle \) from the error term \( O \left( q^{-d/2} \sigma(\tilde{h}) \right) \) is bounded by
\[
T_2 \leq \sum_{n_1, \ldots, n_r \leq Z} \prod_{i=1}^r \Lambda(f_i) \cdot \Lambda(f_r) q^{-d/2} 2^r.
\]
This can be estimated as
\[
(4.9) \quad T_2 \leq q^{-d/2} 2^r (1 + \log Z)^r \ll q^{-d/3}.
\]
The total contribution from the main term \( \prod_{P|h} (1 + |P|^{-1})^{-1} \) is
\[
\sum_{n_1, \ldots, n_r \leq Z} \prod_{i=1}^r q^{-n_i} \Lambda(f_i) \Lambda(f_r) \prod_{P|h} (1 + |P|^{-1})^{-1}.
\]
Removing the restriction that \( \deg f_1, \ldots, \deg f_r \leq Z \) results in an error bounded by
\[
\sum_{\deg h > Z/2} \prod_{P|h} (1 + |P|^{-1})^{-1} |h|^{-2} \sum_{f_1, \ldots, f_r, f_r = h^2} \frac{\Lambda(f_1) \cdots \Lambda(f_r)}{(\deg f_1) \cdots (\deg f_r)}.
\]
Noticing that \( \frac{\Lambda(f_i)}{\deg f_r} \leq 1 \) and \( f_i \)'s are all prime powers, the sum over \( h \) is actually over all monic polynomials \( h \in F[X] \) with \( \omega(h) \leq r \) and \( \deg h > Z/2 \), where \( \omega(h) \) is the function counting the number of distinct prime factors of \( h \). If such an \( h \) is chosen, the number of choices for each \( f_i \) dividing \( h \) which is a prime power is less than \( 2^r \deg h \). Hence the error by removing the restriction that \( \deg f_1, \ldots, \deg f_r \leq Z \) is bounded by
\[
T_3 \leq \sum_{\deg h > Z/2} |h|^{-2} (2r \deg h)^r = \sum_{n > Z/2} q^{-n} (2rn)^r \ll q^{-Z/4}.
\]
Combining these estimates together we obtain
\[
\langle (\triangle Z)^r \rangle = H(r) + T,
\]
where \( T \ll q^{-Z/4} \) and
\[
H(s) = \sum_{n_1, \ldots, n_s \geq 1, i=1} \prod_{i=1}^s q^{-n_i} \Lambda(f_i) \Lambda(f_s) \prod_{P|h} (1 + |P|^{-1})^{-1}.
\]
We write
\[
\langle (N_F)^r \rangle = \langle (\triangle Z)^r \rangle + E_{Z,r},
\]
where
\[ E_{Z,r} = \sum_{l=1}^{r} \binom{r}{l} \langle (\epsilon_Z)^l (\triangle Z)^{r-l} \rangle \ll q^{-Z/4}. \]

Using (4.1) and the above we find that
\[ \langle (N_F)^r \rangle = H(r) + O\left(q^{-Z/4}\right). \]

If \( q \) is fixed and \( d \to \infty \), then for each fixed \( r \),
\[ \lim_{d \to \infty} \langle (N_F)^r \rangle = H(r). \]

Now suppose that \( X \) is a random variable with
\[ \mathbb{E}(X^r) = H(r), \quad \forall r \in \mathbb{N}. \]

For any \( t \in \mathbb{R} \), we can compute the characteristic function \( \phi(t) = \mathbb{E}(e^{itX}) \) of \( X \).

Expanding \( e^{itX} \) by using the identity
\[ e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}, \]
using (4.11) and the expression of \( H(r) \) from Proposition 5.1 which we will prove in the last section, we find that
\[ \phi(t) = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \sum_{r=1}^{\infty} \frac{n!}{2^r r!} \sum_{\lambda_1+\ldots+\lambda_r=n} \prod_{j=1}^{r} u_{P_j}^{\lambda_j} + (-1)^{\lambda_j} v_{P_j}^{\lambda_j} \lambda_j! (1 + |P_j|^{-1}) \]
where for any \( P \in \mathbb{F}_q[X] \),
\[ u_P = -\log(1 - |P|^{-1}), \quad v_P = \log(1 + |P|^{-1}). \]

Changing the order of summation again we obtain
\[ \phi(t) = 1 + \sum_{r=1}^{\infty} \frac{1}{2^r r!} \prod_{P_1,...,P_r \text{ distinct}}^r \left( \sum_{\lambda_j=1}^{\infty} \frac{(it)^{\lambda_j}}{\lambda_j! (1 + |P_j|^{-1})} \right). \]

This implies
\[ \phi(t) = 1 + \sum_{r=1}^{\infty} \frac{1}{2^r r!} \prod_{P_1,...,P_r \text{ distinct}}^r \left( (1 - |P_j|^{-1})^{-it} + (1 + |P_j|^{-1})^{-it} - 2 \right). \]

This completes the proof of (1) of Theorem 1.2.

For the proof of (2) of Theorem 1.2, it is enough to show that as \( q \to \infty \), \( \widetilde{\phi}(t) = \phi(t\sqrt{q}) \to e^{-t^2/2} \), the characteristic function of a standard Gaussian distribution. Notice that
\[ \widetilde{\phi}(t) = \prod_P \left( 1 + \frac{(1 - |P|^{-1})^{-it\sqrt{q}} + (1 + |P|^{-1})^{-it\sqrt{q}} - 2}{2(1 + |P|^{-1})} \right), \]
where the product is over monic irreducible polynomials $P \in \mathbb{F}_q[X]$. It is easy to verify that as $q \to \infty$,

$$\log \hat{\phi}(t) = -t^2/2 + O(q^{-1/2}).$$

This completes the proof of (2) of Theorem 1.2. □

5. Analysis of $H(s)$

5.1. Proposition 1. Let $\mathbb{F}_q$ be a finite field of cardinality $q$. For any positive integer $s$, denote

$$H(s) = \sum_{n_1, \ldots, n_s \geq 1} \prod_{i=1}^{s} q^{-n_i} n_i^{-1} \sum_{\deg f_i = n_i} \Lambda(f_1) \cdots \Lambda(f_s) \prod_{P|h} (1 + |P|^{-1})^{-1}.$$

In this section we derive another representation of $H(s)$ which has been used in the proof of Theorems 1.2.

Proposition 5.1. For any positive integer $s \geq 1$ we have

$$H(s) = \sum_{r=1}^{s} \frac{s!}{2^r r!} \sum_{\lambda_1 + \cdots + \lambda_r = s} \sum_{\text{distinct}} \prod_{i=1}^{r} u_{P_i}^{\lambda_i} + (-1)^{\lambda_i} v_{P_i}^{\lambda_i} \lambda_i! (1 + |P_i|^{-1})^{-1},$$

where the sum on the right side is over all positive integers $\lambda_1, \ldots, \lambda_r$ such that $\lambda_1 + \cdots + \lambda_r = s$ and over all distinct monic irreducible polynomials $P_1, \ldots, P_r \in \mathbb{F}_q[X]$, and

$$u_P = -\log \left(1 - |P|^{-1}\right), \quad v_P = \log \left(1 + |P|^{-1}\right), \quad \forall P \in \mathbb{F}_q[X].$$

Proof. We rewrite $H(s)$ as

$$H(s) = \sum_{h} \prod_{P|h} (1 + |P|^{-1})^{-1} |h|^{-2} \sum_{\substack{f_1, \ldots, f_s \text{ distinct} \atop f_1 \cdots f_s = h^2}} \frac{\Lambda(f_1) \cdots \Lambda(f_s)}{(\deg f_1) \cdots (\deg f_s)}.$$

Since $f_i$’s are prime powers, the sum over $h$ is actually over all monic polynomials $h \in \mathbb{F}_q[X]$ with $\omega(h) \leq r$, where $\omega(h)$ is the number of distinct prime factors of $h$. Hence

$$H(s) = \sum_{r=1}^{s} H(s, r),$$

where

$$H(s, r) = \sum_{h} \prod_{P|h} (1 + |P|^{-1})^{-1} |h|^{-2} \sum_{\substack{f_1, \ldots, f_s \text{ distinct} \atop f_1 \cdots f_s = h^2}} \frac{\Lambda(f_1) \cdots \Lambda(f_s)}{(\deg f_1) \cdots (\deg f_s)}.$$
If $\omega(h) = r$, write explicitly $h = P_1^{a_1} \cdots P_r^{a_r}$ for some distinct primes $P_1, \ldots, P_r$ and exponents $a_1, \ldots, a_r \geq 1$, then

$$H(s, r) = \frac{1}{r!} \sum_{P_1, \ldots, P_r \text{ distinct}} \sum_{a_1, \ldots, a_r \geq 1} \prod_{i=1}^r (1 + |P_i|^{-1})^{-1} |P_i|^{-2a_i} \times \sum_{f_1, \ldots, f_s \text{ distinct}} \Lambda(f_1) \cdots \Lambda(f_s) \left( \frac{(\deg f_1) \cdots (\deg f_s)}{e_1 \cdots e_s} \right).$$

Since each $f_i$ is a prime power and $f_1 \cdots f_s = P_1^{2a_1} \cdots P_r^{2a_r}$, there are finitely many ways to assign prime powers to each $f_i$, according to which we will break $H(s, r)$ into many subsums. With that in mind, for each partition of the set of indexes

$$\{1, 2, \ldots, s\} = \bigcup_{i=1}^r A_i, \quad \#A_i = \lambda_i \geq 1, \forall i,$$

it satisfies the property that

$$\sum_{i=1}^r \lambda_i = s.$$

We say $(A_1, \ldots, A_r)$ is the type of $(f_1, \ldots, f_r)$ with $f_1 \cdots f_r = h^2$, namely whenever $j \in A_i$, then $f_j$ is a power of $P_i$. Suppose that $f_i = Q_i^{e_i}$ for some prime $Q_i \in \{P_1, \ldots, P_r\}$ and exponent $e_i \geq 1$, and the type of $(f_1, \ldots, f_r)$ is $(A_1, \ldots, A_r)$, since $f_1 \cdots f_s = P_1^{2a_1} \cdots P_r^{2a_r}$, comparing the exponents of $P_j$ on both sides we find that

$$\sum_{i \in A_j} e_i = 2a_j \quad \forall 1 \leq j \leq r,$$

and

$$\frac{\Lambda(f_1) \cdots \Lambda(f_s)}{(\deg f_1) \cdots (\deg f_s)} = \frac{1}{e_1 \cdots e_s}.$$

Instead of summing over all integers $a_1, \ldots, a_r$, we sum over all positive integers $e_1, \ldots, e_s$ which satisfy the conditions (5.3). Noting that the value only depends on the vector of integers $(\lambda_1, \ldots, \lambda_r)$ such that

$$\sum_{i=1}^r \lambda_i = s,$$

hence we can write $H(s, r)$ as

$$H(s, r) = \frac{s!}{r!} \sum_{\lambda_1 + \cdots + \lambda_r = s} \sum_{\lambda_i \geq 1} \prod_{i=1}^r \left( \frac{1 + |P_i|^{-1}}{\lambda_i!} \sum_{a_1 + \cdots + a_{\lambda_i} \equiv 0 \pmod{2}} \frac{|P_i|^{-a_1 - \cdots - a_{\lambda_i}}}{a_1 \cdots a_{\lambda_i}} \right).$$
For each prime $P$ and positive integer $\lambda$, denote

$$\eta(\lambda) = \eta_P(\lambda) := \sum_{a_1 + \cdots + a_\lambda \equiv 0 \pmod{2}, a_i \geq 1} \frac{|P|^{-a_1 - \cdots - a_\lambda}}{a_1 \cdots a_\lambda}$$

and

$$\tau(\lambda) = \tau_P(\lambda) := \sum_{a_1 + \cdots + a_\lambda \equiv 1 \pmod{2}, a_i \geq 1} \frac{|P|^{-a_1 - \cdots - a_\lambda}}{a_1 \cdots a_\lambda}.$$  

Since

$$-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1,$$

we find

(5.4) \quad \eta(1) = -\frac{1}{2} \log \left(1 - |P|^{-2}\right),$$

and

(5.5) \quad \eta(\lambda) + \tau(\lambda) = \sum_{a_1, \ldots, a_\lambda \geq 1} \frac{|P|^{-a_1 - \cdots - a_\lambda}}{a_1 \cdots a_\lambda} = (-1)^\lambda \log^\lambda \left(1 - |P|^{-1}\right).$$

Combining (5.4) and (5.5) we have

$$\tau(1) = -\log \left(1 - |P|^{-1}\right) + \frac{1}{2} \log \left(1 - |P|^{-2}\right).$$

For $\lambda \geq 2$, we can write

$$\eta(\lambda) = \sum_{a_2 + \cdots + a_\lambda \equiv 0 \pmod{2}, a_i \geq 1} \left(\prod_{i=1}^{\lambda} \frac{|P|^{-a_i}}{a_i}\right) \eta(1) + \sum_{a_2 + \cdots + a_\lambda \equiv 1 \pmod{2}, a_i \geq 1} \left(\prod_{i=1}^{\lambda} \frac{|P|^{-a_i}}{a_i}\right) \tau(1).$$

This shows that

(5.6) \quad \eta(\lambda) = \eta(1)\eta(\lambda - 1) + \tau(1)\tau(\lambda - 1).$$

Similarly for $\lambda \geq 2$,

(5.7) \quad \tau(\lambda) = \eta(1)\tau(\lambda - 1) + \tau(1)\eta(\lambda - 1).$$

We can assign the initial values

$$\eta(0) = 1, \quad \tau(0) = 0,$$

so that the recursive relations (5.6) and (5.7) hold for any $\lambda \geq 1$. Subtracting these two recursive relations we obtain

$$\eta(\lambda) - \tau(\lambda) = (\eta(1) - \tau(1)) \eta(\lambda - 1) - \tau(\lambda - 1).$$

Applying this relation recursively and using (5.5) we conclude that

$$\eta(\lambda) = \frac{1}{2} \left(u^\lambda_P + (-1)^\lambda v^\lambda_P\right),$$

where

$$u_P = -\log \left(1 - |P|^{-1}\right), \quad v_P = \log \left(1 + |P|^{-1}\right).$$
Therefore $H(s, r)$ can be written as

$$H(s, r) = \frac{s!}{2^r r!} \sum_{\lambda_1 \geq 1} \sum \prod_{i=1}^{r} \frac{u_{P_i}^\lambda_i + (-1)^{\lambda_i} v_{P_i}^\lambda_i}{\lambda_i! (1 + |P_i|^{-1})}.$$ 

Returning to (5.2) completes the proof of Proposition 5.1.

\[\square\]

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