INVARiANCE OF THE JACOBIAN NEWTON DIAGRAM

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ABSTRACT. We prove that the jacobian Newton diagram of the holomorphic mapping $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ depends only on the equisingularity class of the pair of curves $f = 0$ and $g = 0$.

1. Introduction

Write $\mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \}$. The Newton diagram $\Delta_h$ of a power series $h(x, y) = \sum_{i,j} c_{ij} x^i y^j$ is by definition the convex hull of the union $\bigcup \{ (i, j) \in \mathbb{R}_+^2 : (i, j) \neq 0 \}$.

**Example 1.1.** The Newton diagram of $h(x, y) = y^5 + 2xy^3 - x^3y^2 + 3x^4y$ is drawn in the figure. Black dots are the points of the first quadrant $\mathbb{R}_+^2$ corresponding to nonzero monomials of the series $h$.

Let $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping given by $(x, y) = (f(u, v), g(u, v))$. Let $\text{jac} \phi = \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u}$ be the usual jacobian determinant. The direct image of the curve germ $\text{jac} \phi = 0$ by $\phi$ is called the **discriminant curve** of $\phi$ (see [2]). If $D(x, y) = 0$ is an analytic equation of the discriminant curve then the Newton diagram of $D$ is called the **jacobian Newton diagram** of $(f, g)$. We will write $\mathcal{N}_J(f, g)$ for the jacobian Newton diagram.

Let $h = h(u, v) \in \mathbb{C}\{u, v\}$, $h(0, 0) = 0$ be a convergent power series and let $h = h_1^{m_1} \cdots h_s^{m_s}$ be a factorization of $h$ in the ring $\mathbb{C}\{u, v\}$ with $h_i$ irreducible and pairwise co-prime. Then every curve germ $h_i = 0$ is called a branch of the curve $h = 0$ and $m_i$ is called the multiplicity of $h_i = 0$ in $h = 0$.

**Definition 1.1.** Let $\xi, \xi', \nu, \nu'$ be germs of analytic curves in $(\mathbb{C}^2, 0)$. We say that the pairs of curves $\xi, \nu$ and $\xi', \nu'$ are equisingular if there exists a homeomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ preserving for each curve the multiplicity in it of each of its branches such that $\Psi(\xi) = \xi'$ and $\Psi(\nu) = \nu'$.

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2. Main result

**Theorem 2.1.** Let \((f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0), (f, g)^{-1}(0, 0) = \{(0, 0)\}\) be the germ of a holomorphic mapping. Then the jacobian Newton diagram \(N_J(f, g)\) depends only on the equisingularity class of the pair of curves \(f = 0\) and \(g = 0\).

The proof is in the last section.

We give a survey of results related with Theorem 2.1. We need a few notions which will be used only in this section to explain connection between certain analytic factorizations of \(\text{jac}(f, g)\) and the jacobian Newton diagram \(N_J(f, g)\).

The Minkowski sum of Newton diagrams \(\Delta_1\) and \(\Delta_2\) is by definition \(\Delta_1 + \Delta_2 = \{p + q : p \in \Delta_1, q \in \Delta_2\}\). The set of Newton diagrams is a semi-group with respect to Minkowski sum and the generators of this semi-group are elementary Newton diagrams illustrated in Figure 1.

The inclination of the elementary Newton diagram \(\{\frac{a}{b}\}\) is the quotient \(\frac{a}{b}\) (by convention \(\frac{a}{0} = \infty\) and \(\frac{0}{b} = 0\)). For an arbitrary Newton diagram \(\Delta\) represented as a sum of elementary Newton diagrams let us denote \(I(\Delta)\) the set of inclinations of elementary Newton diagrams of the sum. It is easy to check that \(I(\Delta)\) does not depend on the choice of a representation. Coming back to Example 1.1 the Newton diagram \(\Delta_h\) is the sum \(\{\frac{1}{2}\} + \{\frac{3}{2}\} + \{\frac{\infty}{1}\}\) and \(I(\Delta_h) = \{\frac{1}{2}, \frac{3}{2}, \infty\}\).

If \(h\) is any irreducible factor of \(\text{jac}(f, g)\) then \(q(h) = \frac{i_0(g, h)}{i_0(f, h)}\), where \(i_0(\cdot, \cdot)\) stands for the intersection multiplicity, is called the jacobian quotient of \((f, g)\).

**Definition 2.1.** Let \(\text{jac}(f, g) = J_1 \cdots J_n\) be an analytic factorization of the jacobian.

We will call \(J_1 \cdots J_n\) a Hironaka factorization if for every \(J_i\) \((1 \leq i \leq n)\) the jacobian quotient \(q(h)\) is constant for all irreducible factors \(h\) of \(J_i\).

The Hironaka factorization \(J_1 \cdots J_n\) will be called minimal if jacobian quotients of irreducible factors of \(J_l\) and \(J_k\) are different for \(1 \leq l < k \leq n\).

Let \(\text{jac}(f, g) = h_1 \cdots h_n\) be the factorization of the jacobian into irreducible factors. It is easy to check (see \([13]\)) that

\[
N_J(f, g) = \sum_{i=1}^{n} \left\{ \frac{i_0(g, h_i)}{i_0(f, h_i)} \right\}.
\]

It follows directly from the above formula that

- the set of jacobian quotients of \((f, g)\) is the set of inclinations of \(N_J(f, g)\),
- if \(J_1 \cdots J_r\) is a Hironaka factorization of \(\text{jac}(f, g)\) then

\[
N_J(f, g) = \sum_{i=1}^{r} \left\{ \frac{i_0(g, J_i)}{i_0(f, J_i)} \right\}.
\]

Figure 1. Elementary Newton diagrams
• if $N_J(f,g) = \sum_{i=1}^{s} \left( \frac{a_i}{b_i} \right)$ with inclinations $\frac{a_i}{b_i}$ pairwise different then $\text{Jac}(f,g)$ has the minimal Hironaka factorization $J_1 \cdots J_s$ such that $i_0(g,J_i) = a_i$ and $i_0(f,J_i) = b_i$ for $i = 1, \ldots, s$.

Let us consider a germ of a holomorphic mapping $(l,f): (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $f = 0$ is a curve germ without multiple branches and $l = 0$ is a generic smooth curve (here generic means that the curve $l = 0$ is not tangent to any branch of the curve $f = 0$). Under these assumptions $\text{Jac}(l,f) = 0$ is called the polar curve of $f$ and jacobian quotients of $(l,f)$ are called polar quotients. A survey of recent results concerning polar curves is in [6].

Merle in [10] obtained the minimal Hironaka decomposition of the polar curve of the irreducible curve germ $f = 0$. Merle’s result is rewritten in [13] as a formula for the jacobian Newton diagram of $(l,f)$ (see also [6], Theorem 4.1).

In [7], Kuo and Lu described the contact orders of Newton–Puiseux roots of the partial derivative $f'_x(x,y) = 0$ with the Newton–Puiseux roots of $f(x,y) = 0$. They constructed the tree model $T(f)$ which encodes these contact orders. Using the Kuo–Lu tree $T(f)$ one can compute all polar quotients of $(y,f)$. One can also give a formula for the jacobian Newton diagram of $(y,f)$ in terms of $T(f)$ (see the last line before Example 5.2 in [5]).

In [4], the author studied the polar curve of a many-branched curve $f = 0$. He introduced a new type of tree $E(f)$ called now the Eggers tree of $f$. Eggers found the Hironaka factorization of the jacobian $\text{Jac}(l,f)$ such that the factors are indexed by vertices of $E(f)$. He also computed the intersection multiplicities of every factor with $l$ and $f$. Since the Eggers tree $E(f)$ depends only on the equisingularity class of $f$, Theorem 2.1 in this particular case follows from [4].

The papers [7] and [4] provide methods of computing $N_J(l,f)$ using invariants of equisingularity of $f$. Another way to obtain Theorem 2.1 in the polar case is to use a deformation argument. Teissier proved in [14] that for every $\mu^*$-constant family of hypersurfaces with isolated singularities, the jacobian Newton diagram is constant. Since every two plane analytic curves of the same equisingularity type can be joined by a $\mu^*$-constant family of plane curves (see [2], Proposition 5.2 for a direct construction of such a family) we get another proof of Theorem 2.1 in the polar case.

Consider now a general case of a holomorphic mapping germ $(f,g): (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, where $(f,g)^{-1}(0,0) = \{(0,0)\}$.

In [8], the authors additionally assumed that the curve $fg = 0$ has no multiple components. They defined the equivalence relation between vertices of the Kuo–Lu tree $T(fg)$. Then they obtained the Hironaka factorization of $\text{Jac}(f,g)$ such that the factors are indexed by equivalence classes of this relation. However, as Section 5 of [8] shows, the equisingularity class of the pair $f = 0$, $g = 0$ does not determine the intersection multiplicities of some factors with $f$ and $g$. Hence, Theorem 4.1 does not follow from [8].

In [9] and [11], the authors resolved singularities of the curve $fg = 0$. Then they distinguished some subsets of the exceptional divisor called rupture zones and associated a factor of the jacobian $\text{Jac}(f,g)$ with every rupture zone. Maugendre [9] found the set of jacobian quotients using topological methods (see also [3] for an algebraic proof) and Michel [11] completed the work computing the intersection multiplicities of every factor with $f$ and $g$. Since the decomposition of the jacobian obtained by Michel is a
Proof. Our main reference is Chapter III of [12]. Let the pair of curves \( (f,g) \) be a germ of a holomorphic mapping such that \((f,g)^{-1}(0,0) = \{(0,0)\}\). Let \( D(x,y) = 0 \) be the discriminant curve of \((f,g)\). Take any curve germ \( h(x,y) = 0 \) and let \( H(u,v) = h(f(u,v),g(u,v)) \). Then

\[
\mu_0(H) - 1 = i_0(f,g)(\mu_0(h) - 1) + i_0(h, D),
\]

where \( \mu_0(h) \) denotes the Milnor number of the curve \( h = 0 \) at zero.

3. Invariance of a generic curve of the pencil

**Theorem 3.1.** Let \((f,g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)\) be the germ of a holomorphic mapping. Then for all \( t \in \mathbb{C} \) but a finite number the equisingularity class of the curve \( f(x,y) - tg(x,y) = 0 \) depends only on the equisingularity class of the pair of curves \( f = 0 \) and \( g = 0 \).

**Proof.** Our main reference is Chapter III of [12]. Let \( R : M \to (\mathbb{C}^2, 0) \) be the minimal good resolution of singularities of the curve \( f = 0 \). The set \( R^{-1}(\{fg = 0\}) \) can be written as the union of irreducible components \( E_1 \cup \cdots \cup E_n \cup E_{n+1} \cup \cdots \cup E_m \), where \( E = E_1 \cup \cdots \cup E_n \) is the exceptional divisor \( R^{-1}(0) \) and \( E_{n+1}, \ldots, E_m \) are noncompact curves corresponding with branches of the curve \( fg = 0 \). Put \( \tilde{f} = f \circ R \), \( \tilde{g} = g \circ R \) and let \( a_i = \text{order of } f \) along \( E_i \), \( b_i = \text{order of } g \) along \( E_i \) for \( i = 1, \ldots, m \). Then, after renumbering \( E_1, \ldots, E_m \) if necessary, the total dual resolution graph as well as the numbers \( a_i \) and \( b_i \) for \( i = 1, \ldots, m \) depend only on the equisingularity class of the pair of curves \( f = 0 \) and \( g = 0 \).

Consider the meromorphic function \( \tilde{f}/\tilde{g} : M \setminus E \to \mathbb{C} \cup \{\infty\} \). We will check that this function extends analytically to the whole \( M \) with the exception of a finite number of points. Let \( \cdot \) denotes any germ of a holomorphic function \( u(x,y) \) such that \( u(0,0) \neq 0 \).

First take \( P \in E_i \) \((1 \leq i \leq n)\) which is not an intersection point with another component \( E_j \) for \( 1 \leq j \leq m \). Choose a local analytical coordinate system \((x,y)\) centered at \( P \) such that \( E_i \) has the equation \( x = 0 \). In these coordinates \( \tilde{f} = \cdot x^{a_i} \) and \( \tilde{g} = \cdot x^{b_i} \). We get \( \tilde{f}/\tilde{g} = \cdot x^{a_i - b_i} \).

Now take the intersection point \( P \) of \( E_i \) with another component \( E_j \). Choose a local analytical coordinate system \((x,y)\) centered at \( P \) such that \( E_i \) has the equation \( x = 0 \) and \( E_j \) has the equation \( y = 0 \). In these coordinates \( \tilde{f} = \cdot x^{a_i} y^{a_j} \) and \( \tilde{g} = \cdot x^{b_i} y^{b_j} \). We get \( \tilde{f}/\tilde{g} = \cdot x^{a_i - b_i} y^{a_j - b_j} \).

Let \( H \) be an analytic extension of \( \tilde{f}/\tilde{g} \). Divide the set \( \{ E_1, \ldots, E_m \} \) into three subsets \( A_+ = \{ E_i : a_i - b_i > 0 \} \), \( A_0 = \{ E_i : a_i - b_i = 0 \} \) and \( A_- = \{ E_i : a_i - b_i < 0 \} \). It follows from the above description of \( \tilde{f}/\tilde{g} \) near \( E \) that \( H \) is not defined only at the intersection points of components from \( A_+ \) with components from \( A_- \).

Let \( E_i \in A_0 \). Consider the restriction \( H|_{E_i} \) of the meromorphic function \( H \) to \( E_i \). Then \( P \in E_i \) is a zero of \( H|_{E_i} \), if and only if \( \{ P \} = E_i \cap E_j \) for some \( E_j \in A_+ \). Moreover, \( \text{ord}_P H|_{E_i} = a_j - b_j \). Hence the topological degree of \( H|_{E_i} \) is the number \( d_i = \sum (a_j - b_j) \) where the sum runs over all \( j \) such that \( E_j \in A_+ \) and the intersection \( E_i \cap E_j \) is nonempty.
Let $t$ be generic, then, by Theorem 4.3 of [12] there exists a canonical toric resolution of $E$. For every Newton diagram $\Delta$ and for every point $x, y$ on the curve of $\phi$ we have the equation $x = 0$ and $E$ has the equation $y = 0$. In these coordinates $\tilde{f} - \tilde{g} = \alpha x^{a_i-b_i} - t \alpha x^{b_i-b_j} = \alpha x^{a_i-b_i} - y^{b_j-a_j}$, hence $\Gamma$ has the equation $\alpha x^{a_i-b_i} - y^{b_j-a_j} = 0$.

We want to resolve singularities of the curve $\Gamma$ to obtain a good (not necessarily minimal) resolution of singularities of $f - tg = 0$. Every function $h_{i,j}(x, y) = \alpha x^{a_i-b_i} - y^{b_j-a_j}$ is nondegenerate with the Newton diagram $\{ a_i-b_i/b_j-a_j \}$. Hence by Theorem 4.3 of [12] there exists a canonical toric resolution of $h_{i,j}(x, y) = 0$ at the origin, that is the resolution of $\Gamma$ at $P_{i,j}$, which depends only on the Newton diagram of $h_{i,j}$. Applying such a toric resolution at every point $P_{i,j}$ described above we obtain a good resolution of $f - tg = 0$. Moreover, the total dual resolution graph of this resolution depends only on the total dual resolution graph of $R$ and on the numbers $a_i$ and $b_i$ for $i = 1, \ldots, m$.

Since the total dual resolution graph of the plane curve singularity determines its equisingularity class (see [1], Chapter 8.4, Proposition 20) the proof is finished. \hfill $\Box$

4. Proof of the main result

For every Newton diagram $\Delta$ and for every $\vec{v} = (v_1, v_2)$, where $v_1 > 0$, $v_2 > 0$ we define

$$l(\vec{v}, \Delta) = \min \{ v_1i + v_2j : (i, j) \in \Delta \}.$$ 

**Lemma 4.1.** Let $\vec{v} = (m, n)$, where $n$, $m$ are co-prime positive integers. Then for generic $t \in \mathbb{C}$

$$l(\vec{v}, N_f(f, g)) = \mu_0(f^n - ty^m) - i_0(f, g)(m-1)(n-1) - 1.$$

**Proof.** Let $D = 0$, where $D(x, y) = \sum c_{ij}x^iy^j$, be the equation of the discriminant curve of $\phi = (f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Take a curve $x^n - ty^m = 0$.

**Claim.** For generic $t \in \mathbb{C}$ we have $i_0(x^n - ty^m, D) = l(\vec{v}, N_f(f, g))$.

Let $\tau = \sqrt{t}$. Then $x = \tau s^m, y = s^n$ is a parameterization of the branch $x^n - ty^m = 0$. By the classical formula for the intersection multiplicity

$$i_0(x^n - ty^m, D) = \text{ord}_s D(\tau s^m, s^n) = \text{ord}_s \sum c_{ij} \tau^i s^{m_i+n_j} = l(\vec{v}, N_f(f, g))$$

provided $\tau$ is sufficiently general so that the sum $\sum_{m_i+n_j=l(\vec{v}, N_f(f, g))} c_{ij} \tau^i$ is nonzero. The Claim is proved.
The pull-back of the curve $x^n - ty^m = 0$ by $\phi$ has an equation $f^n - tg^m = 0$. Thus by Theorem 2.2 we have

$$\mu_0(f^n - tg^m) - 1 = i_0(f,g)[\mu_0(x^n - ty^m) - 1] + i_0(x^n - ty^m, D),$$

which gives the lemma because $\mu_0(x^n - ty^m) = (m - 1)(n - 1)$. □

Proof of Theorem 2.1. It follows from Lemma 4.1 and Theorem 3.1 that for every vector $\vec{v} = (m, n)$, where $m, n$ are co-prime positive integers, the number $l(\vec{v}, N_{f,g})$ depends only on the equisingularity class of the pair $f = 0$ and $g = 0$. Since every Newton diagram $\Delta$ is equal to the intersection of half-planes determined by $l(\vec{v}, \Delta)$ the theorem is proved. □

References


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