

GLOBAL WELL POSEDNESS AND SCATTERING FOR THE DEFOCUSING, CUBIC NLS IN \mathbb{R}^3

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ABSTRACT. We prove global well-posedness and scattering for the defocusing, cubic NLS on \mathbb{R}^3 with initial data in $H^s(\mathbb{R}^3)$ for $s > 49/74$. The proof combines the ideas of resonance decomposition in [8] and linear–nonlinear decomposition in [9, 14] together with the idea of large time iteration.

1. Introduction

Consider the defocusing cubic nonlinear Schrödinger equations (NLS) in 3D

$$(1.1) \quad \begin{cases} iu_t + \Delta u = |u|^2 u, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0) = u_0 \in H_x^s(\mathbb{R}^3), \end{cases}$$

where $s \geq 1/2$.

It is known that there is the mass conservation law for (1.1), i.e.,

$$(1.2) \quad M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)).$$

If $s \geq 1$, there is also energy conservation law,

$$(1.3) \quad E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{4} \int |u(t, x)|^4 dx = E(u(0)).$$

Moreover, (1.1) is locally well posed for $s > 1/2$. In particular, there is blow-up criteria for (1.1): If $s > 1/2$ and u is the solution to (1.1) with maximal existence interval $[0, T^*)$, then if $T^* < \infty$,

$$(1.4) \quad \lim_{t \uparrow T^*} \|u(t)\|_{H^s} = \infty.$$

Thus global well posedness of (1.1) for $s \geq 1$ (see [5]) follows immediately from energy conservation law. Scattering in energy space or above is proved by Ginibre and Velo in [11]. However, for $s < 1$, there is no energy conservation. More precisely, there is no known coercive quantity that can be used to control the H^s norm, which is the main obstruction for global well posedness and scattering. It was conjectured by the following

Conjecture. Let $s \geq 1/2$, then (1.1) is globally well posed in $H^s(\mathbb{R}^3)$ and there is scattering.

Remark 1.1. The two-dimensional defocusing, cubic NLS analogy of this conjecture has been solved by Dodson [10] recently. He showed that the defocusing, cubic NLS is globally well posed and there is scattering in $L^2(\mathbb{R}^2)$.

The conjecture has attracted much attentions. Previous work can be found in [2, 6, 7, 9, 12]. We state these results briefly.

The breakthrough work was made by Bourgain (see [1–3]). He used the Fourier truncation method to capture the smoothing effect of the nonlinearity. He proved global well posedness for $s > 11/13$ and scattering for radially symmetry data $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/7$.

Inspired by the Fourier truncation method, Colliander *et al.* introduced the I-method (or almost conservation law method) in [6], which is a smoothed version of the Fourier truncation method. By smoothing out the rough data, they can make use of the energy conservation law. Indeed, they proved almost conservation law for the smoothed solution via multilinear estimate, and then proved a polynomial bound for the solution of (1.1) for $s > 5/6$, thus obtained global well posedness for $s > 5/6$, but not the scattering result.

To weaken the regularity requirement in [6] for global well posedness and radical symmetry assumption in [2] for scattering, Colliander *et al.* [7] proved a new type Morawetz inequality. Together with the I-method they are able to bound the solution in $H^s(\mathbb{R}^3)$ and $L^4_{t,x}$ uniformly provided $s > 4/5$, thus they are able to prove global well posedness and scattering for $s > 4/5$.

Recently, Dodson [9] improved the result in [7] via linear–nonlinear decomposition method introduced by Roy [14]. By using linear–nonlinear decomposition, I-method, and together with double layer decomposition, he was able to show global well posedness and scattering for $s > 5/7$.

On the other hand, Kenig and Merle in [13] used the concentration-compactness method to deal with global well posedness and scattering problems at critical regularity. By profile decomposition and concentration compactness/rigidity argument, they showed in [12] that in order to prove conjecture, it suffices to bound the solution in $\dot{H}^{1/2}$.

In this paper, we adopt an idea of large time iteration. Normally, in order to obtain global well posedness, we would obtain local well posedness on a small time interval, and then use the iteration method to extend the local solution to global one. Roughly speaking, for each iteration, we extend the solution on time interval by one unit. Such iteration is ‘slow’ in some sense. Thus, we would like to have a ‘faster’ iteration strategy, where the iterates on time interval are larger than one for each iteration. As a consequence, the number of iterations is heavily reduced.

To see how such an idea works, we combine the idea of linear–nonlinear decomposition used by Dodson in [9] and Roy in [14], the idea of modified energy via resonance decomposition in [8], and the idea of ‘large time iteration’. It is captured that the nonlinear part of the solution enjoys more regularity in high frequency. Thus, we can make use of such a smoothing effect by linear–nonlinear decomposition. Furthermore, by adding a correction term to the energy functional $E(Iu)$, we can obtain a better control of the increment of the energy (see [8] for more discussion). Thus we are able to prove a refined version of almost conservation law. Finally, by large time iteration, we are able to reduce the amount of iterations. The main result of this paper is the following.

Theorem 1.1. *(Equation 1.1) is globally well posed and there is scattering in $H^s(\mathbb{R}^3)$ for $s > 49/74$.*

This paper is organized as follows. In Section 2, we set some notations and recall some preliminary facts. In Sections 3 and 4, we prove a local existence theorem and an smoothing effect of the nonlinear part of the solution, respectively. In Section 5, we recall the construction of modified energy in [8] and prove a refined almost conservation law. Theorem 1.1 will be proved in the last section.

2. Notations and preliminaries

Given $A, B \geq 0$, by $A \lesssim B$ we mean $A \leq C \cdot B$ for some universal constant C . By $A \sim B$ it means $A \lesssim B$ and $B \lesssim A$. The notation $A \gtrsim B$ means $B \lesssim A$. The notation $A \ll B$ means $A \leq K \cdot B$ for some large universal constant K . The notation $A \gg B$ means $A \geq K \cdot B$ for some large constant $K > 0$. The notation $A+$ means $A + \epsilon$ for some universal $0 < \epsilon \ll 1$. Moreover, the notation $A-$ means $A - \epsilon$ for some universal $0 < \epsilon \ll 1$. By $\langle a \rangle$ we mean $(1 + |a|^2)^{1/2}$.

Definition 2.1. Let $1 \leq q, r \leq \infty$, we say that (q, r) is admissible if

$$\frac{2}{q} = 3 \left(\frac{1}{2} - \frac{1}{r} \right).$$

We recall the definition of I -operator, which is a Fourier multiplier.

Definition 2.2. The I -operator $I_N : H^s(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3)$ is defined as

$$\widehat{I_N u}(\xi) = m_N(\xi) \hat{u}(\xi),$$

where m is smooth, radially symmetric, and satisfies

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ (\frac{N}{|\xi|})^{1-s}, & |\xi| > 2N. \end{cases}$$

We abbreviate I_N, m_N as I, m , respectively. For the convenience of the readers, we list some basic facts of the I -operator and explain how the I -method works. For more details, the reader can refer to, for example, [6–9, 14]. We have the estimates

$$\|\nabla Iu\|_{L^2(\mathbb{R}^3)} \lesssim N^{1-s} \|u\|_{\dot{H}^s(\mathbb{R}^3)}.$$

$$\|u\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim \|Iu\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Therefore, $\|u(t)\|_{\dot{H}^s(\mathbb{R}^3)}$ is controlled by $E(Iu(t))$:

$$\|u(t)\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim E(Iu(t)).$$

Thus, we are reduced to controlling the modified energy $E(Iu)$. Note that we can write $E(Iu)$ in a multilinear form:

$$E(Iu) = \Lambda_2(\sigma_2; u) + \Lambda_4(\sigma_4; u),$$

where Λ_2, Λ_4 are some multilinear functionals and σ_2, σ_4 are some symbols, see Section 5.1 for the definition, and see [8] for more details.

To obtain a better control on $E(Iu)$, we add a correction term to $E(Iu)$ to construct another modified energy functional $\tilde{E}(u(t))$ such that $\tilde{E}(u(t))$ has slower energy increment. Similar to [8], we use resonance decomposition to construct \tilde{E} as

$$\tilde{E}(u(t)) := \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u),$$

where $\tilde{\sigma}_4$ is defined via resonance decomposition. See Section 5.1, also see [8] for more details. By such construction, we are reduced to controlling $\tilde{E}(u(t))$.

Let u be a solution to (1.1) on time interval $J = [t_0, T]$ such that $u(t_0) = u_0$. We know that $\forall t \in [t_0, T]$, the Duhamel identity holds:

$$(2.1) \quad u(t) = e^{it\Delta}u_0 - i \int_{t_0}^t e^{i(t-s)\Delta}(|u|^2u)(s) ds.$$

We then decompose u into linear part u_J^l and nonlinear part u_J^{nl} adapted to J , i.e.,

$$(2.2) \quad u_J^l(t) := e^{it\Delta}u(t_0), \quad u_J^{nl}(t) := -i \int_{t_0}^t e^{i(t-s)\Delta}(|u|^2u)(s) ds.$$

In later sections, if there is no cause of confusion, we simply write u_J^l, u_J^{nl} as u^l and u^{nl} , respectively.

We need some Littlewood–Paley theory, see [15, 16] for example. Let $\phi(\xi)$ be a fixed radial bump function adapted to the ball $\{\xi : |\xi| \leq 2\}$ which equals 1 on the ball $\{\xi : |\xi| \leq 1\}$. Let N be a dyadic number. Define the Fourier multipliers

$$\begin{aligned} \widehat{P_{<N}u}(\xi) &:= \phi\left(\frac{\xi}{N}\right) \hat{u}(\xi), \\ \widehat{P_{>N}u}(\xi) &:= \left(1 - \phi\left(\frac{\xi}{N}\right)\right) \hat{u}(\xi), \\ \widehat{P_Nu}(\xi) &:= (\phi(\xi/N) - \phi(2\xi/N)) \hat{u}(\xi). \end{aligned}$$

Similarly, we can define $P_{\geq N}$, $P_{\leq N}$.

In the following, we state some facts that will be used frequently in later sections.

The first one is the Bernstein-type inequalities.

Proposition 2.1. [16] *Let $s \geq 0$ and d a positive integer. $1 \leq p \leq q \leq \infty$. Then*

$$\begin{aligned} \|P_{\geq N}u\|_{L_x^p(\mathbb{R}^d)} &\lesssim_{p,s,d} N^{-s} \|\nabla^s P_{\geq N}u\|_{L_x^p(\mathbb{R}^d)}; \\ \|P_{\leq N} \nabla^s u\|_{L_x^p(\mathbb{R}^d)} &\lesssim_{p,s,d} N^s \|P_{\leq N}u\|_{L_x^p(\mathbb{R}^d)}; \\ \|P_N \nabla^{\pm s} u\|_{L_x^p(\mathbb{R}^d)} &\lesssim_{p,s,d} N^{\pm s} \|P_N u\|_{L_x^p(\mathbb{R}^d)}; \\ \|P_{\leq N}u\|_{L_x^q(\mathbb{R}^d)} &\lesssim_{p,s,d} N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq N}u\|_{L_x^p(\mathbb{R}^d)}; \\ \|P_N u\|_{L_x^q(\mathbb{R}^d)} &\lesssim_{p,s,d} N^{\frac{d}{p} - \frac{d}{q}} \|P_N u\|_{L_x^p(\mathbb{R}^d)}. \end{aligned}$$

Next we state Strichartz estimate, which is fundamental to the study of dispersive equation. The reader can refer to [4, 16] for more details.

Lemma 2.1. *Let (q, r) be admissible. Let u be a solution to (1.1) on time interval $J = [t_0, T]$ with initial data $u(t_0) = u_0$, which satisfies the Duhamel identity,*

$$u(t) = e^{it\Delta}u_0 - i \int_{t_0}^t e^{i(t-s)\Delta}|u|^2u(s) ds.$$

Then we have

(2.3)

$$\|e^{it\Delta}u\|_{L_t^q(J)L_x^r} \lesssim \|u_0\|_{L_x^2}, \quad \left\| \int_J e^{i(t-s)\Delta} |u|^2 u(s) ds \right\|_{L_t^q(J)L_x^r} \lesssim \| |u|^2 u \|_{L_t^{\tilde{q}'}(J)L_x^{\tilde{r}'}} ,$$

where (\tilde{q}, \tilde{r}) is admissible and

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1, \quad \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1.$$

Definition 2.3. Let J be a time interval. Define

$$Z_I(J; u) := \sup_{(q,r) \text{ admissible}} \|\nabla Iu\|_{L_t^q(J)L_x^r(\mathbb{R}^3)}.$$

3. Local existence

We need a simple lemma.

Lemma 3.1. Let $\delta < s$ and (q, r) be admissible pair. Then

$$\|\nabla^\delta P_{\geq N} u\|_{L_t^q L_x^r} \lesssim N^{\delta-1} \|\nabla Iu\|_{L_t^q L_x^r}.$$

The proof is standard by Littlewood–Paley decomposition. We omit the details and leave the proof to the reader.

We also need a local existence result, whose proof can be found in [7].

Lemma 3.2. Consider $u(t, x)$ be as in (1.1) defined on $J \times \mathbb{R}^3$. Assume

$$(3.1) \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} \leq \epsilon,$$

for some small constant $\epsilon > 0$. Assume $u_0 \in C_0^\infty(\mathbb{R}^3)$. Then for $s > 1/2$ and sufficiently large N , we have

$$(3.2) \quad Z_I(J, u) \leq C(\|u_0\|_{\dot{H}^s}).$$

The following local existence is a modification of Lemma 3.2. In Lemma 3.2, the $L_{t,x}^4$ norm is assumed to be small, while, for our purpose, we remove the smallness assumption. In some sense, such a local existence can be viewed as a large-time existence and the iteration based on such a local existence can be viewed as a large-time iteration.

Lemma 3.3 (Modified local existence). Let u be a solution to (1.1) on time interval $J = [0, \tau]$. Assume

$$\sup_{t \in J} E(Iu(t)) \lesssim 1, \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} < \infty.$$

Then for admissible pair (q, r) ,

$$\begin{aligned} Z_I(J; u^l) &\lesssim 1; \\ \|\nabla Iu^{nl}\|_{L_t^q(J)L_x^r} &\lesssim \max\{1, \|u\|_{L_{t,x}^4}^4\}^{1/q}; \\ \|\nabla Iu\|_{L_t^q(J)L_x^r} &\lesssim \max\{1, \|u\|_{L_{t,x}^4}^4\}^{1/q}. \end{aligned}$$

Proof. It is clear that by Strichartz estimate, we have

$$Z_I(J; u^l) \lesssim \|\nabla Iu_0\|_{L_x^2} \lesssim 1.$$

Thus by triangle inequality, it suffices to show that

$$\|\nabla Iu\|_{L_t^q(J)L_x^r} \lesssim \max\{1, \|u\|_{L_{t,x}^4}^4\}^{1/q}.$$

We decompose J into subintervals J_1, \dots, J_m such that for each subinterval we have

$$\|u\|_{L_{t,x}^4(J_k \times \mathbb{R}^3)} \leq \epsilon$$

for some small constant $\epsilon > 0$. Thus, m is essentially $\|u\|_{L_{t,x}^4}^4$. Since for each J_k

$$\|\nabla Iu\|_{L_t^q(J_k)L_x^r}^q \lesssim 1,$$

summing over k yields

$$\|\nabla Iu\|_{L_t^q(J)L_x^r}^q \lesssim \|u\|_{L_{t,x}^4(J \times \mathbb{R})}^4.$$

□

Definition 3.1. We define

$$M(J, u, q) := \max\left\{1, \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)}^4\right\}^{1/q}.$$

4. Smoothing effect of nonlinearity

In this section, we prove a smoothing effect of the nonlinearity, which is crucial to prove the almost conservation law in the next section.

The following lemma was proved by Dodson [9].

Lemma 4.1. *Let u be a solution to (1.1) on time interval $J = [0, T]$ such that*

$$\|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} \leq \epsilon, \quad \|\nabla Iu_0\|_{L_x^2} \leq 1.$$

Let N_j be a dyadic number. Then if $N_j \lesssim N$,

$$(4.1) \quad \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^q L_x^r} \lesssim N_j^{-1/2}, \quad \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^\infty L_x^2} \lesssim N_j^{-1}.$$

and If $N_j \gtrsim N$,

$$(4.2) \quad \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^q L_x^r} \lesssim N^{-1/2}, \quad \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^\infty L_x^2} \lesssim N^{-1}.$$

By Lemma 4.1 and interpolation, we obtain the following smoothing effect.

Theorem 4.1. *Suppose J is an interval such that*

$$(4.3) \quad \sup_{t \in J} E(Iu(t)) \lesssim 1, \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} < \infty.$$

For any admissible pair (q, r) with $q \geq 4$, then if $N_j \lesssim N$,

$$(4.4) \quad \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^q L_x^r} \lesssim N_j^{-\frac{3}{4} - \frac{\theta}{4}} M(J, u, q)$$

and if $N_j \gtrsim N$,

$$(4.5) \quad \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^q L_x^r} \lesssim N^{-\frac{3}{4} - \frac{\theta}{4}} M(J, u, q),$$

where θ satisfies

$$\begin{cases} \frac{1}{q} = \frac{\theta}{\infty} + \frac{1-\theta}{4} = \frac{1}{4} - \frac{\theta}{4}, \\ \frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{3} = \frac{1}{3} + \frac{\theta}{6}. \end{cases}$$

Proof. We only prove the case that $N_j \lesssim N$. First, by the interpolation between $L_t^\infty L_x^2$ and $L_t^2 L_x^6$ with

$$\begin{cases} \frac{1}{4} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1}{2} - \frac{\theta}{2}, \\ \frac{1}{3} = \frac{\theta}{2} + \frac{1-\theta}{6} = \frac{1}{6} + \frac{\theta}{3}, \end{cases}$$

we get $\theta = 1/2$. Thus by the interpolation we have

$$\begin{aligned} \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^4 L_x^3} &\lesssim \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^\infty L_x^2}^{1/2} \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^2 L_x^6}^{1/2} \\ &\lesssim N_j^{-1/2} N_j^{-\frac{1}{2} \times \frac{1}{2}} M(J, u, 2)^{1/2} \\ &\lesssim N_j^{-3/4} M(J, u, 4). \end{aligned}$$

Second, observe that for each admissible pair (q, r) with $q \geq 4$, we have

$$\begin{cases} \frac{1}{q} = \frac{\theta}{\infty} + \frac{1-\theta}{4} = \frac{1}{4} - \frac{\theta}{4}, \\ \frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{3} = \frac{1}{3} + \frac{\theta}{6} \end{cases}$$

for some $0 \leq \theta \leq 1$. Thus

$$\begin{aligned} \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^q L_x^r} &\lesssim \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^\infty L_x^2}^\theta \|P_{>N_j} \nabla Iu^{nl}\|_{L_t^4 L_x^3}^{1-\theta} \\ &\lesssim N_j^{-\theta} N_j^{-\frac{3}{4}(1-\theta)} M(J, u, 4)^{1-\theta} \\ &\lesssim N_j^{-3/4-\theta/4} M(J, u, 4/(1-\theta)) \\ &\lesssim N_j^{-3/4-\theta/4} M(J, u, q). \end{aligned}$$

□

5. Modified energy functional and almost conservation law

In this section, we recall the construction of modified energy functional \tilde{E} in [8]. We prove a refined version of almost conservation law. We show

Theorem 5.1. (*Existence of an almost conserved quantity*) Assume u is a smooth in time, Schwartz in space solution to (1.1) with initial data $u_0 \in H_x^s(\mathbb{R}^3)$ ($s > 1/2$) defined on $J \times \mathbb{R}^3$ such that

$$(5.1) \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} < \infty, \quad \sup_{t \in J} E(Iu(t)) \lesssim 1,$$

then there exists a functional $\tilde{E} = \tilde{E}_N : \mathcal{S}_x(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined on Schwartz functions $u \in \mathcal{S}_x(\mathbb{R}^3)$ with the following properties.

(1) (*Fixed-time bounds*) For any $u \in \mathcal{S}_x(\mathbb{R}^3)$,

$$(5.2) \quad |E(Iu) - \tilde{E}(u)| \lesssim N^{-1/8+}.$$

(2) (*Almost conserved law*)

$$(5.3) \quad \sup_{t \in J} |\tilde{E}(u(t)) - \tilde{E}(u_0)| \lesssim N^{-9/8+} \max \left\{ 1, \frac{M(J, u, 2)}{N^{1-}}, \frac{M(J, u, 1)}{N^{2-}} \right\}.$$

In Section 5.1, we recall the construction of modified energy functional \tilde{E} via resonance decomposition. The proofs of pointwise estimate (5.2) and the almost conservation law (5.3) are given in Sections 5.2 and 5.3, respectively.

5.1. Construction of modified energy via resonance decomposition [8]. In this section, we recall the construction of modified energy via resonance decomposition in [8]. The construction of modified energy functional \tilde{E} in [8] is on \mathbb{R}^2 , which can be extended to \mathbb{R}^3 without any change.

Let k be an integer. Denote the space

$$\Sigma_k := \{(\xi_1, \dots, \xi_k) \in (\mathbb{R}^3)^k \mid \xi_1 + \dots + \xi_k = 0\}.$$

Let $M : \Sigma_k \rightarrow \mathbb{C}$ be a smooth tempered symbol, and $u_1, \dots, u_k \in \mathcal{S}(\mathbb{R}^3)$, define the k -functional

$$\Lambda_k(M; u_1, \dots, u_k) := \operatorname{Re} \int_{\Sigma_k} M(\xi_1, \dots, \xi_k) \widehat{u_1}(\xi_1) \cdots \widehat{u_k}(\xi_k).$$

If k is even, we abbreviate $\Lambda_k(M; u) := \Lambda_k(M; u, \bar{u}, \dots, u, \bar{u})$. Let k be an even number and set $A := \{1, 3, \dots, k-1\}$, $B := \{2, 4, \dots, k\}$. Let h be the operator be defined by

$$h(M(\xi_1, \xi_2, \dots, \xi_{k-1}, \xi_k)) := \overline{M}(\xi_2, \xi_1, \dots, \xi_k, \xi_{k-1}).$$

Let $S(A)$ and $S(B)$ be symmetric groups on A and B , respectively. Let $H := \{h, \operatorname{id}\}$ be a group of two elements, where id is the identity map on Σ_k (hence on the space of tempered symbols). Define G_k to be the group generated by $S(A)$, $S(B)$ and H .

Then $|G_k| = 2(k/2)!(k/2)!$. Define $[M]_{\operatorname{sym}} := \frac{1}{|G_k|} \sum_{g \in G_k} gM$. Then

$$\Lambda_k(M; u) = \Lambda_k([M]_{\operatorname{sym}}; u).$$

Define the extended symbol $X(M)$ by

$$X(M)(\xi_1, \dots, \xi_k) := M(\xi_{123}, \xi_4, \dots, \xi_{k+2}),$$

where $\xi_{123} := \xi_1 + \xi_2 + \xi_3$. Similarly, denote $\xi_{ab} = \xi_a + \xi_b$. Set

$$\alpha_4 := 2\xi_{12} \cdot \xi_{14} = -2|\xi_{12}||\xi_{14}|\cos\angle(\xi_{12}, \xi_{14}), \quad \sigma_2(\xi_1, \xi_2) := \frac{1}{2}|\xi_1|^2 m_1^2.$$

Let θ_0 be a small parameter to be determined later. Define the non-resonant set

$$\Omega_{nr} := \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 \mid \max_{1 \leq j \leq 4} |\xi_j| \leq N \right\}$$

and

$$\Omega_2 = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 \mid |\cos\angle(\xi_{12}, \xi_{14})| \geq \theta_0\}.$$

The symbol $[X(\sigma_2)]_{\operatorname{sym}}$ is given by

$$[2iX(\sigma_2)]_{\operatorname{sym}} = \frac{i}{4} \sum_{j=1}^4 (-1)^{j-1} m_j^2 |\xi_j|^2.$$

Define the modified energy functional

$$(5.4) \quad \tilde{E}(u) := \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u),$$

where

$$(5.5) \quad \tilde{\sigma}_4 := \frac{[2iX(\sigma_2)]_{\text{sym}}}{i\alpha_4} 1_{\Omega_{nr}}.$$

Remark 5.1. Note that

$$E(Iu) = \Lambda_2(\sigma_2; u) + \Lambda_4(\sigma_4; u).$$

Thus

$$(5.6) \quad E(Iu) - \tilde{E}(u) = \Lambda_4(\sigma_4 - \tilde{\sigma}_4; u).$$

Also note that

$$(5.7) \quad \begin{aligned} \tilde{E}(u(t)) - \tilde{E}(u(0)) &= \int_0^t \Lambda_4([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha; u(t')) dt' \\ &\quad + \int_0^t \Lambda_6([4iX(\tilde{\sigma}_4)]_{\text{sym}}; u(t')) dt'. \end{aligned}$$

5.2. Pointwise estimate. In this section, we obtain a pointwise estimate on the modified energy functional \tilde{E} . We prove the following proposition, whose analogy in \mathbb{R}^2 can be found in [8].

Proposition 5.1. *Let $u \in \mathcal{S}(\mathbb{R}^3)$ be a Schwartz function, then we have*

$$(5.8) \quad |E(Iu) - \tilde{E}(u)| \lesssim N^{-1+\theta_0^{-1}} \|\nabla Iu\|_{L_x^2(\mathbb{R}^3)}^4.$$

To prove Proposition 5.1, we need the following lemma, whose proof can be found in [8].

Lemma 5.1. *For any $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4$, we have*

$$|\sigma_4 - \tilde{\sigma}_4| \lesssim \frac{\min(m_1, m_2, m_3, m_4)^2}{\theta_0}.$$

Proof of Proposition 5.1. By (5.6), it suffices to show the following estimate

$$\int_{\Sigma_4} |\sigma_4 - \tilde{\sigma}_4| |\hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4)| \lesssim N^{-1+\theta_0^{-1}} \|\nabla Iu\|_{L_x^2}.$$

To do this, we decompose u into dyadic pieces u_j , where u_j is localized with a smooth cut-off function in spatial frequency space having support $|\xi| \sim 2^{k_j} \equiv N_j, k_j \in \mathbb{Z}$. By symmetry, we can assume $N_1 \geq N_2 \geq N_3 \geq N_4$. Furthermore, we can assume $N_1 \sim N_2 \geq N$.

So it suffices to show that

$$(5.9) \quad I_1 := m(N_1)^2 \int_{\Sigma_4} \prod_{j=1}^4 u_j \leq C(N_1, N_2, N_3, N_4) N^{-1+\theta_0^{-1}} \|\nabla Iu_j\|_{L_x^2},$$

where $C(N_1, N_2, N_3, N_4)$ is sufficient small constant such that we can sum over N_1, N_2, N_3, N_4 . Without loss of generality, we assume $u_i (i = 1, 2, 3, 4)$ is real and non-negative. To this end, we consider the following cases.

Case 1. $N_4 \gtrsim 1$.

$$\begin{aligned}
I_1 &\lesssim m(N_1)^2 \|u_1\|_{L_x^3} \|u_2\|_{L_x^3} \|u_3\|_{L_x^6} \|u_1\|_{L_x^6} \\
&\lesssim m(N_1)^2 \|\nabla^{1/2} u_1\|_{L_x^2} \|\nabla^{1/2} u_2\|_{L_x^2} \|\nabla u_3\|_{L_x^2} \|\nabla u_4\|_{L_x^2} \\
&\lesssim N_1^{-1/2} N_2^{-1/2} m(N_3)^{-1} m(N_4)^{-1} \|\nabla I u_1\|_{L_x^2} \|\nabla I u_2\|_{L_x^2} \|\nabla I u_3\|_{L_x^2} \|\nabla I u_4\|_{L_x^2} \\
&\lesssim N_1^{-1} N^{-1+} \|\nabla I u\|_{L_x^2}^4.
\end{aligned}$$

Case 2. $N_1 \geq N_2 \geq N_3 \gtrsim 1 \gg N_4$.

For each fixed ξ_4 such that $|\xi_4| \sim N_4$, let

$$\Omega_{\xi_4} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \mid \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\}.$$

Then we have

$$\begin{aligned}
I_1 &= m(N_1)^2 \int_{|\xi_4| \sim N_4} \left\{ \int_{\Omega_{\xi_4}} \hat{u}_1 \hat{u}_2 \hat{u}_3 d\xi_1 d\xi_2 d\xi_3 \right\} \hat{u}_4 d\xi_4 \\
&\lesssim m(N_1)^2 \left(\int_{|\xi_4| \sim N_4} \hat{u}_4 d\xi_4 \right) \sup_{\xi_4: |\xi_4| \sim N_4} \left\{ \int_{\Omega_{\xi_4}} \hat{u}_1 \hat{u}_2 \hat{u}_3 d\xi_1 d\xi_2 d\xi_3 \right\} \\
&\lesssim m(N_1)^2 \|u_4\|_{L_x^2} \left[\mu(\{\xi_4 \in \mathbb{R}^3 \mid |\xi_4| \sim N_4\}) \right]^{1/2} \sup_{|\xi_4| \sim N_4} \left\{ \int_{\Omega_{\xi_4}} \hat{u}_1 \hat{u}_2 \hat{u}_3 d\xi_1 d\xi_2 d\xi_3 \right\} \\
&\lesssim m(N_1)^2 N_4^{1/2} \|\nabla I u_4\|_{L_x^2} \|u_1\|_{L_x^{12/5}} \|u_2\|_{L_x^{12/5}} \|u_3\|_{L_x^6} \\
&\lesssim m(N_1)^2 N_4^{1/2} \|\nabla I u_4\|_{L_x^2} \|\nabla^{1/4} u_1\|_{L_x^2} \|\nabla^{1/4} u_2\|_{L_x^2} \|\nabla u_3\|_{L_x^2} \\
&\lesssim N_1^{0-} N_4^{1/2} N^{-3/2+} \|\nabla I u\|_{L_x^2}^4.
\end{aligned}$$

Case 3. $N_3 \ll 1$.

Similar to the argument in Case 2, let

$$\Omega_{\xi_3, \xi_4} := \{(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\}.$$

Then we obtain

$$\begin{aligned}
I_1 &= m(N_1)^2 \int_{|\xi_4| \sim N_4} \int_{|\xi_3| \sim N_4} \left\{ \int_{\Omega_{\xi_3, \xi_4}} \hat{u}_1 \hat{u}_2 d\xi_1 d\xi_2 \right\} \hat{u}_3 \hat{u}_4 d\xi_3 d\xi_4 \\
&\lesssim m(N_1)^2 N_3^{1/2} \|\nabla I u_3\|_{L_x^2} N_4^{1/2} \|\nabla I u_4\|_{L_x^2} \|u_1\|_{L_x^2} \|u_2\|_{L_x^2} \\
&\lesssim N_1^{0-} N_4^{1/2} N^{-2+} \|\nabla I u\|_{L_x^2}^4.
\end{aligned}$$

The proof of Proposition 5.1 is concluded. \square

5.3. Almost conservation law. In this section, we prove an almost conservation law for the modified energy functional \tilde{E} , which is crucial to establish global well posedness and scattering.

Proposition 5.2 (*Almost conservation law*). *Let $J = [0, T]$. Let u be a smooth in time, Schwartz in space solution to (1.1) with initial data $u_0 \in H_x^s(\mathbb{R}^3)$ ($s > 1/2$) defined on $J \times \mathbb{R}^3$ such that*

$$(5.10) \quad \sup_{t \in J} E(Iu(t)) \leq 1, \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} < \infty,$$

then we have the quadrilinear estimate

$$(5.11) \quad \left| \int_0^{t_0} \Lambda_4([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha; \quad u(t)) dt \right| \lesssim N^{-9/8+} \max \left\{ 1, \frac{M(J, u, 2)}{N^{1-}}, \frac{M(J, u, 1)}{N^{2-}} \right\}$$

and the sextilinear estimate

$$(5.12) \quad \left| \int_0^{t_0} \Lambda_6([4iX(\tilde{\sigma}_4)]_{\text{sym}}; \quad u(t)) dt \right| \lesssim N^{-9/8+} \max \left\{ 1, \frac{M(J, u, 2)}{N^{1-}}, \frac{M(J, u, 1)}{N^{2-}} \right\}.$$

5.3.1. Sextilinear estimate. Now we prove the sextilinear estimate. First, we show the following lemma.

Lemma 5.2. *Let $J = [0, T]$. Let u be a smooth in time, Schwartz in space solution to (1.1) with initial data $u_0 \in H_x^s(\mathbb{R}^3)$ ($s > 1/2$) defined on $J \times \mathbb{R}^3$ such that*

$$(5.13) \quad \sup_{t \in J} E(Iu(t)) \leq 1, \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} < \infty,$$

then

$$(5.14) \quad \left| \int_0^T \Lambda_6([4iX(\tilde{\sigma}_4)]_{\text{sym}}; \quad u(t)) dt \right| \lesssim \theta_0^{-1} N^{-2+} \max \left\{ 1, \frac{M(J, u, 2)}{N^{1-}}, \frac{M(J, u, 1)}{N^{2-}} \right\}.$$

Proof. We may assume that $\max_{1 \leq j \leq 6} \{|\xi_j|\} \geq N/3$, otherwise the symbol $[4iX(\tilde{\sigma})]_{\text{sym}}$ vanishes (recall that if $\max_{1 \leq j \leq 6} \{|\xi_j|\} < N/3$, then $4X(\tilde{\sigma}_4) = 1$). With such assumption, we then remove the symmetry of the symbol. It suffices to show that

$$(5.15) \quad \left| \int_0^T \Lambda_6(4iX(\tilde{\sigma}_4); u(t)) dt \right| \lesssim \theta_0^{-1} N^{-2+} \max \left\{ 1, \frac{M(J, u, 2)}{N^{1-}}, \frac{M(J, u, 1)}{N^{2-}} \right\}.$$

By Lemma 5.1, we have

$$|X(\tilde{\sigma}_4)| \lesssim \frac{1}{\theta_0} \min\{m_{123}, m_4, m_5, m_6\}^2.$$

If we arrange ξ_1, \dots, ξ_6 as ξ_1^*, \dots, ξ_6^* such that $|\xi_1^*| \geq |\xi_2^*| \geq \dots \geq |\xi_6^*|$, then we have

$$|X(\tilde{\sigma}_4)| \lesssim \frac{1}{\theta_0} m(\xi_4^*)^2.$$

Thus, we can assume $|\xi_1| \geq |\xi_2| \geq \dots \geq |\xi_6|$. We can also assume $|\xi_1| \sim |\xi_2| \gtrsim N$.

Case 1. $N_1 \sim N_2 \gtrsim N, N_3 \gtrsim 1$.

• *Case 1(a).* $N_6 \gtrsim 1$. Observe that

$$\begin{aligned}
& m(N_4)^2 \int_0^T \int_{\Sigma_6} \prod_{j=1}^6 \hat{u}_j dt \\
& \lesssim m(N_4)^2 \sup_{|\xi_6| \sim N_6, |\xi_5| \sim N_5} \left(\int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} \prod_{j=1}^4 \hat{u}_j dt \right) \int_{|\xi_6| \sim N_6} \hat{u}_6 d\xi_6 \int_{|\xi_5| \sim N_5} \hat{u}_5 d\xi_5 \\
& \lesssim m(N_4)^2 \sup_{|\xi_6| \sim N_6, |\xi_5| \sim N_5} \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} \prod_{j=1}^4 \hat{u}_j dt N_6^{1/2} \|\nabla u_6\|_{L_t^\infty L_x^2} N_5^{1/2} \|\nabla u_5\|_{L_t^\infty L_x^2} \\
& \lesssim N_5 \sup_{|\xi_5| \sim N_5, |\xi_6| \sim N_6} \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} \prod_{j=1}^4 \hat{u}_j dt.
\end{aligned}$$

We decompose u_1 and u_2 into linear–nonlinear components, i.e.,

$$u_i = u_i^l + u_i^{nl}, \quad i = 1, 2$$

In the case of (u_1^l, u_2^l) , we have

$$\begin{aligned}
& N_5 \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} \hat{u}_1^l \hat{u}_2^l \hat{u}_3 \hat{u}_4 dt \\
& \lesssim N_5 \|u_1^l\|_{L_t^2 L_x^6} \|u_2^l\|_{L_t^2 L_x^6} \|u_3\|_{L_t^\infty L_x^2} \|u_4\|_{L_t^\infty L_x^6} \\
& \lesssim N_5 N_1^{-1} N_2^{-1} N_3^{-1} m(N_1)^{-1} m(N_2)^{-1} m(N_3)^{-1} m(N_4)^{-1} \\
& \lesssim N_1^{0-} N^{-2+}.
\end{aligned}$$

If there is one nonlinear term, for example, (u_1^l, u_2^{nl}) , then we obtain

$$\begin{aligned}
& N_5 \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} \hat{u}_1^l \hat{u}_2^{nl} \hat{u}_3 \hat{u}_4 dt \\
& \lesssim N_5 \|u_1^l\|_{L_t^2 L_x^6} \|u_2^{nl}\|_{L_t^\infty L_x^2} \|u_3\|_{L_t^2 L_x^6} \|u_4\|_{L_t^\infty L_x^6} \\
& \lesssim N_5 N_1^{-1} N_2^{-1} N_3^{-1} m(N_1)^{-1} m(N_2)^{-1} m(N_3)^{-1} m(N_4)^{-1} M(J, u, 2) \\
& \lesssim N_1^{0-} N^{-3+} M(J, u, 2).
\end{aligned}$$

If there are two nonlinear terms, then we get

$$\begin{aligned}
& N_5 \int_0^T \int_{\Sigma_6} \hat{u}_1^{nl} \hat{u}_2^{nl} \prod_{j=3}^6 \hat{u}_j dt \lesssim N_5 \|u_1^{nl}\|_{L_t^\infty L_x^2} \|u_2^{nl}\|_{L_t^2 L_x^6} \|u_3\|_{L_t^2 L_x^6} \|u_4\|_{L_t^\infty L_x^6} \\
& \lesssim N_1^- N^{-7/2+} M(J, u, 1).
\end{aligned}$$

• *Case 1(b).* $N_6 \ll 1$. For this case, we need a factor N_6^+ to sum over N_6 . Again we decompose u_1, u_2 into linear–nonlinear components. We can argue exactly as in

Case 1(a) to get

$$\begin{aligned}
m(N_4)^2 \int_0^T \int_{\Sigma_6} \hat{u}_1^l \hat{u}_2^l \prod_{j=3}^6 \hat{u}_j dt &\lesssim N_1^- N_6^{1/2} N^{-2+}; \\
m(N_4)^2 \int_0^T \int_{\Sigma_6} \hat{u}_1^{nl} \hat{u}_2^l \prod_{j=3}^6 \hat{u}_j dt &\lesssim N_1^- N_6^{1/2} N^{-3+} M(J, u, 2); \\
m(N_4)^2 \int_0^T \int_{\Sigma_6} \hat{u}_1^l \hat{u}_2^{nl} \prod_{j=3}^6 \hat{u}_j dt &\lesssim N_1^- N_6^{1/2} N^{-3+} M(J, u, 2); \\
m(N_4)^2 \int_0^T \int_{\Sigma_6} \hat{u}_1^{nl} \hat{u}_2^{nl} \prod_{j=3}^6 \hat{u}_j dt &\lesssim N_1^- N_6^{1/2} N^{-7/2+} M(J, u, 1).
\end{aligned}$$

Case 2. $N_1 \sim N_2 \gtrsim N, N_3 \ll 1$. Similar to Case 1(b), we decompose u_1, u_2 into linear and nonlinear components.

In order to obtain a factor N_6^+ , we interpolate $\|Iu_i\|_{L_t^\infty L_x^6}$ and $\|Iu_i\|_{L_{t,x}^4}$. Note that by Sobolev embedding,

$$\|Iu_i\|_{L_t^\infty L_x^6} \lesssim \|\nabla Iu_i\|_{L_t^\infty L_x^2} \lesssim 1.$$

Then for $0 \leq a \leq 1$, since

$$\begin{cases} \frac{1-a}{4} = \frac{a}{\infty} + \frac{1-a}{4}, \\ \frac{3-a}{12} = \frac{a}{6} + \frac{1-a}{4}, \end{cases}$$

we have

$$\|Iu_i\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}(J \times \mathbb{R}^3)} \lesssim \|Iu_i\|_{L_t^\infty L_x^6}^a \|Iu_i\|_{L_{t,x}^4(J \times \mathbb{R}^3)}^{1-a} \lesssim M(J, u, 4/(1-a)), \quad (**)$$

where the last inequality is by the definition of $M(J, u, q)$. Take $a = 1-$, then $M(J, u, 4/(1-a)) = M(J, u, \infty-)$. Note that $(\frac{4}{1+a}, \frac{6}{2-a})$ is admissible. By Bernstein inequality (Lemma 2.1), inequality (**), we have

$$\begin{aligned}
&m(N_4)^2 \int_0^T \int_{\Sigma_6} \hat{u}_1^l \hat{u}_2^l \prod_{j=3}^6 \hat{u}_j dt \\
&\lesssim N_5^{1/2} N_6^{1/2} \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} \hat{u}_1^l \hat{u}_2^l \hat{u}_3 \hat{u}_4 dt \\
&\lesssim N_5^{1/2} N_6^{1/2} \|u_1^l\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|u_2^l\|_{L_t^2 L_x^6} \|Iu_3\|_{L_t^\infty L_x^{\frac{4}{1+a}}} \|Iu_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\
&\lesssim N_5^{1/2} N_6^{1/2} N_3^{\frac{3(1-a)}{4}} \|u_1^l\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|u_2^l\|_{L_t^2 L_x^6} \|Iu_3\|_{L_t^\infty L_x^2} \|Iu_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\
&\lesssim N_5^{1/2} N_6^{1/2} N_1^{-1} N_2^{-1} N_3^{-1} N_3^{\frac{3(1-a)}{4}} m(N_1)^{-1} m(N_2)^{-1} M(J, u, 4/(1-a)) \\
&\lesssim N_1^- N_6^{\frac{3(1-a)}{4}} N^{-2+} M(J, u, 4/(1-a)) \\
&\lesssim N_1^- N_6^+ N^{-2+} M(J, u, \infty-),
\end{aligned}$$

Remark 5.2. The presence of $M(J, u, \infty -)$ is not essential. As we can see in section 6, $M(J, u, \infty -) \sim N^-$, so $M(J, u, \infty -)N^{-1+} \sim N^{-1++}$. Thus, we omit the factor $M(J, u, \infty -)$ throughout this paper.

Use similar argument as the above, we obtain

$$\begin{aligned}
& m(N_4)^2 \int_0^T \int_{\Sigma_6} u_1^{nl} \hat{u}_2^{nl} \prod_{j=3}^6 \hat{u}_j dt \\
& \lesssim N_5^{1/2} N_6^{1/2} \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} u_1^{nl} \hat{u}_2^{nl} \hat{u}_3 \hat{u}_4 dt \\
& \lesssim N_5^{1/2} N_6^{1/2} \|u_1^{nl}\|_{L_t^\infty L_x^{\frac{6}{2+a}}} \|u_2^{nl}\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|u_3\|_{L_t^2 L_x^{\frac{12}{1+a}}} \|u_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\
& \lesssim N_5^{1/2} N_6^{1/2} N_1^{(1-a)/2} N_1^{-1} N_2^{-1} N_3^{-1} N_3^{(1-a)/4} m(N_1)^{-1} m(N_2)^{-1} \\
& \quad \times \|\nabla I u_1^{nl}\|_{L_t^\infty L_x^2} \|\nabla I u_2^{nl}\|_{L_t^{2+} L_x^{6-}} \|\nabla I u_3\|_{L_t^2 L_x^6} M(J, u, 4/(1-a)) \\
& \lesssim N_1^{0-} N_6^+ N^{-3+} M(J, u, 2).
\end{aligned}$$

The (u_1^{nl}, u_2^{nl}) case:

$$\begin{aligned}
& m(N_4)^2 \int_0^T \int_{\Sigma_6} u_1^{nl} \hat{u}_2^{nl} \prod_{j=3}^6 \hat{u}_j dt \\
& \lesssim N_5^{1/2} N_6^{1/2} \int_0^T \int_{\sum_{j=1}^4 \xi_j = -\xi_5 - \xi_6} u_1^{nl} \hat{u}_2^{nl} \hat{u}_3 \hat{u}_4 dt \\
& \lesssim N_5^{1/2} N_6^{1/2} \|u_1^{nl}\|_{L_t^\infty L_x^{\frac{6}{2+a}}} \|u_2^{nl}\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|u_3\|_{L_t^2 L_x^{\frac{12}{1+a}}} \|u_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\
& \lesssim N_5^{1/2} N_6^{1/2} N_1^{(1-a)/2} N_1^{-1} N_2^{-1} N_3^{-1} N_3^{(1-a)/4} m(N_1)^{-1} m(N_2)^{-1} \\
& \quad \times \|\nabla I u_1^{nl}\|_{L_t^\infty L_x^2} \|\nabla I u_2^{nl}\|_{L_t^{2+} L_x^{6-}} \|\nabla I u_3\|_{L_t^2 L_x^6} M(J, u, 4/(1-a)) \\
& \lesssim N_1^{0-} N_6^+ N^{-7/2+} M(J, u, 1).
\end{aligned}$$

This ends the proof of Lemma 5.2. \square

5.3.2. Quadrilinear estimate. We prove the quadrilinear estimate. We first show the following lemma.

Lemma 5.3. *Let $u(x, t)$ be a smooth in time, Schwartz in space solution to (1.1) with initial data $u_0 \in H_x^s(\mathbb{R}^3)$ ($s > 1/2$) defined on $J \times \mathbb{R}^3$ such that*

$$(5.16) \quad \sup_{t \in J} E(Iu(t)) \leq 1, \quad \|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} < \infty,$$

then

$$\begin{aligned}
(5.17) \quad & \left| \int_0^{t_0} \Lambda_4([-2iX(\sigma_2)]_{sym} + i\tilde{\sigma}_4 \alpha; u(t)) dt \right| \\
& \lesssim \max \left\{ \frac{\theta_0}{N^{1/2-}}, N^{-3/2+}, \frac{M(J, u, 2)}{N^{5/2-}}, \frac{M(J, u, 1)}{N^{13/4-}}, \theta_0 \frac{M(J, u, 2)}{N^{7/4-}}, \theta_0 \frac{M(J, u, 1)}{N^{9/4-}} \right\}.
\end{aligned}$$

Proof. From (5.5) we have

$$([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha_4)(\xi) = [-2iX(\sigma_2)]_{\text{sym}}1_{\Omega_{\text{res}}} = \frac{i}{4} \sum_{j=1}^4 (-1)^{j+1} m_j^2 |\xi_j|^2 1_{\Omega_{\text{res}}},$$

where the resonant set

$$\Omega_{\text{res}} := \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 \mid \max_{1 \leq i \leq 4} \{|\xi_i|\} > N; |\cos \angle(\xi_{12}, \xi_{14})| < \theta_0\}.$$

As in the above, we decompose u_i ($i = 1, 2, 3, 4$) into dyadic pieces such that $|\xi_i| \sim N_i$. By symmetry, we may assume that $N_1 \geq N_2, N_3, N_4$, and $N_2 \geq N_4$. Thus, we can further assume $N_2 \geq N_3 \geq N_4$ by symmetry argument. Denote

$$\Omega_r = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 \mid N_1 > N; N_1 \sim N_2; N_1 \geq N_2 \geq N_3 \geq N_4, \right. \\ \left. |\cos \angle(\xi_{12}, \xi_{14})| < \theta_0 \right\}.$$

Then it suffices to show

(5.18)

$$\int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_j)^2 |\xi_j|^2 \right) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) \\ \lesssim \max \left\{ \frac{\theta_0}{N^{1/2-}}, N^{-3/2+}, \frac{M(J, u, 2)}{N^{5/2-}}, \frac{M(J, u, 1)}{N^{13/4-}}, \theta_0 \frac{M(J, u, 2)}{N^{7/4-}}, \theta_0 \frac{M(J, u, 1)}{N^{9/4-}} \right\}.$$

Observe that on Ω_r ,

$$|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 = 2|\xi_{12}||\xi_{14}||\cos \angle(\xi_{12}, \xi_{14})| \lesssim |\xi_{12}||\xi_{14}|\theta_0.$$

Also note that

$$|\xi_1|^2 - |\xi_2|^2 = (|\xi_1| + |\xi_2|)(|\xi_1| - |\xi_2|) \geq |\xi_1 + \xi_2|(|\xi_1| - |\xi_2|) = |\xi_{12}|(|\xi_1| - |\xi_2|)$$

and

$$|\xi_3|^2 - |\xi_4|^2 = (|\xi_3| + |\xi_4|)(|\xi_3| - |\xi_4|) \geq |\xi_3 + \xi_4|(|\xi_3| - |\xi_4|) = |\xi_{12}|(|\xi_3| - |\xi_4|).$$

Thus we have

$$(5.19) \quad |\xi_1| - |\xi_2| \lesssim |\xi_1|\theta_0, \quad |\xi_3| - |\xi_4| \lesssim |\xi_1|\theta_0.$$

To finish the proof of (5.18), we consider four cases.

Case I. $N_1 \geq N_2 \geq N_3 \geq N_4 \gtrsim N$. Then we have

$$\sum_{j=1}^4 (-1)^{j+1} m(\xi_j)^2 |\xi_j|^2 \lesssim \frac{N^{2-2s}}{|\xi_1|^{2-2s}} |\xi_1|^2 - \frac{N^{2-2s}}{|\xi_2|^{2-2s}} |\xi_2|^2 + \frac{N^{2-2s}}{|\xi_3|^{2-2s}} |\xi_3|^2 - \frac{N^{2-2s}}{|\xi_4|^{2-2s}} |\xi_4|^2 \\ \lesssim N^{2-2s} \left[(|\xi_1|^{2s} - |\xi_2|^{2s}) + (|\xi_3|^{2s} - |\xi_4|^{2s}) \right] \\ \lesssim N^{2-2s} (|\xi_1|^{2s-1} (|\xi_1| - |\xi_2|) + |\xi_3|^{2s-1} (|\xi_3| - |\xi_4|)) \\ \lesssim N^{2-2s} (|\xi_1|^{2s-1} |\xi_1| \theta_0 + |\xi_3|^{2s-1} |\xi_1| \theta_0) \\ \lesssim N^{2-2s} N_1^{2s} \theta_0.$$

We decompose u_1, u_2, u_3 and obtain

$$\begin{aligned}
& \int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_i)^2 |\xi_i|^2 \right) \hat{u}_1^l(\xi_1) \hat{u}_2^l(\xi_2) \hat{u}_3^l(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N^{2-2s} N_1^{2s} \theta_0 \int_0^T \int_{\Sigma_4} \hat{u}_1^l(\xi_1) \hat{u}_2^l(\xi_2) \hat{u}_3^l(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N^{2-2s} N_1^{2s} \theta_0 \|u_1^l\|_{L_t^\infty L_x^2} \|u_2^l\|_{L_t^2 L_x^6} \|u_3^l\|_{L_t^2 L_x^6} \|u_4\|_{L_t^\infty L_x^6} \\
& \lesssim N_1^- N^{-1+\theta_0}.
\end{aligned}$$

Next if there is one nonlinear term, for example, (u_1^{nl}, u_2^l, u_3^l) , then

$$\begin{aligned}
& \int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_i)^2 |\xi_i|^2 \right) \hat{u}_1^{nl}(\xi_1) \hat{u}_2^l(\xi_2) \hat{u}_3^l(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N^{2-2s} N_1^{2s} \theta_0 \int_0^T \int_{\Sigma_4} \hat{u}_1^{nl}(\xi_1) \hat{u}_2^l(\xi_2) \hat{u}_3^l(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N^{2-2s} N_1^{2s} \theta_0 \|u_1^{nl}\|_{L_t^\infty L_x^2} \|u_2^l\|_{L_t^2 L_x^6} \|u_3^l\|_{L_t^2 L_x^6} \|u_4\|_{L_t^\infty L_x^6} \\
& \lesssim N_1^- N^{-2+\theta_0}.
\end{aligned}$$

If there are two nonlinear terms, for example, $(u_1^{nl}, u_2^{nl}, u_3^l)$, take $L_t^\infty L_x^2, L_t^2 L_x^6, L_t^2 L_x^6, L_t^\infty L_x^6$ for u_1, u_2, u_3, u_4 , respectively, then the above argument implies that

$$\begin{aligned}
& \int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_i)^2 |\xi_i|^2 \right) \hat{u}_1^{nl}(\xi_1) \hat{u}_2^{nl}(\xi_2) \hat{u}_3^l(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N_1^- N^{-5/2+\theta_0} M(J, u, 2).
\end{aligned}$$

If there are three nonlinear terms, say, $(u_1^{nl}, u_2^{nl}, u_3^{nl})$, take $L_t^\infty L_x^2, L_t^2 L_x^6, L_t^2 L_x^6, L_t^\infty L_x^6$ for u_1, u_2, u_3, u_4 , respectively, then we have

$$\begin{aligned}
& \int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_i)^2 |\xi_i|^2 \right) \hat{u}_1^{nl}(\xi_1) \hat{u}_2^{nl}(\xi_2) \hat{u}_3^{nl}(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N_1^- N^{-3+\theta_0} M(J, u, 1).
\end{aligned}$$

Case II. $N_3 \gtrsim N, 1 \lesssim N_4 \ll N$. For this case we have

$$\begin{aligned}
\sum_{j=1}^4 (-1)^{j+1} m(\xi_1)^2 |\xi_1|^2 & \lesssim \frac{N^{2-2s}}{|\xi_1|^{2-2s}} |\xi_1|^2 - \frac{N^{2-2s}}{|\xi_2|^{2-2s}} |\xi_2|^2 + \frac{N^{2-2s}}{|\xi_3|^{2-2s}} |\xi_3|^2 - |\xi_4|^2 \\
& \lesssim N^{2-2s} (|\xi_1|^{2s} - |\xi_2|^{2s}) + (|\xi_3|^{2s} - |\xi_4|^{2s}) \\
& \lesssim N^{2-2s} (|\xi_1|^{2s-1} (|\xi_1| - |\xi_2|)) + |\xi_1| |\xi_3| \theta_0 \\
& \lesssim N^{2-2s} N_1^{2s} \theta_0 + N_1 N_3 \theta_0.
\end{aligned}$$

By the same argument as in Case I, we obtain

$$\begin{aligned} & \int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_1)^2 |\xi_1|^2 \right) \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\ & \lesssim N_1^- \theta_0 (N^{-3/2+} + N^{-5/2+} M(J, u, 2) + N^{-3+} M(J, u, 1)). \end{aligned}$$

Case III. $N_3 \gtrsim N$, $N_4 \ll 1$. The argument is similar to Cases I and II except that we can obtain an N_4^+ factor to sum over N_4 directly. More precisely,

$$\begin{aligned} & \int_0^T \int_{\Omega_r} \left(\sum_{j=1}^4 (-1)^{j+1} m(\xi_1)^2 |\xi_1|^2 \right) \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\ & \lesssim N_1^- N_4^+ \theta_0 (N^{-3/2} + N^{-5/2+} M(J, u, 2) + N^{-3+} M(J, u, 1)). \end{aligned}$$

Case IV. $N_4 \leq N_3 \ll N$. For this case we need the following lemma in [8]. The reader may refer to [8] for the proof.

Lemma 5.4. *Let $N_1 \geq N_2 \geq N_3 \geq N_4$, $N_1 \sim N_2 \gtrsim N$, $N_3 \ll N$. Let $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_r$ be such that $|\xi_j| \sim N_j$ ($j = 1, 2, 3, 4$). Then*

$$\begin{aligned} (5.20) \quad & |m^2(\xi_1)|\xi_1|^2 - m^2(\xi_2)|\xi_2|^2 + m^2(\xi_3)|\xi_3|^2 - m^2(\xi_4)|\xi_4|^2| \\ & \lesssim m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2. \end{aligned}$$

Case IV is divided into three subcases.

Case IV(a). $N_3 \ll 1$. We argue similar to Case 2 of Lemma 5.2. We decompose u_1 and u_2 into linear and nonlinear parts. Again we use the estimate

$$\|Iu_i\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}(J \times \mathbb{R}^3)} \lesssim \|Iu_i\|_{L_t^\infty L_x^6}^a \|Iu_i\|_{L_{t,x}^4(J \times \mathbb{R}^3)}^{1-a} \lesssim M(J, u, 4/(1-a)), \quad (**)$$

Let $a = 1-$, for (u_1^l, u_2^l) , we have

$$\begin{aligned} & m(N_1)^2 N_1 N_3 \theta_0 \int_0^T \int_{\Omega_r} \hat{u}_1^l(\xi_1) \hat{u}_2^l(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\ & \lesssim m(N_1)^2 N_1 N_3 \theta_0 \|u_1^l\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{6}{2-a}}} \|u_2^l\|_{L_t^2 L_x^6} \|Iu_3\|_{L_t^\infty L_x^{\frac{24}{7+5a}}} \|Iu_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{24}{5-a}}} \\ & \lesssim \theta_0 N_3^{\frac{5(1-a)}{8}} N_4^{\frac{1-a}{8}} M(J, u, 4/(1-a)) \|\nabla Iu_1^l\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|Iu_2^l\|_{L_t^2 L_x^6} \|\nabla Iu_3\|_{L_t^\infty L_x^2} \\ & \quad \times \|Iu_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\ & \lesssim N_1^- N^{-1+} N_4^+ \theta_0. \end{aligned}$$

If only one nonlinear term appears, then we argue similarly. For example,

$$\begin{aligned}
& m(N_1)^2 N_1 N_3 \theta_0 \int_0^T \int_{\Omega_r} \hat{u}_1^{nl}(\xi_1) \hat{u}_2^l(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim m(N_1)^2 N_1 N_3 \theta_0 \|u_1^{nl}\|_{L_t^\infty L_x^{\frac{6}{2+a}}} \|u_2^l\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|u_3\|_{L_t^2 L_x^{\frac{24}{3+a}}} \|u_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{24}{5-a}}} \\
& \lesssim \theta_0 N_1^+ N_3^+ N_4^+ M(J, u, \infty-) \|\nabla I u_1^{nl}\|_{L_t^\infty L_x^2} \|I u_2^l\|_{L_t^{2+} L_x^{6-}} \|\nabla I u_3\|_{L_t^2 L_x^6} \|I u_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\
& \lesssim N_1^- N^{-2+} N_4^+ \theta_0 M(J, u, 2).
\end{aligned}$$

If two nonlinear terms appear, then

$$\begin{aligned}
& m(N_1)^2 N_1 N_3 \theta_0 \int_0^T \int_{\Omega_r} \hat{u}_1^{nl}(\xi_1) \hat{u}_2^{nl}(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim m(N_1)^2 N_1 N_3 \theta_0 \|u_1^{nl}\|_{L_t^\infty L_x^{\frac{6}{2+a}}} \|u_2^l\|_{L_t^{\frac{4}{1+a}} L_x^{\frac{6}{2-a}}} \|u_3\|_{L_t^2 L_x^{\frac{24}{3+a}}} \|u_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{24}{5-a}}} \\
& \lesssim \theta_0 N_1^+ N_3^+ N_4^+ M(J, u, \infty-) \|\nabla I u_1^{nl}\|_{L_t^\infty L_x^2} \|I u_2^l\|_{L_t^{2+} L_x^{6-}} \|\nabla I u_3\|_{L_t^2 L_x^6} \|I u_4\|_{L_t^{\frac{4}{1-a}} L_x^{\frac{12}{3-a}}} \\
& \lesssim N_1^- N^{-5/2+} N_4^+ \theta_0 M(J, u, 1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& m(N_3)^2 N_3^2 \int_0^T \int_{\Omega_r} \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim N_1^- N_4^+ (N^{-2+} + N^{-3+} M(J, u, 2) + N^{-7/2+} M(J, u, 1)).
\end{aligned}$$

Case IV(b). $N_4 \ll 1, N_3 \gtrsim 1$.

- If $1 \lesssim N_3 \ll N^{1/2}$

First estimate

$$I_2 := m(N_1)^2 N_1 N_3 \theta_0 \int_0^T \int_{\Omega_r} \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4).$$

Again we decompose u_1, u_2 into linear and nonlinear components. The cases (u_1^l, u_2^l) , (u_1^{nl}, u_2^l) , (u_1^l, u_2^{nl}) are easy to deal with. For example, we have

$$\begin{aligned}
& m(N_3)^2 N_1 N_3 \theta_0 \int_0^T \int_{\Omega_r} \hat{u}_1^l(\xi_1) \hat{u}_2^{nl}(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim m(N_3)^2 N_1 N_3 \theta_0 \|u_1^l\|_{L_t^{2+} L_x^{6-}} \|u_2^{nl}\|_{L_t^\infty L_x^2} \|u_3\|_{L_t^2 L_x^6} \|u_4\|_{L_t^\infty L_x^{6+}} \\
& \lesssim N_1^- N_4^+ N^{-2+} \theta_0 M(J, u, 2).
\end{aligned}$$

It remains to deal with the case (u_1^{nl}, u_2^{nl}) . We have

$$\begin{aligned}
& m(N_3)^2 N_1 N_3 \theta_0 \int_0^T \int_{\Omega_r} \hat{u}_1^{nl}(\xi_1) \hat{u}_2^{nl}(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\
& \lesssim m(N_3)^2 N_1 N_3 \theta_0 \|u_1^{nl}\|_{L_t^\infty L_x^2} \|u_2^{nl}\|_{L_t^{2+} L_x^{6-}} \|u_3\|_{L_t^2 L_x^6} \|u_4\|_{L_t^\infty L_x^{6+}} \\
& \lesssim N_1^- N_4^+ N^{-5/2+} \theta_0 M(J, u, 1).
\end{aligned}$$

Next, since $1 \leq N_3 \ll N^{1/2}$, by decomposing u_1, u_2 and u_3 , we get

$$\begin{aligned} & m(N_3)^2 N_3^2 \int_0^T \int_{\Omega_r} \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\ & \lesssim N_1^- N_4^+ (N^{-3/2+} + N^{-11/4+} M(J, u, 2) + N^{-13/4+} M(J, u, 1)). \end{aligned}$$

- If $N_3 \gtrsim N^{1/2}$.

We use the bound

$$\sum_{j=1}^4 (-1)^{j+1} m(\xi_1)^2 |\xi_1|^2 \lesssim N^{2-2s} N_1^{2s} \theta_0 + N_1 N_3 \theta_0.$$

The argument in Case I indeed gives that

$$\begin{aligned} & (N^{2-2s} N_1^{2s} \theta_0 + N_1 N_3 \theta_0) \int_0^T \int_{\Omega_r} \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \hat{u}_4(\xi_4) \\ & \lesssim N_1^- N_4^+ \theta_0 (N^{-1/2+} + N^{-7/4+} M(J, u, 2) + N^{-9/4+} M(J, u, 1)). \end{aligned}$$

Case IV(c). $N_4 \gtrsim 1$, $1 \lesssim N_3 \ll N$. Just argue similarly.

This ends the proof of Lemma 5.3. \square

Proof of Theorem 5.1. Take $\theta_0 = N^{-7/8}$, then Theorem 5.1 follows from Proposition 5.1 and Proposition 5.2. \square

6. Global well posedness and scattering

We prove Theorem 1.1.

Proof of Theorem 1.1. Choose $\lambda \sim N^{\frac{1-s}{s-1/2}}$ such that $E(Iu_0^{(\lambda)}) \leq 1/4$. Define

$$(6.1) \quad W := \{T \in [0, \infty) : \sup_{0 \leq t \leq T} E(Iu^{(\lambda)}(t)) \leq 1/2\}.$$

Then $W \neq \emptyset$ since $0 \in W$. Also W is closed by dominated convergence theorem. Note that if $T \in W$, then we obtain

$$\begin{aligned} & \|u^{(\lambda)}\|_{L_{t,x}^4([0,T] \times \mathbb{R}^3)} \\ & \leq C(\|u_0\|_{L_x^2}) \left(\lambda^{3/8} \sup_{0 \leq t \leq T} \|\nabla Iu^{(\lambda)}(t)\|_{L_x^2}^{1/4} + \lambda^{1/4} \sup_{0 \leq t \leq T} \|\nabla Iu^{(\lambda)}(t)\|_{L_x^2}^{1/4s} \right) \\ & \leq C(\|u_0\|_{L_x^2}) \left(\frac{1}{2} \lambda^{3/8} + \frac{1}{2} \lambda^{1/4} \right) \\ & \leq C(\|u_0\|_{L_x^2}) \lambda^{3/8}. \end{aligned}$$

Thus $\|u^{(\lambda)}\|_{L_{t,x}^4([0,T] \times \mathbb{R}^3)}$ is uniformly bounded for any $T \in W$.

We show that W is open so that $W = [0, \infty)$. Assume $T \in W$. By continuity, there exists $\delta > 0$ such that for each $T' \in (T - \delta, T + \delta) \cap [0, \infty)$,

$$\sup_{t \in [0, T']} E(Iu^{(\lambda)}(t)) \leq 1, \quad \|u^{(\lambda)}\|_{L_{t,x}^4([0, T'] \times \mathbb{R}^3)} \leq 2C(\|u_0\|_{L_x^2}) \lambda^{3/8}.$$

Now we decompose $[0, T']$ into $\lambda^{27/50}$ subintervals $\{J_m\}_{m=1}^{\lambda^{27/50}}$ such that for each J_m ,

$$\|u^{(\lambda)}\|_{L_{t,x}^4(J_m \times \mathbb{R}^3)}^4 \lesssim \lambda^{24/25}.$$

Note that $\lambda^{24/25} \leq N^2$ provided $s \geq 49/74$. Thus if we choose $s > 49/74$, then we have

$$\max \left\{ 1, \frac{\max\{1, \lambda^{12/25}\}}{N^{1-}}, \frac{\max\{1, \lambda^{24/25}\}}{N^{2-}} \right\} \lesssim 1.$$

Thus, we can choose N so large such that

$$\sup_{t \in [0, T']} |\tilde{E}(Iu^{(\lambda)}(t)) - \tilde{E}(Iu^{(\lambda)}(0))| \leq 1/8.$$

By choosing N large enough, we obtain

$$\begin{aligned} & |E(Iu(t)) - E(Iu(0))| \\ & \leq |E(Iu(t)) - \tilde{E}(u(t))| + |\tilde{E}(u(t)) - \tilde{E}(u(0))| + |\tilde{E}(u(0)) - E(u(0))| \\ & \lesssim N^{-1/8+} + 1/8 \lesssim 1/4. \end{aligned}$$

Thus

$$\sup_{t \in [0, T']} E(Iu^{(\lambda)}(t)) \leq 1/2.$$

Hence $T' \in W$. So W is open, which implies that $W = [0, \infty)$.

Scattering follows from standard argument. □

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References

- [1] J. Bourgain, *Refinements of Strichartz inequality and applications to 2D-NLS with critical nonlinearity*, Int. Math. Res. Not. **5** (1998), 253–283.
- [2] ———, *Scattering in the energy space and below in 3D NLS*, J. Anal. Math. **75** (1998), 267–297.
- [3] ———, *Global solutions of nonlinear Schrödinger equations*, Amer. Math. Soc., Providence, RI, 1999.
- [4] T. Cazenave, *Nonlinear Schrödinger equations*, Courant Lecture Notes in Math., **10**, New York University, Amer. Math. Soc., Providence, RI, 2003.
- [5] T. Cazenave and F. Weissler, *The Cauchy problem for the nonlinear Schrödinger equation in H^1* , Manuscript Math. **61** (1988), 477–494.
- [6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation*, Math. Res. Lett. **9**(5–6) (2002), 659–682.
- [7] ———, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on \mathbb{R}^3* , Commun. Pure Appl. Math. **21** (2004), 987–1014.
- [8] ———, *Global well-posedness and scattering for the cubic nonlinear Schrödinger equation in $H^s(\mathbb{R}^2)$ for $s > 1/2$* , Discrete Contin. Dyn. Syst. **21**(3) (2008), 665–686.
- [9] B. Dodson, *Global well-posedness and scattering for the defocusing, cubic, nonlinear Schrödinger equation when $n = 3$ via a linear–nonlinear decomposition*, 2009, [arXiv:0910.2260](#).
- [10] ———, *Global well-posedness and scattering for the defocusing, L^2 -critical, nonlinear Schrödinger equation when $d = 2$* , 2010, [arXiv:1006.1375](#).
- [11] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations (9)*, J. Math. Pure. Appl. **64**(4) (1985), 363–401.

- [12] C. Kenig and F. Merle, *Scattering for $\dot{H}^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions*, Trans. Amer. Math. Soc. **362**(4) (2010), 1937–1962.
- [13] ———, *Global well-posedness, scattering and blow-up for the energy critical, focusing, nonlinear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), 645–675.
- [14] T. Roy, *Adapted linear–nonlinear decomposition and global well-posedness for solutions to the defocusing cubic wave equations in \mathbb{R}^3* , Discrete Contin. Dyn. Syst. **24**(4) (2009), 1307–1323.
- [15] E. Stein, *Harmonic analysis: real variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43. Princeton University Press, Princeton, NJ, 1993.
- [16] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conference Series in Mathematics, 106. American Mathematical Society, Providence, RI, 2006.

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