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ON THE HASSE PRINCIPLE FOR FINITE GROUP SCHEMES OVER GLOBAL FUNCTION FIELDS

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Abstract. Let K be a global function field of characteristic $p > 0$ and let M be a (commutative) finite and flat K-group scheme. We show that the kernel of the canonical localization map $H^1(K, M) \longrightarrow \prod_{\text{all } v} H^1(K_v, M)$ in flat (fppf) cohomology can be computed solely in terms of Galois cohomology. We then give applications to the case where M is the kernel of multiplication by p^m on an abelian variety defined over K.

1. Statement of the main theorem

Let K be a global field and let $\bar{K}$ be a fixed algebraic closure of K. Let K^{s} be the separable closure of K in $\bar{K}$ and set $G_K = \mathrm{Gal}(K^{\mathrm{s}}/K)$. Further, for each prime v of K, let $\bar{K}_v$ be the completion of $\bar{K}$ at a fixed prime $\bar{v}$ of $\bar{K}$ lying above v and let K_v^{s} denote the completion of K^{s} at the prime of K^{s} lying below $\bar{v}$. Set $G_v = \mathrm{Gal}(K_v^{\mathrm{s}}/K_v)$. If M is a commutative, finite and flat K-group scheme, let $H^i(G_K, M)$ (respectively, $H^i(G_v, M)$) denote the Galois cohomology group $H^i(G_K, M(K^{\mathrm{s}}))$ (respectively, $H^i(G_v, M(K_v^{\mathrm{s}}))$). The validity of the Hasse principle for $M(K^{\mathrm{s}})$, i.e., the injectivity of the canonical localization map $H^1(G_K, M) \to \prod_{\text{all } v} H^1(G_v, M)$ in Galois cohomology, has been discussed in [6, 12]. See also [9, Section I.9, pp. 117–120]. However, if K is a global function field of characteristic $p > 0$ and M has p-power order, the injectivity of the canonical localization map $\beta^1(K, M)\colon H^1(K, M) \to \prod_{\text{all } v} H^1(K_v, M)$ in flat (fppf) cohomology has not been discussed before (but see [7, Lemma 1] for some particular cases). In this paper we investigate this problem and show that the injectivity of $\beta^1(K, M)$ depends only on the finite G_K-module $M(K^{\mathrm{s}})$, which may be regarded as the maximal étale K-subgroup scheme of M. Indeed, let $Ш^1(K, M) = \mathrm{Ker}\, \beta^1(K, M)$ and set $Ш^1(G_K, M) = \mathrm{Ker}\,[H^1(G_K, M) \to \prod_{\text{all } v} H^1(G_v, M)]$. Then the following holds.

Main theorem. *Let K be a global function field of characteristic $p > 0$ and let M be a commutative, finite and flat K-group scheme. Let v be any prime of K. Then the inflation map $H^1(G_K, M) \to H^1(K, M)$ induces an isomorphism*

$$\mathrm{Ker}[H^1(G_K, M) \to H^1(G_v, M)] \simeq \mathrm{Ker}[H^1(K, M) \to H^1(K_v, M)].$$

In particular, $Ш^1(K, M) \simeq Ш^1(G_K, M)$.

Thus, the Hasse principle holds for M, i.e., $Ш^1(K, M) = 0$, if, and only if, the Hasse principle holds for the G_K-module $M(K^{\mathrm{s}})$. An interesting example is the following. Let A be an ordinary abelian variety over K such that the Kodaira–Spencer map has

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maximal rank. Then $A_{p^m}(K^{\rm s})=0$ for every integer $m\geq 1$ [17, Proposition, p.1093], and we conclude that the Hasse principle holds for A_{p^m}.

The theorem is proved in the next section. In Section 3 we develop some applications of the theorem and Section 4 contains some concluding remarks.

2. Proof of the main theorem

We keep the notation introduced in the previous section. In addition, we will write $(\operatorname{Spec} K)_{\rm fl}$ for the flat site on $\operatorname{Spec} K$ as defined in [8, II.1, p.47], and $H^r(K,M)$ will denote $H^r((\operatorname{Spec} K)_{\rm fl},M)$. We will see presently that, in fact, $H^r(K,M)\simeq H^r((\operatorname{Spec} K)_{\rm fppf},M)$.

By a theorem of M. Raynaud (see [14] or [1], Theorem 3.1.1, p. 110), there exist abelian varieties A and B defined over K and an exact sequence of K-group schemes

$$0\to M\xrightarrow{\iota} A\xrightarrow{\psi} B\to 0, \tag{1}$$

where ι is a closed immersion. For $r\geq 0$, let $\iota^{(r)}\colon H^r(K,M)\to H^r(K,A)$ and $\psi^{(r)}\colon H^r(K,A)\to H^r(K,B)$ be the maps induced by ι and ψ. The long exact flat cohomology sequence associated to (1) yields an exact sequence

$$0\to \operatorname{Coker}\psi^{(r-1)}\to H^r(K,M)\to \operatorname{Ker}\psi^{(r)}\to 0,$$

where $r\geq 1$. Since the groups $H^r(K,A)$ and $H^r(K,B)$ coincide with the corresponding étale and fppf cohomology groups by [8, Theorem III.3.9, p.114] and [5, Theorem 11.7, p.180], we conclude that $H^r(K,M)=H^r((\operatorname{Spec} K)_{\rm fppf},M)$.

Lemma 2.1. *Let v be any prime of K. If A is an abelian variety defined over K, then $A(K^{\rm s})=A(\bar K)\cap A(K_v^{\rm s})$, where the intersection takes place inside $A(\bar K_v)$.*

Proof. Let F/K be a finite subextension of $K^{\rm s}/K$ and let F_v denote the completion of F at the prime of F lying below $\bar v$. Choose an element $t\in F$ such that $F_v=k((t))$, where k is the field of constants of F, and let $m\geq 1$ be an integer. Then $F_v^{p^{-m}}=k((t^{p^{-m}}))=\sum_{i=1}^{p^m}(t^{p^{-m}})^iF_v\subset F^{p^{-m}}F_v$, whence $F_v^{p^{-m}}=F^{p^{-m}}F_v$. Now let $a\in\bar K$ be inseparable over K. Then there exists an integer $m\geq 1$ and an extension F/K as above such that $K(a)=F^{p^{-m}}$. Consequently, $a\in K(a)\cdot F_v=F^{p^{-m}}F_v=F_v^{p^{-m}}$, whence a is also inseparable over K_v. This shows that $K^{\rm s}=\bar K\cap K_v^{\rm s}$. Now let $V\subset\mathbb{A}^n_K$ be an affine K-variety and let $P=(x_1,\dots,x_n)\in V(\bar K\,)\cap V(K_v^{\rm s})$. Then each $x_i\in\bar K\cap K_v^{\rm s}=K^{\rm s}$, whence $P\in V(K^{\rm s})$. Thus $V(K^{\rm s})=V(\bar K\,)\cap V(K_v^{\rm s})$, and the lemma is now clear since A is covered by affine K-varieties. □

If v is a prime of K, we will write $\psi_v=\psi\times_{\operatorname{Spec} K}\operatorname{Spec} K_v$. Since $H^1(K^{\rm s},A)=H^1(K_v^{\rm s},A)=0$, the exact sequence (1) yields a commutative diagram

$$\begin{array}{ccc} B(K^{\rm s})/\psi(A(K^{\rm s})) & \xrightarrow{\sim} & H^1(K^{\rm s},M) \\ \downarrow & & \downarrow \\ B(K_v^{\rm s})/\psi_v(A(K_v^{\rm s})) & \xrightarrow{\sim} & H^1(K_v^{\rm s},M). \end{array} \tag{2}$$

Lemma 2.2. *Let v be a prime of K. Then the canonical map*

$$B(K^{\rm s})/\psi(A(K^{\rm s}))\to B(K_v^{\rm s})/\psi_v(A(K_v^{\rm s}))$$

is injective.

Proof. Write $M = \operatorname{Spec} R$, where R is a finite K-algebra, and identify $M(\bar{K}_v)$ with $\operatorname{Hom}_{K_v}(K_v \otimes_K R, \bar{K}_v)$. If $s \in M(\bar{K}_v)$, then the image of the composition $\tilde{f}\colon R \to K_v \otimes_K R \xrightarrow{s} \bar{K}_v$ is a finite K-algebra and so, in fact, a finite field extension of K. Consequently, $\tilde{f}$ factors through some $f \in \operatorname{Hom}_K(R, \bar{K}) = M(\bar{K})$. This implies that $M(\bar{K}_v) = M(\bar{K})$. Now let $P \in B(K^{\mathrm{s}}) \cap \psi_v(A(K_v^{\mathrm{s}})) \subseteq B(K_v^{\mathrm{s}})$ and let $Q \in A(K_v^{\mathrm{s}})$ be such that $P = \psi_v(Q)$. Since $A(\bar{K}) \xrightarrow{\psi} B(\bar{K})$ is surjective, there exists an $R \in A(\bar{K})$ such that $\psi(R) = P$. Then $R - Q \in M(\bar{K}_v) = M(\bar{K})$. This shows that $Q \in A(\bar{K}) \cap A(K_v^{\mathrm{s}}) = A(K^{\mathrm{s}})$, by the previous lemma. Thus $P = \psi(Q) \in \psi(A(K^{\mathrm{s}}))$, as desired. □

The above lemma and diagram (2) show that the localization map $H^1(K^{\mathrm{s}}, M) \to H^1(K_v^{\mathrm{s}}, M)$ is injective. The main theorem is now immediate from the exact commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(G_K, M) & \longrightarrow & H^1(K, M) & \longrightarrow & H^1(K^{\mathrm{s}}, M) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^1(G_v, M) & \longrightarrow & H^1(K_v, M) & \longrightarrow & H^1(K_v^{\mathrm{s}}, M), \end{array}$$

whose rows are the inflation-restriction exact sequences in flat cohomology [16, p.422, line -12].

3. Applications

Let K and M be as in the previous section. We will write $K(M)$ for the subfield of K^{s} fixed by $\operatorname{Ker}[G_K \to \operatorname{Aut}(M(K^{\mathrm{s}}))]$. We note that the Hasse principle is known to hold for $M(K^{\mathrm{s}})$ under any of the following hypotheses:

(a) $\operatorname{Gal}(K(M)/K) \subseteq \operatorname{Aut}(M(K^{\mathrm{s}}))$ is cyclic. See [9, Lemma I.9.3, p.118].

(b) $M(K^{\mathrm{s}})$ is a simple G_K-module such that $pM(K^{\mathrm{s}}) = 0$ and $\operatorname{Gal}(K(M)/K)$ is a p-solvable group, i.e., $\operatorname{Gal}(K(M)/K)$ has a composition series whose factors of order divisible by p are cyclic. See [9, Theorem I.9.2(a), p.117].

(c) There exists a set T of primes of K, containing the set S of all primes of K which split completely in $K(M)$, such that $T \setminus S$ has Dirichlet density zero and $[K(M) : K] = \text{l.c.m.}\{[K(M)_v : K_v] \colon v \in T\}$. See [11, Theorem 9.1.9(iii), p.528].

In this section we focus on case (a) above when $M = A_{p^m}$ is the p^m-torsion subgroup scheme of an abelian variety A defined over K. More precisely, we are interested in the class of abelian varieties A such that $A_{p^m}(K^{\mathrm{s}})$ is cyclic, for then $\operatorname{Gal}(K(A_{p^m})/K) \hookrightarrow \operatorname{Aut}(A_{p^m}(K^{\mathrm{s}}))$ is cyclic as well if p is odd or $m \leq 2$ and (a) applies. Clearly, this class contains all ordinary abelian varieties A such that the associated Kodaira–Spencer map has maximal rank since, as noted in Section 1, $A_{p^m}(K^{\mathrm{s}})$ is in fact zero. To find more examples, recall that $A_{p^m}(\bar{K}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^f$ for some integer f (called the *p-rank of* A) such that $0 \leq f \leq \dim A$. Thus, if $f \leq 1$, then $A_{p^m}(K^{\mathrm{s}})$ is cyclic. Clearly, the condition $f \leq 1$ holds if A is an elliptic curve, but there exist higher-dimensional abelian varieties A for which $f \leq 1$. See [13, Section 4].

Remark 3.1. Clearly, $\operatorname{Gal}(K(A_{p^m})/K)$ may be cyclic even if $A_{p^m}(K^{\mathrm{s}})$ is not. For example, let k be the (finite) field of constants of K, let A_0 be an abelian variety defined over k and let $A = A_0 \times_{\operatorname{Spec} k} \operatorname{Spec} K$ be the constant abelian variety over K defined by A_0. Then $A(K^{\mathrm{s}})_{\mathrm{tors}} = A_0(\overline{k})$, and therefore $\operatorname{Gal}(K(A_{p^m})/K) \simeq \operatorname{Gal}(k'/k)$ for some finite extension k' of k. Consequently, $\operatorname{Gal}(K(A_{p^m})/K)$ is cyclic and the Hasse principle holds for $A_{p^m}(K^{\mathrm{s}})$.

We will write $A\{p\}$ for the p-divisible group attached to A, i.e., $A\{p\} = \varinjlim_m A_{p^m}$. If B is an abelian group, $B(p) = \cup_m B_{p^m}$ is the p-primary component of its torsion subgroup, $B^{\wedge} = \varprojlim_m B/p^m$ is the p-adic completion of B and $T_pB = \varprojlim_m B_{p^m}$ is the p-adic Tate module of B. Further, if B is a topological abelian group, B^D will denote $\operatorname{Hom}_{\mathrm{cont.}}(B, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology, where $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology.

Let X denote the unique smooth, projective and irreducible curve over the field of constants of K having function field K. If A is an abelian variety over K, we will write $\mathscr{A}$ for the Néron model of A over X.

The following statement is immediate from the main theorem and the above remarks.

Proposition 3.2. *Let A be an abelian variety defined over K and let m be a positive integer. Assume that $A_{p^m}(K^{\mathrm{s}})$ is cyclic. Assume, in addition, that $m \leq 2$ if $p = 2$. Then the localization map in flat cohomology*

$$H^1(K, A_{p^m}) \to \prod_{\text{all } v} H^1(K_v, A_{p^m})$$

is injective. □

The next lemma confirms a long-standing and widely-held expectation.

Lemma 3.3. $H^2(K, A) = 0$.

Proof. Since $H^2(K_v, A) = 0$ for every v [9, Theorem III.7.8, p.285], it suffices to check that $Ш^2(A) = 0$. For any integer n, there exists a canonical exact sequence of flat cohomology groups

$$0 \to H^1(K, A)/n \to H^2(K, A_n) \to H^2(K, A)_n \to 0.$$

Since the Galois cohomology groups $H^i(K, A)$ are torsion in degrees $i \geq 1$ and $\mathbb{Q}/\mathbb{Z}$ is divisible, the direct limit over n of the above exact sequences yields a canonical isomorphism $H^2(K, A) = \varinjlim_n H^2(K, A_n)$. An analogous isomorphism exists over K_v for each prime v of K, and we conclude that $Ш^2(A)$ is canonically isomorphic to $\varinjlim_n Ш^2(A_n)$. Now, by Poitou–Tate duality [9, Theorem I.4.10(a), p.57] and [4, Theorem 4.8], the latter group is canonically isomorphic to the Pontryagin dual of $\varprojlim_n Ш^1(A_n^t)$, where A^t is the dual abelian variety of A. Thus, it suffices to show that $\varprojlim_n Ш^1(A_n^t) = 0$. Let U be the largest open subscheme of X such that A_n^t extends to a finite and flat U-group scheme $\mathscr{A}_n^t$. For each closed point v of U, let $\mathcal{O}_v$ denote the completion of the local ring of U at v. Now let V be any non-empty open subscheme of U. By the computations at the beginning of [9, III.7, p.280] and the localization sequence [8, Proposition III.1.25, p.92], there exists an exact sequence

$$0 \to H^1(U, \mathscr{A}_n^t) \to H^1(V, \mathscr{A}_n^t) \to \bigoplus_{v \in U \setminus V} H^1(K_v, A_n^t)/H^1(\mathcal{O}_v, \mathscr{A}_n^t).$$

Taking the direct limit over V in the above sequence and using [4, Lemma 2.3], we obtain an exact sequence

$$0 \to H^1(U, \mathscr{A}_n^t) \to H^1(K, A_n^t) \to \prod_{v \in U} H^1(K_v, A_n^t)/H^1(\mathcal{O}_v, \mathscr{A}_n^t),$$

where the product extends over all closed points of U. The exactness of the last sequence shows the injectivity of the right-hand vertical map in the diagram below

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(U, \mathscr{A}_n^t) & \longrightarrow & H^1(K, A_n^t) & \longrightarrow & H^1(K, A_n^t)/H^1(U, \mathscr{A}_n^t) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \prod_{v \in U} H^1(\mathcal{O}_v, \mathscr{A}_n^t) & \longrightarrow & \prod_{v \in U} H^1(K_v, A_n^t) & \longrightarrow & \prod_{v \in U} H^1(K_v, A_n^t)/H^1(\mathcal{O}_v, \mathscr{A}_n^t).
\end{array}$$

We conclude that $Ш_U^1(A_n^t) := \mathrm{Ker}\,[H^1(K, A_n^t) \to \prod_{v \in U} H^1(K_v, A_n^t)]$ equals

$$\bar{H}^1(U, \mathscr{A}_n^t) := \mathrm{Ker}\left[H^1(U, \mathscr{A}_n^t) \to \prod_{v \in U} H^1(\mathcal{O}_v, \mathscr{A}_n^t)\right].$$

Now, it is shown in [10, Propositions 5 and 6], that $\varprojlim_n \bar{H}^1(U, \mathscr{A}_n^t) = 0$, whence $\varprojlim_n Ш_U^1(A_n^t) = 0$. Now the exact sequence

$$0 \to Ш^1(A_n^t) \to Ш_U^1(A_n^t) \to \prod_{v \notin U} H^1(K_v, A_n^t)$$

shows that $\varprojlim_n Ш^1(A_n^t) = 0$, as desired. □

Remark 3.4. With the notation of the above proof, the kernel–cokernel exact sequence [9, Proposition I.0.24, p.16] for the pair of maps

$$H^1(K, A_n^t) \to \bigoplus_{\text{all } v} H^1(K_v, A_n^t) \to \bigoplus_{\text{all } v} H^1(K_v, A^t)$$

yields an exact sequence

$$0 \to Ш^1(A_n^t) \to \mathrm{Sel}(A^t)_n \to \bigoplus_{\text{all } v} H^0(K_v, A^t)/n,$$

where $\mathrm{Sel}(A^t)_n := \mathrm{Ker}\,[H^1(K, A_n^t) \to \oplus_{\text{all } v} H^1(K_v, A^t)]$. Since $\varprojlim_n Ш^1(A_n^t) = 0$ as shown above, the inverse limit over n of the preceding sequences yields an injection

$$\varprojlim_n \mathrm{Sel}(A^t)_n \hookrightarrow \prod_{\text{all } v} \varprojlim_n H^0(K_v, A^t)/n.$$

This injectivity was claimed in [3, p. 300, line -8], but the "proof" given there is inadequate and should be replaced by the above one.

Proposition 3.5. *Let A be an abelian variety over K and let A^t be the corresponding dual abelian variety. Assume that $A_{p^m}^t(K^{\mathrm{s}})$ is cyclic, where m is a positive integer such that $m \leq 2$ if $p = 2$. Then the localization maps*

$$H^2(K, A_{p^m}) \to \bigoplus_{\text{all } v} H^2(K_v, A_{p^m})$$

and

$$H^1(K,A)/p^m \to \bigoplus_{\text{all } v} H^1(K_v,A)/p^m$$

are injective.

Proof. The injectivity of the first map is immediate from Proposition 3.2 and Poitou–Tate duality [4, Theorem 4.8]. On the other hand, the lemma and the long exact flat cohomology sequence associated to $0 \to A_{p^m} \to A \overset{p^m}{\to} A \to 0$ over K and over K_v for each v identifies the second map of the statement with the first, thereby completing the proof. □

Remark 3.6. When A is an elliptic curve over a number field and p is any prime, the injectivity of the second map in the above proposition was first established in [2, Lemma 6.1, p.107].

Let $\prod'_{\text{all } v} H^1(K_v, A_{p^m})$ denote the restricted product of the groups $H^1(K_v, A_{p^m})$ with respect to the subgroups $H^1(\mathcal{O}_v, \mathscr{A}_{p^m})$.

Proposition 3.7. *Let A be an abelian variety over K and let m be a positive integer such that $m \leq 2$ if $p = 2$. Assume that both $A_{p^m}(K^{\rm s})$ and $A^t_{p^m}(K^{\rm s})$ are cyclic. Then there exist exact sequences*

$$0 \to A_{p^m}(K) \to \prod_{\text{all } v} A_{p^m}(K_v) \to H^2(K, A^t_{p^m})^D \to 0,$$

$$0 \to H^1(K, A_{p^m}) \to {\prod_{\text{all } v}}' H^1(K_v, A_{p^m}) \to H^1(K, A^t_{p^m})^D \to 0$$

and

$$0 \to H^2(K, A_{p^m}) \to \bigoplus_{\text{all } v} H^2(K_v, A_{p^m}) \to A^t_{p^m}(K) \to 0.$$

Proof. This is immediate from Propositions 3.2 and 3.5 and the Poitou–Tate exact sequence in flat-cohomology [4, Theorem 4.11]. □

Now set

$$\mathrm{Sel}(A)_{p^m} = \mathrm{Ker}\left[H^1(K, A_{p^m}) \to \bigoplus_{\text{all } v} H^1(K_v, A)\right]$$

and define $T_p \,\mathrm{Sel}(A) = \varprojlim_m \mathrm{Sel}(A)_{p^m}$. Further, recall the group

$$Ш^1(A) = \mathrm{Ker}\left[H^1(K, A) \to \bigoplus_{\text{all } v} H^1(K_v, A)\right].$$

Further, set $H^1(K, T_pA^t) = \varprojlim_m H^1(K, A^t_{p^m})$.

Corollary 3.8. *Under the hypotheses of the proposition, there exist canonical exact sequences*

$$0 \to Ш^1(A)(p) \to \frac{\prod_{\text{all } v} A(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p}{A(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \to H^1(K, T_pA^t)^D \to (T_p \,\mathrm{Sel}(A^t))^D \to 0,$$

$$0 \to T_pШ^1(A) \to \left(\prod_{\text{all } v} A(K_v)^\wedge\right)/A(K)^\wedge \to H^1(K, A^t\{p\})^D$$

and

$$0 \to T_p \operatorname{Sel}(A) \to \prod_{\text{all } v} A(K_v)^{\wedge} \to H^1(K, A^t\{p\})^D.$$

Proof. Let $m \geq 1$ be an integer. The exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(K)/p^m & \longrightarrow & H^1(K, A_{p^m}) & \longrightarrow & H^1(K, A)_{p^m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{\text{all } v} A(K_v)/p^m & \longrightarrow & \prod'_{\text{all } v} H^1(K_v, A_{p^m}) & \longrightarrow & \bigoplus_{\text{all } v} H^1(K_v, A)_{p^m} & \longrightarrow & 0,
\end{array}$$

yields an exact sequence of profinite abelian groups

$$0 \to \text{Ш}^1(A)_{p^m} \to \frac{\prod_{\text{all } v} A(K_v)/p^m}{A(K)/p^m} \to H^1(K, A^t_{p^m})^D \to \text{Б}_m(A) \to 0, \tag{3}$$

where $\text{Б}_m(A) = \operatorname{Coker}\big[H^1(K, A)_{p^m} \to \oplus_{\text{all } v} H^1(K_v, A)_{p^m}\big]$. By the main theorem of [3], $\varinjlim_m \text{Б}_m(A) \simeq (T_p \operatorname{Sel}(A^t))^D$ and the first exact sequence of the statement follows by taking the direct limit over m in (3). On the other hand, since the inverse limit functor is exact on the category of profinite groups [15, Proposition 2.2.4, p. 32], the inverse limit over m of the sequences (3) is the second exact sequence of the statement. The third exact sequence follows from the second and the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(K)^{\wedge} & \longrightarrow & T_p \operatorname{Sel}(A) & \longrightarrow & T_p\text{Ш}^1(A) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(K)^{\wedge} & \longrightarrow & \prod_{\text{all } v} A(K_v)^{\wedge} & \longrightarrow & \left(\prod_{\text{all } v} A(K_v)^{\wedge}\right)/A(K)^{\wedge} & \longrightarrow & 0.
\end{array}$$

□

4. Concluding remarks

Let K and M be as in Section 1 and assume that M has p-power order. Further, let M^* denote the Cartier dual of M. Since $\text{Ш}^1(K, M^*) \simeq \text{Ш}^1(G_K, M^*)$ and there exists a perfect pairing of finite groups

$$\text{Ш}^1(K, M^*) \times \text{Ш}^2(K, M) \to \mathbb{Q}/\mathbb{Z} \tag{4}$$

by [4], Theorem 4.8, it is natural to expect a Galois-cohomological description of $\text{Ш}^2(K, M)$. Note, however, that the natural guess $\text{Ш}^2(K, M) \simeq \text{Ш}^2(G_K, M)$ is incorrect since the latter group is zero (because the p-cohomological dimension of G_K is ≤ 1). To obtain the correct answer, we proceed as follows. Since $H^i(K^{\text{s}}, A) = H^i(K^{\text{s}}, B) = 0$ for all $i \geq 1$, the exact sequence (1) shows that $H^i(K^{\text{s}}, M) = 0$ for all $i \geq 2$. On the other hand, $H^i(G_K, M) = 0$ for all $i \geq 2$ as well, since $\operatorname{cd}_p(G_K) \leq 1$. Now the exact sequence of terms of low degree belonging to the Hochschild–Serre spectral sequence $H^i(G_K, H^j(K^{\text{s}}, M)) \Rightarrow H^{i+j}(K, M)$ yields a canonical isomorphism

$$H^2(K, M) \simeq H^1(G_K, H^1(K^{\text{s}}, M)) \simeq H^1(G_K, B(K^{\text{s}})/\psi(A(K^{\text{s}}))).$$

Analogous isomorphisms exist over K_v for every prime v of K, and we conclude that

$$Ш^2(K, M) \simeq Ш^1(G_K, B/\psi(A)).$$

For example, if $M = A_{p^m}$ for an abelian variety A over K, then $Ш^2(K, A_{p^m}) \simeq Ш^1(G_K, A/p^m)$ and the pairing (4) takes the form

$$Ш^1(G_K, A^t_{p^m}) \times Ш^1(G_K, A/p^m) \to \mathbb{Q}/\mathbb{Z}.$$

In particular, if $A^t_{p^m}(K^{\rm s})$ is cyclic with $m \leq 2$ if $p = 2$, then Proposition 3.2 applied to A^t and the perfectness of the above pairing yield $Ш^1(G_K, A/p^m) = 0$, i.e., the Hasse principle holds for the G_K-module $A(K^{\rm s})/p^m$.

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