LOWER BOUNDS ON THE HAUSDORFF MEASURE OF NODAL SETS II

CHRISTOPHER D. SOGGE AND STEVE ZELDITCH

Abstract. We give a very short argument showing how the main identity (0.2) from our earlier paper [12] immediately leads to the best lower bound currently known [2] for the Hausdorff measure of nodal sets in dimensions $n \geq 3$.

Let $(M,g)$ be a compact smooth Riemannian manifold of dimension $n$ and let $e_\lambda$ be real-valued eigenfunction of the associated Laplacian, i.e.,

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x)$$

with frequency $\lambda > 0$. Recent papers have been concerned with lower bounds for the $(n-1)$-dimensional Hausdorff measure, $|Z_\lambda|$, of the nodal set of $e_\lambda$,

$$Z_\lambda = \{ x \in M : e_\lambda(x) = 0 \}$$

in dimensions $n \geq 3$. When $n = 2$ the sharp lower bound by the frequency, $\lambda \lesssim |Z_\lambda|$, was obtained by Brüning in [1] and independently by Yau. For all dimensions, in the analytic case, the sharp upper and lower bounds $|Z_\lambda| \approx \lambda$ were obtained by Donnelly and Fefferman [4,5].

Until recently, the best known lower bound when $n \geq 3$ seems to have been $e^{-c\lambda} \lesssim |Z_\lambda|$ (see [6]). Using a variation (0.2) of an identity of Dong [3], the authors showed in [12] that this can be improved to be $\lambda^{\frac{7}{4} - \frac{3}{n}} \lesssim |Z_\lambda|$. Independently Colding and Minicozzi [2] obtained the more favorable lower bound

$$\lambda^{1-\frac{n-1}{2}} \lesssim |Z_\lambda|$$

(0.1)

by a different method. Subsequently, the first author and Hezari [7] were also able to obtain the lower bound (0.1) by an argument which was in the spirit of [12]. The purpose of this sequel to [12] is to show that the lower bound (0.1) can also be derived by a very small modification (indeed a simplification) of the original argument of [12].

The lower bounds of [7,12] are based on the identity

$$\lambda^2 \int_M |e_\lambda| \, dV = 2 \int_{Z_\lambda} |\nabla_g e_\lambda|_g \, dS,$$

from [12] and the (sharp) lower bound for $L^1$-norms

$$\lambda^{-\frac{n+1}{2}} \lesssim \int_M |e_\lambda| \, dV,$$

which was also established in [12]. Here, $dV$ is the volume element of $(M,g)$.

The lower bound (0.1) is a very simple consequence of the identity (0.2) and the following lemma (which was implicit in [12]).
Lemma 1. If $\lambda > 0$ then

$$\|\nabla g e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{1+\frac{n-1}{2}} \|e_\lambda\|_{L^1(M)}.$$  

Indeed if we use (0.2) and then apply Lemma 1, we obtain

$$\lambda^2 \int_M |e_\lambda| dV = 2 \int_{Z_\lambda} |\nabla g e_\lambda| dS \leq 2 |Z_\lambda| \|\nabla g e_\lambda\|_{L^\infty(M)} \lesssim 2 |Z_\lambda| \lambda^{1+\frac{n-1}{2}} \|e_\lambda\|_{L^1(M)},$$

which of course implies (0.1).

Lemma 1 improves the upper bound on the integral given in Lemma 1 of [12], and its proof is almost the same as the proof of (0.3) in Proposition 2 of [12]:

Proof. For $\rho \in C_0^\infty(\mathbb{R})$, we define the $\lambda$-dependent family of operators

$$\chi_\lambda f = \int_0^\infty \rho(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}f dt = \hat{\rho}(\lambda - \sqrt{-\Delta_g})f = \sum_{j=0}^\infty \hat{\rho}(\lambda - \lambda_j)E_j f,$$

on $L^2(M,dV)$ with $E_j f$ denoting the projection of $f$ onto the $j$th eigenspace of $\sqrt{-\Delta_g}$. Here $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are its eigenvalues, and if $\{e_j\}_{j=0}^\infty$ is the associated orthonormal basis of eigenfunctions (i.e. $\sqrt{-\Delta_g} e_j = \lambda_j e_j$), then

$$E_j f = \left(\int_M f \overline{e_j} dV\right) e_j.$$

We denote the kernel of $\chi_\lambda$ by $K_\lambda(x,y)$, i.e.,

$$\chi_\lambda f(x) = \int_M K_\lambda(x,y)f(y)dV(y), \quad (f \in C(M)).$$

If the Fourier transform of $\rho$ satisfies $\hat{\rho}(0) = 1$, then $\chi_\lambda e_\lambda = e_\lambda$, or equivalently

$$\int_M K_\lambda(x,y)e_\lambda(y)dV(y) = e_\lambda(x).$$

Thus, $K_\lambda$ is a reproducing kernel for $e_\lambda$ if $\hat{\rho}(0) = 1$.

As in Section 5.1 in [10], we choose $\rho$ so that the reproducing kernel $K_\lambda(x,y)$ is uniformly bounded by $\frac{\lambda^{n-1}}{\pi}$ on the diagonal as $\lambda \to +\infty$. This is essential for the proof of (0.4). If we assume that $\rho(t) = 0$ for $|t| \notin [\varepsilon/2,\varepsilon]$, with $\varepsilon > 0$ being a fixed number which is smaller than the injectivity radius of $(M,g)$, then it is proved in Lemma 5.1.3 of [10] that

$$K_\lambda(x,y) = \frac{\lambda^{n-1}}{\pi} a_\lambda(x,y)e^{i\lambda r(x,y)},$$

where $a_\lambda(x,y)$ is bounded with bounded derivatives in $(x,y)$ and where $r(x,y)$ is the Riemannian distance between points. This WKB formula for $K_\lambda(x,y)$ is known as a parametrix and may be obtained from the Hörmander parametrix for $e^{it\sqrt{-\Delta}}$ in [8] or from the Hadamard parametrix for $\cos t\sqrt{-\Delta}$. We refer to [10,11] for the background.

It follows from (0.7) that

$$|\nabla_g K_\lambda(x,y)| \leq C\lambda^{1+\frac{n-1}{2}},$$
and therefore,
\[
\sup_{x \in M} |\nabla_g \chi_\lambda f(x)| = \sup_x \left| \int f(y) \nabla_g K_\lambda(x, y) \, dV \right|
\leq \left\| \nabla_g K_\lambda(x, y) \right\|_{L^\infty(M \times M)} \|f\|_{L^1} \\
\leq C \lambda^{1 + \frac{n-1}{2}} \|f\|_{L^1}.
\]

To complete the proof of the Lemma, we set \( f = e_\lambda \) and use that \( \chi_\lambda e_\lambda = e_\lambda \). \( \square \)

We note that \( K_\lambda(x, y) \) has quite a different structure from the kernels of the spectral projection operators \( E_{[\lambda, \lambda+1]} = \sum_{j: \lambda_j \in [\lambda, \lambda+1]} E_j \) and the estimate in Lemma 1 is quite different from the sup norm estimate in Lemma 4.2.4 of [10]. Indeed, in a \( \lambda^{-1} \) neighborhood of the diagonal, the spectral projections kernel \( E_{[\lambda, \lambda+1]}(x, y) \) is of size \( \lambda^{n-1} \). For instance, in the case of the standard sphere \( S^n \), the kernel of the orthogonal projection \( E_k \) onto the space of spherical harmonics of degree \( k \) \( \simeq \lambda \) is the constant \( E_k(x, x) = \frac{\lambda^{-n-1}}{\Vol(S^n)} \) on the diagonal. We are able to choose the test function \( \rho \) above, so that the reproducing kernel \( K_\lambda(x, y) \) is uniformly of size \( \lambda^{\frac{n-1}{2}} \) (as in [9] and [10] Section 5.1) because we only need it to reproduce eigenfunctions \( e_\lambda \) of one eigenvalue and because it does not matter how \( K_\lambda \) acts on eigenfunctions of other eigenvalues. From the viewpoint of Lagrangian distributions, the Lagrangian manifold \( \Lambda_x \) associated to both \( E_{[\lambda, \lambda+1]}(x, y) \) and \( K_\lambda(x, y) \), for fixed \( x \) is the flowout \( \Lambda_x = \bigcup_{t \in \supp \rho} G^t S_\times^* M \subset S^* M \) of the unit-cosphere \( S_\times^* M \) under the geodesic flow \( G^t \).

The natural projection of \( \Lambda_x \) to \( M \) has a large singularity along \( S_\times^* M \) which causes the \( \lambda^{n-1} \) blowup of \( E_{[\lambda, \lambda+1]}(x, y) \) at \( y = x \), but the projection is a covering map for the part of \( \Lambda_x \) where \( t \in [\varepsilon, 2\varepsilon] = \supp \rho \). The parametrix (0.7) reflects the fact that the test function \( \rho \) cuts out all of \( \Lambda_x \) except where its projection to \( M \) is a covering map. For further discussion of the geometry underlying Lagrangian distributions we refer to [10, 11, 13].

Finally, we briefly compare the proof of (0.1) in this note with the estimates in [12]:

- Instead of Lemma 0.1, the estimate \( \|\nabla_e \|_{L^\infty(M)} \lesssim \lambda^{1 + \frac{n-1}{2}} \|e_\lambda\|_{L^2} \) was used in [12]. The latter estimate is a consequence of the pointwise local Weyl law for \( |\nabla e_\lambda(x)|^2 \).
- In [12] the authors proved the lower bounds (0.3) by showing that
  \[
  \|e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{\frac{n-1}{2}} \|e_\lambda\|_{L^1(M)},
  \]
  by essentially the same argument as in Lemma 1. In the proof given in this note, (0.3) is not used in the proof of (0.1) since the factor \( \|e_\lambda\|_{L^1(M)} \) cancels out in the left and right sides.

References


Department of Mathematics, Johns Hopkins University, Baltimore, MD, USA

E-mail address: csogge@math.jhu.edu

Department of Mathematics, Northwestern University, Evanston, IL, USA

E-mail address: zelditch@math.northwestern.edu