VARIANCE OF THE EXONENTS OF ORBIFOLD LANDAU–GINZBURG MODELS

WOLFGANG EBELING AND ATSUSHI TAKAHASHI

Abstract. We prove a formula for the variance of the set of exponents of a non-degenerate weighted homogeneous polynomial with an action of a diagonal subgroup of SL_n(C).

Introduction

Let X be a smooth compact Kähler manifold of dimension n. The Hodge numbers h^{p,q}(X) := \dim_C H^q(X, \Omega_X^p), p, q \in \mathbb{Z}, are some of the most important numerical invariants of X. They satisfy

\[ h^{p,q}(X) = h^{q,p}(X), \quad p, q \in \mathbb{Z}, \]

and the Serre duality

\[ h^{p,q}(X) = h^{n-p,n-q}(X), \quad p, q \in \mathbb{Z}. \]

The Euler number \( \chi(X) \) can also be written in terms of the Hodge numbers as

\[ \chi(X) = \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} h^{p,q}(X). \]

One can easily calculate the expectation value of the distribution \( \{q \in \mathbb{Z} \mid h^{p,q}(X) \neq 0\} \), which is given by the formula

\[ \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} q \cdot h^{p,q}(X) = \frac{1}{2} n \cdot \chi(X). \]

Equivalently, this can be rewritten as

\[ \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \left( q - \frac{n}{2} \right) h^{p,q}(X) = 0. \]

This means nothing else but that the mean of the distribution \( \{q \in \mathbb{Z} \mid h^{p,q}(X) \neq 0\} \) is \( n/2 \). It is then natural to ask what is the variance of this distribution. A formula for this variance was given by Libgober and Wood [9] and Borisov [2]:

Theorem 1 (Libgober–Wood, Borisov). One has

\[ \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(X) = \frac{1}{12} n \cdot \chi(X) + \frac{1}{6} \int_X c_1(X) \cup c_{n-1}(X), \]

where \( c_i(X) \) denotes the \( i \)th Chern class of \( X \).

Received by the editors June 22, 2012.
2010 Mathematics Subject Classification. 32S25, 32S35, 14L30.
If the first Chern class, $c_1(X)$ is numerically zero, then the above formula becomes

\[(0.2) \sum_{p, q \in \mathbb{Z}} (-1)^{p+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(X) = \frac{1}{12} n \cdot \chi(X). \]

Similar phenomena were discovered in singularity theory. Let us consider a polynomial $f(x_1, \ldots, x_n)$ with an isolated singularity at the origin. There, the analogue of the set $\{ q \in \mathbb{Z} \mid h^{p,q}(X) \neq 0 \}$ above will be the set of the exponents of $f(x_1, \ldots, x_n)$, which is a set of rational numbers and is also one of the most important numerical invariants defined by the mixed Hodge structure associated to $f(x_1, \ldots, x_n)$. Let us give two important examples.

First, suppose that $f(x_1, \ldots, x_n)$ is a non-degenerate weighted homogeneous polynomial, namely, a polynomial with an isolated singularity at the origin with the property that there are positive rational numbers $w_i, i = 1, \ldots, n$, such that
\[f(\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n) = \lambda f(x_1, \ldots, x_n), \quad \lambda \in \mathbb{C}\setminus\{0\}.\]

We have the following properties of the exponents of $f$:

**Theorem 2** (cf. [10]). Let $q_1 \leq q_2 \leq \cdots \leq q_\mu$ be the exponents of $f$, where $\mu$ is the Milnor number of $f$ defined by
\[\mu := \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n] / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).\]

Then one has
\[\mu = (-1)^n \prod_{i=1}^{n} \left( 1 - \frac{1}{w_i} \right)\]
and
\[\sum_{i=1}^{\mu} y^{q_i - \frac{n}{2}} = (-1)^n \prod_{i=1}^{n} \frac{y^{\frac{1}{2}} - y^{w_i - \frac{1}{2}}}{1 - y^{w_i}}.\]

In particular, one has a duality of exponents $q_i + q_{\mu-i+1} = n, i = 1, \ldots, \mu$, and hence
\[\sum_{i=1}^{\mu} q_i = \frac{1}{2} n \cdot \mu.\]

The following formula was proven by Hertling [6] in the context of Frobenius manifolds and an elementary proof was given by Dimca [3].

**Theorem 3** (Hertling, Dimca). Let $q_1 \leq q_2 \leq \cdots \leq q_\mu$ be the exponents of $f$. One has
\[\sum_{i=1}^{\mu} \left( q_i - \frac{n}{2} \right)^2 = \frac{1}{12} \hat{c} \cdot \mu, \quad \hat{c} := n - 2 \sum_{i=1}^{n} w_i.\]

Next, consider the polynomial $f(x_1, x_2, x_3) := x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3} - x_1 x_2 x_3$ such that $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 < 1$. We have the following properties of the exponents of $f$:
Theorem 4 (cf. [1]). The set of exponents \( \{ q_i \} \) of \( f \) is given by
\[
\left\{ \frac{1}{\alpha_1} + 1, \frac{2}{\alpha_1} + 1, \ldots, \frac{\alpha_1 - 1}{\alpha_1} + 1, \frac{1}{\alpha_2} + 1, \frac{2}{\alpha_2} + 1, \ldots, \frac{\alpha_2 - 1}{\alpha_2} + 1, \frac{1}{\alpha_3} + 1, \frac{2}{\alpha_3} + 1, \ldots, \frac{\alpha_3 - 1}{\alpha_3} + 1, 2 \right\}.
\]
In particular, one has
\[
\sum_{i=1}^{\mu} \left( q_i - \frac{3}{2} \right)^2 = \frac{1}{12} \mu + \frac{1}{6} \chi, \quad \chi := 2 + \sum_{i=1}^{3} \left( \frac{1}{\alpha_i} - 1 \right).
\]

The purpose of this paper is to generalize these results to pairs \((f, G)\), where \( G \subset \text{SL}_n(\mathbb{C}) \) is a finite abelian subgroup leaving \( f \) invariant. If \( f \) is weighted homogeneous, such a pair is also called an orbifold Landau–Ginzburg model because \( f \) is the potential of such a model. Our main theorem in this paper is Theorem 19. The generalization of Theorem 4 is given as Theorem 21. The similarity between smooth compact Kähler manifolds and isolated hypersurface singularities with a group action is not an accident but a matter of course. Mirror symmetry predicts a correspondence between Landau–Ginzburg models and (non-commutative) Calabi–Yau orbifolds. For example, a mirror partner of a weighted homogeneous polynomial with a group action is a fractional Calabi–Yau manifold of dimension \( \hat{c} \), which has lead us to the statement of Theorem 19.

1. Basic properties of E-functions

Let \( G \) be a finite abelian subgroup of \( \text{SL}_n(\mathbb{C}) \) acting diagonally on \( \mathbb{C}^n \). For \( g \in G \), we denote by \( \text{Fix} g := \{ x \in \mathbb{C}^n \mid g \cdot x = x \} \) the fixed locus of \( g \) and by \( n_g := \dim \text{Fix} g \) its dimension.

We first introduce the notion of the age of an element of a finite group as follows:

Definition ([8]). Let \( g \in G \) be an element and \( r \) be the order of \( g \). Then \( g \) has a unique expression of the following form:
\[
g = \text{diag}(e[a_1/r], \ldots, e[a_n/r]) \quad \text{with } 0 \leq a_i < r,
\]
where \( e[-] = e^{2\pi \sqrt{-1} \cdot -} \). Such an element \( g \) is often simply denoted by
\[
g = \frac{1}{r}(a_1, \ldots, a_n).
\]

The age of \( g \) is defined as
\[
\text{age}(g) := \frac{1}{r} \sum_{i=1}^{n} a_i.
\]

Since we assume that \( G \subset \text{SL}_n(\mathbb{C}) \), the number \( \text{age}(g) \) is a non-negative integer for all \( g \in G \).

Definition. An element \( g \in G \) of age 1 with \( \text{Fix} g = \{0\} \) is called a junior element. The number of junior elements is denoted by \( j_G \).

Let \( f = f(x_1, \ldots, x_n) \) be a polynomial with an isolated singularity at the origin, which is invariant under the natural action of \( G \). For \( g \in G \), set \( f^g := f|_{\text{Fix} g} \).
Proposition 5. The function \( f^g \) has an isolated singularity at the origin.

Proof. Since \( G \) acts diagonally on \( \mathbb{C}^n \), we may assume that \( \text{Fix} g = \{ x_{n_g+1} = \cdots = x_n = 0 \} \) by a suitable renumbering of indices. Since \( f \) is invariant under \( G \), \( g \cdot x_i \neq x_i \) for \( i = n_g+1, \ldots, n \) and \( \frac{\partial f}{\partial x_{n_g+1}}, \ldots, \frac{\partial f}{\partial x_n} \) form a regular sequence, we have

\[
\left( \frac{\partial f}{\partial x_{n_g+1}}, \ldots, \frac{\partial f}{\partial x_n} \right) \subset \left( x_{n_g+1}, \ldots, x_n \right).
\]

Therefore, we have

\[
\dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_{n_g}\} / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n_g}} \right) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\} / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n_g}}, x_{n_g+1}, \ldots, x_n \right) \leq \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\} / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) < \infty.
\]

\[ \square \]

We shall associate to \( f \) the following bi-graded vector space:

**Definition.** Let \( H^{n-1}(Y_\infty, \mathbb{C}) \) be the vanishing cohomology of \( f \) on which Steenbrink constructed a canonical mixed Hodge structure in [10]. Denote by \( F^* \) the Hodge filtration on \( H^{n-1}(Y_\infty, \mathbb{C}) \).

Define the bi-graded vector space \( H_f := \bigoplus_{p,q \in \mathbb{Q}} H_f^{p,q} \) as

(i) If \( p + q \neq n \), then \( H_f^{p,q} := 0 \).

(ii) If \( p + q = n \) and \( p \in \mathbb{Z} \), then

\[
H_f^{p,q} := \text{Gr}_{F^p} H^{n-1}(Y_\infty, \mathbb{C})_1.
\]

(iii) If \( p + q = n \) and \( p \notin \mathbb{Z} \), then

\[
H_f^{p,q} := \text{Gr}_{F^p}^\lfloor p \rfloor H^{n-1}(Y_\infty, \mathbb{C})_{e^{2\pi i p}}
\]

where \( \lfloor p \rfloor \) is the largest integer less than \( p \).

We shall use the fact that \( H_f^g \) admits a natural \( G \)-action by restricting the \( G \)-action on \( \mathbb{C}^n \) to \( \text{Fix} g \) (which is well-defined since \( G \) acts diagonally on \( \mathbb{C}^n \)).

To the pair \((f,G)\) we can associate the following bi-graded vector space:

**Definition.** Define the bi-graded \( \mathbb{C} \)-vector space \( H_{f,G} \) as

\[
H_{f,G} := \bigoplus_{g \in G} (H_{f^g})^G(-\text{age}(g), -\text{age}(g)),
\]

where \((H_{f^g})^G\) denotes the \( G \)-invariant subspace of \( H_{f^g} \).

Since the bi-graded vector space \( H_{f,G} \) is the analog of \( \bigoplus_{p,q \in \mathbb{Z}} H^q(X, \Omega^p_X) \) for a smooth compact Kähler manifold \( X \), we introduce the following notion:

**Definition.** The Hodge numbers for the pair \((f,G)\) are

\[
h^{p,q}(f,G) := \dim_{\mathbb{C}} H_{f,G}^{p,q}, \quad p, q \in \mathbb{Q}.
\]
**Definition.** The rational number \( q \) with \( H^{p,q}_{f,G} \neq 0 \) is called an *exponent* of the pair \((f,G)\). The *set of exponents* of the pair \((f,G)\) is the multi-set of exponents

\[
\{ q \ast h^{p,q}(f,G) \mid p, q \in \mathbb{Q}, \ h^{p,q}(f,G) \neq 0 \},
\]

where by \( u \ast v \) we denote \( v \) copies of the rational number \( u \).

Note that \( p + q \in \mathbb{Z} \) for the rational number \( q \) with \( h^{p,q}(f,G) \neq 0 \) since \( G \subset \text{SL}_n(\mathbb{C}) \).

**Definition.** The E-function for the pair \((f,G)\) is

\[
E(f,G)(t,\bar{t}) := \sum_{p,q \in \mathbb{Q}} (-1)^{(p-n)+q} h^{p,q}(f,G) \cdot t^p \bar{t}^{\frac{q}{2}} \bar{t}^{-\frac{q}{2}}.
\]

**Definition.** The Milnor number for the pair \((f,G)\) is

\[
\mu(f,G) := E(f,G)(1,1) = \sum_{p,q \in \mathbb{Q}} (-1)^{(p-n)+q} h^{p,q}(f,G).
\]

**Theorem 6.** Assume that \( f \) is a non-degenerate weighted homogeneous polynomial. Write \( g \in G \) in the form \((\lambda_1(g), \ldots, \lambda_n(g))\) where \( \lambda_i(g) = e[a_iw_i] \). The E-function for the pair \((f,G)\) is given by the following formula:

\[
E(f,G)(t,\bar{t}) = \sum_{g \in G} E_g(f,G)(t,\bar{t}),
\]

\[
E_g(f,G)(t,\bar{t}) := (-1)^n \prod_{a_i, w_i \in \mathbb{Z}} (t\bar{t})^{w_ia_i - [w_ia_i] - \frac{1}{2}}
\]

\[
\cdot \frac{1}{|G|} \sum_{h \in G} \prod_{a_i, w_i \in \mathbb{Z}} \left( \frac{h}{\bar{t}} \right)^{\frac{1}{2}} - \lambda_i(h) \left( \frac{h}{\bar{t}} \right)^{w_i - \frac{1}{2}}
\]

\[
\cdot \frac{1}{1 - \lambda_i(h) \left( \frac{h}{\bar{t}} \right)^{w_i}}.
\]

Here \([a] \) for \( a \in \mathbb{Q} \) denotes the largest integer less than or equal to \( a \).

**Proof.** Theorem 2 enables us to obtain \( E_g(f,G)(t,\bar{t}) \). In particular, the term

\[
\frac{1}{|G|} \sum_{h \in G} (-1)^{n_g} \prod_{a_i, w_i \in \mathbb{Z}} \left( \frac{h}{\bar{t}} \right)^{\frac{1}{2}} - \lambda_i(h) \left( \frac{h}{\bar{t}} \right)^{w_i - \frac{1}{2}}
\]

calculates the \( G \)-invariant part of \( E(f^g, \{1\})(t,\bar{t}) \) and the term

\[
(-1)^{n-g} \prod_{w_i, a_i \in \mathbb{Z}} (t\bar{t})^{w_ia_i - [w_ia_i] - \frac{1}{2}}
\]

gives the contribution from the age shift \((-\text{age}(g), -\text{age}(g))\). \( \square \)

We have the following properties of the Hodge numbers \( h^{p,q}(f,G) \).

**Corollary 7.** Assume that \( f \) is a non-degenerate weighted homogeneous polynomial. We have

\[
h^{p,q}(f,G) = h^{q,p}(f,G), \quad p, q \in \mathbb{Q}.
\]

In other words, we have

\[
E(f,G)(t,\bar{t}) = E(f,G)(\bar{t},t).
\]
Proof. This is shown by an elementary direct calculation. □

**Corollary 8.** Assume that \( f \) is a non-degenerate weighted homogeneous polynomial. The Hodge numbers satisfy the “Serre duality”

\[
h^{p,q}(f, G) = h^{n-p,n-q}(f, G), \quad p, q \in \mathbb{Q}.
\]

In other words, we have

\[
E(f, G)(t, \bar{t}) = E(f, G)(t^{-1}, \bar{t}^{-1}).
\]

Proof. By using the formula

\[
w_i(-a_i) - [w_i(-a_i)] - \frac{1}{2} = -w_i a_i + [w_i a_i] + \frac{1}{2},
\]

an easy calculation yields the formula. □

**Corollary 9.** Assume that \( f \) is a non-degenerate weighted homogeneous polynomial. The mean of the set of exponents of \( (f, G) \) is \( n/2 \). Namely, we have

\[
\sum_{p, q \in \mathbb{Q}} (-1)^{(p-n)q} \left( q - \frac{n}{2} \right) h^{p,q}(f, G) = 0.
\]

Proof. This is obvious from the previous corollary. □

**Definition.** Define the variance of the set of exponents of \( (f, G) \) by

\[
\text{Var}_{(f,G)} := \sum_{p, q \in \mathbb{Q}} (-1)^{(p-n)q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(f, G).
\]

In order to state our formula for the variance, we introduce the following notion of dimension for a polynomial \( f \) with an isolated singularity at the origin.

**Definition.** The non-negative rational number \( \hat{c} \) defined as the difference of the maximal exponent of the pair \( (f, \{1\}) \) and the minimal exponent of the pair \( (f, \{1\}) \) is called the dimension of \( f \).

**Proposition 10.** Assume that \( f \) is a non-degenerate weighted homogeneous polynomial. The dimension \( \hat{c} \) of \( f \) is given by

\[
\hat{c} := n - 2 \sum_{i=1}^{n} w_i.
\]

Proof. It easily follows from Theorem 2 that the maximal exponent and the minimal exponent are given by \( n - \sum_{i=1}^{n} w_i \) and \( \sum_{i=1}^{n} w_i \), respectively. □

It is natural from the mirror symmetry point of view to expect that the variance of the set of exponents of \( (f, G) \) should be given by

\[
\text{Var}_{(f,G)} = \frac{1}{12} \hat{c} \cdot \mu_{(f,G)}.
\]

This will be proved in the next section.
2. Variance of the exponents

**Definition.** The $\chi_y$-genus for the pair $(f,G)$ is

$$\chi(f,G)(y) := E(f,G)(1, y).$$

We have

$$\chi(f,G)(y) = (-1)^n \sum_{g \in G} \left( y^{\text{age}(g)} - n \cdot \frac{1}{|G|} \sum_{h \in G} \prod_{\lambda_i(g) = 1} \frac{y^{\frac{1}{2}} - \lambda_i(h)y^{w_i} - \frac{1}{2}}{1 - \lambda_i(h)y^{w_i}} \right).$$

One has

$$\mu_{(f,G)} = \lim_{y \to 1} \chi(f,G)(y),$$

$$\text{Var}_{(f,G)} = \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} \chi(f,G)(y) \right).$$

**Proposition 11.** Let

$$p_i(y) := \frac{y^{\frac{1}{2}} - \lambda_i(h)y^{w_i} - \frac{1}{2}}{1 - \lambda_i(h)y^{w_i}}.$$  

(i) For $\lambda_i(h) = 1$ one has

$$\lim_{y \to 1} p_i(y) = 1 - \frac{1}{w_i}, \quad \lim_{y \to 1} \frac{d}{dy} p_i(y) = 0, \quad \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} p_i(y) \right) = \frac{1 - 2w_i}{12}.$$  

(ii) For $\lambda_i(h) \neq 1$ one has

$$\lim_{y \to 1} p_i(y) = 1, \quad \lim_{y \to 1} \frac{d}{dy} p_i(y) = \frac{1 + \lambda_i(h)}{2(1 - \lambda_i(h))},$$

$$\lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} p_i(y) \right) = \frac{(1 - 2w_i)\lambda_i(h)}{(1 - \lambda_i(h))^2}.$$  

**Proof.** For (i) see the proof of [3, Proposition 5.2]. Statement (ii) follows from a similar elementary but tedious computation. □

Let $I_0 := \{1, \ldots, n\}$ and let $H \subset G$ be a subgroup of $G$. For a subset $I \subset I_0$ ($I = \emptyset$ is admitted) let $H^I$ be the maximal subgroup of $H$ fixing the coordinates $x_i$, $i \in I$.

**Lemma 12.** Let $H \subset G$ be a subgroup of $G$ and $i \in I_0$. Then

$$\sum_{h \in H \setminus H^{(i)}} \frac{1 + \lambda_i(h)}{1 - \lambda_i(h)} = 0.$$

**Proof.** One has

$$\sum_{h \in H \setminus H^{(i)}} \frac{1 + \lambda_i(h)}{1 - \lambda_i(h)} = \sum_{h \in H \setminus H^{(i)}} \frac{1}{1 - \lambda_i(h)} + \sum_{h \in H \setminus H^{(i)}} \frac{1}{\lambda_i(h^{-1}) - 1} = 0.$$ □
Proposition 13. Let \( r \in \mathbb{Z}, r \geq 2, \) and \( \zeta_r = e^{1/r} \) be a primitive \( r \)th root of unity. Then one has
\[
- \sum_{k=1}^{r-1} \frac{\zeta_r^k}{(1 - \zeta_r^k)^2} = \frac{r^2 - 1}{12}.
\]

Proof. One has
\[
- \sum_{k=1}^{r-1} \frac{\zeta_r^k}{(1 - \zeta_r^k)^2} = \lim_{t \to 1} q'(t) \text{ where } q(t) := - \sum_{k=1}^{r-1} \frac{1}{1 - \zeta_r^k t}.
\]
One can easily see that
\[
q(t) = -r \left( \sum_{k=0}^{r-2} t^k \right) + \sum_{k=0}^{r-2} (k+1) t^k \sum_{k=0}^{r-1} t^k.
\]
This implies
\[
\lim_{t \to 1} q'(t) = \frac{1}{r^2} \left[ \sum_{k=1}^{r-2} k(k - r + 1)r - \left( \sum_{k=1}^{r-1} (k - r) \right) \left( \sum_{k=1}^{r-1} k \right) \right] = \frac{r^2 - 1}{12}.
\]

Corollary 14. Let \( H \subset G \) be a subgroup of \( G \) and \( i \in I_0 \). Then
\[
- \sum_{h \in H \setminus H^{(i)}} \frac{\lambda_i(h)}{(1 - \lambda_i(h))^2} = \frac{|H \cap H^{(i)}| (|H/H \cap H^{(i)}|^2 - 1)}{12}.
\]

Proof. The image of the factor group \( H/H \cap H^{(i)} \) under the induced character \( \lambda_i : H/H \cap H^{(i)} \to \mathbb{C}^* \) is a finite abelian subgroup of the unit circle \( S^1 \) and hence cyclic. Therefore, the formula follows from Proposition 13.

Let
\[
((x)) := \begin{cases} 
  x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \in \mathbb{R}, x \notin \mathbb{Z}, \\
  0, & \text{if } x \in \mathbb{Z}.
\end{cases}
\]

Proposition 15. Let \( r \in \mathbb{Z}, r \geq 2, \zeta_r = e^{1/r} \) be a primitive \( r \)th root of unity, and \( a, b \) be integers satisfying \( 0 < a, b < r \). Then one has
\[
\frac{1}{4r} \sum_{r_{|ak, bk}}^{r-1} \frac{1 + \zeta_r^a k + \zeta_r^b k}{1 - \zeta_r^a k} = - \sum_{k=1}^{r-1} \left( \left( \frac{ak}{r} \right) \left( \frac{bk}{r} \right) \right).
\]

Remark 16. The right-hand side of the formula of Proposition 15 is a generalized Dedekind sum and Proposition 15 is a slight generalization of [7, 5.2 Theorem 1], since
\[
\frac{1 + e[x]}{1 - e[x]} = \sqrt{-1} \cot \pi x
\]
for any real number \( x \). The difference is that [7, 5.2 Theorem 1] is only formulated for integers \( a, b \) prime to \( r \).
Corollary 17. Let $\Lambda \subset I \subset I_0$. Then

$$\frac{1}{4} \sum_{h \in G^K} \left( \sum_{j \in J \setminus K, \lambda_j(h) \neq 1} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 = -|G^K| \sum_{h \in G^K} \left( \sum_{j \in J \setminus K} (a_j w_j) \right)^2,$$

where $\lambda_j(h) = e[a_j w_j]$ for all $h \in G^K$ and $j \in J \setminus K$.

Proof. This follows from Proposition 15 by the same arguments as in the proof of Corollary 14. 

Proposition 18. For a non-degenerate weighted homogeneous polynomial $f$, one has

$$\mu_{(f,G)} = \frac{(-1)^n}{|G|} \left\{ \sum_{f \subset I_0} \prod_{i \in I} \left( 1 - \frac{1}{w_i} \right) \left[ \sum_{J \subset J \subset I_0} (-1)^{|J|-|I|}|G^J|^2 \right] \right\}.$$  

Proof. Let $J \subset I_0$. Let $G_J$ be the set of elements $g \in G$ with $\lambda_j(g) = 1$ for $j \in J$ and $\lambda_j(g) \neq 1$ for $j \notin J$, i.e., the set of elements of $G$ which fix the coordinates $x_j$, $j \in J$, and only these coordinates. Then

$$|G_J| = \sum_{K, J \subset K \subset I_0} (-1)^{|K|-|J|}|G^K|.$$

Let $I \subset J$. Let $G_{I,J}$ be the set of elements $g \in G$ with $\lambda_i(g) = 1$ for $i \in I$ and $\lambda_j(g) \neq 1$ for $j \notin J \setminus I$ (and $\lambda_k(g)$ arbitrary for $k \in I_0 \setminus J$). Then

$$|G_{I,J}| = \sum_{K, I \subset K \subset J} (-1)^{|K|-|I|}|G^K|.$$
By Proposition 11 one has

$$\lim_{y \to 1} \chi(f, G)(y) = \frac{(-1)^n}{|G|} \sum_{J \subseteq I_0} |G_J| \left( \sum_{I \subseteq J, i \in I} \left( 1 - \frac{1}{w_i} \right) |G_{I,J}| \right)$$

$$= \frac{(-1)^n}{|G|} \sum_{I \subseteq I_0} \prod_{i \in I} \left( 1 - \frac{1}{w_i} \right) \left( \sum_{J \subseteq I, j \in J} |G_J||G_{I,J}| \right).$$

Now let $I \subseteq I_0$ be fixed. Then

$$\sum_{J, I \subseteq J \subseteq I_0} |G_J||G_{I,J}| = \sum_{I \subseteq J \subseteq I_0} \left( \sum_{K \subseteq J \subseteq I_0} (-1)^{|K| - |J||G^K|} \right) \left( \sum_{L \subseteq J \subseteq I_0} (-1)^{|L| - |J||G^L|} \right)$$

$$= \sum_{I \subseteq J \subseteq I_0} \sum_{K \subseteq J \subseteq I_0} \left( \sum_{L \subseteq J \subseteq K} (-1)^{|K| + |L| - |J||G^K|} \right) |G^K||G^L|$$

$$= \sum_{I \subseteq J \subseteq I_0} (-1)^{|K| - |J||G^K|^2},$$

since for fixed $L \subseteq I_0$ and $K \subseteq I_0$ with $L \subseteq K$

$$(2.2) \sum_{L \subseteq J \subseteq K} (-1)^{|K| + |L| - |J||G^K|} = (-1)^{|K| - |J|}(1 - 1)^{|K| - |L|} = \begin{cases} (-1)^{|K| - |J|} & \text{for } L = K, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we are ready to state the main result of our paper.

**Theorem 19.** For a non-degenerate weighted homogeneous polynomial $f$, one has

$$\text{Var}(f, G) = \sum_{p, q \in \mathbb{Q}} (-1)^{(p-n)+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(f, G) = \frac{1}{12} \hat{c} \cdot \mu(f, G).$$

**Proof.** We use the notation introduced in the proof of Proposition 18. By Proposition 11 and Lemma 12 we have

$$\lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} \chi(f, G)(y) \right) = A + B + C,$$

where

$$A := \frac{(-1)^n}{|G|} \sum_{J \subseteq I_0} \sum_{g \in G_J} \left( \text{age}(g) - \frac{n - ng}{2} \right) \left[ \sum_{I \subseteq J, i \in I} \left( 1 - \frac{1}{w_i} \right) |G_{I,J}| \right],$$

and

$$B := \frac{(-1)^n}{|G|} \sum_{J \subseteq I_0} \sum_{g \in G_J} \left( \text{age}(g) - \frac{n - ng}{2} \right) \left[ \sum_{I \subseteq J, i \in I} \left( 1 - \frac{1}{w_i} \right) |G_{I,J}| \right],$$

and

$$C := \frac{(-1)^n}{|G|} \sum_{J \subseteq I_0} \sum_{g \in G_J} \left( \text{age}(g) - \frac{n - ng}{2} \right) \left[ \sum_{I \subseteq J, i \in I} \left( 1 - \frac{1}{w_i} \right) |G_{I,J}| \right].$$
On the other hand, we have by Corollary 17

\[ B := \frac{(-1)^n}{|G|} \sum_{j \in I_0} |G_J| \left[ \sum_{i \in I} \prod_{j \in J} (1 - \frac{1}{w_i}) \sum_{h \in G_{I,J}} \frac{1}{4} \left( \sum_{j \in J \setminus I} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \right], \]

\[ C := \frac{(-1)^n}{|G|} \sum_{j \in I_0} |G_J| \times \left[ \sum_{i \in I} \prod_{j \in J} \left( 1 - \frac{1}{w_i} \right) \left( |G_{I,J}| \left( \sum_{i \in I} \frac{1 - 2w_i}{12} \right) - \sum_{h \in G_{I,J}} \sum_{j \in J} (1 - 2w_j) \lambda_j(h) (1 - \lambda_j(h))^2 \right) \right]. \]

(a) We first show that \( A + B = 0 \). We first take the sums in \( A \) and \( B \) in a different order:

\[ A = \frac{(-1)^n}{|G|} \sum_{i \in I} \prod_{j \in J} \left( 1 - \frac{1}{w_i} \right) A_I, \quad A_I := \sum_{j \in J \subseteq I_0} \sum_{g \in G_J} \left( \text{age}(g) - \frac{n - n_g}{2} \right)^2 |G_{I,J}|, \]

\[ B = \frac{(-1)^n}{|G|} \sum_{i \in I} \prod_{j \in J} \left( 1 - \frac{1}{w_i} \right) B_I, \]

\[ B_I := \sum_{j \in J \subseteq I_0} |G_J| \left( \sum_{h \in G_{I,J}} \frac{1}{4} \left( \sum_{j \in J \setminus I} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \right). \]

Now let \( I \subseteq I_0 \) be fixed. Let \( \lambda_i(g) = e[a_i w_i] \). Then, we have on the one hand:

\[ A_I = \sum_{j \in J \subseteq I_0} |G_{I,J}| \sum_{g \in G_J} \left( \sum_{j \in I_0 \setminus J} (a_j w_j) \right)^2 \]

\[ = \sum_{j \in J \subseteq I_0} |G_{I,J}| \sum_{j \in K \subseteq I_0 \setminus J} (-1)^{|K| - |J|} \sum_{g \in G_K} \left( \sum_{j \in I_0 \setminus K} (a_j w_j) \right)^2. \]

On the other hand, we have by Corollary 17

\[ B_I = \sum_{j \in J \subseteq I_0} |G_J| \sum_{h \in G_{I,J}} \frac{1}{4} \left( \sum_{j \in J \setminus I} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \]

\[ = \sum_{j \in J \subseteq I_0} |G_J| \sum_{h \in G_K} (-1)^{|K| - |J|} \sum_{j \in K \subseteq J} \frac{1}{4} \left( \sum_{j \in J \setminus K} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \]

\[ = - \sum_{j \in J \subseteq I_0} |G_J| \sum_{h \in G_K} (-1)^{|K| - |J|} |G^K| \sum_{j \in J \setminus K} \left( \sum_{j \in K \subseteq J} (a_j w_j) \right)^2. \]
For $I \subset K \subset J \subset I_0$ let

$$s(K, J) := \sum_{g \in G^K} \left( \sum_{j \in J \setminus K} \left( a_j w_j \right) \right)^2.$$ 

Then

$$A_I = \sum_{K, I \subset K} (-1)^{|K|-|J|} |G_{I,J}| s(K, I_0)$$

$$= \sum_{K, I \subset K} (-1)^{|K|-|J|} \left( \sum_{L \subset I \cap K} (-1)^{|L|-|I|} |G^L| \right) |G^K| s(K, I_0)$$

$$= \sum_{L, I \subset L} \left( \sum_{K \subset I \cap L} (-1)^{|K|+|L|-|I|-|J|} \right) |G^L| |G^K| s(K, I_0)$$

by Formula (2.2). On the other hand, we have

$$B_I = - \sum_{K, I \subset K} (-1)^{|K|-|I|} |G_J||G^K| s(K, J)$$

$$= - \sum_{L, I \subset L} \left( \sum_{K \subset I \cap L} (-1)^{|K|+|L|-|I|-|J|} \right) |G^L| |G^K| s(K, J)$$

$$= - \sum_{K, I \subset K} (-1)^{|K|-|I|} |G^K| s(K, I_0) = -A_I,$$

again by Formula (2.2) and since $|G^{I_0}| = 1$. This shows that $A + B = 0$.

(b) We now consider the term $C$. Let $J \subset I_0$, $I \subset J$ and $j \in J$, $j \notin I$. Then it follows from Corollary 14 that

$$- \sum_{h \in G_{I,J}} \frac{\lambda_j(h)}{(1 - \lambda_j(h))^2} = \frac{1}{12} m^J_{I,j},$$

where

$$m^J_{I,j} := \sum_{K \notin K, I \subset K \subset J} (-1)^{|K|-|I||G^K_{\cup\{i\}|} \left( \left| G^K_{\cup\{i\}} \right|^2 - 1 \right).$$
By (a) we have
\[
\lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} \chi(f, G)(y) \right) = C = \frac{(-1)^n}{|G|} \sum_{j, j \in I_0} |G_j|
\]
\[
\times \left[ \sum_{J, i \in I} \prod_{i \notin I} \left( 1 - \frac{1}{w_i} \right) \left( |G_{I, J}| \left( \sum_{i \in I} \frac{1 - 2w_i}{12} \right) + \sum_{j \in j, j \notin I} m_{I, j} \left( \frac{1 - 2w_j}{12} \right) \right) \right]
\]
\[
= \frac{(-1)^n}{|G|} \sum_{J, i \in I_0} \prod_{i \notin I} \left( 1 - \frac{1}{w_i} \right)
\]
\[
\times \left[ \sum_{J, j \in I_0} |G_j| \left( |G_{I, J}| \left( \sum_{i \in I} \frac{1 - 2w_i}{12} \right) + \sum_{j \in j, j \notin I} m_{I, j} \left( \frac{1 - 2w_j}{12} \right) \right) \right]
\].

Now let $I \subset I_0$ and $j \notin I$ be fixed. Then
\[
\sum_{J, j \in J, i \in I_0} |G_j| m_{I, j} = \sum_{J, j \in J, i \in I_0} \left( \sum_{K, j \in K, K \subset I_0} (-1)^{|K| - |J|} |G^K| \right)
\]
\[
\times \left( \sum_{K, j \in K, K \subset I_0} (-1)^{|L| - |I|} |G^{L \cup \{j\}}| \left( \left| G^L / G^{L \cup \{j\}} \right|^2 - 1 \right) \right)
\]
\[
= \sum_{L, j \in L, i \in L} \sum_{K, j \in K, K \subset I_0} \left( \sum_{J, j \in J, i \in J} (-1)^{|K| + |L| - |I| - |J|} \right)
\]
\[
\times |G^K||G^{L \cup \{j\}}| \left( \left| G^L / G^{L \cup \{j\}} \right|^2 - 1 \right).
\]

Since $j \notin L$ but $j \in J$, the case $J = L$ and hence also $K = L$ is excluded in the sum
\[
\sum_{J, j \in J, i \in K} (-1)^{|K| + |L| - |I| - |J|}.
\]

Therefore
\[
\sum_{J, j \in J, i \in K} (-1)^{|K| + |L| - |I| - |J|} = \begin{cases} (-1)^{|I| - |J|} & \text{for } K = L \cup \{j\}, \\ 0 & \text{otherwise.} \end{cases}
\]
Hence, we obtain
\[ \sum_{J,j \in J, I \subset J \subset I_0} (\frac{1}{|G_j|} m_{I_j}) = \sum_{L,j \in L, I \subset L \subset I_0} (-1)^{|L|-|I|} (|G^L/G^{L \cup \{j\}}|^2 - 1). \]

\[ = \sum_{L,j \in L, I \subset L \subset I_0} (-1)^{|L|-|I|} (|G^L|^2 - |G^{L \cup \{j\}}|^2) \]

\[ = \sum_{I \subset K \subset I_0} (-1)^{|K|-|I|} |G^K|^2. \]

Therefore, the statement follows from Proposition 18. \qed

3. Variance of the exponents for cusp singularities with group actions

Let \( f(x_1, x_2, x_3) := x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3} - x_1 x_2 x_3 \) and \( G \) be a finite subgroup of \( SL_n(\mathbb{C}) \) acting diagonally on \( \mathbb{C}^n \) under which \( f \) is invariant. Let \( K_i \subset G \) be the maximal subgroup fixing the coordinate \( x_i, i = 1, 2, 3 \). Define numbers \( \gamma_1, ..., \gamma_s \) by

\( (\gamma_1, ..., \gamma_s) = \left( \frac{\alpha_i}{|G/K_i|} \ast |K_i|, i = 1, 2, 3 \right), \)

where we omit numbers which are equal to one on the right-hand side. Define a number \( \chi(f,G) \) by

\[ \chi(f,G) := 2 - 2jG + \sum_{i=1}^{s} \left( \frac{1}{\gamma_i^2} - 1 \right). \]

Lemma 20. Let the pair \( (f,G) \) be as above.

(i) The Milnor number of the pair \( (f,G) \) is given by

\[ \mu(f,G) = 2 - 2jG + \sum_{i=1}^{s} (\gamma_i - 1). \]

(ii) The set of exponents for the pair \( (f,G) \) is given by

\[ \{1,2\} \prod \left\{ \frac{1}{\gamma_1} + 1, \frac{2}{\gamma_1} + 1, \ldots, \gamma_1 - 1, \gamma_1 + 1 \right\} \]

\[ \prod \left\{ \frac{1}{\gamma_2} + 1, \frac{2}{\gamma_2} + 1, \ldots, \gamma_2 - 1, \gamma_2 + 1 \right\} \prod \ldots \]

\[ \prod \left\{ \frac{1}{\gamma_s} + 1, \frac{2}{\gamma_s} + 1, \ldots, \gamma_s - 1, \gamma_s + 1 \right\} \]

Proof. See Corollary 5.13 and the proof of Theorem 5.12 of [4]. \qed

We have the following formula for the variance. Note that we have \( \hat{c} = 1 \) by Theorem 4.

Theorem 21. Let the pair \( (f,G) \) be as above. The variance of the set of exponents of \( (f,G) \) is given by

\[ \text{Var}(f,G) = \frac{1}{12} \mu(f,G) + \frac{1}{6} \chi(f,G) = \frac{1}{12} \hat{c} \cdot \mu(f,G) + \frac{1}{6} \chi(f,G). \]
Proof. Some elementary calculation yields the statement. □

Note that the pair \((f, G)\) can be considered as a mirror partner of the orbifold curve (Deligne–Mumford stack) \(\mathcal{C}\) which is a smooth projective curve of genus \(j_G\) with \(s\) isotropic points of orders \(\gamma_1, \ldots, \gamma_s\) (cf. Theorem 7.1 of [4]). The above formula for the variance is compatible with this observation. In particular, the dimension of \(\mathcal{C}\) is 1, \(\mu_{(f,G)}\) is the orbifold Euler number \(\chi(\mathcal{C})\) of \(\mathcal{C}\) and \(\chi_{(f,G)}\) is the orbifold Euler characteristic of \(\mathcal{C}\), which is the degree of the first Chern class \(c_1(\mathcal{C})\) of \(\mathcal{C}\). Applying this to the formula in Theorem 1, we recover the equation (3.3).

Acknowledgments

This work was supported by the DFG-program SPP1388 “Representation Theory” (Eb 102/6-1). The second named author is also supported by JSPS KAKENHI Grant Number 24684005. We are very grateful to the anonymous referee for carefully reading our paper, drawing our attention to a serious gap in the proof of the main result in the first version, and for most valuable comments which led to a major revision of the article.

References


Institut für Algebraische Geometrie, Leibniz Universität Hannover, Postfach 6009, D-30060 Hannover, Germany
E-mail address: ebeling@math.uni-hannover.de

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan
E-mail address: takahashi@math.sci.osaka-u.ac.jp