CONGRUENCES BETWEEN HILBERT MODULAR FORMS:
CONSTRUCTING ORDINARY LIFTS, II

THOMAS BARNET-LAMB, TOBY GEE AND DAVID GERAGHTY

Abstract. In this paper, we improve on the results of our earlier paper [BLGG12], proving a near-optimal theorem on the existence of ordinary lifts of a mod \( l \) Hilbert modular form for any odd prime \( l \).

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1. Introduction

Let \( F \) be a totally real field with absolute Galois group \( G_F \), and let \( l \) be an odd prime number. In our earlier paper [BLGG12], we proved a general result on the existence of ordinary modular lifts of a given modular representation \( \overline{\rho} : G_F \to \GL_2(\mathbb{F}_l) \); we refer the reader to the introduction of op. cit. for a detailed discussion of the problem of constructing such a lift, and of our techniques for doing so.

The purpose of this paper is to improve on the hypotheses imposed on \( \overline{\rho} \), removing some awkward assumptions on its image; in particular, if \( l = 3 \) then the results of [BLGG12] were limited to some cases where \( \overline{\rho} \) was induced from a quadratic character, whereas our main theorem is the following.

Theorem A. Suppose that \( l > 2 \) is prime, that \( F \) is a totally real field, and that \( \overline{\rho} : G_F \to \GL_2(\mathbb{F}_l) \) is irreducible and modular. Assume that \( \overline{\rho}|_{G_{F_v}} \) is reducible at all places \( v|l \) of \( F \).

If \( l = 5 \) and the projective image of \( \overline{\rho}|_{G_F(\zeta_5)} \) is isomorphic to \( \PSL_2(\mathbb{F}_5) \), assume further that there is a finite solvable totally real extension \( F'/F \) such that \( \overline{\rho}|_{G_{F'}} \) is conjugate to a representation valued in \( \GL_2(\mathbb{F}_5) \).

Then \( \overline{\rho} \) has a modular lift \( \rho : G_F \to \GL_2(\overline{\mathbb{Q}}_l) \), which is ordinary at all places \( v|l \).

Received by the editors December 21, 2012.
2000 Mathematics Subject Classification. 11F33.
(Note that the assumption that $\rho|_{G_{F_v}}$ is reducible at all places $v|l$ of $F$ is necessary.) Our methods are based on those of [BLGG12]. The reason that we are now able to prove a stronger result is that the automorphy lifting results that we employed in [BLGG12] have since been optimized in [BLGGT10] and [Tho12]; in particular, we make extensive use of the results of the appendix to [BLGG13], which improves on a lifting result of [BLGGT10], and classifies the subgroups of $\text{GL}_2(F_l)$, which are adequate in the sense of [Tho12]. In Section 2, we use these results to prove Theorem A, except in the case that $l = 3$ or $5$ and the projective image of $\overline{\rho}(G_{F(\zeta_l)})$ is isomorphic to $\text{PSL}_2(F_l)$, and certain cases where $\rho$ is dihedral. In the dihedral cases, the result is proved in [All12]. In the remaining cases, the adequacy hypothesis we require fails, but in Section 3 we handle this case completely when $l = 3$ by making use of the Langlands–Tunnell theorem, and we prove a partial result when $l = 5$ using the results of [SBT97].

1.1. Notation. If $M$ is a field, we let $G_M$ denote its absolute Galois group. We write $\overline{\epsilon}$ for the mod $l$ cyclotomic character. We fix an algebraic closure $\overline{Q}$ of $Q$, and regard all algebraic extensions of $Q$ as subfields of $\overline{Q}$. For each prime $p$ we fix an algebraic closure $\overline{Q}_p$ of $Q_p$, and we fix an embedding $\overline{Q} \hookrightarrow \overline{Q}_p$. In this way, if $v$ is a finite place of a number field $F$, we have a homomorphism $G_{F_v} \to G_F$. We also fix an embedding $\overline{Q} \hookrightarrow \mathbb{C}$.

We normalize the definition of Hodge–Tate weights so that all the Hodge–Tate weights of the $l$-adic cyclotomic character $\epsilon$ are $-1$. We refer to a two-dimensional potentially crystalline representation with all pairs of labelled Hodge–Tate weights equal to $\{0, 1\}$ as a weight 0 representation. (The reason for this terminology is that the Galois representations associated to an automorphic representation, which is cohomological of weight 0 have these Hodge–Tate weights.)

If $F$ is a totally real field, then a continuous representation $\overline{\rho} : G_F \to \text{GL}_2(\overline{F}_l)$ is said to be modular if there exists a regular algebraic automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$, such that $\overline{\rho}(\pi) \equiv \overline{\epsilon}$, where $r_l(\pi)$ is the $l$-adic Galois representation associated with $\pi$.

We let $\zeta_l$ be a primitive $l$th root of unity.

2. The adequate case

2.1. The notion of an adequate subgroup of $\text{GL}_n(\overline{F}_l)$ is defined in [Tho12]. We will not need to make use of the actual definition; instead, we will use the following classification result. Note that by definition an adequate subgroup of $\text{GL}_n(\overline{F}_l)$ necessarily acts irreducibly on $\overline{F}_l^n$.

Proposition 2.1.1. Suppose that $l > 2$ is a prime, and that $G$ is a finite subgroup of $\text{GL}_2(\overline{F}_l)$, which acts irreducibly on $\overline{F}_l^2$. Then precisely one of the following is true:
- We have $l = 3$, and the image of $G$ in $\text{PGL}_2(\overline{F}_3)$ is conjugate to $\text{PSL}_2(\overline{F}_3)$.
- We have $l = 5$, and the image of $G$ in $\text{PGL}_2(\overline{F}_3)$ is conjugate to $\text{PSL}_2(\overline{F}_3)$.
- $G$ is adequate.

Proof. This is Proposition A.2.1 of [BLGG13]. □
In the case that $\overline{\rho}(G_{F(\zeta)})$ is adequate, our main result follows exactly as in section 6 of [BLGG12], using the results of Appendix A of [BLGG13] (which in turn build on the results of [BLGGT10]). We obtain the following theorem.

**Theorem 2.1.2.** Suppose that $l > 2$ is prime, that $F$ is a totally real field, and that $\overline{\rho} : G_F \to \text{GL}_2(\mathbb{F}_l)$ is irreducible and modular. Suppose also that $\overline{\rho}(G_{F(\zeta)})$ is adequate. Then:

1. There is a finite solvable extension of totally real fields $L/F$ which is linearly disjoint from $\overline{F}^{\text{ker} \, \overline{\rho}}$ over $F$, such that $\overline{\rho}|_{G_L}$ has a modular lift $\rho_L : G_L \to \text{GL}_2(\overline{\mathbb{Q}}_l)$ of weight 0, which is ordinary at all places $v|l$.
2. If furthermore $\overline{\rho}|_{G_{F' \zeta}}$ is reducible at all places $v|l$, then $\overline{\rho}$ itself has a modular lift $\rho : G_F \to \text{GL}_2(\overline{\mathbb{Q}}_l)$ of weight 0, which is ordinary at all places $v|l$.

**Proof.** First, note that (2) is easily deduced from (1) using the results of Section 3 of [Gee11] (which build on Kisin’s reinterpretation of the Khare–Wintenberger method). Indeed, the proofs of Theorems 6.1.5 and 6.1.7 of [BLGG12] go through unchanged in this case.

Similarly, (1) is easily proved in the same way as Proposition 6.1.3 of [BLGG12] (and in fact the proof is much shorter). First, note that the proof of Lemma 6.1.1 of [BLGG12] goes through unchanged to show that there is a finite solvable extension of totally real fields $L/F$ which is linearly disjoint from $\overline{F}^{\text{ker} \, \overline{\rho}}$ over $F$, such that $\overline{\rho}|_{G_L}$ has a modular lift $\rho' : G_L \to \text{GL}_2(\overline{\mathbb{Q}}_l)$ of weight 0 which is potentially crystalline at all places dividing $l$, and in addition both $\overline{\rho}|_{G_{L_w}}$ and $\overline{\varepsilon}|_{G_{L_w}}$ are trivial for each place $w|l$ (and in particular, $\overline{\rho}|_{G_{L_w}}$ admits an ordinary lift of weight 0), and $\overline{\rho}$ is unramified at all finite places. By Lemma 4.4.1 of [GK12], $\rho'|_{G_{L_w}}$ is potentially diagonalizable in the sense of [BLGGT10] for all places $w|l$ of $L$.

Choose a CM quadratic extension $M/L$ that is linearly disjoint from $L(\zeta)$ over $L$, in which all places of $L$ dividing $l$ split. We can now apply Theorem A.4.1 of [BLGG13] (with $F' = F = M$, $S$ the set of places of $L$ dividing $l$, and $\rho_v$ an ordinary lift of $\overline{\rho}|_{G_{L_w}}$ for each $w|l$) to see that $\overline{\rho}|_{G_M}$ has an ordinary automorphic lift $\rho_M : G_M \to \text{GL}_2(\overline{\mathbb{Q}}_l)$ of weight 0.

The argument of the last paragraph of the proof of Proposition 6.1.3 of [BLGG12] (which uses the Khare–Wintenberger method to compare deformation rings for $\overline{\rho}|_{G_L}$ and $\overline{\rho}|_{G_M}$) now goes over unchanged to complete the proof. \(\square\)

### 3. Inadequate cases

#### 3.1. The first inadequate case

We now consider the case that $l = 3$ and $\overline{\rho}|_{G_{F(\zeta)}}$ is irreducible, but $\overline{\rho}(G_{F(\zeta)})$ is not adequate. By Proposition 2.1.1, this means that the projective image of $\overline{\rho}(G_{F(\zeta)})$ is isomorphic to $\text{PSL}_2(\mathbb{F}_3)$, and is in particular soluble. We now use the Langlands–Tunnell theorem to prove our main theorem in this case.

**Theorem 3.1.1.** Suppose that $F$ is a totally real field, and that $\overline{\rho} : G_F \to \text{GL}_2(\mathbb{F}_3)$ is irreducible and modular. Assume that $\overline{\rho}|_{G_{F \zeta}}$ is reducible at all places $v|3$ of $F$, and that the projective image of $\overline{\rho}(G_{F(\zeta)})$ is isomorphic to $\text{PSL}_2(\mathbb{F}_3)$.

Then $\overline{\rho}$ has a modular lift $\rho : G_F \to \text{GL}_2(\overline{\mathbb{Q}}_3)$ which is ordinary at all places $v|3$. 
3.2. The second inadequate case. We now suppose that \( l = 5 \), that \( \rho \big|_{G_F(\zeta_5)} \) is irreducible but its image is not adequate. Then \( \rho \big|_{G_F(\zeta_5)} \) has projective image conjugate to \( \text{PSL}_2(\mathbb{F}_3) \), and we see that \( \rho(G_F) \) has projective image conjugate to either \( \text{PGL}_2(\mathbb{F}_5) \) or \( \text{PSL}_2(\mathbb{F}_5) \). (This follows from [DDT97, Prop. 2.47].) Thus, after conjugating, we may assume that \( \rho : G_F \rightarrow \text{GL}_2(\mathbb{F}_5) \) takes values in \( \mathbb{F}_5^\times \text{GL}_2(\mathbb{F}_5) \).

In order to apply the results of [SBT97], we need to assume further that there is a finite solvable totally real extension \( F'/F \) such that \( \rho \big|_{G_{F'}} \), is valued in \( \text{GL}_2(\mathbb{F}_5) \). (This condition is not automatic, but it holds if the projective image of \( \rho(G_F) \) is isomorphic to \( \text{PSL}_2(\mathbb{F}_5) \).)

**Theorem 3.2.1.** Suppose that \( F \) is a totally real field, and that \( \rho : G_F \rightarrow \text{GL}_2(\mathbb{F}_5) \) is irreducible and modular. Assume that \( \rho \big|_{G_{F'}} \) is reducible at all places \( v \nmid 5 \) of \( F \), and that the projective image of \( \rho(G_F(\zeta_5)) \) is isomorphic to \( \text{PSL}_2(\mathbb{F}_5) \). Assume further that there is a finite solvable totally real extension \( F'/F \) so that \( \rho \big|_{G_{F'}} \), is conjugate to a representation valued in \( \text{GL}_2(\mathbb{F}_5) \).

Then \( \rho \) has a modular lift \( \rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_5) \) which is ordinary at all places \( v \nmid 5 \).

**Proof.** Since \( \rho \) is totally odd, we can replace \( F'/F \) by a further finite solvable totally real extension and assume that \( \rho \big|_{G_{F'}} \) takes values in \( \text{GL}_2(\mathbb{F}_5) \) and has determinant equal to the cyclotomic character. Now, as in the proof of Theorem 2.1.2, to prove the current theorem, it suffices to show that \( \rho \big|_{G_{F'}} \) has a modular lift of weight 0, which is ordinary at each \( v \nmid 5 \). (The only thing that needs to be checked is that Proposition 3.1.5 of [Gee11] applies to \( \rho \big|_{G_{F'}} \). The only hypothesis which is not immediate is that if the projective image of \( \rho \big|_{G_{F'}} \) is \( \text{PGL}_2(\mathbb{F}_5) \), then \( [F'(\zeta_5) : F'] = 4 \). To see this, note that if \( [F'(\zeta_5) : F'] = 2 \), then since the determinant of \( \rho \big|_{G_{F'}} \) is the mod 5 cyclotomic character, it has image \( \{ \pm 1 \} \). This implies that the projective image is \( \text{PSL}_2(\mathbb{F}_5) \), as required.)

By [SBT97, Theorem 1.2], there exists an elliptic curve \( E/F' \) such that \( E[5] \cong \rho \big|_{G_{F'}} \), and the image of \( G_{F'} \) in \( \text{Aut}(E[3]) \) contains \( \text{SL}_2(\mathbb{F}_2) \) (and hence its image is equal to \( \text{Aut}(E[3]) \) since the determinant is totally odd). We may further suppose that \( E \) has good ordinary reduction at each prime of \( F' \) dividing 5. (To see this, note that we may
incorporate Ekedahl’s effective version of the Hilbert Irreducibility Theorem [Eke90] into the proof of [SBT97, Theorem 1.2] exactly as is done in [Tay03, Lemma 2.3].) By the Langlands–Tunnell theorem, $E[3]$ has a modular lift corresponding to a Hilbert modular form $f_0$ of parallel weight 1. Replacing $F'$ by a finite totally real solvable extension linearly disjoint from $\overline{F'}\ker E[3]$, we may assume that $f_0$ is ordinary at each prime dividing 3. By Hida theory, $E[3]$ then has a modular lift corresponding to a Hilbert modular form of parallel weight 2, which is ordinary at each prime dividing 3. Note that the conditions of the modularity lifting theorem [Gee09, Theorem 1.1], applied to $\rho := T_3E$, are satisfied. (For the third condition, note that $E[3]|_{G_F'}$ has non-dihedral image.) It follows that $T_3E$ is modular and hence that $T_3E$ is modular. Thus we have exhibited a modular lift of $\overline{\rho}|_{G_F'} \cong E[5]$ which has weight 0 and is ordinary at each prime above 5.

Finally, we deduce our main result from Theorems 2.1.2, 3.1.1 and 3.2.1.

**Proof of Theorem A.** If $\overline{\rho}|_{G_F(\zeta_l)}$ is reducible, then $\overline{\rho}$ is dihedral, and the result follows from Lemma 5.1.2 of [All12]. If $l = 3$ (respectively $l = 5$) and the projective image of $\overline{\rho}(G_F(\zeta_l))$ is isomorphic to $\text{PSL}_2(\mathbb{F}_l)$, then the result follows from Theorem 3.1.1 (respectively, from Theorem 3.2.1). In all other cases, we see from Proposition 2.1.1 that $\overline{\rho}(G_F(\zeta_l))$ is adequate and the result follows from Theorem 2.1.2(2). □

**Acknowledgment**

We would like to thank Vincent Pilloni for pointing out to us that we could make use of the results of [SBT97].

**References**


Department of Mathematics, Brandeis University, 415 South St, Waltham, MA 02453, USA
E-mail address: tbl@brandeis.edu

Department of Mathematics, Imperial College London, South Kensington Campus, Exhibition Rd, London SW7 2AZ, UK
E-mail address: toby.gee@imperial.ac.uk

Princeton University and Institute for Advanced Study, 1 Einstein Dr, Princeton, NJ 08540, USA
E-mail address: geraghty@math.princeton.edu