A PRESENTATION FOR THE PURE HILDEN GROUP

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Abstract. Consider the half ball, $B_3^+$, containing $n$ unknotted and unlinked arcs $a_1, a_2, \ldots, a_n$ such that the boundary of each $a_i$ lies in the plane. The Hilden (or Wicket) group is the mapping class group of $B_3^+$ fixing the arcs $a_1 \cup a_2 \cup \cdots \cup a_n$ setwise and fixing the half sphere $S^2_+$ pointwise. This group can be considered as a subgroup of the braid group on $2n$ strands. The pure Hilden group is defined to be the intersection of the Hilden group and the pure braid group. In a previous paper, we computed a presentation for the Hilden group using an action of the group on a cellular complex. This paper uses the same action and complex to calculate a finite presentation for the pure Hilden group.

The framed braid group acts on the pure Hilden group by conjugation and this action is used to reduce the number of cases.

1. Introduction

Let $B_3^+$ be a half ball in upper half space and let $S^2_+$ be the half sphere contained in its boundary. The half ball and half sphere intersect the plane in a 2-ball $B^2$ and a circle $S^1$. We can embed $n$ disjoint semi-circular arcs, $a_1, a_2, \ldots, a_n$, into $B_3^+$ so that the arcs are disjoint from $S^2_+$ and only intersect $B^2$ at their end points (see Figure 1). We will write $a = a_1 \cup a_2 \cup \cdots \cup a_n$. The Hilden group $H_{2n}$ is the group of isotopy classes of self-homeomorphisms of $B_3^+$ which preserve $a$ setwise and $S^2_+$ pointwise. The inclusion $(B^2, \partial a, S^1) \hookrightarrow (B_3^+, a, S^2_+)$ induces the embedding $H_{2n} \hookrightarrow B_{2n}$ of the Hilden group in the braid group. We define the pure Hilden group to be the intersection of the Hilden group and the pure braid group.

$$PH_{2n} = P_{2n} \cap H_{2n}$$

Generators for the corresponding subgroup of the braid group of the sphere were found by Hilden [4] and a finite presentation for the Hilden group was calculated independently by the Tawn [6, 7] and Brendle and Hatcher [2].

In [6, 7] we define an action of the Hilden group on a cellular complex and then use the method of Hatcher and Thurston [3], Wajnryb [8, 9, 10], etc., to compute a presentation of the group from this action. In this paper, we will use the same method with the same complex and action to prove the following theorem.

**Theorem 1.** The pure Hilden group has a finite presentation with generating set $S$ and relations $R$, $PH_{2n} = \langle S \mid R \rangle$, where $S$ and $R$ are as follows.

$$S = \{p_{\{i,j\}}, x_{\{i,j\}}, y_{\{i,j\}}, t_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$$

To simplify notation we will write $p_{ij} = p_{\{i,j\}}$, $x_{ij} = x_{\{i,j\}}$ and $y_{ij} = y_{\{i,j\}}$, so, for example, $p_{ij} = p_{ji}$. The generators $p_{ij}$, $x_{ij}$, $y_{ij}$ and $t_k$ are the following elements of

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The set $R$ of relations is as follows.

(C-pt) $p_{ij} t_k = t_k p_{ij}$
(C-tt) $t_i t_j = t_j t_i$
(C-xt) $x_{ij} t_k = t_k x_{ij}$ $i < j$ $k \neq i$
(C-yt) $y_{ij} t_k = t_k y_{ij}$ $i < j$ $k \neq j$
(C1) $\alpha_{ij} \beta_{kl} = \beta_{kl} \alpha_{ij}$ $\alpha, \beta \in \{p, x, y\}$ $(i, j, k, l)$ cyclically ordered
(C2) $\alpha_{ij} \beta_{ik} \gamma_{jk} = \beta_{ik} \gamma_{jk} \alpha_{ij}$ $(i, j, k)$ cyclically ordered $(\alpha, \beta, \gamma)$ as in Table 1
(C3) $\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1} = p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik}$ $\alpha, \beta \in \{p, x, y\}$ $(i, j, k, l)$ cyclically ordered
(C-xpt) $x_{ij} p_{ij} t_i = p_{ij} t_i x_{ij}$ $i < j$
(C-ypt) $y_{ij} p_{ij} t_j = p_{ij} t_j y_{ij}$ $i < j$

Table 1. The values of $(\alpha, \beta, \gamma)$ for (C2)
All of the relations are asserting that certain pairs of elements commute. The first four relations are the obvious ways in which a $t$ can commute with the other generators. The relations (C1)–(C3) are analogous to the relations in the pure braid group. In fact, Table 1 lists all possible triples for which (C2) holds, these were found using the MAGMA computational algebra system [1]. To see why (C-xt) and (C-yt) hold it is easiest to consider the element $p_{ij} t_i t_j$. If you ignore all the other strings this is a full twist of the $i$th and $j$th pairs of strings and so clearly it commutes with $x_{ij}$ and $y_{ij}$.

We refer the reader to [6, 7] for the definition of the cut–system complex and the method for computing the group presentation. In particular we will use the complex $X_n$ with its associated $H_{2n}$ action and basepoint $v_0 = \langle d_1, d_2, \ldots, d_n \rangle$. From the method we will use the notion of an h-product, the elements $\{r_\lambda\}$ and the notation for the generator sets $S_0$ and $S_1$ and the sets of relations $R_0$, $R_1$, $R_2$ and $R_3$.

In order to use the method to calculate a presentation, we need to show that the action on $X_n$ is transitive on the vertex set and find edge and face orbit representatives.

**Theorem 2.** The action of $PH_{2n}$ on $X^0_n$ is transitive.

**Proof.** This follows from the proof that the action of $H_{2n}$ on $X^0_n$ is transitive given in [6]. All that is needed is to note that the constructed braids do not permute the punctures. □

2. Vertex stabilizer

**Proposition 3.** The stabilizer of the vertex $v_0$ is the framed pure braid group $FP_n$ and so is isomorphic to $P_n \times \mathbb{Z}^n$.

**Proof.** If we restrict our attention to $\mathbb{B}^2$, elements of $PH_{2n}$ can be thought of as motions of the end points of the $a_i$. For elements of the stabilizer of $v_0$ this motion moves the line segments $d_i \cap \mathbb{B}^2$. So this is the fundamental group of configurations of $n$ ordered line segments in the plain, the framed pure braid group. □

The pure braid group has a presentation with generators $p_{ij}$ and relations (C1), (C2) and (C3) (with $\alpha = \beta = \gamma = p$). See, for example, Margalit and McCammond [5].

From this, we see that the vertex stabilizer is generated by the $p_{ij}$ and $t_k$, that all relations between these elements follow from (C-pt), (C-tt), (C1), (C2) and (C3), and hence the $R_0$ relations are included in $R$.

3. Edge orbits

Let $E$ denote the set of all oriented edges that start at $v_0$ the basepoint of $X_n$. We will now find a representative of each orbit of the $FP_n$ action on $E$, thus giving a set of $PH_{2n}$ edge orbit representatives. Given an edge $(v_0, v) \in E$, because $v = \langle D_1, D_2, \ldots, D_n \rangle$ differs from $v_0$ by a simple move, there exists a unique $i$ such that $D_i \neq d_i$. 

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If the edge is of length one then there is a unique $d_j$ under $D_i \cup d_i$. All of the remaining discs, $d_k$ for $k \neq i, j$, can be moved by an element of $\Phi_n$ away from $D_i \cup d_i$ and then back from behind to their original positions. After applying $t_p^i$ for some $p$ we have one of the following possibilities, each of which lie in a different orbit.

Similarly, if the edge is of length two then there exists two discs $d_j$ and $d_k$ under $d_i \cup D_i$. We may assume that $j < k$. As in the previous case there is an element of $\Phi_n$ which takes $(v_0, v)$ to one of the following possibilities, each of which lie in different orbits. There are three possible positions for $d_j$ and $d_k$, either both lie to the right of $d_i$, there is one on either side, or they are both to the left.

**Proposition 4.** The pure Hilden group $\text{PH}_{2n}$ is generated by $p_{ij}$, $t_i$, $x_{ij}$ and $y_{ij}$.

**Proof.** The group $\text{PH}_{2n}$ is generated by the generators of the vertex stabilizer and the set $\{r_\lambda\}$. We have that

$$\{r_\lambda\} = \{x_{ij}, x_{ij}^{-1}, y_{ij}, y_{ij}^{-1} \mid i < j\} \cup \{x_{ij}, x_{ik}, x_{ij}^{-1} x_{ik}^{-1}, x_{jk}, y_{ij}, y_{ij}^{-1} x_{jk}^{-1}, y_{ik}, y_{ik}^{-1} y_{jk}^{-1} \mid i < j < k\}$$

and so all of these generators either are contained in $S$ or can be written in terms of the elements of $S$. $\square$
4. Action of the framed braid group

We have an embedding of the framed braid group on $n$ strings $\text{FB}_n$ in the braid group on $2n$ strings given as follows.

This makes $\text{FB}_n$ a subgroup of $H_{2n}$. It is clear that conjugation by elements of $\text{FB}_n$ preserves the pure Hilden group and hence we have a left action of $\text{FB}_n$ on $\text{PH}_{2n}$. In fact, this action can be defined on the level of reduced words as well. In other words, we have an action of $\langle \sigma_i, \tau_j \rangle$, the free group on the letters $\sigma_i$ and $\tau_j$, on $\langle p_{ij}, x_{ij}, y_{ij}, t_k \rangle$, the free group on the letters $p_{ij}, x_{ij}, y_{ij}, t_k$.

In Section 8, we will construct this homomorphism $\Phi: \langle \sigma_i, \tau_j \rangle \to \text{Aut}(\langle p_{ij}, x_{ij}, y_{ij}, t_k \rangle)$, which we will write as $g \mapsto \Phi_g$, and then show that it satisfies the following properties.

We will write $x = B_{2n} y$ when the words $x$ and $y$ have the same image in $\langle S \mid R \rangle$.

For any word $g \in \langle \sigma_i, \tau_j \rangle$,

(A) the automorphism $\Phi_g$ acts by conjugation at the level of braids,

$$\Phi_g(x) = B_{2n} g x g^{-1} \text{ for each } x \in \langle p_{ij}, x_{ij}, y_{ij}, t_k \rangle$$

(B) the automorphism $\Phi_g$ preserves the subgroup generated by the $p_{ij}$ and $t_k$,

$$\Phi_g(h) \in \langle p_{ij}, t_k \rangle \text{ for each } h \in \langle p_{ij}, t_k \rangle$$

(C) for each $\lambda$ there exists $h_1, h_2 \in \langle p_{ij}, t_k \rangle$ and $r_{\lambda'}$ such that $\Phi_g(r_{\lambda}) = R_{1} h_1 r_{\lambda'} h_2$,

(D) if $x = R_{1} y$ then $\Phi_g(x) = R_{1} \Phi_g(y)$.

We will now assume the existence of such a $\Phi$ and use it to show that $R_1, R_2$ and $R_3$ relations follow from those in $R$.

5. The $R_1$ relations

The $R_1$ relations consist of a relation of the form $r_{\lambda} t_{\lambda}^{-1} = h$ for each edge orbit representative $(v_0, v_0 \cdot r_{\lambda})$, for each $t$ in a generating set of the stabilizer of this edge and for some word $h$ in $\text{FB}_n$.

**Proposition 5.** The stabilizer of the edge $(v_0, v_0 \cdot x_{12})$ is generated as follows.

$$\text{Stab}(v_0, v_0 \cdot x_{12}) = \langle p_{ij} \text{ for } i, j > 2, \quad t_k \text{ for } k > 1, \quad p_{12} t_1, \quad p_{1j} p_{2j} \text{ for } j > 2 \rangle$$

**Proof.** The stabilizer can be viewed as the mapping class group of the disc fixing pointwise both its boundary and the arcs shown in Figure 2. The two arcs connecting the first two punctures form a loop $l$. As this loop is fixed by elements of the stabilizer, this allows us to split the group into the product of two (boundary fixing) mapping class groups. One corresponding to the inside of $l$ and the other to the outside.

For the inside, we get the mapping class group of the annulus, which is cyclic and generated by $t_2$. 
For the outside, we get the mapping class group of a disc with \( n - 1 \) subdiscs removed. This is \( \text{FP}_{n-1} \) the framed pure braid group on \( n - 1 \) strings. The framing of the subdisc bounded by the loop \( l \) is generated by \( p_{12}t_1t_2 \) and for the other subdiscs by \( t_k \) for \( k > 2 \). The braiding of the subdiscs are generated by \( p_{1j}p_{2j} \) for \( j > 2 \) and \( p_{ij} \) for \( i, j > 2 \).

So the \( R_1 \) relations can be chosen as follows.

\[
\begin{align*}
(5.1) & \quad x_{12} p_{ij} x_{12}^{-1} = p_{ij} \quad \text{for } 2 < i < j \\
(5.2) & \quad x_{12} t_k x_{12}^{-1} = t_k \quad \text{for } k > 1 \\
(5.3) & \quad x_{12} p_{12} t_1 x_{12}^{-1} = p_{12} t_1 \\
(5.4) & \quad x_{12} p_{1j} p_{2j} x_{12}^{-1} = p_{1j} p_{2j} \quad \text{for } j > 2
\end{align*}
\]

Relation (5.1) follows from (C1), relation (5.2) follows from (C-xt), relation (5.3) follows from (C-xpt) and relation (5.4) follows from (C2).

**Proposition 6.** The stabilizer of the edge \( (v_0, v_0 \cdot x_{12} x_{13}) \) is generated as follows.

\[
\text{Stab}(v_0, v_0 \cdot x_{12} x_{13}) = \left\langle p_{23}, \ p_{12} p_{13} t_1, \ p_{ij} \quad \text{for } i, j > 3, \right. \\
\left. t_k \quad \text{for } k > 1 , \ p_{1j} p_{2j} p_{3j} \quad \text{for } j > 3 \right\rangle
\]

**Proof.** As with the previous proposition, the stabilizer can be viewed as the mapping class group of a disc fixing pointwise both its boundary and the arcs shown in Figure 3. Again, we have a loop \( l \) formed from the two arcs joining the first two punctures, which allows us to split the group into a product of two mapping class groups.
Inside \( l \) we get the mapping class group of a disc with two subdiscs removed. This is \( \text{FP}_2 \) the framed pure braid group on two strings. The framing of the strings is generated by \( t_2 \) and \( t_3 \) and the braiding is generated by \( p_{23} \).

Outside \( l \) we get the mapping class group of a disc with \( n-2 \) subdiscs removed, i.e. \( \text{FP}_{n-2} \). The framing of the subdisc bounded by \( l \) is generated by \( p_{12} p_{13} t_1 \) \( t_2 t_3 \) and the framing of the other subdiscs is given by \( t_k \) for \( k > 3 \). The braiding of the subdiscs are generated by \( p_{1j} p_{2j} p_{3j} \) for \( j > 3 \) and \( p_{ij} \) for \( i, j > 3 \).

Hence the \( R_1 \) relations can be chosen as follows.

\[
\begin{align*}
(5.5) \quad & x_{12} x_{13} p_{23} (x_{12} x_{13})^{-1} = p_{23} \\
(5.6) \quad & x_{12} x_{13} p_{ij} (x_{12} x_{13})^{-1} = p_{ij} \quad \text{for } i, j > 3 \\
(5.7) \quad & x_{12} x_{13} t_k (x_{12} x_{13})^{-1} = t_k \quad \text{for } k > 1 \\
(5.8) \quad & x_{12} x_{13} p_{12} p_{13} t_1 (x_{12} x_{13})^{-1} = p_{12} p_{13} t_1 \\
(5.9) \quad & x_{12} x_{13} p_{1j} p_{2j} p_{3j} (x_{12} x_{13})^{-1} = p_{1j} p_{2j} p_{3j} \quad \text{for } j > 3
\end{align*}
\]

Relation (5.5) follows from (C2), relation (5.6) follows from two applications of (C1), relation (5.7) follows from two applications of (C-xt). Relation (5.8) follows from the following.

\[
x_{12} x_{13} p_{12} p_{13} t_1
\]

\[
\overset{(C2)}{= x_{12} x_{13} p_{13} p_{23} p_{12} p_{23}^{-1} t_1} = x_{12} x_{13} p_{13} t_1 p_{23} p_{12} p_{23}^{-1} = x_{12} p_{13} t_1 x_{13} p_{23} p_{12} p_{23}^{-1}
\]

\[
\overset{(C2)}{=} x_{12} p_{13} t_1 p_{23} p_{12} x_{13} p_{23}^{-1} = x_{12} p_{13} p_{23} p_{12} t_1 x_{13} p_{23}^{-1} = p_{13} p_{23} x_{12} p_{12} t_1 x_{13} p_{23}^{-1}
\]

\[
\overset{(C2)}{=} p_{13} p_{23} p_{12} t_1 x_{13} x_{13} p_{23}^{-1} = p_{13} p_{23} p_{12} t_1 x_{12} x_{13} x_{23}^{-1} x_{12} x_{13} = p_{12} p_{13} t_1 x_{12} x_{13}
\]

Finally, (5.9) follows from the following.

\[
\overset{(C2)}{= p_{1k} p_{3k} x_{13} p_{3k}^{-1} p_{2k} p_{3k}}
\]

\[
\overset{(C2)}{=} p_{1k} p_{3k} x_{13} p_{23} p_{2k} p_{23}^{-1} = p_{1k} p_{3k} p_{23} p_{2k} p_{23}^{-1} x_{13} = p_{1k} p_{2k} p_{3k} x_{13}
\]

\[
\overset{(C2)}{= p_{1k} p_{2k} p_{3k} x_{12} p_{3k}} = p_{1k} p_{2k} p_{3k} x_{12}
\]
Now consider the edge orbit representative \((v_0, v_0 \cdot r_\lambda)\) for \(r_\lambda \neq x_{12} \) or \(x_{12} x_{13}\). There exist some \(g \in \mathbb{FB}_n\) such that \((v_0, v_0 \cdot r_1) \cdot g = (v_0, v_0 \cdot r_\lambda)\), where \(r_1 = x_{12} \) or \(x_{12} x_{13}\). By property (A) of \(\Phi\), we have \(\Phi_{g^{-1}}(r_1) = \Phi_{g^{-1}}(v_0) = g^{-1} r_1 g\) and by property (C) there exists words \(h_1, h_2 \in \mathbb{FP}_n\) and some \(r_{\lambda'}\) such that

\[
(5.10) \quad \Phi_{g^{-1}}(r_1) = R h_1 r_{\lambda'} h_2.
\]

Combining these we see that \(v_0 \cdot r_1 g = v_0 \cdot r_{\lambda'} h_2\) and hence that \(\lambda = \lambda'\) and \(h_2 \in \text{Stab}(v_0, v_0 \cdot r_\lambda)\).

Let \(T\) be the choice of generators for \(\text{Stab}(v_0, v_0 \cdot r_1)\) chosen above. So for all \(t \in T\) there exists \(h \in \mathbb{FP}_n\) such that \(r_1 t r_1^{-1} = R h\). So by property (D) we have

\[
(5.11) \quad \Phi_{g^{-1}}(r_1 t r_1^{-1}) = R \Phi_{g^{-1}}(h).
\]

Property (B) implies that \(\Phi_{g^{-1}}(t) \in \mathbb{FP}_n\) and \(\Phi_{g^{-1}}(h) \in \mathbb{FP}_n\). Combining (5.10) and (5.11) we get

\[
\begin{align*}
    h_1 r_\lambda h_2 \Phi_{g^{-1}}(t) h_2^{-1} r_{\lambda'}^{-1} h_1^{-1} = & R \Phi_{g^{-1}}(h) \\
\end{align*}
\]

and so \(h_2 \Phi_{g^{-1}}(t) h_2^{-1} \in \text{Stab}(v_0, v_0 \cdot r_\lambda)\).

**Claim.** The set \(\{h_2 \Phi_{g^{-1}}(t) h_2^{-1} \mid t \in T\}\) generates \(\text{Stab}(v_0, v_0 \cdot r_\lambda)\).

**Proof.** As \(h_2 \in \text{Stab}(v_0, v_0 \cdot r_\lambda)\) the set \(\{h_2 \Phi_{g^{-1}}(t) h_2^{-1} \mid t \in T\}\) generates \(\text{Stab}(v_0, v_0 \cdot r_\lambda)\) if and only if the set \(\{\Phi_{g^{-1}}(t) \mid t \in T\}\) generates \(\text{Stab}(v_0, v_0 \cdot r_\lambda)\). This is equivalent to saying that for any \(s \in \text{Stab}(v_0, v_0 \cdot r_\lambda)\) we can find \(t_1, \ldots, t_k \in T\) such that \(s = \Phi_{g^{-1}}(t_1 \ldots t_k)\), in other words that \(\Phi_{g}(s) \in \text{Stab}(v_0, v_0 \cdot r_1)\). Now

\[
(v_0 \cdot r_1) \cdot \Phi_{g}(s) = v_0 \cdot r_1 g s g^{-1} = v_0 \cdot r_{\lambda'} g^{-1} = v_0 \cdot r_\lambda g^{-1} = v_0 \cdot r_1.
\]

Therefore the claim holds. \(\square\)

So for our \(R_1\) relations we can choose

\[
    r_\lambda h_2 \Phi_{g^{-1}}(t) h_2^{-1} r_{\lambda'}^{-1} = h_1^{-1} \Phi_{g^{-1}}(h) h_1
\]

and hence we can choose our \(R_1\) relations so that they all follow from \(R\).

6. The \(R_2\) relations

The \(R_2\) relations consist of a relation of the form \(r_{\lambda'} h r_\lambda = h'\) for each edge orbit representative, where the left-hand side (LHS) is an \(h\)-product for the path \((v_0, v_0 \cdot r_\lambda, v_0)\) and \(h' \in \mathbb{FB}_n\). For each edge \((v_0, v_0 \cdot r_\lambda)\) the edge \((v_0, v_0 \cdot r_\lambda^{-1})\) is in a different orbit. Our choice of \(r_\lambda\) means that for all \(\lambda\) there exists \(\lambda'\) such that \(r_{\lambda}^{-1} = r_{\lambda'}\). This means that for all the \(R_2\) relations we can choose \(r_{\lambda}^{-1} r_\lambda = 1\), i.e. they are all trivial.

7. The \(R_3\) relations

The \(R_3\) relations consist of a relation of the form \(r_{\lambda_k} h_k \ldots r_{\lambda_i} h_1 = h\) for each face orbit representative, where the LHS is an \(h\)-product that represents the boundary of
the face and \( h \in \mathbb{F}_n \). As with the \( R_1 \) relations, we will calculate the relations for some specific orbits first then use \( \Phi \) for the general case. There are three types of faces: triangular, non-nested rectangular and nested rectangular.

We will start with the triangular face \((v_0, v_0 \cdot x_{12} x_{13}, v_0 \cdot x_{13} v_0)\). An \( h \)-product for this path is \( x_{13}^{-1} x_{12}^{-1} (x_{12} x_{13}) \). So the \( R_3 \) relations is \( x_{13}^{-1} x_{12}^{-1} (x_{12} x_{13}) = 1 \) and so it is trivial.

Next consider the non-nested rectangular face \((v_0, v_0 \cdot x_{12}, v_0 \cdot x_{34} x_{12}, v_0 \cdot x_{34}, v_0)\). An \( h \)-product that represents this path is \( x_{34}^{-1} x_{12}^{-1} x_{34} x_{12} \). So the \( R_3 \) relations is \( x_{34}^{-1} x_{12}^{-1} x_{34} x_{12} = 1 \), which follows from (C1).

Now consider the nested rectangular face \((v_0, v_0 \cdot x_{23}, v_0 \cdot x_{12} x_{13}, v_0 \cdot x_{12} x_{13} x_{23}, v_0 \cdot x_{12} x_{13} x_{23}, v_0)\). An \( h \)-product that represents this path is \( (x_{12} x_{13})^{-1} x_{23}^{-1} (x_{12} x_{13}) x_{23} \). So the \( R_3 \) relations is \( (x_{12} x_{13})^{-1} x_{23}^{-1} (x_{12} x_{13}) x_{23} = 1 \), which follows from (C2).

Given any other face orbit representative \((v_0 = u_0, u_1, \ldots, u_k = v_0)\) then by the classification of face orbits given in [6] there exist some \( g \in \mathbb{F}_n \) such that \((u_0, u_1, \ldots, u_k) = (v_0, v_1, \ldots, v_k) \cdot g \) where \((v_0, v_1, \ldots, v_k)\) is the boundary of one of the three faces whose \( R_3 \) relations we calculated above. Suppose the relation from \((v_0, v_1, \ldots, v_k)\) is the following.

\[
\lambda_k h_k \ldots \lambda_1 h_1 = h
\]

By property (C), for each \( \lambda_i \) there exists \( h_{i1}, h_{i2} \in \mathbb{F}_n \) and \( \lambda_i' \) such that

\[
\Phi_{g^{-1}}(\lambda_i) = R h_{i1} \lambda_i' h_{i2}.
\]

**Claim.** The following \( h \)-product represents the path \((u_0, u_1, \ldots, u_k)\).

\[
r_{\lambda_i'} h_{k2} \Phi_{g^{-1}}(h_k) h_{(k-1)1} \ldots r_{\lambda_1} h_{11} \Phi_{g^{-1}}(h_1)
\]

**Proof.** The \( i \)th vertex of the path associated to the \( h \)-product is given as follows.

\[
v_0 \cdot r_{\lambda_i'} h_{i2} \Phi_{g^{-1}}(h_i) h_{(i-1)1} \ldots r_{\lambda_1} h_{11} \Phi_{g^{-1}}(h_1)
= v_0 \cdot \Phi_{g^{-1}}(r_{\lambda_i} h_i \ldots r_{\lambda_1} h_1)
= v_0 \cdot r_{\lambda_i} h_i \ldots r_{\lambda_1} h_1 g
= v_i \cdot g
= u_i
\]

\[\Box\]
Therefore, for our $R_3$ relation we may choose
\[ r'_{k_1} h_{k_2} \Phi_{y^{-1}}(h_{k_1}) h_{(k_1-1)} \cdots r'_{k_1} h_{11} \Phi_{y^{-1}}(h_{1}) = h_{k_1}^{-1} \Phi_{y^{-1}}(h), \]
which follows from $R$ by property (D).

8. Construction and properties of $\Phi$

All that remains to prove Theorem 1 is to construct $\Phi$ and show that it satisfies properties (A)–(D).

Define $\Phi$, the action of $F\langle \sigma_i, \tau_j \rangle$ on $F\langle p_{ij}, x_{ij}, y_{ij}, t_k \rangle$, as follows. For $\alpha \in \{p, x, y\}$
\[
\begin{align*}
\Phi_{\sigma_i}(\alpha_{kl}) &= \alpha_{kl} & \text{for } i \neq k-1, k, l-1, l \\
\Phi_{\sigma_i}(\alpha_{ij}) &= \alpha_{i+1,j} & \text{for } i + 1 < j \\
\Phi_{\sigma_i}(\alpha_{i+1,j}) &= p_{i,i+1} \alpha_{ij} p_{i,i+1}^{-1} & \text{for } i + 1 < j \\
\Phi_{\tau_j}(\alpha_{i,j+1}) &= p_{j,j+1} \alpha_{ij} p_{j,j+1}^{-1} & \text{for } i + 1 < j \\
\Phi_{\sigma_i}(p_{i,i+1}) &= p_{i,i+1} & \Phi_{\sigma_i}(x_{i,i+1}) = x_{i,i+1} \\
\Phi_{\sigma_i}(y_{i,i+1}) &= t_{i+1}^{-1} y_{i,i+1} t_{i+1} & \Phi_{\sigma_i}(y_{i,i+1}) = x_{i,i+1} \\
\Phi_{\sigma_i}(t_j) &= \begin{cases} t_j & \text{if } j \neq i, i + 1 \\ t_{j+1} & \text{if } j = i \\ t_i & \text{if } j = i + 1 \end{cases} \\
\Phi_{\tau_i}(p_{kl}) &= p_{kl} \\
\Phi_{\tau_i}(x_{kl}) &= \begin{cases} x_{kl} & \text{if } i \neq k \\ x_{kl}^{-1} p_{kl} & \text{if } i = k \end{cases} & \text{for } k < l \\
\Phi_{\tau_i}(y_{kl}) &= \begin{cases} y_{kl} & \text{if } i \neq l \\ y_{kl}^{-1} p_{kl} & \text{if } i = l \end{cases} & \text{for } k < l \\
\Phi_{\tau_i}(t_j) &= t_j
\end{align*}
\]

Proposition 7. The map $\Phi$ is a well-defined action of $F(\tau_i, \sigma_i)$ on $F(p_{ij}, t_i, x_{ij}, y_{ij})$.

Proof. All that needs to be checked is that $\Phi_{\sigma_i}$ and $\Phi_{\tau_i}$ are invertible. The inverses are as follows.
\[
\begin{align*}
\Phi_{\sigma_i}^{-1}(\alpha_{kl}) &= \alpha_{kl} & \text{for } i \neq k-1, k, l-1, l \\
\Phi_{\sigma_i}^{-1}(\alpha_{ij}) &= p_{i,i+1}^{-1} \alpha_{i+1,j} p_{i,i+1} & \text{for } i + 1 < j \\
\Phi_{\sigma_i}^{-1}(\alpha_{i+1,j}) &= \alpha_{ij} & \text{for } i + 1 < j \\
\Phi_{\sigma_i}^{-1}(\alpha_{i,j+1}) &= \alpha_{ij} & \text{for } i + 1 < j \\
\Phi_{\tau_i}^{-1}(\alpha_{ij}) &= p_{j,j+1}^{-1} \alpha_{i,j+1} p_{j,j+1} & \text{for } i + 1 < j \\
\Phi_{\tau_i}^{-1}(p_{kl}) &= p_{kl} \\
\Phi_{\tau_i}^{-1}(x_{kl}) &= \begin{cases} x_{kl} & \text{if } i \neq k \\ x_{kl} p_{kl}^{-1} & \text{if } i = k \end{cases} & \text{for } k < l \\
\Phi_{\tau_i}^{-1}(y_{kl}) &= \begin{cases} y_{kl} & \text{if } i \neq l \\ y_{kl} p_{kl}^{-1} & \text{if } i = l \end{cases} & \text{for } k < l \\
\Phi_{\tau_i}^{-1}(t_j) &= t_j
\end{align*}
\]
\[ \Phi_{\sigma_i}^{-1}(p_{ki,i+1}) = p_{ki,i+1} \]
\[ \Phi_{\sigma_i}^{-1}(x_{i,i+1}) = y_{i,i+1} \]
\[ \Phi_{\sigma_i}^{-1}(y_{i,i+1}) = t_i x_{i,i+1} t_i^{-1} \]

\[ \Phi_{\tau_i}^{-1}(t_j) = t_j \]
\[ \Phi_{\tau_i}^{-1}(t_j) = \begin{cases} 
  t_j & \text{if } j \neq i, i + 1 \\
  t_{j+1} & \text{if } j = i \\
  t_{j-1} & \text{if } j = i + 1 
\end{cases} \]
\[ \Phi_{\tau_i}^{-1}(p_{kl}) = p_{kl} \]
\[ \Phi_{\tau_i}^{-1}(x_{kl}) = \begin{cases} 
  x_{kl} & \text{if } i \neq k \\
  p_{kl} x_{kl}^{-1} & \text{if } i = k 
\end{cases} \quad \text{for } k \leq l \]
\[ \Phi_{\tau_i}^{-1}(y_{kl}) = \begin{cases} 
  y_{kl} & \text{if } i \neq l \\
  p_{kl} y_{kl}^{-1} & \text{if } i = l 
\end{cases} \quad \text{for } k \leq l \]

We will need the following lemma.

**Lemma 8.** For \( x \in F(p_{ij}, t_i, x_{ij}, y_{ij}) \) we have
\[ \Phi_{\sigma_m}^{-1}(x) = R p_{m,m+1}^{-1} x p_{m,m+1} \quad \Phi_{\tau_m}^{-1}(x) = R t_m^{-1} x t_m. \]

It is easy to check the \( \Phi \) satisfies property (A), i.e. that for every word \( g \in F(\sigma_i, \tau_j) \) and for each \( x \in F(p_{ij}, t_i, x_{ij}, y_{ij}) \) we have that \( \Phi_g(x) = g x g^{-1} \) as braids. It is also clear that \( \Phi \) satisfies property (B). That is that for any word \( g \in F(\sigma_i, \tau_j) \) and for any word \( h \in F(p_{ij}, t_k) \) we have \( \Phi_g(h) \in F(p_{ij}, t_k) \).

**Proposition 9.** The map \( \Phi \) satisfies property (C), i.e. for any word \( g \in F(\sigma_i, \tau_j) \) and any \( r_\lambda \) we have a relation \( \Phi_g(r_\lambda) = h_1 r_\lambda^1 h_2 \) for some \( h_1, h_2 \in F(p_{ij}, t_k) \) and some \( r_\lambda^1 \) that can be deduced from the relations in \( R \).

**Proof.** First note that for each word \( h \in F(p_{ij}, t_k) \), by property (B), the map \( \Phi_g \) takes \( h \) to another word in \( F(p_{ij}, t_k) \). Therefore, it suffices to check \( \Phi_g \) for \( g = \tau_m, \sigma_m, \tau_m^{-2} \) and \( \sigma_m^{-2} \). By Lemma 8, property (C) is satisfied for \( g = \tau_m^{-2} \) and \( \sigma_m^{-2} \).

For \( r_\lambda = x_{ij}, x_{ij}^{-1}, y_{ij}, y_{ij}^{-1} \), this follows immediately from the definition of \( \Phi \) given above.

Now consider \( \Phi_{\sigma_m}(r_\lambda) \) for \( r_\lambda = x_{ij} x_{ik}, x_{jk} y_{ij} \) or \( y_{ik} y_{jk} \). The only cases when \( \Phi_{\sigma_m}(r_\lambda) \neq r_\lambda \) are \( m = i - 1, m = i \) and \( j = i + 1, j = i + 1, j = i + 1, m = j - 1 \) and \( i < j - 1, m = j \) and \( k = j + 1, m = j \) and \( j < k - 1, k < k - 1, m = k \).

\[ m = i - 1 \]
\[ \Phi_{\sigma_{i-1}}(x_{ij} x_{ik}) = p_{i-1,i} x_{i-1,j} x_{i-1,k} p_{i-1,i}^{-1} \]
\[ \Phi_{\sigma_{i-1}}(x_{jk} y_{ij}) = \frac{x_{jk} p_{i-1,i} y_{i-1,j} p_{i-1,i}^{-1}}{x_{jk} y_{i-1,j} p_{i-1,i}^{-1}} \]
\[ \Phi_{\sigma_{i-1}}(y_{ik} y_{jk}) = \frac{p_{i-1,i} y_{i-1,k} p_{i-1,i}^{-1} y_{jk}}{p_{i-1,i} y_{i-1,k} y_{jk} p_{i-1,i}^{-1}} \] (C1)
\[ m = i \text{ and } j = i + 1 \]
\[
\Phi_{\sigma_i}(x_{ij} x_{ik}) = t_j^{-1} y_{ij} t_j x_{jk}
\]  
\[= t_j^{-1} y_{ij} p_{jk}^{-1} x_{jk} p_{jk} t_j \]  
\[= t_j^{-1} p_{jk}^{-1} p_{ik}^{-1} y_{ij} p_{ik} x_{jk} p_{jk} t_j \]  
\[= t_j^{-1} p_{jk}^{-1} x_{jk} y_{ij} p_{jk} t_j \]  
\[\Phi_{\sigma_i}(x_{jk} y_{ij}) = p_{ij} x_{ik} p_{ij}^{-1} x_{ij} \]  
\[= p_{jk}^{-1} x_{ij} x_{ik} p_{jk} \]  
\[\Phi_{\sigma_i}(y_{ik} y_{jk}) = y_{ik} p_{ij} y_{ik} p_{ij}^{-1} \]  
\[= y_{ik} y_{jk} \]

\[ m = i \text{ and } j > i + 1 \]
\[
\Phi_{\sigma_i}(x_{ij} x_{ik}) = x_{i+1,j} x_{i+1,k} \]
\[\Phi_{\sigma_i}(x_{jk} y_{ij}) = x_{jk} y_{i+1,j} \]
\[\Phi_{\sigma_i}(y_{ik} y_{jk}) = y_{i+1,k} y_{jk} \]

\[ m = j-1 \text{ and } i < j - 1 \]
\[
\Phi_{\sigma_{j-1}}(x_{ij} x_{ik}) = p_{j-1,j} x_{i,j-1} p_{j-1,j}^{-1} x_{ik} \]  
\[= p_{j-1,j} x_{i,j-1} x_{ik} p_{j-1,j}^{-1} \]  
\[\Phi_{\sigma_{j-1}}(x_{jk} y_{ij}) = p_{j-1,j} x_{j-1,k} y_{i,j-1} p_{j-1,j}^{-1} \]  
\[\Phi_{\sigma_{j-1}}(y_{ik} y_{jk}) = y_{ik} p_{j-1,j} y_{j-1,k} p_{j-1,j}^{-1} \]  
\[= p_{j-1,j} y_{ik} y_{j-1,k} p_{j-1,j}^{-1} \]

\[ m = j \text{ and } k = j + 1 \]
\[
\Phi_{\sigma_j}(x_{ij} x_{ik}) = x_{ik} p_{jk} x_{ij} p_{jk}^{-1} \]  
\[= x_{ij} x_{ik} \]
\[\Phi_{\sigma_j}(x_{jk} y_{ij}) = t_k^{-1} y_{jk} t_k y_{ik} \]  
\[= t_k^{-1} p_{jk} t_k y_{jk} t_k^{-1} p_{jk}^{-1} t_k y_{ik} \]  
\[= p_{jk} y_{jk} p_{jk}^{-1} y_{ik} \]  
\[= p_{jk} y_{jk} p_{jk}^{-1} y_{ik} p_{ik}^{-1} p_{jk}^{-1} \]  
\[= p_{jk} y_{ik} y_{jk} p_{jk}^{-1} \]  
\[\Phi_{\sigma_j}(y_{ik} y_{jk}) = p_{jk} y_{ij} p_{jk}^{-1} x_{jk} \]  
\[= p_{jk} y_{ij} p_{ik} x_{jk} \]  
\[= x_{jk} y_{ij} \]  

\[ m = j \text{ and } k > j + 1 \]
\[
\Phi_{\sigma_j}(x_{ij} x_{ik}) = x_{i,j+1} x_{ik} \]
\[\Phi_{\sigma_j}(x_{jk} y_{ij}) = x_{j+1,k} y_{i,j+1} \]
\[\Phi_{\sigma_j}(y_{ik} y_{jk}) = y_{ik} y_{j+1,k} \]
\[ m = k - 1 \text{ and } j < k - 1 \quad \Phi_{\sigma_{k-1}}(x_{ij} x_{ik}) = x_{ij} p_{k-1,k} x_{i,k-1} p_{k-1,k}^{-1} \quad \Phi_{\sigma_{k-1}}(x_{jk} y_{ij}) = x_{jk} p_{k-1,k} x_{j,k-1} p_{k-1,k}^{-1} y_{ij} \quad \Phi_{\sigma_{k-1}}(y_{ik} y_{jk}) = p_{k-1,k} y_{i,k-1} y_{j,k-1}^{-1} p_{k-1,k}^{-1} \]

\[ m = k \quad \Phi_{\sigma_k}(x_{ij} x_{ik}) = x_{ij} x_{i,k+1}, \quad \Phi_{\sigma_k}(x_{jk} y_{ij}) = x_{j,k+1} y_{ij}, \quad \Phi_{\sigma_k}(y_{ik} y_{jk}) = y_{i,k+1} y_{j,k+1}. \]

For \( \Phi_{\tau_m} \) we only have three cases where \( \Phi_{\tau_m}(r_{\lambda}) \neq r_{\lambda} \) these are when \( m = i \) and \( r_{\lambda} = x_{ij} x_{ik}, m = j \) and \( r_{\lambda} = x_{jk} y_{ij}, \) and \( m = k \) and \( r_{\lambda} = y_{ik} y_{jk}. \)

\[ \Phi_{\tau_1}(x_{ij} x_{ik}) = x_{ij}^{-1} p_{ij} x_{ik}^{-1} p_{ik} = x_{ij}^{-1} p_{ij}^{-1} x_{ik}^{-1} p_{ik} = x_{ij}^{-1} x_{ik}^{-1} p_{ij} p_{ik} \]

\[ \Phi_{\tau_2}(x_{jk} y_{ij}) = x_{jk}^{-1} p_{jk}^{-1} y_{ij} p_{ij} = x_{jk}^{-1} p_{jk}^{-1} y_{ij} p_{ij} = y_{ij}^{-1} x_{jk}^{-1} p_{jk} p_{ij} \]

\[ \Phi_{\tau_3}(y_{ik} y_{jk}) = y_{ik}^{-1} p_{ik}^{-1} y_{jk}^{-1} p_{jk} = y_{ik}^{-1} p_{ik}^{-1} y_{jk}^{-1} p_{jk} = y_{ik}^{-1} y_{jk}^{-1} p_{ik} p_{jk} \]

For \( r_{\lambda} = x_{ik}^{-1}, y_{ij}^{-1} x_{ik}^{-1} \) and \( y_{ij}^{-1} y_{ik}^{-1} \) we have shown that for some \( h_1, h_2 \in \mathbb{F}_n \) and some \( r_{\lambda}' \) we have that \( \Phi_g(r_{\lambda}') = h_1 r_{\lambda}' h_2. \) Hence, we have \( \Phi_g(r_{\lambda}) = R \)

For \( r_{\lambda} = x_{ik}^{-1}, y_{ij}^{-1} x_{ik}^{-1} \) and \( y_{ij}^{-1} y_{ik}^{-1} \) we have shown that for some \( h_1, h_2 \in \mathbb{F}_n \) and some \( r_{\lambda}' \) we have that \( \Phi_g(r_{\lambda}') = h_1 r_{\lambda}' h_2. \) Hence, we have \( \Phi_g(r_{\lambda}) = R \)

\[ \Phi_{\tau_m}(r_{\lambda}) \neq r_{\lambda} \text{ if } m = i \text{ and } \Phi_{\tau_m}(r_{\lambda}) \neq r_{\lambda} \text{ if } m = j \text{ and } \Phi_{\tau_m}(r_{\lambda}) \neq r_{\lambda} \text{ if } m = k. \]

**Proposition 10.** The map \( \Phi \) satisfies property (D). In other words, for any word \( g \in F(\sigma_i, \tau_j) \) and any relation \( x = R y \) we have that \( \Phi_g(x) = R \Phi_g(y). \)

**Proof.** As in the proof of property (C), it suffices to show this for \( g \) in a monoidal generating set for \( F(\sigma_i, \tau_j). \) For \( g = \sigma_i^{-2} \) and \( \tau_j^{-2} \) this follows from Lemma 8, so it remains to show it for \( g = \sigma_i \) and \( \tau_j. \)

For any relation only involving \( p_{ij} \)'s and \( t_k \)'s the image under \( \Phi_g \) will still only involve \( p_{ij} \)'s and \( t_k \)'s and hence, by Section 2, the new relation will follow from those in \( R. \)

We will now consider the action of \( \Phi_{\sigma_q} \) and \( \Phi_{\tau_q} \) on each of the relations. For any relation \( x = R y, \) we will say that the deduction of \( \Phi_g(x) = \Phi_g(y) \) is trivial if \( \Phi_g(x) = \Phi_g(y) \) is a relation in \( R \) of the same type.

**Case 1.** (C-xt) \( x_{ij} t_k = t_k x_{ij} \) \( k \neq i \), \( i < j \)

First consider \( \Phi_{\sigma_q}. \) Start with \( q = 1 \) and then increase it. The first non-trivial case is when \( q = i - 1. \) The next case is when \( q = i \) and this is only non-trivial if \( j = i + 1. \) The next case is when \( q = j - 1 \) and \( j \neq i + 1. \) The remaining values are all trivial.
When \( q = i - 1 \) we have that \( \Phi_{\sigma_q}(t_k) = t_k' \) where \( k' \neq i - 1 \).

\[
\Phi_{\sigma_q}(x_{ij} \ t_k) = p_{i-1,i} x_{i-1,j} p_{i-1,i}^{-1} t_k' \tag{C-xt} = p_{i-1,i} x_{i-1,j} t_k' p_{i-1,i}^{-1}
\]

When \( q = i \) and \( j = i + 1 \) we have that \( \Phi_{\sigma_q}(t_k) = t_k' \) where \( k' \neq j \).

\[
\Phi_{\sigma_q}(x_{ij} \ t_k) = t_j^{-1} y_{ij} t_j \tag{C-tt} = t_j^{-1} y_{ij} t_j' \tag{C-xt} = t_k' t_j^{-1} y_{ij} t_j = \Phi_{\sigma_q}(t_k x_{ij})
\]

When \( q = j - 1 \) and \( j \neq i + 1 \) we have that \( \Phi_{\sigma_q}(t_k) = t_k' \) where \( k' \neq i \).

\[
\Phi_{\sigma_q}(x_{ij} \ t_k) = p_{j-1,j} x_{i,j-1} p_{j-1,j}^{-1} t_k' \tag{C-xt} = p_{j-1,j} x_{i,j-1} t_k' p_{j-1,j}^{-1}
\]

Now consider \( \Phi_{\tau_q} \), the only non-trivial case is when \( q = i \).

\[
\Phi_{\tau_q}(x_{ij} \ t_k) = x_{ij}^{-1} p_{ij} t_k \tag{C-pi} = x_{ij}^{-1} t_k p_{ij} \tag{C-xt} = t_k x_{ij}^{-1} p_{ij} = \Phi_{\tau_q}(t_k x_{ij})
\]

**Case 2.** (C-yt) \( y_{ij} t_k = t_k y_{ij} \ \ k \neq j, \ i < j \)

First consider \( \Phi_{\sigma_q} \), the non-trivial cases are \( q = i - 1, q = i \) and \( j = i + 1 \), and \( q = j - 1 \) and \( j \neq i + 1 \).

When \( q = i - 1 \), we have that \( \Phi_{\sigma_q}(t_k) = t_k' \), where \( k' \neq j \).

\[
\Phi_{\sigma_q}(y_{ij} \ t_k) = p_{i-1,i} y_{i-1,j} p_{i-1,i}^{-1} t_k' \tag{C-xt} = p_{i-1,i} y_{i-1,j} t_k' p_{i-1,i}^{-1}
\]

When \( q = i \) and \( j = i + 1 \), we have that \( \Phi_{\sigma_q}(t_k) = t_k' \) where \( k' \neq i \).

\[
\Phi_{\sigma_q}(y_{ij} \ t_k) = x_{ij} t_k' \tag{C-xt} = t_k' x_{ij} = \Phi_{\sigma_q}(t_k y_{ij})
\]

When \( q = j - 1 \) and \( j \neq i + 1 \), we have that \( \Phi_{\sigma_q}(t_k) = t_k' \) where \( k' \neq j - 1 \).

\[
\Phi_{\sigma_q}(y_{ij} \ t_k) = p_{j-1,j} y_{i,j-1} p_{j-1,j}^{-1} t_k' \tag{C-xt} = p_{j-1,j} y_{i,j-1} t_k' p_{j-1,j}^{-1}
\]

Now consider \( \Phi_{\tau_q} \), the only non-trivial case is when \( q = j \).

\[
\Phi_{\tau_q}(y_{ij} \ t_k) = y_{ij}^{-1} p_{ij} t_k \tag{C-pi} = y_{ij}^{-1} t_k p_{ij} \tag{C-xt} = t_k y_{ij}^{-1} p_{ij} = \Phi_{\tau_q}(t_k y_{ij})
\]

**Case 3.** (C1) \( \alpha_{ij} \beta_{kl} = \beta_{kl} \alpha_{ij} \) \( (i, j, k, l) \) cyclically ordered

First consider \( \Phi_{\sigma_q} \). The non-trivial cases are \( q = i - 1 \) and \( i \neq l + 1 \), \( q = i \) and \( j = i + 1 \), \( q = j - 1 \) and \( j \neq i + 1 \), \( q = j \) and \( k = j + 1 \), \( q = k - 1 \) and \( j \neq k - 1 \), \( q = k \) and \( l = k + 1 \), \( p = l - 1 \) and \( l \neq k + 1 \), and \( p = l \) and \( i = l + 1 \).
When $q = i - 1$ and $i \neq l + 1$, we have the following.

$$
\Phi_{\sigma}(\alpha_{ij}\beta_{kl}) = p_{i-1,i}^{-1} \alpha_{i-1,j}^{-1} p_{i-1,i} \beta_{kl}^{-1} \tag{C1}
= p_{i-1,i}^{-1} \beta_{kl}^{-1} \alpha_{i-1,j}^{-1} p_{i-1,i} \tag{C1}
= \Phi_{\sigma}(\beta_{kl}^{-1} \alpha_{ij})
$$

When $q = i$ and $j = i + 1$, the only non-trivial case is when $\alpha = x$.

$$
\Phi_{\sigma}(x_{ij}\beta_{kl}) = t^{-1}_j y_{ij} t_j \beta_{kl}^{-1} \tag{C-\alpha t}
= t^{-1}_j y_{ij} \beta_{kl} t_j \tag{C-\alpha t}
= \beta_{kl} t_j^{-1} y_{ij} t_j = \Phi_{\sigma}(\beta_{kl} x_{ij})
$$

When $q = j - 1$ and $j \neq i + 1$, we have the following.

$$
\Phi_{\sigma}(\alpha_{ij}\beta_{kl}) = p_{j-1,j} \alpha_{i-,j}^{-1} p_{j-1,j} \beta_{kl}^{-1} \tag{C1}
= p_{j-1,j} \beta_{kl}^{-1} \alpha_{i-,j}^{-1} p_{j-1,j} \tag{C1}
= \Phi_{\sigma}(\beta_{kl} \alpha_{ij})
$$

When $q = j$ and $k = j + 1$, we have the following.

$$
\Phi_{\sigma}(\alpha_{ij}\beta_{kl}) = \alpha_{ik} p_{jk} \beta_{ji} p_{jk}^{-1} \tag{C3}
= p_{jk} \beta_{ji} p_{jk}^{-1} \alpha_{ik} \tag{C3}
= \Phi_{\sigma}(\beta_{kl} \alpha_{ij})
$$

When $q = k - 1$ and $j \neq k - 1$, we have the following.

$$
\Phi_{\sigma}(\alpha_{ij}\beta_{kl}) = \alpha_{ij} p_{k-1,k} \beta_{k-1,l} p_{k-1,k}^{-1} \tag{C1}
= p_{k-1,k} \beta_{k-1,l} \alpha_{ij} p_{k-1,k}^{-1} \tag{C1}
= \Phi_{\sigma}(\beta_{kl} \alpha_{ij})
$$

When $q = k$ and $l = k + 1$, the only non-trivial case is when $\beta = x$.

$$
\Phi_{\sigma}(\alpha_{ij} x_{kl}) = \alpha_{ij} t^{-1}_l y_{kl} t_l \tag{C-\alpha t}
= t^{-1}_l \alpha_{ij} y_{kl} t_l \tag{C-\alpha t}
= \Phi_{\sigma}(x_{kl} \alpha_{ij})
$$

When $q = l - 1$ and $l \neq k + 1$, we have the following.

$$
\Phi_{\sigma}(\alpha_{ij}\beta_{kl}) = \alpha_{ij} p_{l-1,l} \beta_{k,l-1} p_{l-1,l}^{-1} \tag{C1}
= p_{l-1,l} \alpha_{ij} \beta_{k,l-1} p_{l-1,l}^{-1} \tag{C1}
= \Phi_{\sigma}(\beta_{kl} \alpha_{ij})
$$

Finally, when $q = l$ and $i = l + 1$, we have the following.

$$
\Phi_{\sigma}(\alpha_{ij}\beta_{kl}) = p_{il} \alpha_{jl} p_{il}^{-1} \beta_{ik} \tag{C3}
= \beta_{ik} p_{il} \alpha_{jl} p_{il}^{-1} \tag{C3}
= \Phi_{\sigma}(\beta_{kl} \alpha_{ij})
$$

Now consider $\Phi_{r_q}$, there are two non-trivial cases. In the first case $\Phi_{r_q}(\alpha_{ij}) = \alpha_{ij}^{-1} p_{ij}$ and we have the following.

$$
\Phi_{r_q}(\alpha_{ij}\beta_{kl}) = \alpha_{ij}^{-1} p_{ij} \beta_{kl}^{-1} \tag{C1}
= \alpha_{ij}^{-1} \beta_{kl} p_{ij}^{-1} \tag{C1}
= \Phi_{r_q}(\beta_{kl} \alpha_{ij})
$$

In the second case $\Phi_{r_q}(\beta_{kl}) = \beta_{kl}^{-1} p_{kl}$ and we have the following.

$$
\Phi_{r_q}(\alpha_{ij}\beta_{kl}) = \alpha_{ij} p_{kl} \tag{C1}
= \beta_{kl}^{-1} \alpha_{ij} p_{kl} \tag{C1}
= \Phi_{r_q}(\beta_{kl} \alpha_{ij})
$$
Case 4. (C2) \( \alpha_{ij} \beta_{ik} \gamma_{jk} = \beta_{ik} \gamma_{jk} \alpha_{ij} \) \((i,j,k)\) cyclically ordered, 
\((\alpha, \beta, \gamma)\) as in Table 1

First consider \( \Phi_{\sigma_q} \). The only non-trivial cases are when \( q = i - 1 \) and \( i \neq k + 1 \), \( q = i \) and \( j = i + 1 \), \( q = j - 1 \) and \( j \neq i + 1 \), \( q = j \) and \( k = j + 1 \), \( q = k - 1 \) and \( k \neq j + 1 \), and \( q = k \) and \( i = k + 1 \).

When \( q = i - 1 \) and \( i \neq k + 1 \), we have the following.
\[
\Phi_{\sigma_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = p_{i-1,i} \alpha_{i-1,j} \beta_{i-1,k} p_{i-1,i}^{-1} \gamma_{jk} \\
= p_{i-1,i} \alpha_{i-1,j} \beta_{i-1,k} p_{i-1,i}^{-1} \gamma_{jk} \\
= p_{i-1,i} \beta_{i-1,k} \gamma_{jk} \alpha_{i-1,j} p_{i-1,i}^{-1} \\
= p_{i-1,i} \beta_{i-1,k} p_{i-1,i}^{-1} \gamma_{jk} \alpha_{i-1,j} p_{i-1,i}^{-1} = \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]

When \( q = i \) and \( j = i + 1 \), we have two cases. Except for when \( i < j < k \) and \((\alpha, \beta, \gamma) = (x, x, p)\) or \( k < i < j \) and \((\alpha, \beta, \gamma) = (x, y, p)\), we have the following deduction. Let \( \bar{t}_j \) and \( \bar{\alpha}_{ij} \) be defined as follows.
\[
\bar{t}_j = \begin{cases} 
  t_j & \text{if } \alpha = x \\
  1 & \text{if } \alpha \neq x
\end{cases}
\]
\[
\bar{\alpha}_{ij} = \begin{cases} 
  p_{ij} & \text{if } \alpha = p \\
  y_{ij} & \text{if } \alpha = x \\
  x_{ij} & \text{if } \alpha = y
\end{cases}
\]

So we have that \( \Phi_{\sigma_q}(\alpha_{ij}) = \bar{t}_j^{-1} \bar{\alpha}_{ij} \bar{t}_j \).
\[
\Phi_{\sigma_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = \bar{t}_j^{-1} \bar{\alpha}_{ij} \bar{t}_j \beta_{jk} p_{ij} \gamma_{ik} p_{ij}^{-1} \\
= \bar{t}_j^{-1} \bar{\alpha}_{ij} \bar{t}_j \beta_{jk} p_{ij} \gamma_{ik} p_{ij}^{-1} \bar{t}_j \\
= \bar{t}_j^{-1} \gamma_{ik} \beta_{jk} \bar{\alpha}_{ij} \bar{t}_j \\
= \gamma_{ik} \beta_{jk} p_{ij} p_{ij}^{-1} \bar{t}_j^{-1} \bar{\alpha}_{ij} \bar{t}_j \\
= \beta_{jk} p_{ij} \gamma_{ik} p_{ij}^{-1} \bar{t}_j^{-1} \bar{\alpha}_{ij} \bar{t}_j = \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]

When \( i < j < k \) and \((\alpha, \beta, \gamma) = (x, x, p)\) or \( k < i < j \) and \((\alpha, \beta, \gamma) = (x, y, p)\), we have the following deduction with \( \beta = x \) or \( y \) respectively.
\[
\Phi_{\sigma_q}(x_{ij} \beta_{jk} p_{jk}) = t_j^{-1} y_{ij} t_j \beta_{jk} p_{ij} p_{ij}^{-1} \\
= t_j^{-1} y_{ij} t_j p_{ij} p_{ij}^{-1} \beta_{jk} \\
= p_{ij} y_{ij} p_{ik} \beta_{jk} p_{ij}^{-1} \\
= p_{ij} p_{ik} \beta_{jk} y_{ij} p_{ij}^{-1} \\
= \beta_{jk} p_{ij} p_{ik} y_{ij} p_{ij}^{-1} \\
= \beta_{jk} p_{ij} p_{ik} p_{ij}^{-1} t_j^{-1} p_{ij} y_{ij} p_{ij}^{-1} \\
= \beta_{jk} p_{ij} p_{ik} p_{ij}^{-1} t_j^{-1} y_{ij} p_{ij}^{-1} t_j p_{ij}^{-1} \\
= \beta_{jk} p_{ij} p_{ik} p_{ij}^{-1} t_j^{-1} y_{ij} p_{ij}^{-1} = \Phi_{\sigma_q}(\beta_{jk} \gamma_{jk} x_{ij})
\]
When $q = j - 1$ and $j \neq i + 1$, we have the following.

\[
\Phi_{\sigma_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = p_{j-1,j} \alpha_{i,j-1} p_{j-1,j}^{-1} \beta_{ik} \alpha_{ij} p_{j-1,j}^{-1} \gamma_{j-1,k} p_{j-1,j}^{-1} = p_{j-1,j} \alpha_{i,j-1} \beta_{ik} \gamma_{j-1,k} p_{j-1,j}^{-1} = p_{j-1,j} \beta_{ik} \gamma_{j-1,k} \alpha_{i,j-1} p_{j-1,j}^{-1} = \beta_{ik} p_{j-1,j} \gamma_{j-1,k} \alpha_{i,j-1} p_{j-1,j}^{-1} = \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]  

When $q = j$ and $k = j + 1$, we have two cases. Except for when $\gamma = x$, i.e. when $i < j < k$ and $(\alpha, \beta, \gamma) = (y, p, x)$ or when $j < k < i$ and $(\alpha, \beta, \gamma) = (x, p, x)$, we have the following. Here

\[
\tilde{\gamma}_{jk} = \begin{cases} 
p_{jk} & \text{if } \gamma = p \\
_{jk} & \text{if } \gamma = y
\end{cases}
\]

\[
\Phi_{\sigma_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = \alpha_{ik} p_{jk} \beta_{ij} p_{jk}^{-1} \tilde{\gamma}_{jk} = \alpha_{ik} p_{jk}^{-1} \beta_{ij} p_{ik} \tilde{\gamma}_{jk} = \alpha_{ik} \tilde{\gamma}_{jk} \beta_{ij} = \alpha_{ik} \tilde{\gamma}_{jk} \beta_{ij} \alpha_{ik} = p_{jk} \beta_{ij} p_{jk}^{-1} \tilde{\gamma}_{jk} \alpha_{ik} = \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]  

When $i < j < k$ and $(\alpha, \beta, \gamma) = (y, p, x)$ or when $j < k < i$ and $(\alpha, \beta, \gamma) = (x, p, x)$, we have

\[
\Phi_{\sigma_q}(\alpha_{ij} p_{ik} x_{jk}) = \alpha_{ik} p_{jk} p_{ij}^{-1} \tilde{t}_{k}^{-1} y_{jk} t_{k} = \alpha_{ik} p_{jk} p_{ij}^{-1} \tilde{t}_{k}^{-1} y_{jk} t_{k} = p_{jk} p_{ij} \tilde{t}_{k}^{-1} y_{jk} t_{k} \alpha_{ik} = \Phi_{\sigma_q}(p_{ik} x_{jk} \alpha_{ij})
\]  

When $q = k - 1$ and $k \neq j + 1$, we have the following.

\[
\Phi_{\sigma_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = \alpha_{ij} p_{k-1,k} \beta_{i,k-1} \gamma_{j,k-1} p_{k-1,k} = p_{k-1,k} \alpha_{ij} \beta_{i,k-1} \gamma_{j,k-1} p_{k-1,k} = p_{k-1,k} \beta_{i,k-1} \gamma_{j,k-1} \alpha_{ij} p_{k-1,k} = p_{k-1,k} \beta_{i,k-1} \gamma_{j,k-1} \alpha_{ij} p_{k-1,k} = \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]
Finally, when \( q = k \) and \( i = k + 1 \), we have the following two cases. If \( \beta \neq x \) then we have the following. Here

\[
\bar{\beta}_{ik} = \begin{cases} 
  p_{ik} & \text{if } \beta = p \\
  x_{ik} & \text{if } \beta = y 
\end{cases}
\]

\[
\Phi_{\sigma_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = p_{ik} \alpha_{jk} p_{ik}^{-1} \bar{\beta}_{ik} \gamma_{ij} = p_{ij}^{-1} \alpha_{jk} p_{ij} \bar{\beta}_{ik} \gamma_{ij}
\]

\[
= \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]

And if \( \beta = x \) then we have the following.

\[
\Phi_{\sigma_q}(\alpha_{ij} x_{ik} \gamma_{jk}) = p_{ik} \alpha_{jk} p_{ik}^{-1} t_{i}^{-1} y_{ik} t_{i} \gamma_{ij}
\]

\[
\begin{align*}
&= p_{ik} \alpha_{jk} t_{i}^{-1} p_{ik}^{-1} y_{ik} t_{i} \gamma_{ij} \\
&= p_{ik} \alpha_{jk} y_{ik} t_{i}^{-1} p_{ik}^{-1} t_{i} \gamma_{ij} \\
&= p_{ik} \alpha_{jk} y_{ik} p_{ik}^{-1} \gamma_{ij}
\end{align*}
\]

\[
= p_{ik} \alpha_{jk} y_{ik} p_{ik}^{-1} \gamma_{ij} = p_{ik} \gamma_{ij} y_{ik} \alpha_{jk} p_{ik}^{-1}
\]

\[
= p_{ik} y_{ik} p_{jk} \gamma_{ij} p_{jk}^{-1} \alpha_{jk} p_{ik}^{-1}
\]

\[
= p_{ik} y_{ik} p_{ik}^{-1} \gamma_{ij} p_{ik} \alpha_{jk} p_{ik}^{-1}
\]

\[
= p_{ik} y_{ik} t_{i}^{-1} p_{ik}^{-1} t_{i} \gamma_{ij} p_{ik} \alpha_{jk} p_{ik}^{-1}
\]

\[
= t_{i}^{-1} y_{ik} t_{i} \gamma_{ij} p_{ik} \alpha_{jk} p_{ik}^{-1} = \Phi_{\sigma_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]

Now consider \( \Phi_{\tau_q} \), the non-trivial cases are as follows.

\[
q = i \quad i < j < k \\
\begin{align*}
& (x, p, p) \\
& (x, y, y) \\
& (x, x, p)
\end{align*}
\]

\[
q = j \quad i < j < k \\
\begin{align*}
& (y, p, p) \\
& (y, y, y) \\
& (y, p, x)
\end{align*}
\]

\[
q = k \quad i < j < k \\
\begin{align*}
& (p, y, y) \\
& (x, y, y) \\
& (y, y, y)
\end{align*}
\]

For the first two columns of the cases \( q = i \) and \( q = j \), we have the following.

\[
\Phi_{\tau_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = \alpha_{ij}^{-1} p_{ij} \beta_{ik} \gamma_{jk} = \alpha_{ij}^{-1} \beta_{ik} \gamma_{jk} p_{ij}
\]

\[
= \beta_{ik} \gamma_{jk} \alpha_{ij}^{-1} p_{ij} = \Phi_{\tau_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]
For the third column in the case \( q = i \), we have the following.

\[
\Phi_{\tau_q}(\alpha_{ij} \beta_{ik} p_{jk}) = \alpha_{ij}^{-1} p_{ij} \beta_{ik}^{-1} p_{ik} p_{jk} = \alpha_{ij}^{-1} p_{ij} \beta_{ik}^{-1} p_{ij}^{-1} p_{ik} p_{jk} p_{ij} = \alpha_{ij}^{-1} p_{ij} \beta_{ik}^{-1} p_{ijk} p_{ij} p_{ij} = \beta_{ik}^{-1} p_{ijk} \alpha_{ij}^{-1} p_{ij} = \Phi_{\tau_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]

For the third column in the case \( q = j \), we have the following.

\[
\Phi_{\tau_q}(\alpha_{ij} p_{ik} \gamma_{jk}) = \alpha_{ij}^{-1} p_{ij} p_{ik} \gamma_{jk}^{-1} p_{jk} = \alpha_{ij}^{-1} \gamma_{jk}^{-1} p_{ij} p_{ik} p_{jk} = \alpha_{ij}^{-1} \gamma_{jk}^{-1} p_{ij}^{-1} p_{ik} p_{jk} p_{ij} = \alpha_{ij}^{-1} \gamma_{jk}^{-1} p_{ik} p_{jk} \alpha_{ij} = \Phi_{\tau_q}(\alpha_{ij} \gamma_{jk} \alpha_{ij})
\]

For the case when \( q = k \), we have the following.

\[
\Phi_{\tau_q}(\alpha_{ij} \beta_{ik} \gamma_{jk}) = \alpha_{ij} \beta_{ik}^{-1} p_{ik} \gamma_{jk}^{-1} p_{jk} = \alpha_{ij} \beta_{ik}^{-1} p_{ij}^{-1} \gamma_{jk}^{-1} p_{ij} p_{ik} p_{jk} = \alpha_{ij} \beta_{ik}^{-1} \gamma_{jk}^{-1} p_{ij} p_{ik} p_{jk} = \gamma_{jk}^{-1} \beta_{ik}^{-1} p_{ik} p_{jk} \alpha_{ij} = \gamma_{jk}^{-1} \beta_{ik}^{-1} p_{ik} p_{jk} \alpha_{ij} = \Phi_{\tau_q}(\beta_{ik} \gamma_{jk} \alpha_{ij})
\]

**Case 5. (C3)** \( \alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1} = p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik} \quad (i, j, k, l) \) cyclically ordered

First consider \( \Phi_{\tau_q} \). As before the only non-trivial cases are when \( q = i - 1 \) and \( i \neq l + 1, q = i \) and \( j = i + 1, q = j - 1 \) and \( j \neq i + 1, q = j \) and \( k = j + 1, q = k - 1 \) and \( k \neq j + 1, q = k \) and \( l = k + 1 \), \( q = l - 1 \) and \( l \neq k - 1 \), and \( q = l \) and \( i = l + 1 \).

When \( q = i - 1 \), we have the following.

\[
\Phi_{\tau_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = p_{i-1,i} \alpha_{i-1,k} p_{i-1,i}^{-1} p_{jk} \beta_{jl} p_{jk}^{-1} = p_{i-1,i} p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{i-1,k} p_{i-1,i}^{-1} = p_{i-1,i} \beta_{jl} p_{jk}^{-1} \alpha_{i-1,k} p_{i-1,i}^{-1} = \Phi_{\tau_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})
\]

When \( q = i \) and \( j = i + 1 \), we have the following. (Here the (C2)s hold because we are in either of the bottom two rows of Table 1, both of which contain \((\alpha, p, p)\) for \( \alpha = p, \ x, \) and \( y \).)

\[
\Phi_{\tau_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = p_{ij} \alpha_{ij} p_{ik} \beta_{il} p_{ik}^{-1} p_{ij}^{-1} = p_{ij} \alpha_{ij} \beta_{il} p_{ik}^{-1} p_{ij}^{-1} = p_{ij} \beta_{il} p_{ik}^{-1} p_{ij}^{-1} \alpha_{ij} = \Phi_{\tau_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})
\]
When $q = j - 1$ and $j \neq i + 1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{ik} p_{j-1,k} p_{j-1,l}^{-1} p_{j-1,k}^{-1} p_{j-1,j}^{-1} = \alpha_{ik} p_{j-1,k} \beta_{j-1,l}^{-1} p_{j-1,k}^{-1} p_{j-1,j}^{-1} \alpha_{ik} p_{j-1,j}^{-1} = \Phi_{\sigma_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})$$

(C1)

When $q = j$ and $k = j + 1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{ik} p_{j-1,k} p_{j-1,l}^{-1} p_{j-1,k}^{-1} p_{j-1,j}^{-1} \alpha_{ik} p_{j-1,j}^{-1} = \Phi_{\sigma_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})$$

(C1)

When $q = k - 1$ and $k \neq j + 1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = p_{k-1,k} \alpha_{ik} p_{j-1,k} p_{j-1,l}^{-1} p_{j-1,k}^{-1} p_{j-1,j}^{-1} = \alpha_{ik} p_{j-1,k} \beta_{j-1,l}^{-1} p_{j-1,k}^{-1} p_{j-1,j}^{-1} \alpha_{ik} p_{j-1,j}^{-1} = \Phi_{\sigma_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})$$

(C1)

Finally, when $q = l$ and $i = l + 1$, we have the following. (Here the (C2)s hold because they always hold for the triples $(\beta, p, p)$ for $\beta = p$, $x$, and $y$.)

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{il} p_{jl} p_{kl} \beta_{jl} p_{kl}^{-1} p_{jl}^{-1} = \alpha_{il} \beta_{jl}$$

(C1)

When $q = l - 1$ and $l \neq k + 1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{ik} p_{l-1,l} p_{l-1,l}^{-1} \beta_{j,l-1} p_{l-1,l}^{-1} p_{l-1,l}^{-1} = \alpha_{ik} p_{l-1,l} \beta_{j,l-1}^{-1} p_{l-1,l}^{-1} \alpha_{ik} p_{l-1,l}^{-1} = \Phi_{\sigma_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})$$

(C1)(C1)(C1)

Finally, when $q = l$ and $i = l + 1$, we have the following. (Here the (C2)s hold because they always hold for the triples $(\alpha, p, p)$ and $(\beta, p, p)$.)

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{il} p_{jl} p_{kl} \beta_{jl} p_{kl}^{-1} p_{jl}^{-1} = \alpha_{il} \beta_{jl}$$

(C1)(C1)(C1)

Now consider $\Phi_{\sigma_q}$, there are two non-trivial cases. In the first case $\Phi_{\sigma_q}(\alpha_{ik}) = \alpha_{ik}^{-1} p_{ik}$ and we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{ik}^{-1} p_{ik} p_{jk} \beta_{jl} p_{jk}^{-1} p_{ik} = \alpha_{ik}^{-1} p_{jk} \beta_{jl} p_{jk}^{-1} p_{ik}$$

(C3)

Note that we have $p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik}^{-1} p_{ik} = \Phi_{\sigma_q}(p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik})$.
In the second case \( \Phi_\tau (\beta_{jl}) = \beta_{jl}^{-1} p_{jl} \) and we have the following.

\[
\Phi_\tau (\alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1}) = \alpha_{ik} p_{jk} \beta_{jl} p_{jk}^{-1} = p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik} p_{jk} p_{jk}^{-1} = p_{jk} \beta_{jl} p_{jk}^{-1} \alpha_{ik}
\]

**(C-xpt)** \( x_{ij} p_{ij} t_i = p_{ij} t_i x_{ij} \quad i < j \)

First consider \( \Phi_{\sigma_q} \). The only non-trivial cases are when \( q = i - 1, q = i \) and \( j = i + 1 \), and \( q = j - 1 \) and \( j \neq i + 1 \).

When \( q = i - 1 \), we have the following.

\[
\Phi_{\sigma_q} (x_{ij} p_{ij} t_i) = p_{i-1, i} x_{i-1, j} p_{i-1, j} p_{i-1, i}^{-1} t_{i-1}
\]

**(C-pt)** \( p_{i-1, i} x_{i-1, j} p_{i-1, j} t_{i-1, i} p_{i-1, j}^{-1} = p_{i-1, i} p_{i-1, j} t_{i-1, i} x_{i-1, j} p_{i-1, i}^{-1} \)

**(C-pt)** \( p_{i-1, i} p_{i-1, j} p_{i-1, i}^{-1} t_{i-1, i} p_{i-1, i}^{-1} = p_{i-1, i} p_{i-1, j} x_{i-1, j} p_{i-1, i}^{-1} = \Phi_{\sigma_q} (p_{ij} t_i x_{ij}) \)

When \( q = i \) and \( j = i + 1 \), we have the following.

\[
\Phi_{\sigma_q} (x_{ij} p_{ij} t_j) = t_j^{-1} y_{ij} t_j p_{ij} t_j = t_j^{-1} y_{ij} p_{ij} t_j t_j
\]

**(C-ypt)** \( t_j^{-1} p_{ij} t_j y_{ij} t_j = t_j^{-1} p_{ij} t_j y_{ij} t_j = \Phi_{\sigma_q} (p_{ij} t_i x_{ij}) \)

When \( q = j - 1 \) and \( j \neq i + 1 \), we have the following.

\[
\Phi_{\sigma_q} (x_{ij} p_{ij} t_i) = p_{j-1, j} x_{i,j-1} p_{i,j-1}^{-1} p_{j-1, j} t_i
\]

**(C-pt)** \( p_{j-1, j} x_{i,j-1} p_{i,j-1}^{-1} t_i p_{j-1, j}^{-1} = p_{j-1, j} p_{i,j-1} t_i x_{i,j-1} p_{j-1, j}^{-1} \)

**(C-pt)** \( p_{j-1, j} p_{i,j-1} p_{j-1, j} t_i p_{j-1, j} x_{i,j-1} p_{j-1, j}^{-1} = \Phi_{\sigma_q} (p_{ij} t_i x_{ij}) \)

Now consider \( \Phi_\tau \), the only non-trivial case is when \( q = i \).

\[
\Phi_\tau (x_{ij} p_{ij} t_i) = x_{ij}^{-1} p_{ij} t_i p_{ij} = x_{ij}^{-1} p_{ij} t_i = p_{ij} x_{ij}^{-1} = \Phi_\tau (p_{ij} t_i x_{ij})
\]

**(C-ypt)** \( y_{ij} p_{ij} t_j = p_{ij} t_j y_{ij} \quad i < j \)

First consider \( \Phi_{\sigma_q} \). The only non-trivial cases are when \( q = i - 1, q = i \) and \( j = i + 1 \), and \( q = j - 1 \) and \( j \neq i + 1 \).

When \( q = i - 1 \), we have the following.

\[
\Phi_{\sigma_q} (y_{ij} p_{ij} t_j) = p_{i-1, i} y_{i-1, j} p_{i-1, j} p_{i-1, i}^{-1} t_j
\]

**(C-ypt)** \( p_{i-1, i} y_{i-1, j} p_{i-1, j} t_j p_{i-1, i}^{-1} = p_{i-1, i} p_{i-1, j} t_j y_{i-1, j} p_{i-1, i}^{-1} \)

**(C-pt)** \( p_{i-1, i} p_{i-1, j} p_{i-1, i}^{-1} t_j p_{i-1, i} y_{i-1, j} p_{i-1, i}^{-1} = \Phi_{\sigma_q} (p_{ij} t_j y_{ij}) \)

When \( q = i \) and \( j = i + 1 \), we have the following.

\[
\Phi_{\sigma_q} (y_{ij} p_{ij} t_j) = t_j p_{ij} t_i x_{ij} = \Phi_{\sigma_q} (p_{ij} t_j y_{ij})
\]
When \( q = j - 1 \) and \( j \neq i + 1 \), we have the following.

\[
\Phi_{\sigma_q}(y_{ij} p_{ij} t_j) = p_{j-1,j} y_{i,j-1} p_{i,j-1} p_{j-1,j}^{-1} t_j^{-1}
\]

\[
= p_{j-1,j} y_{i,j-1} p_{i,j-1} t_j^{-1} p_{j-1,j}^{-1} = p_{j-1,j} p_{i,j-1} t_j^{-1} y_{i,j-1} p_{j-1,j}^{-1}
\]

Now consider \( \Phi_{\tau_q} \), the only non-trivial case is when \( q = j \).

\[
\Phi_{\tau_q}(y_{ij} p_{ij} t_j) = y_{ij}^{-1} p_{ij} t_j^{-1}
\]

\[
= y_{ij}^{-1} p_{ij} t_j p_{ij} t_j p_{ij} = p_{ij} t_j y_{ij}^{-1} p_{ij} = \Phi_{\sigma_q}(p_{ij} t_j y_{ij})
\]

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References


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