ON COMPLEX SPACES WITH PRESCRIBED SINGULARITIES

M. COLTOIU, K. DIEDERICH, C. JOITA

To the memory of our unforgettable teacher and shining example Hans Grauert

Abstract. For a given complex space \( Y \) we construct a complex space \( X \) such that \( \text{Sing}(X) = Y \).

1. Introduction

For a reduced complex space \( X \) we denote by \( \text{Sing}(X) \) the set of singular points of \( X \). In this paper we are dealing with the following question: given a reduced complex space \( Y \), does there exists a reduced complex space \( X \) such that \( \text{Sing}(X) = Y \). We show that the answer is “yes”. Namely we prove the following theorem:

Theorem 1. Let \( Y \) be a reduced complex space. Then there exists a reduced complex space \( X \) such that:

1. \( \text{Sing}(X) = Y \), \( \dim(X) = \dim(Y) + 2 \).
2. along \( \text{Reg}(Y) \), the complex space \( X \) has only quadratic singularities, (i.e., the product of a complex manifold of dimension \( n = \dim(Y) \) and a surface with an isolated quadratic 2-dimensional singularity).

Moreover, if \( Y \) is normal then \( X \) can be chosen to be normal and if \( Y \) is locally irreducible then \( X \) can be chosen to be locally irreducible.

If \( Y \) is a complex manifold the proof is trivial because one can choose \( X = Y \times S \) where \( S \) has only one singular point. Obviously this argument does not work if \( \text{Sing}(Y) \neq \emptyset \) because \( \text{Sing}(Y \times S) = \text{Sing}(Y) \times S \cup Y \times \text{Sing}(S) \). To prove our main theorem we consider a resolution of singularities \( \pi : \tilde{Y} \to Y \) (which exists by the results of Bierstone and Milman [3], and Aroca, et al. [1]) and over \( \tilde{Y} \) we consider a rank 2 vector bundle \( E \to \tilde{Y} \), which is relatively negative. On each fiber of \( E \) we have the equivalence relation \( x \sim (-x) \). If we let \( F := E/\sim \) we obtain a locally trivial fibration \( \tau : F \to \tilde{Y} \) with typical fiber \( \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2 = z_3^2 \} \), which has a quadratic two-dimensional isolated singularity. From \( F \) we get the desired complex space \( X \) by applying the relative Remmert quotient theorem (see [11]) and Wiegmann quotient theorem [15].

In the embedded case, i.e., if \( Y \) is a complex subspace of a complex manifold \( Z \), we give another construction of \( X \) using only Wiegmann quotient theorem. In this particular case, we obtain:

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**Theorem 2.** Suppose that $Z$ is a complex manifold and $Y$ is a closed subspace of $Z$. Then there exists a complex space $X$ with the following properties:

1. $\text{Sing}(X) = Y$ and $\dim(X) = \dim(Z) + 1$.
2. $X$ is locally irreducible.
3. The normalization of $X$ is smooth and therefore $X$ is not normal at any point of $Y$.
4. If $Z$ is connected then $X$ is irreducible.

**2. Preliminaries**

Throughout this paper all complex spaces are assumed to be reduced.

We recall that a complex space $X$ is called holomorphically convex if the holomorphically convex hull of every compact subset is compact.

**Definition 1.** A holomorphic map of complex spaces $\pi : X \to S$ is called holomorphically convex if for any point $s \in S$ there exists an open neighborhood $U$ of $s$ such that $X(U) := \pi^{-1}(U)$ is holomorphically convex. If for any point $s$ we can find $U$ such that $X(U)$ is Stein then $\pi$ is called a Stein morphism.

Knorr and Schneider in [11] proved the following result:

**Theorem 3.** Suppose that $\pi : X \to S$ is a holomorphically convex map between two complex spaces. Then there exists a complex spaces $R$ and a holomorphic map $\rho : X \to R$, called the relative Remmert reduction of $\pi$, such that $\rho_* \mathcal{O}_X = \mathcal{O}_R$ (so $\rho$ is proper, surjective, and has connected fibers) and a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\rho} & R \\
\downarrow{\pi} & & \downarrow{\sigma} \\
S & & 
\end{array}
$$

with $\sigma$ being a Stein morphism.

Throughout this paper a complex space $X$ is called 1-convex if there exists a smooth exhaustion function $\phi : X \to \mathbb{R}$ which is strictly plurisubharmonic outside a compact subset $K \subset X$.

**Definition 2.** A holomorphic map $\pi : X \to S$ is called 1-convex if for any $s \in S$ there exists an open neighborhood $U$ of $s$ such that $X(U) := \pi^{-1}(U)$ is 1-convex, a $C^\infty$ function $\phi : X(U) \to \mathbb{R}$ and a real number $c_0 \in \mathbb{R}$ such that:

1. $\phi_{\{x \in X(U) : \phi(x) > c_0\}}$ is 1-convex,
2. for every $c \in \mathbb{R}$ we have that $\pi_{\{x \in X(U) : \phi(x) \leq c\}}$ is a proper map.

The following Theorem is Satz. 3.4 in [11], see also [14].

**Theorem 4.** Every 1-convex map is holomorphically convex.

We recall the definition of a relatively exceptional set given in [11].

**Definition 3.** Suppose that $\pi : X \to S$ is a holomorphic map between two complex spaces and $A \subset X$ is a closed analytic subset such that $\pi|_A$ is proper and has nowhere
discrete fibers. A is called relatively exceptional with respect to $\pi$ if there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\Phi} & Y \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S & \xrightarrow{\pi'} & S \\
\end{array}
$$

where $Y$ is a complex space and $\pi'$ and $\Phi$ are holomorphic maps, such that:

(i) $\pi'_{|\Phi(A)}$ has discrete fibers,
(ii) $\Phi$ induces a biholomorphism $X \setminus A \to Y \setminus \Phi(A),$
(iii) $\Phi^*({\mathcal{O}}_X) = {\mathcal{O}}_Y.$

**Definition 4.** If $\pi : X \to S$ is a holomorphic map between two complex spaces and $A$ is a closed analytic subset of $X$, then $A$ is called maximally proper over $S$ if $\pi_{|A}$ is proper, has nowhere discrete fibers and for any closed analytic subset $A'$ of $X$ with these two properties we have $A' \subset A$.

The following result is Satz 5.4 of [11].

**Proposition 1.** Suppose that $\pi : X \to S$ is a holomorphic map and $A \subset X$ is a closed analytic subspace of $X$. We assume that $A$ has a neighborhood $W$ such that $\pi_{|W}$ is 1-convex and $A$ is maximally proper over $S$ in $W$. Then $A$ is relatively exceptional with respect to $S$.

We identify a vector bundle with the sheaf of germs of local sections in the bundle. Suppose that $X$ is a compact complex space and $p : E \to X$ is a holomorphic vector bundle of rank $r$. We let $\pi : \mathbb{P}(E) \to X$ be the holomorphic fiber bundle for which $\pi^{-1}(x)$ is the space of all $(r-1)$-dimensional linear subspaces of $p^{-1}(x)$. In general, for a coherent sheaf $\mathcal{F}$ on $X$ one can associate a projective variety over $X$, $\mathbb{P}(\mathcal{F})$, obtaining in this way a contravariant functor. For details we refer to [8] and [5], Chapter 1. For the proof of the following theorem see [7] and [10].

**Theorem 5.** The following statements are equivalent:

(a) $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ is ample.
(b) For every coherent sheaf $\mathcal{F}$ on $X$ there exists a positive integer $m_0$ such that $H^q(X, \mathcal{F} \otimes S^m(E)) = 0$ for every $q \geq 1, m \geq m_0$ ($S^m(E)$ denotes the $m$-th symmetric power of $E$).
(c) For every coherent sheaf $\mathcal{F}$ on $X$ there exists a positive integer $m_0$ such that $\mathcal{F} \otimes S^m(E)$ is spanned by its global sections.
(d) The zero section of $E^*$ is exceptional.
(e) The zero section of $E^*$ has a strongly pseudoconvex neighborhood.

A vector bundle is called ample if the above equivalent conditions are satisfied. A vector bundle is called negative if its dual is ample.

We will need the following generalization in the relative case. Suppose that $\pi : X \to S$ is a proper holomorphic map and $p : E \to X$ is a holomorphic vector bundle.

**Definition 5.** (a) $E$ is called relatively negative if its restriction to every fiber of $\pi^{-1}(s)$ is negative in the sense of Grauert, i.e., the null-section has a strictly pseudoconvex neighborhood.
(b) $E$ is called relatively ample if its dual $E^*$ is relatively negative.

(c) $\pi : X \rightarrow S$ is called relatively ample if there exists a relatively ample line bundle $p : L \rightarrow X$.

For the next Lemma see Corollary 2.7 in [13]

**Lemma 1.** Suppose that $s_0$ is a point in $S$ and $E|_{\pi^{-1}(s_0)}$ is negative. Then there exists a neighborhood $U$ of $s_0$ such that $\pi \circ p$ is a 1-convex morphism on $p^{-1}(\pi^{-1}(U))$.

**Corollary 1.** If $\pi$ has nowhere discrete fibers then $E$ is relatively negative iff its null-section is relatively exceptional.

**Remark:** For more general results concerning the relative blowing down of complex spaces, see [6].

Suppose now that $X$ and $Y$ are complex spaces, $f : X \rightarrow Y$ is a proper holomorphic map, and $L \rightarrow X$ a holomorphic line bundle. It was proved in [13], Theorem 3.6, (using the results on 1-convex morphisms obtained in [11]) that $L$ is relatively ample with respect to $f$ if and only if for every coherent sheaf $\mathcal{F}$ on $X$ and every compact set $K \subset Y$ there exists a positive integer $n_0 = n_0(K, \mathcal{F})$ such that $R^q f_* (\mathcal{F}(n)) = 0$ on $K$ for every $n \geq n_0$ and every $q \geq 1$ ($\mathcal{F}(n)$ stands for $\mathcal{F} \otimes L^n$). At the same time in [2], chapter 4, Théorème 4.1, it was shown that this last property implies that for every point $y \in Y$ there exists a neighborhood $V$ of $y$ and a large enough positive integer $n$ such that, on $f^{-1}(V)$, the canonical morphism $f^{-1}(V) \rightarrow \mathbb{P}(f_*(L^n))$ is an embedding. Moreover, in the proof of this theorem of [2] (page 179) it was shown that by further increasing $n$ we obtain that for every relatively compact open subset $U$ of $Y$ the canonical morphism $f^{-1}(U) \rightarrow \mathbb{P}(f_*(L^n))$ is an embedding for $n$ large enough ($n$ depending on $U$). Therefore putting together Theorem 3.6 in [13] and Theorem 4.1, chapter 4 in [2], when $X$ and $Y$ are compact, we have:

**Theorem 6.** If $X$ and $Y$ are compact complex spaces, $f : X \rightarrow Y$ is a holomorphic map, and $L \rightarrow X$ a holomorphic line bundle, the following are equivalent:

(a) $L$ is relatively ample with respect to $f$.

(b) There exists $n_0$ such that $R^q f_* (\mathcal{F}(n)) = 0$ for every $n \geq n_0$ and every $q \geq 1$.

(c) There exists $n_0$ such that the canonical morphism $f^* f_* \mathcal{F}(n) \rightarrow \mathcal{F}(n)$ is surjective for every $n \geq n_0$.

(d) There exists $n_1$ such that $X \rightarrow \mathbb{P}(f_*(L^n))$ is an embedding for $n \geq n_1$.

**Remark.** From (c) we have an embedding $X \rightarrow \mathbb{P}(f^* f_* L^n) = \mathbb{P}(f_* L^n) \times_Y X$, hence a map $X \rightarrow \mathbb{P}(f_* L^n)$. Condition (d) means that increasing $n$ this map becomes an embedding.

The following Lemma is a folklore result (see e.g. [9] Exercise 5.12). For reader’s convenience we provide a proof.

**Lemma 2.** Suppose that $X$ and $Y$ are compact complex spaces, $f : X \rightarrow Y$ a holomorphic map, $G \rightarrow Y$ an ample line bundle and $L \rightarrow X$ a relatively ample line bundle with respect to $f$. Then $L \otimes f^* G$ is ample on $X$. 
Proof. Using Theorem 6, we choose a positive integer \( n \) such that we have an embedding \( j \) over \( Y \):

\[
\begin{array}{c}
X \\
\downarrow j \\
\downarrow \ \\
\pi \\
\downarrow \\
\mathbb{P}(f_* L^n) \\
\end{array}
\]

such that \( L^n = j^*(\mathcal{O}(1)) \). By [8], Proposition 1.5, if \( \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) is a sheaf epimorphism then one has an embedding \( \mathbb{P}(\mathcal{F}_2) \hookrightarrow \mathbb{P}(\mathcal{F}_1) \) over \( Y \), which is linear over each fiber. Since \( G \) is ample it follows that, for \( \nu \) large enough, \( f_* L^n \otimes G^\nu \) is generated by global sections. Hence we have an epimorphism \( \mathcal{O}_Y^k \rightarrow f_* L^n \otimes G^\nu \) for some \( k \). Because \( G \) is a line bundle we have that \( \mathbb{P}(f_* L^n \otimes G^\nu) = \mathbb{P}(f_* L^n) \). Passing to the associated projective spaces, we get an embedding \( h : \mathbb{P}(f_* L^n) \hookrightarrow Y \times \mathbb{P}^{k-1} \) over \( Y \) such that \( \mathcal{O}(1) \) over \( \mathbb{P}(f_* L^n) \) is the pull-back by \( h \) of the hypersection bundle of \( \mathbb{P}^{k-1} \). Composing with \( j \) and using again the ampleness of \( G \) we get that \( L^n \otimes f^* G^\mu \) is ample for every \( \mu \). In particular it is ample for \( \mu = n \) and this in turn implies that \( L \otimes f^* G \) is ample. \( \Box \)

We will briefly recall some facts about desingularization of complex spaces (see [3]).

Let \( X \) be a complex space and \( Z \subset X \) a smooth closed complex subspace. For any point \( x_0 \in X \) we choose \( U \) an open neighborhood of \( x_0 \) together with a closed embedding \( U \hookrightarrow B \subset \mathbb{C}^N \) where \( B \) is an open ball in \( \mathbb{C}^N \). Then \( Z \) corresponds to a complex submanifold \( W \) of \( B \) and we consider the blow-up of \( B \) with center \( W \). In this blow-up we consider the proper transform of \( U \) and in this way we obtain the blow-up of \( U \) with center \( U \cap Z \). This construction does not depend on the local embedding and the local blow-ups patch-up to get the blow-up of \( X \) with (smooth) center \( Z \).

The following result (Theorem 13.4 of [3]) is the fundamental theorem of global desingularization of complex spaces.

**Theorem 7.** Any complex space \( X \) admits a desingularization \( \pi : \tilde{X} \rightarrow X \) such that \( \pi \) is the composition of a locally finite sequence of blow-ups with smooth centers and \( \pi^{-1}(\text{Sing}(X)) \) is a divisor with normal crossings in \( \tilde{X} \).

In this theorem locally finite means that on compact sets all but finitely many blow-ups are trivial.

**Corollary 2.** The desingularization \( \pi : \tilde{X} \rightarrow X \) given by Theorem 7 is relatively ample, the relatively ample line bundle \( p : L \rightarrow \tilde{X} \) corresponding to the exceptional divisor of \( \pi \).

**Proof.** Let

\[
\cdots \rightarrow X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X
\]

be the sequence of blow-ups given by Theorem 7 and \( L_j \rightarrow X_j \) the line bundle corresponding to the exceptional divisor of \( \pi_j \). Each \( L_j \) is relatively ample with respect to \( \pi_j \).

Suppose that \( x \) is a point in \( X \). We consider the restrictions of \( L_1 \) and \( L_2 \) to \( \pi_1^{-1}(x) \) and, respectively, \( (\pi_1 \circ \pi_2)^{-1}(x) \) and we denote them by \( L_1 \rightarrow \pi_1^{-1}(x) \) and \( L_2 \rightarrow (\pi_1 \circ \pi_2)^{-1}(x) \). We have that \( L_1 \rightarrow \pi_1^{-1}(x) \) is ample and \( L_2 \rightarrow (\pi_1 \circ \pi_2)^{-1}(x) \)
is relatively ample with respect to \( \pi_2 \). We apply Lemma 2 and we deduce that \( L_2 \otimes \pi_2^*(L_1) \rightarrow (\pi_1 \circ \pi_2)^{-1}(x) \) is ample.

We conclude that \( L_2 \otimes \pi_2^*(L_1) \rightarrow X \) is relatively ample with respect to \( \pi_1 \circ \pi_2 \). We continue inductively this procedure and we obtain that the line bundle \( L \) defined, by abuse of notation, by \( L = \otimes_{i \in \mathbb{N}} L_i \rightarrow X \) is relatively ample with respect to \( \pi \).

The infinite tensor product of line bundles (and the entire construction) makes sense since the sequence of blow-ups is locally finite. \( \square \)

**Definition 6.** ([15]) Suppose that \((X, \mathcal{O}_X)\) is a complex space, \(F\) is a subset of \(\mathcal{O}_X(X)\) and let \(\phi_F : X \rightarrow \mathbb{C}^F, \phi_F(x) = (f(x))_{f \in F}\).

(a) \((X, \mathcal{O}_X)\) is called \(F\)-separable if \(\phi_F\) is injective.

(b) \((X, \mathcal{O}_X)\) is called \(F\)-convex if \(\phi_F\) is proper.

\(F\)-separable means that functions in \(F\) separate the points of \(X\) and \(F\)-convex means that for every discrete sequence \(\{x_n\}\) in \(X\) there exists a function \(f \in F\) such that \(\{|f(x_n)|\}\) is unbounded.

The following theorem, generalizing a result of Remmert, was proved by Wiegmann [15].

**Theorem 8.** Suppose that \((X, \mathcal{O}_X)\) is a reduced complex space and \(F\) is a subalgebra of \(\mathcal{O}_X(X)\) such that \((X, \mathcal{O}_X)\) is \(F\)-convex. Then there exists an \(F\)-convex and \(F\)-separable reduced Stein space \((Y, \mathcal{O}_Y)\) together with a proper surjective holomorphic mapping \(p : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)\) such that if \(\pi : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)\) is the induced morphisms of \(\mathbb{C}\)-algebras then \(\pi(\mathcal{O}_Y(Y)) \supset F\). Moreover, \((Y, \mathcal{O}_Y)\) is unique, up to isomorphism, with these properties, if \(F\) is closed in \(\mathcal{O}_X(X)\) then \(\pi(\mathcal{O}_Y(Y)) = F\) and if \(F = \mathcal{O}_X(X)\) then \(\pi\) is an isomorphism.

The complex space \((Y, \mathcal{O}_Y)\) is called the Remmert reduction of \((X, \mathcal{O}_X)\) with respect to \(F\) and is denoted by \(R_F(X, \mathcal{O}_X)\). Note that Remmert’s theorem corresponds to the case \(F = \mathcal{O}_X(X)\).

For a complex space \((Z, \mathcal{O}_Z)\) we let \(T(Z, \mathcal{O}_Z)\) be the underlying topological space \(Z\) and, for an open subset \(U\) of \(Z\), \(\Gamma_U(Z, \mathcal{O}_Z) = \mathcal{O}_Z(U)\). We recall briefly Wiegmann’s construction. The topological space \(T(R_F(X, \mathcal{O}_X))\) is defined as \(T(R_F(X, \mathcal{O}_X)) = X/\sim\) and \(p\) is the quotient map, where, for \(x_1, x_2 \in X, x_1 \sim x_2\) if and only if \(f(x_1) = f(x_2)\) for every \(f \in F\). The structure sheaf is defined as follows. For \(y \in T(R_F(X, \mathcal{O}_X))\) let \(m_y\) by the ideal of \(F\) that contains all function \(f \in F\) that vanish on \(p^{-1}(y)\). For every open subset \(U\) of \(T(R_F(X, \mathcal{O}_X))\), \(\Gamma_U(R_F(X, \mathcal{O}_X))\) is the algebra of all functions \(g \in \mathcal{O}_X(p^{-1}(U))\) such that for every point \(y \in U\) there exists a positive integer \(k\), a convergent power series \(\sum_{i_1, \ldots, i_k} c_{i_1, \ldots, i_k} T_1^{i_1} \cdots T_k^{i_k} \in \mathbb{C}[\langle T_1, \ldots, T_k \rangle]\) and \(f_1, \ldots, f_k \in m_y\) such that \(\sum_{i_1, \ldots, i_k} c_{i_1, \ldots, i_k} f_1^{i_1} \cdots f_k^{i_k}\) converges uniformly to \(g\) on a neighborhood of \(p^{-1}(y)\).

**Lemma 3.** Suppose that \((X, \mathcal{O}_X)\) is a reduced complex space, \(F\) and \(G\) are two subalgebras of \(\mathcal{O}_X(X)\) such that \((X, \mathcal{O}_X)\) is \(F\)-convex, \(F \subset G\) and \(F\) is dense in \(G\). Then the canonical morphism \(R_F(X, \mathcal{O}_X) \rightarrow R_G(X, \mathcal{O}_X)\) is an isomorphism.

**Proof.** It follows from the discussion after Theorem 8 that if \(F\) is a dense subset of \(G\) then \(T(R_F(X, \mathcal{O}_X)) = T(R_G(X, \mathcal{O}_X))\) and that for every open subset \(U\) of \(T(R_F(X, \mathcal{O}_X))\) we have \(\Gamma_U(R_F(X, \mathcal{O}_X)) \subset \Gamma_U(R_G(X, \mathcal{O}_X))\).
Let \( \overline{F} \) be the closure of \( F \) (hence \( G \subset \overline{F} \)) and let \( Y := T(R_F(X, \mathcal{O}_X)) \). We have then \( F \subset \Gamma_Y(R_F(X, \mathcal{O}_X)) \subset \Gamma_Y(R_G(X, \mathcal{O}_X)) \subset \Gamma_Y(R_{\overline{F}}(X, \mathcal{O}_X)) = \overline{F} \). As \( \Gamma_Y(R_F(X, \mathcal{O}_X)) \) and \( \Gamma_Y(R_G(X, \mathcal{O}_X)) \) are closed in \( \mathcal{C}(Y) \) (the algebra of continuous functions on \( Y \)) and \( \overline{F} \) is the smallest closed subset containing \( F \), it follows that the map \( \Gamma_Y(R_F(X, \mathcal{O}_X)) \to \Gamma_Y(R_G(X, \mathcal{O}_X)) \) is bijective. As both \( R_F(X, \mathcal{O}_X) \) and \( R_G(X, \mathcal{O}_X) \) are reduced Stein spaces it follows that the canonical morphism \( R_F(X, \mathcal{O}_X) \to R_G(X, \mathcal{O}_X) \) is an isomorphism. \( \square \)

In Wiegmann’s theorem one needs \( X \) to be \( F \)-convex. In particular, \( X \) has to be \( \mathcal{O}_X(X) \)-convex which is a strong global condition. On the other hand, it may happen that \( \mathcal{O}_X(X) = \mathbb{C} \) (e.g., if \( X \) is compact) and then the Remmert reduction is just a point. For our purpose we need to apply Wiegmann’s theorem locally. To be able do this, we need a “patching” result. This is the purpose of the following proposition.

**Proposition 2.** Suppose that \( (X, \mathcal{O}_X) \) is a reduced complex space and \( \{V_i\}_{i \in \mathbb{N}} \) is a locally finite open covering of \( X \). Let \( F_i \) be a closed subalgebra of \( \mathcal{O}_X(V_i) \), \( \sim_i \) be the equivalence relation on \( V_i \) induced by \( F_i \); \( f_1 \sim_i f_2 \) if and only if \( f_1(x) = f_2(x) \) for all \( x \in V_i \), and \( F_{ij} = F_i \cap F_j \) is a closed subalgebra of \( \mathcal{O}_X(V_i \cap V_j) \). We assume that:

(a) \( \mathcal{O}_X|V_i \) is \( F_i \)-convex,

(b) \( F_i|V_i \cap V_j \) is a dense subset of \( F_{ij} \) for every \( i, j \in \mathbb{N} \),

(c) \( V_i \cap V_j \) is saturated with respect to \( \sim_i \) for every \( i, j \in \mathbb{N} \).

Then there exists a reduced complex space \( (Y, \mathcal{O}_Y) \), a proper holomorphic map \( p : X \to Y \) and an open covering \( \{U_i\}_i \) of \( Y \) such that \( (U_i, \mathcal{O}_Y|U_i) \) is isomorphic to \( R_{F_i}(V_i, \mathcal{O}_X|V_i) \) and \( p|U_i \) is the canonical morphism given by Theorem 8.

**Proof.** We define the following relation on \( X \): \( x \sim y \) if and only if there exists \( i \in \mathbb{N} \) such that \( x, y \in V_i \) and \( x \sim_i y \). Note that if \( x \in V_i \), \( y \in V_i \cap V_j \) and \( x \sim_i y \) then using (c) we get that \( x \in V_i \cap V_j \) and by (b) and Lemma 3 we get that \( x \sim_j y \). This shows that \( \sim \) is an equivalence relation. Moreover, each \( V_i \) is saturated with respect to \( \sim \).

Let \( Y = X/\sim \), endowed with the quotient topology, and \( p : X \to Y \) be the quotient map. We set \( U_i = p(V_i) \) which is an open subset of \( Y \). By Wiegmann’s construction of \( R_{F_i}(V_i, \mathcal{O}_X|V_i) \) explained above we have that \( T(R_{F_i}(V_i, \mathcal{O}_X|V_i)) = U_i \). We define the structure sheaf \( \mathcal{O}_Y \) as follows: if \( \Omega \) is an open subset of \( Y \) and \( f \in \mathcal{C}(\Omega) \) then \( f \in \mathcal{O}_Y(\Omega) \) if and only if for every point \( y \in U_i \) for some \( i \in I \) there exists \( D \) an open subset of \( Y \) such that \( D \subset \Omega \cap U_i \) and \( f|_D \in \Gamma_D(R_{F_i}(V_i, \mathcal{O}_X|V_i)) \). By Lemma 3 this definition does not depend on the choice of \( i \). The fact that \( (U_i, \mathcal{O}_Y|U_i) \) is isomorphic to \( R_{F_i}(V_i, \mathcal{O}_X|V_i) \) follows from the construction of the relative Remmert reduction. \( \square \)

**Example.** Suppose that \( X = \mathbb{P}^1 \). Let \( B_1, B_2, B_3 \) be three balls (in local coordinate charts) such that \( B_1 \cup B_2 \cup B_3 = \mathbb{P}^1 \) and \( B_i \cap B_j \) is Runge in \( B_i \) for every \( i, j \in \{1, 2, 3\} \). We assume that \( a := [0 : 1] \in B_1 \setminus (\overline{B_2} \cup \overline{B_3}) \). Let \( F_2 = F_{22} = \mathcal{O}(B_2) \), \( F_3 = F_{33} = \mathcal{O}(B_3) \), \( F_1 = F_{11} = \{ f \in \mathcal{O}(B_1) : f'(a) = 0 \} \) and, for \( i \neq j \), \( F_{i,j} = \mathcal{O}(B_i \cap B_j) \). Then we are in the hypothesis of Proposition 2. Note that a holomorphic function \( f \), defined in a neighborhood of the origin \( 0 \in \mathbb{C} \), satisfies \( f'(0) = 0 \) if and only if there exists a holomorphic function \( F \) of two variables, defined in a neighborhood of the origin in \( \mathbb{C}^2 \), such that \( f(z) = F(z^3, z^2) \) and the map \( z \to (z^3, z^2) \) is a parameterization of the cusp singularity \( \{(x, y) \in \mathbb{C}^2 : x^2 = y^3\} \).
3. The results

Lemma 4. If $X$ is a complex space then any open covering has a locally finite open refinement $\{\Omega_m\}_{m \in \mathbb{N}}$ such that $\Omega_m$ is Stein for every $m \in \mathbb{N}$ and the pair $(\Omega_{m_1}, \Omega_{m_2} \cap \Omega_{m_2})$ is Runge for every $m_1, m_2 \in \mathbb{N}$.

Proof. We consider $\{W_j\}_{j \in \mathbb{N}}, \{V_j\}_{j \in \mathbb{N}}, \{U_j\}_{j \in \mathbb{N}}$ locally finite countable open covering of $X$ such that $\{U_j\}_{j \in \mathbb{N}}$ is a refinement of the given covering, $W_j \Subset V_j \Subset U_j$ and $U_j$ is Stein for every $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ and each $x \in \overline{W}_j$ we choose $\phi_{j,x} : U_j \to [0, \infty)$ a plurisubharmonic function such that:

(a) $\phi_{j,x}(x) = 0$ and $\{z \in U_j : \phi_{j,x}(z) < 1\} \subset V_j$,

(b) if, for some $k \in \mathbb{N}$, $\{z \in U_j : \phi_{j,x}(z) < 1\} \cap V_k \neq \emptyset$ then $\{z \in U_j : \phi_{j,x}(z) < 1\} \subset U_k$.

Then $\{z \in U_j : \phi_{j,x}(z) < 1\}_{x \in \overline{W}_j}$ is an open covering of $\overline{W}_j$. We extract a finite subcovering $\{z \in U_j : \phi_{j,s}(z) < 1\}_{s \in A_j}$ where $A_j$ is a finite set and we set $\Omega_{j,s} := \{z \in U_j : \phi_{j,s}(z) < 1\}$. The $\{\Omega_{j,s}\}_{j,s}$ is a locally finite open covering of $X$. Since $\phi_{j,x}$ is plurisubharmonic on $U_j$ each $\Omega_{j,s}$ is Stein. On the other hand, if $\Omega_{j,s} \cap \Omega_{k,t} \neq \emptyset$, as $\Omega_{k,t} \subset V_k$, we have that $\Omega_{j,s} \cap V_k \neq \emptyset$ and hence by property (b) above we have that $\Omega_{j,s} \subset U_k$. This implies that $\Omega_{j,s} \cap V_k = \{z \in \Omega_{j,s} : \phi_{k,t}(z) < 1\}$ which is Runge in $\Omega_{j,s}$, see [12]. If we choose a bijection $\chi : \mathbb{N} \to \{(j, s) : j \in \mathbb{N}, s \in A_j\}$ and we set $\Omega_m := \chi(m)$ we get the desired family. \(\square\)

Proof of Theorem 1. Let $\nu : Y_1 \to Y$ be the normalization map and $\tau : Z \to Y_1$ be a desingularization map which is relatively ample. Let $p : L \to Z$ be a relatively negative line bundle (which exists by Corollary 2) and set $E := L \oplus L$.

Let $\sigma : \mathbb{C}^2 \to \mathbb{C}^2, \sigma(w) = -w$. Clearly $\sigma \circ \sigma$ is the identity of $\mathbb{C}^2$ and therefore we obtain a linear action of $\mathbb{Z}_2$ on $\mathbb{C}^2$. It is easy to see that $\mathbb{C}^2/\mathbb{Z}_2$ is isomorphic to $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2 = z_3^2\}$ which is a normal surface with only one singular point of quadratic type. By linearity we obtain an action of $\mathbb{Z}_2$ on any vector bundle and in particular on the vector bundle $E$ defined above. Let $\tilde{E}$ be the quotient space of $E$ through this action. We get then a locally trivial fibration $\tilde{p} : \tilde{E} \to Z$ with typical fiber $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2 = z_3^2\}$. Note that $Sing(\tilde{E}) = Z$ (the zero section). The composition $f := \tau \circ \tilde{p} : \tilde{E} \to Y_1$ is 1-convex, and hence is a holomorphically convex map. Thus we can consider the relative Remmert quotient associated to $f$. We obtain a complex space $W_1$ together with a map $g : \tilde{E} \to W_1$ such that $g_*\mathcal{O}_{\tilde{E}} = \mathcal{O}_{W_1}$. We get then a closed embedding $\sigma : Y_1 \hookrightarrow W_1$. Via this embedding $Y_1$ is the image through $g$ of the null-section of $\tilde{E}$. Note that $g$ is biholomorphic outside the null-section and hence $W_1$ has singularities precisely on $Y_1$. There is a natural holomorphic retraction $r : W_1 \to Y_1$, which is a Stein morphism, corresponding to the projection map $f : \tilde{E} \to Y_1$. Over the regular part of $Y_1$ the space $W_1$ has only quadratic singularities.

At this moment we reduced the proof of Theorem 1 to the following Lemma (relative contraction for finite maps), which will be applied to the normalization map.
Lemma 5. Let $A$ and $B$ be complex spaces and $m : A \to B$ be a finite surjective holomorphic map. We assume that $A$ is a closed complex space of a complex space $S$ and $m$ admits a holomorphic extension $\tilde{m} : S \to B$ which is a Stein morphism. Then there exists a complex space $T$ and a holomorphic map $\alpha : S \to T$ such that $T$ contains $B$ as a closed complex subspace, $\alpha|_A = m$ and, outside $B$, $\alpha$ is a biholomorphism between $S \setminus A$ and $T \setminus B$.

Proof. Using Lemma 4 we choose a locally finite Stein covering $\{D_i\}_{i \in \mathbb{N}}$ of $B$ such that $D_i \cap D_j$ is Runge in $D_i$ and in $D_j$ for every $i, j \in \mathbb{N}$ and $\tilde{m}^{-1}(D_i) \subset S$ is Stein. Therefore $\tilde{m}^{-1}(D_i \cap D_j)$ is Runge in $\tilde{m}^{-1}(D_i)$ and in $\tilde{m}^{-1}(D_j)$ for every $i, j \in \mathbb{N}$. On $\tilde{m}^{-1}(D_i)$ we consider the set $F_i$ of all holomorphic functions $f \in \mathcal{O}(\tilde{m}^{-1}(D_i))$ such that $f|_{A \cap \tilde{m}^{-1}(D_i)}$ comes from a holomorphic function on $D_i$, i.e., there exists a holomorphic function $g \in \mathcal{O}(D_i)$ with $f|_{A \cap \tilde{m}^{-1}(D_i)} = g \circ m$. Then $F_i$ is a subalgebra of $\mathcal{O}(\tilde{m}^{-1}(D_i))$ and $\tilde{m}^{-1}(D_i)$ is $F_i$-holomorphically convex. Similarly, we define the set $F_{ij}$ of all holomorphic functions $f \in \mathcal{O}(\tilde{m}^{-1}(D_i \cap D_j))$ such that $f|_{A \cap \tilde{m}^{-1}(D_i \cap D_j)}$ comes from a holomorphic function on $D_i \cap D_j$. Applying Wiegmann quotient theorem to the subalgebras $F_i$ we get a Stein complex space $T_i$ containing $D_i$ as a closed complex subspace. Using Proposition 2, these complex spaces $\{T_i\}_{i \in \mathbb{N}}$ can be glued together and we get the desired complex space $T$. This concludes the proof of Lemma 5 and of Theorem 1. □

Proof of Theorem 2. Suppose that $\Omega$ is a Stein manifold and $A$ is a closed analytic subset of $\Omega$. We denote by $\pi : \Omega \times \mathbb{C} \to \Omega$ the standard projection and we identify a holomorphic function $f \in \mathcal{O}(\Omega)$ with $f \circ \pi$. Hence we have $\mathcal{O}(\Omega) \subset \mathcal{O}(\Omega \times \mathbb{C})$. Let $\lambda$ be the coordinate function on $\mathbb{C}$ and $F := \{f \in \mathcal{O}(\Omega \times \mathbb{C}) : \partial f/\partial x \equiv 0 \text{ on } A \times \{0\}\}$. Then:

- $F$ is a closed subalgebra of $\mathcal{O}(\Omega \times \mathbb{C})$ and $F \supset \mathcal{O}(\Omega)$,
- if $f \in \mathcal{O}(\Omega \times \mathbb{C})$ and $f|_{A \times \{0\}} \equiv 0$ then $f^2 \in F$.

Suppose that $K$ is a compact subset of $\Omega \times \mathbb{C}$. Then $\hat{K}^F$, the holomorphically convex hull of $K$ with respect to $F$ is a subset of $\hat{K}^{\mathcal{O}_{\Omega \times \mathbb{C}} \cup A}$. Indeed, if $z \in \Omega \times \mathbb{C} \backslash (\hat{K}^{\mathcal{O}_{\Omega \times \mathbb{C}} \cup A})$ then there exists $f \in \mathcal{O}_{\Omega \times \mathbb{C}}$ such that $f|_{A \times \{0\}} \equiv 0$ and $|f(z)| > \|f\|_K$. It follows that $|f^2(z)| > \|f^2\|_K$ and $f^2 \in F$. At the same time from $\mathcal{O}(\Omega) \subset F$ we get that $\hat{K}^F \subset \pi^{-1}(\pi(K)^{\mathcal{O}_{\Omega}})$. Hence, $\hat{K}^F \subset (\hat{K}^{\mathcal{O}_{\Omega \times \mathbb{C}} \cup A}) \cap \pi^{-1}(\pi(K)^{\mathcal{O}_{\Omega}})$, which implies that $\hat{K}^F$ is compact and hence $\Omega \times \mathbb{C}$ is $F$-convex.

Similarly, we can show that $\Omega \times \mathbb{C}$ is $F$-separable. Namely, for any two points $x, y \in \Omega \times \mathbb{C}$, if $x, y \in A \times \{0\}$ then we can choose $f \in \mathcal{O}(\Omega)$ with $f(x) \neq f(y)$ and if at least one of them is not in $A$ we can choose $f \in \mathcal{O}(\Omega \times \mathbb{C})$ such that $f^2(x) \neq f^2(y)$. Let $(Y, \mathcal{O}_Y) = R_F(\Omega \times \mathbb{C}, \mathcal{O}_{\Omega \times \mathbb{C}})$, $p : \Omega \times \mathbb{C} \to Y$ the canonical morphism and $B = p(A \times \{0\})$, which is a closed analytic subset of $Y$. Since $\Omega \times \mathbb{C}$ is $F$-separable it follows that $p$ is a homeomorphism.

We want to show next that $p : \Omega \times \mathbb{C} \setminus A \times \{0\} \to Y \setminus B$ is a biholomorphism and hence, in particular $\text{Sing}(Y) \subset B$. It suffices to show that for any open subset $U$ of $\Omega \times \mathbb{C} \setminus A \times \{0\}$ and any $x \in U$ we have that every holomorphic function $f$ on $U$ can be approximated, uniformly on a neighborhood of $x$ by functions in $F$ (this will imply that the functions in $F$ give local coordinates outside $A \times \{0\}$). Let $c \in \mathbb{C}$ be such that $f(x) + c \neq 0$. We choose an open neighborhood $V$ of $x$ such that $V \subset U$,
\( \nabla \cap A = \emptyset, \nabla \) is holomorphically convex and there exists a holomorphic function \( g \) defined on a neighborhood of \( \nabla \) such that \( g^2 = f + c \). It follows that we can find \( \{h_j\}_{j \geq 0}, h_j \in \mathcal{O}(\Omega) \) such that \( h_j|_{A \times \{0\}} \equiv 0 \) and \( h_j \to g \) uniformly on \( \nabla \). It remains to notice that \( h^2_j - c \in F \) and \( h^2_j - c \to f \) uniformly on \( \nabla \).

Note also that \( F \supset \mathcal{O}(\Omega) \) implies that \( p|_{\mathcal{O}(\Omega)}: \Omega \times \{0\} \to p(\Omega \times \{0\}) \) is a biholomorphism and hence \( p|_{A}: A \to B \) is a biholomorphism.

We claim now that \( B \subset \text{Sing}(Y) \). Let \( y \in B \) and \( x = p^{-1}(y) \in A \). If \( Y \) were smooth in \( y \), it would be normal in \( y \), hence it would be normal in a neighborhood of \( y \), and therefore we could find \( U \subset X \) an open neighborhood of \( x \) and \( W \subset Y \) an open neighborhood of \( y \) such that \( p(U) = W \) and \( p: U \to W \) is a biholomorphism. Therefore for every holomorphic function \( f: U \to \mathbb{C} \) we would have that \( f \circ p^{-1} \) is holomorphic on \( W \). This would imply that we can approximate \( f \), uniformly on a neighborhood of \( x \), with functions from \( F \). However, the coordinate function \( \lambda: U \to \mathbb{C} \) does not satisfy this property.

**Lemma 6.** Let \( M \) be a Stein manifold, \( A \subset M \) a closed analytic subset and \( U \subset M \) a Runge open subset of \( M \). Then the restriction map \( f \to f|_{U \times \mathbb{C}} \) from \( \{f \in \mathcal{O}(M \times \mathbb{C}) : \frac{\partial f}{\partial x} \equiv 0 \text{ on } A \times \{0\}\} \) to \( \{f \in \mathcal{O}(U \times \mathbb{C}) : \frac{\partial f}{\partial x} \equiv 0 \text{ on } A \cap U \times \{0\}\} \) has dense image in the topology of uniform convergence on compacts. Here, \( \lambda \) is the coordinate function on \( \mathbb{C} \).

**Proof.** Let \( f: U \times \mathbb{C} \to \mathbb{C} \) be a holomorphic function such that \( \frac{\partial f}{\partial x} \equiv 0 \) on \( A \cap U \times \{0\} \). Because \( U \times \mathbb{C} \) is Runge in \( M \times \mathbb{C} \) there exists a sequence of holomorphic functions \( \{g_n\}_{n \geq 1}, g_n \in \mathcal{O}(M \times \mathbb{C}) \), such that \( g_n \equiv 0 \) on \( A \cap U \times \{0\} \) and \( \{g_n|_{A \times \{0\}}\}_{n \geq 1} \) converges to \( \frac{\partial f}{\partial x} \). At the same time there exists a sequence \( \{h_n\}_{n \geq 1}, h_n \in \mathcal{O}(M) \) such that \( \{h_n|_{U}\}_{n \geq 1} \) converges to \( f(z,0) \). For each \( n \geq 1 \) we consider the following primitive with respect to \( \lambda \) of \( g_n \): \( f_n(z,\lambda) = \int_0^1 g_n(z,\xi)d\xi + h_n(z) \), where \( \gamma: [0,1] \to \mathbb{C} \) is a path that joins \( 0 \in \mathbb{C} \) with \( \lambda \). For \( \gamma(t) = tz \) we get \( f_n(z,\lambda) = \int_0^1 g_n(z,t\lambda)\lambda dt + h_n(z) \). We have then \( \frac{\partial f_n}{\partial x} = g_n \equiv 0 \) on \( A \times \{0\} \). At the same time, since both \( f \) and \( \int_0^1 \frac{\partial f}{\partial x}(z,t\lambda)\lambda dt \) are primitives for \( \frac{\partial f}{\partial x} \), we have \( f(z,\lambda) = \int_0^1 \frac{\partial f}{\partial x}(z,t\lambda)\lambda dt + f(z,0) \). Hence

\[
f_n(z,\lambda) - f(z,\lambda) = \int_0^1 \left( g_n(z,t\lambda) - \frac{\partial f}{\partial \lambda}(z,t\lambda) \right) \lambda dt + (h_n(z) - f(z,0)).
\]

Now, if \( K \subset M \times \mathbb{C} \) is a compact set, we choose \( K_0 \), a compact subset of \( M \), and \( B \subset \mathbb{C} \) a compact disk centered at the origin such that \( K \subset K_0 \times B \). Using \( \|g_n - \frac{\partial f}{\partial x}\|_{K_0 \times B} \to 0 \) and \( \|h_n - f(z,0)\|_{K_0} \to 0 \), we obtain easily that \( \|f_n - f\|_{K_0} \to 0 \).

Let now \( Z \) be a complex manifold and \( Y \) a closed complex subspace of \( Z \). We use Lemma 4 and we choose an open Stein covering \( \{\Omega_i\}_{i \in \mathbb{N}} \) of \( Z \) such that the pair \( (\Omega_i, \Omega_i \cap \Omega_j) \) is Runge for every \( i, j \in \mathbb{N} \). Let \( F_i := \{f \in \mathcal{O}(\Omega_i \times \mathbb{C}) : \frac{\partial f}{\partial x} \equiv 0 \text{ on } Y \times \{0\}\} \) and, similarly, \( F_{ij} := \{f \in \mathcal{O}((\Omega_i \cap \Omega_j) \times \mathbb{C}) : \frac{\partial f}{\partial x} \equiv 0 \text{ on } Y \times \{0\}\} \).

We apply Wiegmann’s quotient theorem to \( F_i \) and we use Proposition 2, to glue together the complex spaces thus obtained and we get the desired complex space \( X \). Note that because a positive codimension analytic subset does not disconnect a complex manifold it follows that \( X \) is locally irreducible and, if \( Z \) is connected, \( X \) is
irreducible. At the same time it follows from our proof that the normalization of $X$ is $Z \times \mathbb{C}$.

Remarks:

1. In [4] the following result was proved: given a closed analytic subset $A$ of $\mathbb{C}^n$, $\text{codim}(A) \geq 2$, there exists an irreducible analytic hypersurface $H \subset \mathbb{C}^n$ such that $\text{Sing}(H) = A$. This shows, in particular, that one can prescribe singularities for Stein spaces. However the construction in [4] cannot be used for arbitrary singularities since it is not functorial and the local models cannot be glued together to obtain a complex space with prescribed singularities.

2. The following problem was raised to the first author by C. Bănică in connection with the duality on complex spaces: could every complex space $Z$ of bounded Zariski dimension be embedded as a closed analytic subset of a complex manifold?

3. The following problem remains open: suppose that $Y$ is a reduced complex space, not necessarily normal. Is it possible to find a normal complex space $X$ such that $\text{Sing}(X) = Y$?

4. If $Y$ is a projective algebraic variety then one can construct a normal projective algebraic variety $X$ such that $\text{Sing}(X) = Y$. We would like to thank Iustin Coandă for this remark.

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References


Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit 3, P.O. Box 1-764, Bucharest 014700, Romania

E-mail address: Mihnea.Coltoiu@imar.ro, Cezar.Joita@imar.ro

Mathematik, Universität Wuppertal, Gausstr. 20, D-42095 Wuppertal, Germany

E-mail address: klas@math.uni-wuppertal.de