MULTI-WINDOW GABOR FRAMES IN AMALGAM SPACES

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ABSTRACT. We show that multi-window Gabor frames with windows in the Wiener algebra $W(L^\infty, \ell^1)$ are Banach frames for all Wiener amalgam spaces. As a by-product of our results we positively answer an open question that was posed by Krishtal and Okoudjou [28] and concerns the continuity of the canonical dual of a Gabor frame with a continuous generator in the Wiener algebra. The proofs are based on a recent version of Wiener’s $1/f$ lemma.

1. Introduction

A Gabor system is a collection of functions $G(g, \Lambda) = \{ \pi(\lambda)g \mid \lambda \in \Lambda \}$, where $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ is a lattice, $g \in L^2(\mathbb{R}^d)$, and the time-frequency shifts of $g$ are given by

$$\pi(x, \omega)g(y) = e^{2\pi i \omega \cdot y} g(y - x) \quad (y \in \mathbb{R}^d).$$

This system is called a frame if $\|f\|_2^2 \approx \sum_{\lambda} |\langle f, \pi(\lambda)g \rangle|^2$. In this case, there exists a dual Gabor system $G(\tilde{g}, \Lambda) = \{ \pi(\lambda)\tilde{g} \mid \lambda \in \Lambda \}$ providing the $L^2$-expansions

$$f = \sum_{\lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\tilde{g} = \sum_{\lambda} \langle f, \pi(\lambda)\tilde{g} \rangle \pi(\lambda)g. \quad (1.1)$$

It is known that under suitable assumptions on $g$ and $\tilde{g}$ that expansion extends to $L^p$ spaces [3, 17, 20, 21]. To some extent, these results parallel the theory of Gabor expansions on modulation spaces [14, 18]. However, since modulation spaces are defined in terms of time–frequency concentration — and are indeed characterized by the size of the numbers $\langle f, \pi(\lambda)g \rangle$ — Gabor expansions are also available in a more irregular context, where $\Lambda$ does not need to be a lattice. In contrast, the theory of Gabor expansions in $L^p$ spaces relies on the strict algebraic structure of $\Lambda$. Indeed, as shown in [30], Poisson summation formula implies that the frame operator $Sf := \sum_{\lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$ can be written as

$$Sf(x) = \frac{1}{\beta^d} \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \left( g(x - j/\beta - \alpha k)g(x - \alpha k) \right) f(x - j/\beta). \quad (1.2)$$

This expression allows one to transfer spatial information about $g$ to boundedness properties of $S$ and is at the core of the $L^p$-theory of Gabor expansions.

One often has explicit information only about $g$, while the existence of $\tilde{g}$ is merely inferred from the frame inequality. It is then important to know whether certain good properties of $g$ are also inherited by $\tilde{g}$, so as to deduce the validity of (1.1) in various
function spaces. The key technical point is showing the $S$ in invertible not only in $L^2$ but also in the other relevant spaces. This was proved for modulation spaces in [19,22] and for $L^p$ spaces in [26]. In this latter case the analysis relies on the fact that $S^{-1}$ is the frame operator associated with the dual Gabor system $G(\tilde{g}, \Lambda)$ and thus admits an expansion like the one in (1.2).

The objective of this article is to extend the $L^p$-theory of Gabor expansions to multi-window Gabor systems (see [2,23]),

$$G(\Lambda^1, \ldots, \Lambda^n, g^1, \ldots, g^n) = \{ \pi(\lambda^i)g^i \mid \lambda^i \in \Lambda^i, 1 \leq i \leq n \},$$

where $\Lambda^1, \ldots, \Lambda^n \subseteq \mathbb{R}^{2d}$ are lattices $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ and $g^1, \ldots, g^n : \mathbb{R}^d \to \mathbb{C}$. The challenge in doing so is that, in contrast to the case of a single lattice $\Lambda$, the corresponding dual system does not consist of lattice time–frequency translates of a certain family of functions $\tilde{g^1}, \ldots, \tilde{g^n}$. The main technical point of this article is to show that, nevertheless, $S^{-1}$ admits a generalized expansion

$$(1.3) \quad S^{-1}f(x) = \sum_k G_k(x)f(x - x_k),$$

where now the family of points $\{x_k\}_k$ may not be contained in a lattice. We then prove that certain spatial localization properties of $g^1, \ldots, g^n$ imply corresponding localization properties for the family $\{G_k\}_k$, and deduce that $S^{-1}$ is bounded on $L^p$-spaces. For technical reasons we work in the more general context of Wiener amalgam spaces, that are spaces of functions that belong locally to $L^q$ and globally to $L^p$.

To achieve this, we study a Banach algebra of operators admitting an expansion like in (1.3) with a suitable summability condition. We then resort to a recent Wiener-type result on non-commutative almost-periodic Fourier series [4] to prove that this algebra is spectral within the class of bounded operators on $L^p$. This means that if an operator from that algebra is invertible on $L^p$, then the inverse operator necessarily belongs to the algebra. This approach is now common in time–frequency analysis [1,4–7,10,14,19,22,24,25,29] but its application to spaces that are not characterized by time–frequency decay is rather subtle. As a by-product, we obtain consequences that are new even for the case of one generator. We prove that if all the functions $g^i$ are continuous, so is every function in the dual system. This question was posed in [26].

This paper is organized as follows. In Section 2 we define Wiener amalgam spaces and recall their characterization via Gabor frames. In Section 3 we present the main technical result of this paper: a spectral invariance theorem for a sub-algebra of weighted-shift operators in $B(L^p(\mathbb{R}^d))$. In Section 4, we use the result of the previous section to extend the theory of multi-window Gabor frames to the class of Wiener amalgam spaces. In particular, this last section contains a Wiener-type lemma for multi-window Gabor frames.

2. Amalgam spaces and Gabor expansions

Before introducing the Wiener amalgam spaces, we first set the notation that will be used throughout the paper.
Given $x, \omega \in \mathbb{R}^d$, the translation and modulation operators act on a function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$T_x f(y) := f(y - x), \quad M_\omega f(y) := e^{2\pi i \omega \cdot y} f(y),$$

where $\omega \cdot y$ is the usual dot product. The time-frequency shift associated with the point $\lambda = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ is the operator $\pi(\lambda) = \pi(x, \omega) := M_\omega T_x$.

Given two non-negative functions $f, g$, we write $f \lesssim g$ if $f \leq Cg$, for some constant $C > 0$. If $E$ is a Banach space, we denote by $B(E)$ the Banach algebra of all bounded linear operators on $E$.

We use the following normalization of the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$:

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i \omega \cdot x} \, dx.$$

### 2.1. Definition and properties of the amalgam spaces.

A function $w : \mathbb{R}^d \to (0, +\infty)$ is called a weight if it is continuous and symmetric (i.e., $w(x) = w(-x)$). A weight $w$ is submultiplicative if

$$w(x + y) \leq w(x)w(y), \quad x, y \in \mathbb{R}^d.$$

Prototypical examples are given by the polynomial weights $w(x) = (1 + |x|)^s$, which are submultiplicative if $s \geq 0$. The main results in this article require to consider an extra condition on the weights. A weight $w$ is called admissible if $w(0) = 1$, it is submultiplicative and satisfies the Gelfand–Raikov–Shilov condition

$$\lim_{k \to \infty} w(kx)^{1/k} = 1, \quad x \in \mathbb{R}^d.$$

Note that this condition, together with the submultiplicativity, implies that $w(x) \geq 1$, $x \in \mathbb{R}^d$.

Given a submultiplicative weight $w$, a second weight $v : \mathbb{R}^d \to (0, +\infty)$ is called $w$-moderate if there exists a constant $C_v > 0$ such that

$$v(x + y) \leq C_v w(x)v(y), \quad x, y \in \mathbb{R}^d. \quad (2.1)$$

For polynomial weights $v(x) = (1 + |x|^t)^s$, $w(x) = (1 + |x|^s)^s$, $v$ is $w$-moderate if $|t| \leq s$. If $v$ is $w$-moderate, it follows from (2.1) and the symmetry of $w$ that $1/v$ is also $w$-moderate (with the same constant).

Let $w$ be a submultiplicative weight and let $v$ be $w$-moderate. This will be the standard assumption in this article. We will keep the weight $w$ fixed and consider classes of function spaces related to various weights $v$. For $1 \leq p, q \leq +\infty$, we define the Wiener amalgam space $W(L^p, L^q_v)$ as the class of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{W(L^p, L^q_v)} := \left( \sum_{k \in \mathbb{Z}^d} \|f\|^q_{L^p([0,1]^d + k)} v(k)^q \right)^{1/q} < \infty \quad (2.2)$$

with the usual modifications when $q = +\infty$. As with Lebesgue spaces, we identity two functions if they coincide almost everywhere. For a study of this class of spaces in a much broader context see [12, 13, 16]. We only point out that, as a consequence of the assumptions on the weights $v$ and $w$, it can be shown that the partition $\{[0, 1]^d + k : \}$. 
$k \in \mathbb{Z}^d$ in (2.2) can be replaced by more general coverings yielding an equivalent norm.

Weighted amalgam spaces are solid. This means that if $f \in W(L^p, L^q_v)$ and $m \in L^\infty(\mathbb{R}^d)$, then $mf \in W(L^p, L^q_v)$ and

$$(2.3) \quad \|mf\|_{W(L^p, L^q_v)} \leq \|m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{W(L^p, L^q_v)}.$$  

In addition, using the fact that $v$ is $w$-moderate, it follows that $W(L^p, L^q_v)$ is closed under translations and

$$(2.4) \quad \|T_x f\|_{W(L^p, L^q_v)} \leq C_v w(x) \|f\|_{W(L^p, L^q_v)},$$

where $C_v$ is the constant in (2.1).

The Köthe-dual of $W(L^p, L^q_v)$ is the space of all measurable functions $g : \mathbb{R}^d \to \mathbb{C}$ such that $g \cdot W(L^p, L^q_v) \subseteq L^1(\mathbb{R}^d)$. It is equal to $W(L^{p'}, L^{q'}_v)$, where $1/p + 1/p' = 1/q + 1/q' = 1$ for all $1 \leq p, q \leq \infty$. In particular, the pairing

$$(\cdot, \cdot) : W(L^p, L^q_v) \times W(L^{p'}, L^{q'}_v) \to \mathbb{C}, \quad (f, g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, dx$$

is bounded. The functionals arising from integration against functions in $W(L^{p'}, L^{q'}_v)$ determine a topology in $W(L^p, L^q_v)$ denoted by $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_v))$.

### 2.2. Gabor expansions on amalgam spaces

We now recall the theory of Gabor expansions on Wiener amalgam spaces as developed in [15, 17, 20, 21]. Let $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ be a (separable) lattice which will be used to index time-frequency shifts. For convenience we assume that $\alpha, \beta > 0$. We point out that the theory depends heavily on the assumption that $\Lambda$ is a separable lattice $\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$.

We first recall the definition of the family of sequence spaces corresponding to amalgam spaces via Gabor frames. For a weight $v$ and $1 \leq p, q \leq +\infty$ we define the sequence space $S_v^{p,q}(\Lambda)$ in the following way. We let $\mathcal{F}L^p([0, 1/\beta)^d)$ stand for the image of $L^p([0, 1/\beta)^d)$ under the discrete Fourier transform. More precisely, a sequence $c \equiv \{c_j \mid j \in \beta \mathbb{Z}^d\} \subseteq \mathbb{C}$ belongs to $\mathcal{F}L^p([0, 1/\beta)^d)$ if there exists a (unique) function $f \in L^p([0, 1/\beta)^d)$ such that

$$c_j = \hat{f}(j) = \beta^d \int_{[0, 1/\beta)^d} f(x)e^{-2\pi ijx} \, dx, \quad j \in \beta \mathbb{Z}^d.$$  

The space $\mathcal{F}L^p([0, 1/\beta)^d)$ is given by the norm $\|c\|_{\mathcal{F}L^p([0, 1/\beta)^d)} := \|f\|_{L^p([0, 1/\beta)^d)}$.

We now let $S_v^{p,q}(\Lambda)$ be the set of all sequences $c \equiv \{c_\lambda \mid \lambda \in \Lambda\} \subseteq \mathbb{C}$ such that, for each $k \in \alpha \mathbb{Z}^d$, the sequence $(c_{k,j})_{j \in \beta \mathbb{Z}^d}$ belongs to $\mathcal{F}L^p([0, 1/\beta)^d)$ and

$$\|c\|_{S_v^{p,q}(\Lambda)} := \left( \sum_{k \in \alpha \mathbb{Z}^d} \left( \sum_{j \in \beta \mathbb{Z}^d} \|c_{k,j}\|_{\mathcal{F}L^p([0, 1/\beta)^d)} v(k)^q \right)^q \right)^{1/q} < +\infty$$

with the usual modifications when $q = \infty$. When $1 < p < +\infty$ this is simply

$$\|c\|_{S_v^{p,q}(\Lambda)} := \left( \sum_{k \in \alpha \mathbb{Z}^d} \left( \sum_{j \in \beta \mathbb{Z}^d} c_{k,j} e^{2\pi ij} \|c_{k,j}\|_{L^p([0, 1/\beta)^d)} v(k)^q \right)^q \right)^{1/q} < +\infty,$$

and the usual modifications hold for $q = \infty$. 
The following theorem from [21] introduces the analysis and synthesis operators, clarifies their precise meaning and gives their mapping properties.

**Theorem 1** ([21], Theorem 3.2). Let \( w \) be a submultiplicative weight, \( v \) a \( w \)-moderate weight, \( g \in W(L^\infty, L^1_w) \) and \( 1 \leq p, q \leq +\infty \). Then the following properties hold:

(a) The analysis (coefficient) operator

\[
C_g,\Lambda : W(L^p, L^q_v) \to S^{p,q}_v(\Lambda), \quad C_g,\Lambda(f) := \langle f, \pi(\lambda)g \rangle_{\lambda \in \Lambda}
\]

is bounded with a bound that only depends on \( \alpha, \beta, \|g\|_{W(L^\infty, L_w^1)} \), and the constant \( C_v \) in (2.1).

(b) Let \( c \in S^{p,q}_v(\Lambda) \) and \( m_k \in L^p([0,1/\beta]^d) \) be the unique functions such that \( \hat{m}_k(j) = c_{k,j} \). Then the series

\[
R_{g,\Lambda}(c) := \sum_{k \in \alpha \mathbb{Z}^d} m_k T_k g
\]

converges unconditionally in the \( \sigma(W(L^p, L^q_v), W(L^p', L^{q'}_1)) \)-topology and, moreover, unconditionally in the norm topology of \( W(L^p, L^q_v) \) if \( p, q < \infty \).

(c) The synthesis operator \( R_{g,\Lambda} : S^{p,q}_v(\Lambda) \to W(L^p, L^q_v) \) is bounded with a bound that depends only on \( \alpha, \beta, \|g\|_{W(L^\infty, L_w^1)} \), and the constant \( C_v \) in (2.1).

The definition of the operator \( R_{g,\Lambda} \) is rather abstract. As shown in [15], the convergence can be made explicit by means of a summability method. For \( g \in W(L^\infty, L^1_w) \), a sequence \( c \in S^{p,q}_v(\Lambda) \), and \( N,M \geq 0 \) let us consider the partial sums

\[
R_{N,M}(c)(x) := \sum_{|k|_\infty \leq \alpha N} \sum_{|j|_\infty \leq \beta M} c_{k,j} e^{2\pi i j x} g(x-k).
\]

In the conditions \( |k|_\infty \leq N, |j|_\infty \leq M \) above we consider elements \( (k,j) \in \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \); it is important that we use the max norm. We also consider the regularized partial sums

\[
\sigma_{N,M}(c)(x) := \sum_{|k|_\infty \leq \alpha N} \sum_{|j|_\infty \leq \beta M} r_{j,M} c_{k,j} e^{2\pi i j x} g(x-k),
\]

where the regularizing weights are given by

\[
r_{j,M} := \prod_{h=1}^d \left( 1 - \frac{|j_h|}{\beta(M+1)} \right).
\]

We then have the following convergence result [15, 21].

**Theorem 2.** Let \( w \) be a submultiplicative weight, \( g \in W(L^\infty, L^1_w) \), \( v \) a \( w \)-moderate weight and \( 1 \leq p, q \leq +\infty \). Then the following properties hold:

(a) If \( 1 < p < \infty \) and \( q < \infty \), then

\[
R_{N,M}(c) \to R_{g,\Lambda}(c) \quad \text{as } N, M \to \infty
\]

in the norm of \( W(L^p, L^q_v) \).
(b) For each \(c \in S_{p,q}^{v}(\Lambda)\),
\[
\sigma_{N,M}(c) \to R_{g,\Lambda}(c) \quad \text{as } N, M \to \infty
\]
in the \(\sigma(W(L^p, L^q_0), W(L^{p'}, L^q_1/^1_v))\)-topology and also in the norm of \(W(L^p, L^q_v)\) if \(p, q < +\infty\).

**Remark 1.** A more refined convergence statement, with more general summability methods, can be found in [15]. We will only need the norm and weak convergence of Gabor expansions but we point out that the problem of pointwise summability has also been extensively studied [15, 17, 20, 21, 31].

**Proof.** Part (a) is proved in [21, Proposition 4.6]. The case \(p < +\infty\) of (b) is proved in [15, Theorem 4], where only unweighted amalgam spaces are considered. The same proof extends with simple modifications to the weighted case and weak*-convergence for \(p = \infty\). \(\square\)

We now present a representation of Gabor frame operators that will be essential for the results to come. For proofs see [30] or [21, Theorem 4.2 and Lemma 5.2] for the weighted version.

**Theorem 3.** Let \(w\) be a submultiplicative weight, \(v\) a \(w\)-moderate weight, \(g, h \in W(L^\infty, L^1_w)\) and \(1 \leq p, q \leq +\infty\). Then the operator \(R_{h,\Lambda}C_{g,\Lambda} : W(L^p, L^q_v) \to W(L^p, L^q_v)\) can be written as
\[
R_{h,\Lambda}C_{g,\Lambda}f = \beta^{-d} \sum_{j \in \mathbb{Z}^d} G_j T_{j/\beta} f,
\]
where
\[
G_j(x) := \sum_{k \in \mathbb{Z}^d} g(x - j/\beta - \alpha k) h(x - \alpha k), \quad x \in \mathbb{R}^d.
\]
In addition, the functions \(G_j : \mathbb{R}^d \to \mathbb{C}\) satisfy
\[
\sum_{j \in \mathbb{Z}^d} \|G_j\|_{\infty}w(j/\beta) \lesssim \|g\|_{W(L^\infty, L^1_w)} \|h\|_{W(L^\infty, L^1_w)} < +\infty.
\]
As a consequence, the series in (2.6) converges absolutely in the norm of \(W(L^p, L^q_v)\).

### 3. The algebra of \(L^\infty\)-weighted shifts

**3.1. \(L^\infty\)-weighted shifts.** Guided by (2.6), we will now introduce a Banach*-algebra of operators on function spaces that will be the key technical object of the article. For an admissible weight \(w\) we let \(A_w\) be the set of all families \(\mathcal{M} = (m_x)_{x \in \mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d))\) with the standard Banach space norm
\[
\|\mathcal{M}\|_{A_w} = \sum_{x \in \mathbb{R}^d} \|m_x\|_{L^\infty(\mathbb{R}^d)}w(x) < +\infty.
\]

The algebra structure and the involution on \(A_w\), however, will be non-standard. They will come from the identification of \(A_w\) with the class of operators on function spaces of the form
\[
f \mapsto \sum_{x \in \mathbb{R}^d} m_x f(x - x).
\]
Observe that due to (3.1) the family $\mathcal{M} = (m_x)_{x \in \mathbb{R}^d}$ has countable support and also that the operator in (3.2) is well defined and bounded on all $L^p(\mathbb{R}^d)$, $p \in [1, \infty]$ (recall that the admissibility of $w$ implies that $\rho \geq 1$).

With a slight abuse of notation, given a function $m \in L^\infty(\mathbb{R}^d)$ we also denote by $m$ the multiplication operator $f \mapsto mf$. It is then convenient to write $\mathcal{M} \in \mathcal{A}_w$ as

$$\mathcal{M} = \sum_{x \in \mathbb{R}^d} m_x T_x, \quad (m_x)_{x \in \mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d)),$$

and endow $\mathcal{A}_w$ with the product and involution inherited from $B(L^2(\mathbb{R}^d))$. More precisely, the product on $\mathcal{A}_w$ is given by

$$\left( \sum_x m_x T_x \right) \left( \sum_x n_x T_x \right) = \sum_x \left( \sum_y m_y n_{x-y} (\cdot - y) \right) T_x$$

and the involution – by

$$\left( \sum_x m_x T_x \right)^* = \sum_x m_x (\cdot + x) T_{-x} = \sum_x m_{-x} (\cdot - x) T_x.$$

It is straightforward to verify that with this structure $\mathcal{A}_w$ is, indeed, a Banach*-algebra which embeds continuously into $B(L^2(\mathbb{R}^d))$. We shall establish a number of other continuity properties of the operators defined by families in $\mathcal{A}_w$ in Proposition 1 below. These will be useful in dealing with Gabor expansions on amalgam spaces.

Before that, we mention that the identification of families in $\mathcal{A}_w$ and operators on $B(L^p(\mathbb{R}^d))$ given by the operator in (3.2) is one to one; this follows from the characterization of $\mathcal{A}_w$ in the following subsection and can easily be proved directly. Because of this we shall no longer distinguish between the families in $\mathcal{A}_w$ and operators generated by them. We will write $\mathcal{A}_w \subset B(L^p(\mathbb{R}^d))$ if we need to highlight that we treat members of $\mathcal{A}_w$ as operators on $L^p(\mathbb{R}^d)$. We also point out that for $m \in L^\infty(\mathbb{R}^d)$ and $x, w \in \mathbb{R}^d$

$$M_w m T_x M_{-w} = e^{2\pi i \omega \cdot x} m T_x.$$  

**Proposition 1.** Let $1 \leq p, q \leq +\infty$ and let $v$ be a $w$-moderate weight. Then the following statements hold:

(a) $\mathcal{A}_w \hookrightarrow B(W(L^p, L^q_w))$. More precisely, every $\mathcal{M} = \sum_x m_x T_x \in \mathcal{A}_w$ defines a bounded operator on $W(L^p, L^q_w)$ given by the formula

$$\mathcal{M}(f) := \sum_x m_x f (\cdot - x).$$

The series defining $\mathcal{M} : W(L^p, L^q_w) \to W(L^p, L^q_w)$ converges absolutely in the norm of $W(L^p, L^q_w)$ and $\|\mathcal{M}\|_{B(W(L^p, L^q_w))} \leq C_v \|\mathcal{M}\|_{\mathcal{A}_w}$, where $C_v$ is the constant in (2.1).

(b) For every $\mathcal{M} \in \mathcal{A}_w$, $f \in W(L^p, L^q_w)$ and $g \in W(L^p, L^q_w)$,

$$\langle \mathcal{M}(f), g \rangle = \langle f, \mathcal{M}^*(g) \rangle.$$

(c) For every $\mathcal{M} \in \mathcal{A}_w$, the operator $\mathcal{M} : W(L^p, L^q_w) \to W(L^p, L^q_w)$ is continuous in the $\sigma(W(L^p, L^q_w), W(L^p, L^q_w))$-topology.
Then the discrete topology. The normalized Haar measure on 

\[(3.4)\]

\(\varepsilon > 0\) for every algebra of those operators for which the Fourier series in (3.5) is continuous and almost periodic in the sense of Bohr. For every \(\varepsilon > 0\) \(\rho_{y}(\mathcal{M}) := M_{y}\mathcal{M}M_{-y}\). Explicitly,

\[\rho(y)\mathcal{M}f(x) = e^{2\pi iy \cdot x}(\mathcal{M}g)(x), \quad g(x) = e^{-2\pi iy \cdot x}f(x).\]

The map \(\rho : \mathbb{R}^{d} \to B(L^{p}(\mathbb{R}^{d}))\) defines an isometric representation of \(\mathbb{R}^{d}\) on the algebra \(B(L^{p}(\mathbb{R}^{d}))\). This means that \(\rho\) is a representation of \(\mathbb{R}^{d}\) on the Banach space \(B(L^{p}(\mathbb{R}^{d}))\) and, in addition, for each \(y \in \mathbb{R}^{d}\), \(\rho(y)\) is an algebra automorphism and an isometry.

A continuous map \(Y : \mathbb{R}^{d} \to B(L^{p}(\mathbb{R}^{d}))\) is almost-periodic in the sense of Bohr if for every \(\varepsilon > 0\) there is a compact \(K = K_{\varepsilon} \subset \mathbb{R}^{d}\) such that for all \(x \in \mathbb{R}^{d}\)

\[(x + K) \cap \{y \in \mathbb{R}^{d} | ||Y(g + y) - Y(g)|| < \varepsilon, \forall g \in \mathbb{R}^{d}\} \neq \emptyset.\]

Then \(Y\) extends uniquely to a continuous map of the Bohr compactification \(\hat{R}_{c}^{d}\) of \(\mathbb{R}^{d}\), also denoted by \(Y\). Thus, now \(Y : \hat{R}_{c}^{d} \to B(L^{p}(\mathbb{R}^{d}))\), where \(\hat{R}_{c}^{d}\) represents the topological dual group (i.e., the group of characters) of \(\mathbb{R}^{d}\) when \(\mathbb{R}^{d}\) is endowed with the discrete topology. The normalized Haar measure on \(\hat{R}_{c}^{d}\) is denoted by \(\bar{\mu}(dy)\).

For each \(\mathcal{M} \in \hat{B}(L^{p}(\mathbb{R}^{d}))\), we consider the map,

\[(3.4) \quad \widehat{\mathcal{M}} : \mathbb{R}^{d} \to B(L^{p}(\mathbb{R}^{d})), \quad \widehat{\mathcal{M}}(y) := \rho(y)\mathcal{M} = M_{y}\mathcal{M}M_{-y}.\]

An operator \(\mathcal{M} \in B(L^{p}(\mathbb{R}^{d}))\) is said to be \(\rho\)-almost periodic if the map \(\widehat{\mathcal{M}}\) is continuous and almost periodic in the sense of Bohr. For every \(\rho\)-almost periodic operator \(\mathcal{M}\), the function \(\widehat{\mathcal{M}}\) admits a \(B(L^{p}(\mathbb{R}^{d}))\)-valued Fourier series

\[(3.5) \quad \widehat{\mathcal{M}}(y) \sim \sum_{x \in \mathbb{R}^{d}} e^{2\pi iy \cdot x}C_{x}(\mathcal{M}) \quad (y \in \mathbb{R}^{d}).\]

The coefficients \(C_{x}(\mathcal{M}) \in B(L^{p}(\mathbb{R}^{d}))\) in (3.5) are uniquely determined by \(\mathcal{M}\) via

\[(3.6) \quad C_{x}(\mathcal{M}) = \int_{\hat{R}_{c}^{d}} \widehat{\mathcal{M}}(y)e^{-2\pi iy \cdot x}\bar{\mu}(dy) = \lim_{T \to \infty} \frac{1}{(2T)^{d}} \int_{[-T,T]^{d}} \widehat{\mathcal{M}}(y)e^{-2\pi iy \cdot x}dy\]

and, therefore, satisfy

\[(3.7) \quad \rho(y)C_{x}(\mathcal{M}) = e^{2\pi iy \cdot x}C_{x}(\mathcal{M}).\]

Hence, they are eigenvectors of \(\rho\) (see [4] for details).

Within the class of \(\rho\)-almost periodic operators we consider \(AP_{w}^{p}(\rho)\), the subclass of those operators for which the Fourier series in (3.5) is \(w\)-summable, where \(w\) is an
admissible weight. More precisely, a \( \rho \)-almost periodic operator \( \mathcal{M} \) belongs to \( AP^p_w(\rho) \) if its Fourier coefficients with respect to \( \rho \) satisfy

\[
\| \mathcal{M} \|_{AP^p_w(\rho)} := \sum_{x \in \mathbb{R}^d} \| C_x(\mathcal{M}) \|_{L^p(\mathbb{R}^d)} w(x) < +\infty.
\]

By the submultiplicativity of \( \rho \) if its Fourier coefficients with respect to \( \rho \) follow from the theory of almost-periodic series that \( M \in \rho \) if its Fourier coefficients with respect to \( \rho \).

Conversely, if \( M \) is given by (3.10), with the coefficients \( C_x \) satisfying (3.8) and (3.7). In particular, for \( y = 0 \), it follows that each \( M \in AP^p_w(\rho) \) can be written as

\[
M = \sum_{x \in \mathbb{R}^d} C_x(\mathcal{M}).
\]

Theorem 3.2 from [4] establishes the spectral invariance of \( AP^p_w(\rho) \rightarrow B(L^p(\mathbb{R}^d)) \), \( p \in [1, \infty] \) (the result there applies to a more general context). Our goal here is to establish connection between \( A_w \) and \( AP^p_w(\rho) \) and prove a spectral invariance result for \( A_w \).

To achieve this goal we first characterize the eigenvectors \( C_x \) of the representation \( \rho \).

**Lemma 1.** For any \( 1 \leq p \leq \infty \) and any \( m \in L^\infty(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), \( C_x = mT_x \) is an eigenvector of \( \rho : \mathbb{R}^d \rightarrow B(L^p(\mathbb{R}^d)) \). For \( 1 \leq p < \infty \) these are the only eigenvectors.

**Proof.** If \( C_x = mT_x \), then, according to (3.3), it satisfies (3.8).

The converse works only for \( 1 \leq p < \infty \). Suppose that \( C_x \in B(L^p(\mathbb{R}^d)) \) satisfies (3.8). Using (3.3) once again we have

\[
\rho(y)(C_x T_{-x}) = e^{2\pi iy \cdot x} C_x e^{-2\pi iy \cdot x} T_{-x} = C_x T_{-x}.
\]

It follows that \( C_x T_{-x} \) commutes with every modulation \( M_y \). Hence, \( C_x T_{-x} \) must be a multiplication operator \( m \), so \( C_x = mT_x \). \( \square \)

For \( p = \infty \) there are eigenvectors of \( \rho \) which are not of the form \( mT_x \). An example of such an eigenvector is given in [27, Section 5.1.11]. Hence, one would need additional conditions to conclude that \( C_x = mT_x \) for some \( m \in L^\infty(\mathbb{R}^d) \).

From the discussion above, \( AP^p_w(\rho) \) consists of all the operators \( \mathcal{M} = \sum_{x \in \mathbb{R}^d} C_x \), with \( C_x \) satisfying (3.8) and (3.7). In addition, by the previous lemma, for \( 1 \leq p < \infty \) an operator \( C_x \) satisfies (3.7) if and only if it is of the form \( C_x = mT_x \), for some function \( m \in L^\infty(\mathbb{R}^d) \). In this case, \( \| C_x \|_{B(L^p(\mathbb{R}^d))} = \| m \|_{\infty} \) and, thus, (3.8) reduces to (3.1). Hence we obtained

**Proposition 2.** For \( p \in [1, \infty) \) the class \( A_w \subset B(L^p(\mathbb{R}^d)) \) coincides with \( AP^p_w(\rho) \), the class of \( \rho \)-almost periodic elements, having \( w \)-summable Fourier coefficients.

For \( p = \infty \), the two classes are different. Nevertheless, the results we have obtained so far are sufficient to prove our main technical result.
Theorem 4. Let \( w \) be an admissible weight. Then, the embedding \( A_w \hookrightarrow B(L^p(\mathbb{R}^d)) \), \( p \in [1, \infty) \) is spectral. In other words, if \( M \in A_w \) defines an invertible operator \( \sum_x m_x T_x \in B(L^p(\mathbb{R}^d)) \) for some \( p \in [1, \infty) \), then \( M^{-1} \in A_w \).

Proof. For \( 1 \leq p < \infty \) the result follows from Proposition 2 and [4, Theorem 3.2]. This last result states that \( A^p_w(\rho) \) is spectral.

For \( p = \infty \) we follow a different path. Given an operator

\[
M = \sum_{x \in \mathbb{R}^d} m_x T_x \in A_w \subset B(L^\infty(\mathbb{R}^d))
\]

with \( \sum_{x \in \mathbb{R}^d} w(x) \| m_x \|_{L^\infty(\mathbb{R}^d)} < \infty \), we consider the operator

\[
N = \sum_{x \in \mathbb{R}^d} T_x (m_x) T_x = \sum_{x \in \mathbb{R}^d} m_{-x} (\cdot - x) T_x \in A_w \subset B(L^1(\mathbb{R}^d)),
\]

which is well defined since \( \| T_x (m_{-x}) \|_{L^\infty(\mathbb{R}^d)} = \| m_x \|_{L^\infty(\mathbb{R}^d)} \). By direct computation, the transpose (Banach adjoint) of \( N : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d) \) is precisely \( M : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \). Thus, \( M = N' \) and by Lax [28, Theorem 3, Chapter 20] it follows that \( N \) is invertible when \( M \) is invertible. Now, by spectrality of \( A_w \) in \( B(L^1(\mathbb{R}^d)) \) (as obtained earlier) and [28, Theorem 8(ii), Chapter 15], we obtain that \( M^{-1} = (N^{-1})' \in A_w \), that is \( M^{-1} = \sum_{x \in \mathbb{R}^d} n_x T_x \) for some bounded functions \( n_x \) such that \( \sum_{x \in \mathbb{R}^d} w(x) \| n_x \|_{L^\infty(\mathbb{R}^d)} < \infty \). \( \square \)

Remark 2. In concrete terms, Theorem 4 says that if \( M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) is an invertible operator of the form \( M = \sum_{x \in \mathbb{R}^d} m_x T_x \) with \( \{ m_x : x \in \mathbb{R}^d \} \subseteq L^\infty(\mathbb{R}^d) \) and \( \sum_x \| m_x \|_w < +\infty \), for an admissible weight \( w \), then \( M^{-1} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) can also be written as \( M^{-1} = \sum_{x \in \mathbb{R}^d} n_x T_x \), for some measurable functions \( n_x, x \in \mathbb{R}^d \) satisfying \( \sum_{x \in \mathbb{R}^d} \| n_x \|_w < +\infty \).

Remark 3. In [26] two of us used a special case of Theorem 4 for \( \rho \)-periodic (rather than \( \rho \)-almost periodic) operators in \( B(L^2(\mathbb{R}^d)) \). In [26, Example 2.1], however, we neglected to mention this restriction and erroneously implied that all of the operators in \( B(L^2(\mathbb{R}^d)) \) were \( \rho \)-periodic.

3.3. Corollaries of spectral invariance. Let us denote by \( \sigma_p(M) \) and \( \sigma_{A_w}(M) \) the spectra of the operator \( M \in A_w \) in the algebras \( B(L^p(\mathbb{R}^d)) \), \( p \in [1, \infty) \), and \( A_w \), respectively.

Corollary 1. Consider \( M = \sum_x m_x T_x \in A_w \). Then \( \sigma_p(M) = \sigma_{A_w}(M) \) for all \( p \in [1, \infty] \).

We conclude the section with the following very important result.

Theorem 5. Assume that \( M \in A_w \) satisfies \( M^* = M = \sum_x m_x T_x \) and \( A_r \| f \|_r \leq \| Mf \|_r \) for some \( A_r > 0 \) and all \( f \in L^r(\mathbb{R}^d) \) for some \( r \in [1, \infty) \). Then \( M^{-1} \in A_w \).

Moreover, suppose that \( E \subseteq W(L^p, L^q) \), \( 1 \leq p, q \leq +\infty \), is a closed subspace (in the norm of \( W(L^p, L^q) \)) such that \( ME \subseteq E \). Then \( M^{-1} E \subseteq E \) and, as a consequence, \( ME = E \).

Proof. From Corollary 1 we deduce that \( \sigma_{A_w}(M) = \sigma_r(M) = \sigma_2(M) \subset \mathbb{R} \) since \( M \in B(L^2(\mathbb{R}^d)) \) is self-adjoint. Recall that in Banach algebras every boundary point of the spectrum belongs to the approximative spectrum. The boundedness below
condition, however, implies that 0 does not belong to the approximative spectrum of $\mathcal{M} \in B(L^r(\mathbb{R}^d))$. Hence, $0 \notin \sigma_r(\mathcal{M})$ and, by Theorem 4, $\mathcal{M}^{-1} \in A_w$.

To prove the second part, let $A_w(E)$ be the subalgebra of $A_w$ formed by all those operators $S$ such that $SE \subseteq E$. Since $E$ is closed in $W(L^p, L^q_w)$ and $A_w \hookrightarrow B(W(L^p, L^q_w))$ by Proposition 1, it follows that $A_w(E)$ is a closed subalgebra of $A_w$ (we do not claim that it is closed under the involution). From the first part of the proof it follows that the set $\mathbb{C} \setminus \sigma_{A_w}(\mathcal{M})$ is connected. Consequently (see for example [11, Theorem VII 5.4]), $\sigma_{A_w(E)}(\mathcal{M}) = \sigma_{A_w}(\mathcal{M})$. Finally, $0 \notin \sigma_{A_w}(\mathcal{M}) = \sigma_{A_w(E)}(\mathcal{M})$ which proves that $\mathcal{M}^{-1} \in A_w(E)$, as desired. \qed

4. Dual Gabor frames on amalgam spaces

4.1. Multi-window Gabor frames. Let $\Lambda = \Lambda^1 \times \cdots \times \Lambda^n$ be the Cartesian product of separable lattices $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ and let $g^1, \ldots, g^n \in W(L^\infty, L^1_w)$. We consider the (multi-window) Gabor system

$$\mathcal{G} = \{ g^i_{\lambda^i} := \pi(\lambda^i)g^i \mid \lambda^i \in \Lambda^i, 1 \leq i \leq n \}.$$ 

We consider the system $\mathcal{G}$ as an indexed set, hence $\mathcal{G}$ might contain repeated elements. The frame operator of the system $\mathcal{G}$ is given by

$$S_{\mathcal{G}} = S_{g^1, \Lambda^1} + \cdots + S_{g^n, \Lambda^n},$$

where $S_{g^i, \Lambda^i} = R_{g^i, \Lambda^i} C_{g^i, \Lambda^i}$ (see Section 2.2). For $1 \leq p, q \leq +\infty$ and a $w$-moderate weight $v$, we define the space $S_{v, q}^p(\Lambda) := S_{v, q}^p(\Lambda^1) \times \cdots \times S_{v, q}^p(\Lambda^n)$ endowed with the norm

$$\|c = (c^1, \ldots, c^n)\|_{S_{v, q}^p(\Lambda)} := \sum_{i=1}^n \|c^i\|_{S_{v, q}^p(\Lambda^i)}.$$ 

The analysis map is $W(L^p, L^q_w) \ni f \mapsto C_{\mathcal{G}}(f) := (C_{g^i, \Lambda^i}(f))_{1 \leq i \leq n} \in S_{v, q}^p(\Lambda)$, while the synthesis map is $S_{v, q}^p \ni c \mapsto R_{\mathcal{G}}(c) := \sum_{i=1}^n R_{g^i, \Lambda^i}(c^i) \in W(L^p, L^q_w)$. With these definitions, the boundedness results in Theorem 1 extend immediately to the multi-window case. The frame expansions are however more complicated since the dual system of a frame of the form of $\mathcal{G}$ may not be a multi-window Gabor frame. We now investigate this matter.

4.2. Invertibility of the frame operator and expansions.

**Theorem 6.** Let $w$ be an admissible weight, $g^1, \ldots, g^n \in W(L^\infty, L^1_w)$ and $\Lambda = \Lambda^1 \times \cdots \times \Lambda^n$, with $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ separable lattices. Suppose that the Gabor system

$$\mathcal{G} = \{ g^i_{\lambda^i} := \pi(\lambda^i)g^i \mid \lambda^i \in \Lambda^i, 1 \leq i \leq n \}$$

is such that its frame operator $S_{\mathcal{G}}$ is bounded below in some $L^r(\mathbb{R}^d)$ for some $r \in [1, \infty]$, i.e.,

$$A_r \|f\|_r \leq \|S_{\mathcal{G}}f\|_r, \quad A_r > 0 \quad \text{for all } f \in L^r(\mathbb{R}^d).$$

Then the frame operator $S_{\mathcal{G}}$ is invertible on $W(L^p, L^q_w)$ for all $1 \leq p, q \leq +\infty$ and every $w$-moderate weight $v$. Moreover, the inverse operator $S_{\mathcal{G}}^{-1} : W(L^p, L^q_w) \rightarrow W(L^p, L^q_w)$ is continuous both in $\sigma(W(L^p, L^q_w), W(L^p, L^q_{1/v}))$ and the norm topologies.
Proof. For each \(1 \leq i \leq n\), the frame operator \(S_{g^i,\Lambda^i} = R_{g^i,\Lambda^i}C_{g^i,\Lambda^i}\) belongs to the algebra \(A_w\) as a consequence of the Walnut representation in Theorem 3. Hence, \(S_g = S_{g^1,\Lambda^1} + \cdots + S_{g^n,\Lambda^n} \in A_w\). Since \(S_g\) is bounded below in \(L^r(\mathbb{R}^d)\), Theorem 5 implies that \(S_g^{-1} \in A_w\). The conclusion now follows from Proposition 1. \(\square\)

We now derive the corresponding Gabor expansions.

**Theorem 7.** Under the conditions of Theorem 6, define the dual atoms by \(\tilde{g}_\lambda^i := S_g^{-1}(g_\lambda^i)\). Let \(1 \leq p,q \leq +\infty\) and \(v\) be a \(w\)-moderate weight. Then the following expansions hold:

(a) For every \(f \in W(L^p, L^q_v)\),

\[
\lim_{N,M \to \infty} \sum_{i=1}^{n} \sum_{|i| \leq N} \sum_{|j| \leq M} r_{\beta,i,M} \langle f, \tilde{g}_\lambda^i \rangle \tilde{g}_\lambda^i \in L^p \cap L^q_v
\]

(b) If \(1 < p < +\infty\) and \(q < +\infty\), for every \(f \in W(L^p, L^q_v)\),

\[
\lim_{N,M \to \infty} \sum_{i=1}^{n} \sum_{|i| \leq N} \sum_{|j| \leq M} \langle f, \tilde{g}_\lambda^i \rangle \tilde{g}_\lambda^i \in L^p \cap L^q_v
\]

\(\square\)

**Remark 4.** A more refined convergence statement including more sophisticated summability methods can be obtained using the results in [15].

**Proof.** Theorem 2 implies that for all \(f \in W(L^p, L^q_v)\),

\[
S_g(f) = \lim_{N,M \to \infty} \sum_{i=1}^{n} \sum_{|i| \leq N} \sum_{|j| \leq M} r_{\beta,i,M} \langle f, g_\lambda^i \rangle g_\lambda^i
\]

with the kind of convergence required in (a). Since \(S_g^{-1} \in A_w\), Proposition 1 implies that \(S_g^{-1} : W(L^p, L^q_v) \to W(L^p, L^q_v)\) is continuous both in the norm and \(\sigma(W(L^p, L^q_v), W(L^p, L^q_v))\)-topology. Consequently, we can apply \(S_g^{-1}\) to both sides of (4.1) to obtain the first expansion in (a). The second one follows by applying (4.1) to the function \(S_g^{-1}(f)\) and using Proposition 1 to get

\[
\langle S_g^{-1}(f), g_\lambda^i \rangle = \langle f, S_g^{-1}(g_\lambda^i) \rangle = \langle f, \tilde{g}_\lambda^i \rangle.
\]

The statement in (b) follows similarly, this time using the corresponding statement in Theorem 2. \(\square\)
4.3. Continuity of dual generators. We now apply Theorem 5 to Gabor expansions.

**Theorem 8.** In the conditions of Theorem 6, let \(1 \leq p, q \leq +\infty\) and let \(v\) be a \(w\)-moderate weight. Let \(E \subseteq W(L^p, L^q_v)\) be a closed subspace (in the norm of \(W(L^p, L^q_v)\)) such that \(S\overline{\mathcal{G}}E \subseteq E\). Suppose that the atoms \(g^1, \ldots, g^n \in E\). Then the dual atoms, \(\tilde{g}^\lambda_i = S\overline{\mathcal{G}}^{-1}(g^\lambda_i) \in E\).

**Proof.** As seen in the proof of Theorem 6, \(S\overline{\mathcal{G}} \in A_w\). Hence, the conclusion follows from Theorem 5. \(\square\)

As an application of Theorem 8 we obtain the following corollary, which was one of our main motivations. The case \(n = 1\) was an open problem in [26].

**Corollary 2.** In the conditions of Theorem 6, if all the atoms \(g^1, \ldots, g^n\) are continuous functions, so are all the dual atoms \(\tilde{g}^\lambda_i = S\overline{\mathcal{G}}^{-1}(g^\lambda_i)\).

**Proof.** We apply Theorem 8 to the subspace \(W(C_0, L^1_w)\) formed by the functions of \(W(L^\infty, L^1_w)\) that are continuous. To this end we need to observe that \(S\overline{\mathcal{G}}W(C_0, L^1_w) \subseteq W(C_0, L^1_w)\). Since \(S\overline{\mathcal{G}} = S\overline{\mathcal{G}}^{g^1, \Lambda^1} + \cdots + S\overline{\mathcal{G}}^{g^n, \Lambda^n}\), it suffices to show that each \(S\overline{\mathcal{G}}^{g^i, \Lambda^i}\) maps \(W(C_0, L^1_w)\) into \(W(C_0, L^1_w)\).

Let \(f \in W(C_0, L^1_w)\). The Walnut representation of \(S\overline{\mathcal{G}}^{g^i, \Lambda^i}\) in Theorem 3 gives \(S\overline{\mathcal{G}}^{g^i, \Lambda^i}(f) = \beta^{-d} \sum_j G^i_j T_j / \beta_i f\) with absolute convergence in the norm of \(W(L^\infty, L^1_w)\). Hence it suffices to observe that each of the functions \(G^i_j\) is continuous. According to Theorem 3 these are given by

\[
G^i_j(x) := \sum_{k \in \mathbb{Z}^d} g^i(x - j / \beta_i - \alpha_i k)g^i(x - \alpha_i k).
\]

Since the function \(g^i\) is continuous it suffices to note that in the last series the convergence is locally uniform. This is an easy consequence of the fact that \(\|g^i\|_{W(L^\infty, L^1_w)} < +\infty\). \(\square\)

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