ON THE QUOTIENT OF $\mathbb{C}^4$ BY A FINITE PRIMITIVE GROUP OF TYPE (I)

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Abstract. We study rationality problem for the quotient of $\mathbb{C}^4$ by a finite primitive group $G$ of Type (I). We prove that this quotient is a rational variety for any such $G$.

1. Introduction

Given a complex affine space $\mathbb{C}^n = \text{Spec}(\mathbb{C}[x_1, \ldots, x_n])$ and a finite group $G$ acting linearly on $\mathbb{C}^n$, one of the fundamental questions to ask is whether the field of $G$-invariant rational functions on $\mathbb{C}^n$ is also a purely transcendental extension of $\mathbb{C}$, or, in other words, whether variety $\mathbb{C}^n/G$ is rational (see [4] (and references therein) for an extensive overview of the current state of the problem). By a simple argument (see [4, Proposition 1.2]), one can show that $\mathbb{C}^n/G$ is birationally isomorphic to $(\mathbb{P}(\mathbb{C}^n)/G) \times \mathbb{P}^1$, and hence $n = 4$ is the first non-trivial issue, since the L"uroth problem has a positive solution for $n \leq 3$. The case of $n = 4$ has been treated in detail in [4]. However, for some of the groups $G$ (non-)rationality of $\mathbb{C}^4/G$ was not established.

Namely, let $\mathcal{O}, \mathcal{I} \subset SL_2(\mathbb{C})$ be the octahedron and icosahedron subgroups, respectively. Identify $U_0 := \mathbb{C}^4$ with the space of $(2 \times 2)$-matrices $A := \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, $X_i \in \mathbb{C}$, and consider the action of the group $G := \mathcal{O} \times \mathcal{I}$ on $U_0$ such that $\mathcal{O}$ and $\mathcal{I}$ act by multiplying $A$ from the left and right, respectively. Furthermore, by the above argument in order to establish rationality of $U_0/G$, one may assume that $G := (\mathcal{O} \times \mathcal{I}) \cdot \mathbb{C}^*$ for the standard diagonal action of $\mathbb{C}^*$ on $U_0$. Then for such group action, we prove the following:

**Theorem 1.1.** The 3-fold $U_0/G$ is rational.

Theorem 1.1 settles the remaining case in [4] of quotients of $\mathbb{P}^3$ (or, equivalently, $\mathbb{C}^4$) by finite primitive groups of Type (I) (see [4, Section 2] for the description of these).

Let us outline the proof of Theorem 1.1. Recall that in [4], after taking the $\mathbb{C}^*$-quotient of $U_0$ and passing to the projectivized $G$-action on $\mathbb{P}^3$, with $G$ now equal $\mathcal{O} \times \mathcal{I}$, one can notice that $\mathbb{P}^3/G$ is birationally isomorphic to $SL_2(\mathbb{C})/G$ for the induced $G$-action on $SL_2(\mathbb{C}) \subset U_0$. Further, compactifying $SL_2(\mathbb{C})$ by a smooth Fano 3-fold $W$ with either $\mathcal{O}$- or $\mathcal{I}$-action, one might try to prove that the corresponding quotient of $W$ is rational by finding an equivariant birational map of $W$ onto a product of positive-dimensional varieties (see [4, Section 2], where this idea worked perfectly well for all finite primitive groups of Type (I), except for the given $G$).

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Our approach is more direct (and simpler in a sense). Namely, let the group \( \mathbb{Z}/2\mathbb{Z} \) act on \( U_0 \) by multiplying every \( X_i \) by \(-1\), so that the \( G \)-action descends to \( U_0/(\mathbb{Z}/2\mathbb{Z}) \). A natural generalization of the construction of \( \mathbb{P}^1 \) leads to a projective compactification \( V' \) of \( U_0/(\mathbb{Z}/2\mathbb{Z}) \) (see Section 2 below).\(^1\) This \( V' \) turns out to be a Fano 4-fold with isolated terminal singularities, of Picard number 1 and Fano index 4, i.e., \( V' \) is a quadratic cone in \( \mathbb{P}^5 \) by a result of T. Fujita (see Lemma 2.15). Furthermore, the \( G \)-action on \( U_0/(\mathbb{Z}/2\mathbb{Z}) \) extends to a regular action on \( V' \), and \( V' \subset \mathbb{P}^5 \) happens to have three linearly independent \( G \)-invariant hyperplane sections (see Lemma 3.3). Then, considering the corresponding \( G \)-equivariant linear projection \( V \dashrightarrow \mathbb{P}^2 \), we split the threefold \( V'/G \) birationally into a product of positive-dimensional varieties, thus proving rationality of \( V'/G \) (see Lemma 3.4). It is now easy to see that \( U_0/G \) is also rational (see Lemma 3.5).

**Remark 1.2.** Instead of \( \mathcal{O} \times \mathcal{I} \) one may take any other finite primitive group \( G \) of Type (I) and prove that the corresponding quotient \( \mathbb{C}^4/G \) is rational, repeating literally the arguments in Sections 2 and 3 below. This gives another proof of Theorem 2.1 in [4].

**Notation.** We use standard notions and facts from [3]. Also throughout the paper we use the following notation:

1. Given two varieties \( X \) and \( Y \), \( X \approx Y \) denotes birational equivalence between them. For an algebraic group \( G \) acting regularly on both \( X \) and \( Y \), we write \( X \approx_G Y \) if there exists a \( G \)-equivariant birational map \( X \dashrightarrow Y \).

### 2. One explicit compactification

**2.1.** Take another copy \( U_1 \) of \( \mathbb{C}^4 \). Identify \( U_1 \) with the space of \((2 \times 2)\)-matrices, as \( U_0 \) above. Let \( \varphi_1 : U_0 \dashrightarrow U_1 \) be birational map induced by the morphism \( GL_2(\mathbb{C}) \rightarrow GL_2(\mathbb{C}) \) which sends every invertible matrix \( A \in U_0 \) to \( A^{-1} \in U_1 \). Set \( X_1^{(1)} := \varphi_1^{-1}(X_i), \ 1 \leq i \leq 4 \). These extend to affine coordinates on \( U_1 \). Put also \( \Delta_0 := \det A \) and \( \Delta_1 := \varphi_1^{-1}(\Delta_0) \).

Further, let \( l_{\alpha, \beta} \) be the linear automorphism of \( U_0 \) which permutes \( X_\alpha \) and \( X_\beta \) in \( A \) with \( \alpha + \beta \neq 5 \). Take another copy \( U_{\alpha, \beta} \) of \( \mathbb{C}^4 \), as \( U_0 \) and \( U_1 \) above, and consider birational map \( \varphi_{\alpha, \beta} := \varphi_1 \circ l_{\alpha, \beta} : U_0 \dashrightarrow U_{\alpha, \beta} \). Set \( X_1^{(\alpha, \beta)} := \varphi_{\alpha, \beta}^{-1}(X_i) \). These extend to affine coordinates on \( U_{\alpha, \beta} \). Put also \( \Delta_{\alpha, \beta} := \varphi_{\alpha, \beta}^{-1}(\Delta_0) \).

Now glue \( U_0, U_1, U_{\alpha, \beta} \) together via the maps \( \varphi_1, \varphi_{\alpha, \beta} \) for various \( \alpha, \beta \). We get a smooth complex 4-fold \( V \) so that \( U_0, U_1, U_{\alpha, \beta} \) are analytic domains covering \( V \). Note that \( \Delta_1 = \Delta_0^{-1} \) on \( U_0 \cap U_1 \) and \( \Delta_{\alpha, \beta} = l_{\alpha, \beta}^*(\Delta_0) \) on \( U_0 \cap U_{\alpha, \beta} \).

**Lemma 2.2.** \( V = G(2, 4) \), the Grassmanian of 2-planes in \( \mathbb{C}^4 \).

**Proof.** Evident (by definition of the complex structure on \( G(2, 4) \)). \( \square \)

**2.3.** Let us now replace each of \( U_i \) and \( U_{\alpha, \beta} \) in 2.1 by \( \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z}) \), where \( \mathbb{Z}/2\mathbb{Z} \) acts via \( X_i \mapsto -X_i, \ 1 \leq i \leq 4 \). Note that the gluing maps \( \varphi_1 \) and \( \varphi_{\alpha, \beta} \) are \((\mathbb{Z}/2\mathbb{Z})\)-equivariant, hence we can glue the six copies of \( \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z}) \) together via \( \varphi_1, \varphi_{\alpha, \beta} \) as above. We get an algebraic space \( V' \) (with \( \{U_0, U_1, U_{\alpha, \beta}\}_{\alpha, \beta} \) being an open cover of \( V' \) in the orbifold topology).

\(^1\)By “\( V' \) compactifies \( U_0/(\mathbb{Z}/2\mathbb{Z}) \)” we mean that \( C(V') = C(U_0/(\mathbb{Z}/2\mathbb{Z})) \) for the fields of meromorphic functions.
Remark 2.4. Note that the gluing maps \( \varphi_1, \varphi_{1,2}, \ldots \) on \( V' \) are rather algebraic (see [1, Chapter 1]) than analytic. Indeed, \( \varphi_1, \varphi_{1,2}, \ldots \), when lifted to the universal covers of the charts \( U_0 := \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z}) \), are only \( \mathbb{Z}/2\mathbb{Z} \)-equivariant, but not \( \mathbb{Z}/2\mathbb{Z} \)-invariant. It is easy to see, however, that the complex (scheme) structure on \( V' \) is provided by the charts \( U_0 \cup U_1, U_0 \cup U_{1,2}, \ldots \) (but not by \( \{ U_0, U_1, U_{\alpha,\beta} \\}_{\alpha,\beta} \)), glued from \( U_0, U_1, U_{1,2}, \ldots \), via \( \varphi_1, \varphi_{1,2}, \ldots \).

**Lemma 2.5.** \( V' \) is compact.

**Proof.** Let \( \Delta \subset \mathbb{C} \) be a small disk around 0. We have to prove that any (analytic) family of points \( O_t \in V' \), parameterized by \( \Delta \setminus \{ 0 \} \ni t \), extends to a family at \( t = 0 \). This follows from Lemma 2.2 and the fact that the gluing maps \( \varphi_1, \varphi_{1,2}, \ldots \) are \( \mathbb{Z}/2\mathbb{Z} \)-equivariant. \( \square \)

The next lemma is straightforward from the construction of \( V' \) (cf. Remark 2.4):

**Lemma 2.6.** \( \mathbb{C}(V') = \mathbb{C}(U_0) \).

**Remark 2.7.** One can easily see that the quotient map \( \mathbb{C}^4 \to U_0 := \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z}) \) does not induce a regular map \( V = G(2,4) \to V' \). Thus, in view of Lemma 2.6, \( V' \) is only birationally a quotient \( V/(\mathbb{Z}/2\mathbb{Z}) \).

**2.8.** Let \( D_0 \) be a divisor on \( V' \) with local equations \( \Delta_0 = 0 \) on \( U_0 \) and \( \Delta_{\alpha,\beta} = 0 \) on \( U_{\alpha,\beta} \) for all \( \alpha, \beta \) (cf. 2.1). Note that the defining equations of \( D \), when lifted to the universal covers of \( U_0, U_1, \ldots \), are \( (\mathbb{Z}/2\mathbb{Z}) \)-invariant (cf. Remark 2.4). Then the sheaf property (see [1, Chapter 2]) implies that \( D_0 \) is a Cartier divisor on \( V' \). Let \( \mathcal{L} := \mathcal{O}_{V'}(D_0) \) be the corresponding line bundle.

**Lemma 2.9.** \( D_0 \) is irreducible and \( \mathcal{L} \) carries a Hermitian metric \( | \cdot | \) such that \( 1 = | \Delta_0 | = | \Delta_{\alpha,\beta} | \) on \( U_0 \cap U_1 \) and \( U_0 \cap U_{\alpha,\beta} \) for all \( \alpha, \beta \).

**Proof.** Evident. \( \square \)

**Proposition 2.10.** \( D_0 \) is ample.

**Proof.** Let \( \theta \in H^0(V', \mathcal{L}) \) be the global section such that \( (\theta)_0 = D_0 \). Put \( \theta_0 := \theta|_{U_0} \), \( \theta_1 := \theta|_{U_1} \), \( \theta_{\alpha,\beta} := \theta|_{U_{\alpha,\beta}} \).

Restrict \( \mathcal{L} \) to \( U_0 \) and define a Hermitian metric \( h_0 \) on \( \mathcal{L}|_{U_0} \) as follows:

\[
h_0 := (1 + |X_1|^2)|\theta_0|.
\]

Then on \( U_0 \cap U_1 \), we have

\[
|\theta_1| = |\theta_0| \frac{1}{|\Delta_0|} = |\theta_0|
\]

and hence

\[
h_0 = |\theta_1| + \frac{|X_1|^2}{|\Delta_0|^2}|\theta_1| = \left( 1 + |X_1^{(1)}|^2 \right) |\theta_1|.
\]

This extends \( h_0 \) to a metric on \( \mathcal{L} \) over \( U_0 \cup U_1 \). Repeating the same construction, with \( U_1 \) replaced by \( U_{\alpha,\beta} \), we obtain a global metric on \( \mathcal{L} \), equal

\[
(1 + |X_1^{(\alpha,\beta)}|^2)|\theta_{\alpha,\beta}|
\]
on each $U_{\alpha,\beta}$. Moreover, starting with the metric
\[
h := |\theta_0| \prod_{i=1}^{4} (1 + |X_i|^2)^{1/4}
\]
on $\mathcal{L}$ over $U_0$, the same argument yields to a metric\(^2\) on $\mathcal{L}$ over $X$ which extends $h$. Let us again denote this new metric by $h$ and consider the $(1,1)$-form $\Theta := \sqrt{-1} \partial \bar{\partial} \log h \in c_1(\mathcal{L})$. Then from the Nakai–Moishezon criterion (see [2, Theorem 5.1]), we get the following:

**Lemma 2.11.** If $\sqrt{-1} \Theta > 0$, then $D_0$ is ample.

Further, the condition $\sqrt{-1} \Theta > 0$ is local, so we restrict ourselves to the chart $U_0$ (the argument is the same for $U_1$ and $U_{\alpha,\beta}$), and on $U_0$ we have
\[
\sqrt{-1} \Theta = \frac{1}{8\pi} \sum_{i=1}^{4} \frac{dX_i \wedge d\bar{X}_i}{(1 + |X_i|^2)^2} > 0.
\]

\(\square\)

2.12. There is a unique (prime) Cartier divisor $D_\infty \sim D_0$ on $V'$ with equation $\Delta_1 = 0$ on $U_1$. Indeed, one can define $D_\infty$ by taking the closure of the locus $(\Delta_1 = 0) \subset U_1$ in $V'$, and $D_\infty \sim D_0$ because of the rational map $V' \dashrightarrow \mathbb{P}^1$ which extends the map $A \mapsto \det A$ on $U_0$. Equivalently, one can notice that the divisors $D_\infty$ and $D_0 + (f)$ determine the same valuations on the function field $\mathbb{C}(V')$, where $f$ is a rational function on $V'$, equal $\Delta_0^{-1}$ on $U_0$ (cf. Remark 2.14 below). Note also that $D_0 \neq D_\infty$ (cf. the similar construction of $\mathbb{P}^1$ and of the divisors $0, \infty \in \mathbb{P}^1$).

**Lemma 2.13.** $K_{V'} \sim -4D_0$.

**Proof.** Let us start with the form $\omega := dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4$ on $U_0$. We have
\[
\hat{X}_j := d \left( \frac{X_j}{X_1X_4 - X_2X_3} \right) = \frac{dX_j}{X_1X_4 - X_2X_3} - \frac{X_j d(X_1X_4 - X_2X_3)}{(X_1X_4 - X_2X_3)^2}
\]
for all $j$, and it is easy to see that
\[
\hat{X}_1 \wedge \hat{X}_2 \wedge \hat{X}_3 \wedge \hat{X}_4 = \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{(X_1X_4 - X_2X_3)^4} - \sum_{1 \leq j \leq 4} X_j d(X_1X_4 - X_2X_3) dX_1 \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge dX_4
\]
\[
= \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{(X_1X_4 - X_2X_3)^4}.
\]
Then we get
\[
dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4 = \frac{1}{\Delta_1^4} dX_1^{(1)} \wedge dX_2^{(1)} \wedge dX_3^{(1)} \wedge dX_4^{(1)}
\]
\(^2\)Equal $|\theta_{\alpha,\beta}| \prod_{i=1}^{4} (1 + |X_i^{(\alpha,\beta)}|^2)^{1/4}$ on $U_{\alpha,\beta}$.\]
on $U_0 \cap U_1$. This extends $\omega$ to a meromorphic form on $U_0 \cup U_1$. Note that $K_{V'} = -4D_\infty \sim -4D_0$ on $U_0 \cup U_1$.

Repeating the same construction, with $U_1$ replaced by $U_{\alpha,\beta}$, we obtain a global meromorphic section of the line bundle $\mathcal{O}_{V'}(K_{V'})$, equal

$$\frac{1}{l_{\alpha,\beta}^* (\Delta_{\alpha,\beta})^4} dX_1^{(\alpha,\beta)} \wedge dX_2^{(\alpha,\beta)} \wedge dX_3^{(\alpha,\beta)} \wedge dX_4^{(\alpha,\beta)}$$

on $U_0 \cap U_{\alpha,\beta}$ for all $\alpha, \beta$. Hence $K_{V'} = -4D_\infty \sim -4D_0$ on $V'$.

Remark 2.14. It follows from the proof of Lemma 2.13 that the equation of the divisor $D_\infty$ on $U_{\alpha,\beta}$ is $l_{\alpha,\beta}^* (\Delta_{\alpha,\beta}) = 0$ for all $\alpha, \beta$.

Lemma 2.15. $V'$ is a quadratic cone with a unique singular point.

Proof. Firstly, $V'$ has only isolated terminal singularities (by definition of the latter and construction of $V'$). Now the assertion follows from Lemma 2.13, Proposition 2.10 and [3, Theorem 3.1.14].

3. Proof of Theorem 1.1

3.1. Consider $V'$ as in Section 2. Let us show that the $G$-action extends from $U_0 = \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$ to a regular action on $V'$ (note $G$ is obviously defined on $U_0$).

By construction of $V'$, every $g \in G$ determines a birational automorphism $g : V' \to V'$, regular and bijective on $U_0 \cup U_1$. Furthermore, we have $V' \setminus (U_0 \cup U_1) \subseteq D_0 \cup D_\infty$, since

$$U_0 \cup U_1 \supseteq V' \setminus (D_0 \cup D_\infty) = U_0 \cap U_1 \cap \bigcap_{\alpha,\beta} U_{\alpha,\beta}$$

(cf. 2.1 and the equations of $D_0$, $D_\infty$). Then, since $g(D \cap U_0) = D \cap U_0$, $g(D_\infty \cap U_1) = D_\infty \cap U_1$ and $D_0$, $D_\infty$ are irreducible, we obtain that $g$ is an isomorphism in codimension 2 on $V'$, and hence $g_* (D) = D$, $g_* (D_\infty) = D_\infty$ in Pic($V'$). This implies that $g$ is induced by an automorphism of $\mathbb{P}^5 \supset V'$. Thus, we get $g \in \text{Aut}(V')$ and $U_0/G \approx V'/G$ (cf. Lemma 2.6).

Remark 3.2. Note that given the embedding $U_0 := \mathbb{C}^4 \subseteq G(2,4) =: V$, the $G$-action extends from $U_0$ to $V$ by similar arguments as for $V'$ above. There is also another construction (communicated by Yu. Prokhorov) of $V$ and $G \subseteq \text{Aut}(V)$ such that compactification $V \supset U_0$ is $G$-equivariant. Indeed, take the standard compactification of $U_0 := \mathbb{C}^4$ by $\mathbb{P}^4$, with the divisor $B \subseteq \mathbb{P}^4$ at infinity, and extend the $G$-action to $\mathbb{P}^4$ in the usual way. Then there is a $G$-invariant smooth quadric $S \subset B = \mathbb{P}^3$. Let $\sigma : Y \to \mathbb{P}^4$ be the blow up of $S$ with the exceptional divisor $E := \sigma^{-1}(S)$. It is easy to see that the linear system $|2L - E|$, $L := \sigma^*(B)$, determines a birational contraction $\tilde{\sigma} : Y \to \tilde{Y}$, mapping the proper transform $\sigma^{-1}_*(B) \sim L - E$ of the divisor $B$ to a point. Moreover, since the normal bundle of $\sigma^{-1}_*(B) \simeq \mathbb{P}^3$ on $Y$ is $\mathcal{O}_{\mathbb{P}^3}(-1)$, one immediately gets that $\tilde{\sigma}$ is the blow up of a smooth point on $\tilde{Y}$. Furthermore, $\tilde{Y}$ is a (smooth) Fano 4-fold, with Pic($\tilde{Y}$) $= \mathbb{Z} \cdot \tilde{\sigma}_*(L)$ and such that $\tilde{\sigma}^*(K_Y) = K_{\tilde{Y}} - 3\sigma^{-1}_*(B) = -4L$, i.e., the Fano index of $\tilde{Y}$ is 4. Hence, by Iskovskikh and Prokhorov [3, Theorem 3.1.14], $\tilde{Y}$ is a smooth quadric in $\mathbb{P}^5$. Finally, the construction of $\tilde{Y}$ implies that both $\sigma$ and $\tilde{\sigma}$ are $G$-equivariant. Hence $\tilde{Y} (= V)$ is a $G$-equivariant
compactification of $U_0$. However, we could not obtain similar (“Italian”) construction for $V'$, since the way we have built $V'$ is not actually birational. Yet we need $V'$ to have, for instance, such properties as Lemma 3.3 below (which does not hold for the smooth quadric $V$).

**Lemma 3.3.** The space $H^0(V', \mathcal{O}_{V'}(D_0))$ contains three linearly independent $G$-invariant elements.

**Proof.** Note that $D_0$ and $D_{\infty}$ are $G$-invariant. Moreover, since $D_0$ and $D_{\infty}$ are hyperplane sections of $V' \subset \mathbb{P}^5$ which pass through the vertex $O \in V'$, there is also a smooth $G$-invariant hyperplane section $H$ of $V'$. Indeed, consider the linear projection $V' \rightarrow Q$ from $O$, with $Q \subset \mathbb{P}^4$ being a smooth quadric (cf. Lemma 2.15). Let also $f : V'' \rightarrow V'$ be the blow up of $O$. Then we get $V'' = \mathbb{P}(\mathcal{E})$ for some $\mathbb{C}^2$-vector bundle $\mathcal{E}$ over $Q$ such that the natural projection $V'' \rightarrow Q$ is $G$-equivariant.

Further, since both $\mathbb{O}, \mathbb{I} \subset G$ are simple and commute with $\mathbb{C}^*$, the class of $\mathcal{E}$ in $H^1(Q, GL_2(\mathcal{O}_Q))$ is $G$-invariant. Hence the $G$-action on $V''$ extends to the one on $\mathcal{E}$. Now, $\mathcal{E}$ admits two $G$-invariant sections, the 0-section and the one corresponding to the exceptional divisor of $f$. This implies that the $G$-action on the fibers of the projection $V'' \rightarrow Q$ coincides with the $\mathbb{C}^*$-action. The existence of the above $H$ is now evident.

Finally, $D_0, D_{\infty}$ and $H$ are (obviously) linearly independent in $H^0(V', \mathcal{O}_{V'}(D_0))$. □

**Lemma 3.4.** The 3-fold $V'/G$ is rational.

**Proof.** By Lemma 3.3, we may assume the equation of $V' \subset \mathbb{P}^5 = \text{Proj}(\mathbb{C}[x_0, \ldots, x_5])$ to be $x_0x_1 + x_2x_3 + x_3^2 = 0$, with $\mathbb{C}^* \subset G$ acting diagonally and $\mathbb{O} \times \mathbb{I} \subset G$ fixing $x_0, x_1, x_5$. Let $V' \rightarrow \mathbb{P}^2$ be the restriction to $V'$ of the linear projection from the $G$-invariant plane $\Pi := (x_2 = x_3 = x_4 = 0)$. Note that $V' \cap \Pi$ is a pair of distinct lines (with trivial $\mathbb{O} \times \mathbb{I}$-action). Then, blowing up $V'$ at $V' \cap \Pi$, we get a normal 4-fold $V'' \approx_G V'$ together with a $G$-equivariant morphism $V'' \rightarrow \mathbb{P}^2$ which has at least three $G$-invariant sections and generic fiber $\approx [\text{a quadratic cone}]$. In particular, we get

$$V' \approx_G [\text{quadratic cone with trivial } (\mathbb{O} \times \mathbb{I})\text{-action}] \times \mathbb{P}^2,$$

which implies that $V'/G$ is rational. □

**Lemma 3.5.** The 3-fold $V/G$ is rational.

**Proof.** We have

$$\mathbb{C}^4/G = \mathbb{C}^4/(\mathbb{O} \times \mathbb{I} \times \mathbb{C}^*) \simeq \mathbb{C}^4/(\mathbb{O} \times \mathbb{I} \times \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}) = U_0/G$$

for the (non-canonical) isomorphism $\mathbb{C}^* \simeq \mathbb{C}^*/(\mathbb{Z}/2\mathbb{Z})$. Now the statement follows from Lemma 3.4 because $\mathbb{C}(V'/G) = \mathbb{C}(U_0/G)$. □

Lemma 3.5 proves Theorem 1.1.

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3Indeed, we have $D_0 \cap U_0 = (X_1X_4 - X_2X_3 = 0)$, hence $O \in D_0$, and similarly for $D_{\infty}$ on $U_1$.
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