ON THE $U_p$ OPERATOR IN CHARACTERISTIC $p$

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Abstract. For a perfect field $k$ of characteristic $p > 0$, a positive integer $N$ not divisible by $p$, and an arbitrary subgroup $\Gamma$ of $\GL_2(\mathbf{Z}/N\mathbf{Z})$, we prove (with mild additional hypotheses when $p \leq 3$) that the $U$-operator on the space $M_k(\mathcal{P}_\Gamma/\kappa)$ of (Katz) modular forms for $\Gamma$ over $k$ induces a surjection $U : M_k(\mathcal{P}_\Gamma/\kappa) \twoheadrightarrow M_{k'}(\mathcal{P}_\Gamma/\kappa)$ for all $k \geq p + 2$, where $k' = (k - k_0)/p + k_0$ with $2 \leq k_0 \leq p + 1$ the unique integer congruent to $k$ modulo $p$. When $\kappa = \mathbf{F}_p$, $p \geq 5$, $N \neq 2, 3$, and $\Gamma$ is the subgroup of upper-triangular or upper-triangular unipotent matrices, this recovers a recent result of Dewar [3].

1. Introduction

Fix a prime $p$, an integer $N > 0$ with $p \nmid N$, and a subgroup $\Gamma$ of $\GL_2(\mathbf{Z}/N\mathbf{Z})$. Let $\tilde{\Gamma}$ be the preimage in $\SL_2(\mathbf{Z})$ of $\Gamma_0 := \Gamma \cap \SL_2(\mathbf{Z}/N\mathbf{Z})$, and write $\tilde{M}_k(\tilde{\Gamma})$ for the space of weight $k$ mod $p$ modular forms for $\tilde{\Gamma}$ (in the sense of Serre [8, Section 1.2]). When $N = 1$, a classical result of Serre [8, Section 2.2, Théorème 6] asserts that the $U_p$ operator is a contraction: for $k \geq p + 2$, the map $U_p : \tilde{M}_k(\Gamma(1)) \rightarrow \tilde{M}_k(\Gamma(1))$ has image contained in $\tilde{M}_{k'}(\Gamma(1))$ for some $k' < k$ satisfying $pk' \leq k + p^2 - 1$. In fact, Serre’s result may be generalized and significantly sharpened:

Theorem 1.1. Let $k$ be a perfect field of characteristic $p$ and denote by $M_k(\mathcal{P}_\Gamma/\kappa)$ the space of weight $k$ Katz modular forms for $\Gamma$ over $\kappa$ (see Section 3). Let $k_0$ be the unique integer between $2$ and $p + 1$ congruent to $k$ modulo $p$, and if $p \leq 3$, assume that $N > 4$ and that $\Gamma_0$ is a subgroup of the upper-triangular unipotent matrices. Then for $k \geq p + 2$, the $U$-operator (see Section 3) acting on $M_k(\mathcal{P}_\Gamma/\kappa)$ induces a surjection $U : M_k(\mathcal{P}_\Gamma/\kappa) \twoheadrightarrow M_{k'}(\mathcal{P}_\Gamma/\kappa)$, for $k' := (k - k_0)/p + k_0$.

When $\tilde{\Gamma} = \Gamma_*(N)$ for $* = 0, 1$ and $\kappa = \mathbf{F}_p$, the endomorphism $U$ coincides with the usual Atkin operator $U_p$ (see Corollary 3.3). In particular, if $p \geq 5$, so (by Theorems 1.7.1, and 1.8.1–1.8.2 of [5]) $\tilde{M}_k(\tilde{\Gamma}) \simeq M_k(\mathcal{P}_\Gamma/\mathbf{F}_p)$ and $N \neq 2, 3$, Theorem 1.1 is due to Dewar [3]. The proofs of both Serre’s original result and Dewar’s refinement of it rely on a delicate analysis of the interplay between the operators $U_p, V_p,$ and $\theta$ acting on mod $p$ modular forms. In the present note, we take an algebro-geometric perspective, and show how Theorem 1.1 follows immediately from a (trivial extension of a) general theorem of Tango\footnote{Tango’s paper, which appeared the year prior to Serre’s [8], is perhaps not as well-known as it should be.} [9] on the behavior of vector bundles under the Frobenius map. In this optic, the contractivity of $U_p$ in characteristic $p$ is simply an instance of the “Dwork Principle” of analytic continuation along Frobenius. In

Received by the editors December 1, 2013.
2010 Mathematics Subject Classification. Primary: 11F33, 11G18.
Key words and phrases. Mod $p$ modular forms, Atkin $U_p$-operator.

particular, we use neither the \( \theta \)-operator, nor the notion of “filtration” of a mod \( p \) modular form. Moreover, our formulation of Theorem 1.1 and its proof totally avoid the use of \( q \)-expansions, so should be readily adaptable to the Shimura curve setting.

2. **Tango’s Theorem**

Fix a perfect field \( \kappa \) of characteristic \( p \), and write \( \sigma : \kappa \to \kappa \) for the \( p \)-power Frobenius automorphism of \( \kappa \). Let \( X \) be a smooth, proper, and geometrically connected curve over \( \kappa \) of genus \( g \). Attached to \( X \) is its **Tango number**:

\[
(2.1) \quad n(X) := \max \left\{ \sum_{x \in X(\pi)} \left\lfloor \frac{\ord_x(df)}{p} \right\rfloor : f \in \pi(X) \setminus \pi(X)^p \right\},
\]

where \( \pi(X) \) is the function field of \( X_{\kappa} \). As in Lemma 10 and Proposition 14 of [9], it is easy to see that \( n(X) \) is a well-defined integer satisfying \(-1 \leq n(X) \leq \lfloor (2g - 2)/p \rfloor \), with the lower bound an equality if and only if \( g = 0 \).

**Proposition 2.1** (Tango). Let \( S \neq X \) be a reduced closed subscheme of \( X \) with ideal sheaf \( \mathcal{I}_S \subseteq \mathcal{O}_X \), and let \( \mathcal{L} \) be a line bundle on \( X \). If \( \deg \mathcal{L} > n(X) \) then the natural \( \sigma \)-linear map

\[
(2.2) \quad F^* : H^1(X, \mathcal{L}^{-1} \otimes \mathcal{I}_S) \to H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S)
\]

induced by pullback by the absolute Frobenius of \( X \) is injective, and the natural \( \sigma^{-1} \)-linear “trace map”

\[
(2.3) \quad F_* : H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathcal{L}^p) \to H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathcal{L})
\]

given by the Cartier operator ([1], [7, Section 10]) is surjective.

**Proof.** The formation of (2.2) and (2.3) is compatible, via \( \sigma \)- (respectively \( \sigma^{-1} \))-linear extension, with any scalar extension \( \kappa \to \kappa' \) to a perfect field \( \kappa' \); we may therefore assume that \( \kappa \) is algebraically closed. When \( g = 0 \) we have \( X \cong \mathbb{P}_\kappa^1 \) and the proposition is easily verified by direct calculation, so we may further assume that \( g > 0 \). As the two assertions are dual\(^2\) by Serre duality [7, Section 10, Proposition 9], it suffices to prove the injectivity of (2.2). The case \( S = \emptyset \) is Tango’s Theorem [9, Theorem 15]. In general, as \( \deg(\mathcal{L}) > 0 \) and \( \mathcal{O}_X/\mathcal{I}_S^j \) is a skyscraper sheaf for all \( j > 0 \), one finds a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & H^0(X, \mathcal{O}_X/\mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-1} \otimes \mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-1}) & \to & 0 \\
& & \downarrow F^* & \downarrow F^* & \downarrow F^* & & \\
0 & \to & H^0(X, \mathcal{O}_X/\mathcal{I}_S^p) & \to & H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S^p) & \to & H^1(X, \mathcal{L}^{-p}) & \to & 0 \\
& & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\
0 & \to & H^0(X, \mathcal{O}_X/\mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-p} \otimes \mathcal{I}_S) & \to & H^1(X, \mathcal{L}^{-p}) & \to & 0.
\end{array}
\]

in which the lower vertical arrows are induced by the inclusion \( \mathcal{I}_S^p \subseteq \mathcal{I}_S \). Using that \( \kappa = \pi \) and identifying \( H^0(X, \mathcal{O}_X/\mathcal{I}_S) \) with \( \kappa^S \), the left vertical composite is easily

\(^2\)Note that \( \kappa \)-linear duality interchanges \( \sigma \)-linear maps with \( \sigma^{-1} \)-linear ones.
seen to coincide with the map \( \oplus S \sigma : \kappa^S \to \kappa^S \) which is \( \sigma \) on each factor; it is therefore injective. As the right vertical composite map is injective by Tango’s Theorem, an easy diagram chase finishes the proof. \( \square \)

3. Modular forms mod \( p \) as differentials on the Igusa curve

In order to apply Tango’s Theorem to prove Theorem 1.1, we must recall Katz’s geometric definition of mod \( p \) modular forms, and Serre’s interpretation of them as certain meromorphic differentials on the Igusa curve.

Let us write \( R_{\Gamma} := (\mathbb{Z}[\zeta_N])^{\text{det}(\Gamma)} \), and for any \( R_{\Gamma} \)-algebra \( A \) denote by \( \mathcal{P}_{\Gamma}/A \) the moduli problem \( ([\Gamma(N)]/\Gamma)^{R_{\Gamma}-\text{can}} \otimes_{R_{\Gamma}} A \) on \( (\text{Ell}/A) \) (see Section 3.1, Section 7.1, 9.4.2, and 10.4.2 of [6]) and by \( M_k(\mathcal{P}_{\Gamma}/A) \) the space of weight \( k \) Katz modular forms for \( \mathcal{P}_{\Gamma}/A \) (e.g., [10, Section 6]) that are holomorphic at \( \infty \) in the sense of [5, Section 1.2]. Equivalently, \( M_k(\mathcal{P}_{\Gamma}/A) \) is the \( A \)-submodule of level \( N \), weight \( k \) modular forms in the sense of [2, VII.3.6] that are invariant under the natural action of \( \Gamma_0 := \Gamma \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \). Viewing \( \mathcal{C} \) as an \( R_{\Gamma} \)-algebra via \( \zeta_N \to \exp(2\pi i/N) \), we note that \( M_k(\mathcal{P}_{\Gamma}/\mathcal{C}) \) is the “classical” space of weight \( k \) modular forms for \( \Gamma \) over \( \mathcal{C} \) defined via the transcendental theory [2, VII.4].

Now fix a ring homomorphism \( R_{\Gamma} \to \kappa \) with \( \kappa \) a perfect field of characteristic \( p \). From here until the end of this section we will assume that \( \mathcal{P}_{\Gamma}/\kappa \) is representable and that \( \kappa \) acts without fixed points on the space of cusp-labels for \( \Gamma \) (see [6, Section 10.6] and cf. [6, 10.13.7–8]). We will later explain how to relax these hypotheses to those of Theorem 1.1. We write \( Y_{\Gamma} \) (respectively \( X_{\Gamma} \)) for the associated (compactified) moduli scheme; by [6, 10.13.12], one knows that \( X_{\Gamma} \) is a proper, smooth, and geometrically connected curve over \( \kappa \). Writing \( \rho : \mathcal{E} \to Y_{\Gamma} \) for the universal elliptic curve, our hypothesis that \( \kappa \) acts without fixed points on the cusp labels for \( \Gamma \) ensures that the line bundle \( \omega_{\Gamma} := \rho_* \Omega^1_{\mathcal{E}/Y_{\Gamma}} \) on \( Y_{\Gamma} \) admits a canonical extension, again denoted \( \omega_{\Gamma} \), to a line bundle on \( X_{\Gamma} \) [6, 10.13.4, 10.13.7]. By definition, \( M_k(\mathcal{P}_{\Gamma}/\kappa) = H^0(X_{\Gamma}, \omega_{\Gamma}^k) \).

Let \( I_{\Gamma} \) be the Igusa curve of level \( p \) over \( X_{\Gamma} \); by definition, \( I_{\Gamma} \) is the compactified moduli scheme associated to the simultaneous problem \( (\mathcal{P}_{\Gamma}/\kappa, [\text{Ig}(p)]) \) on \( (\text{Ell}/\kappa) \) [6, Section 12]. By [6, 12.7.2], the Igusa curve is proper, smooth, and geometrically connected, and the natural map \( \pi : I_{\Gamma} \to X_{\Gamma} \) is finite étale and Galois with group \( (\mathbb{Z}/p\mathbb{Z})^{\times} \) outside the supersingular points and totally ramified over every supersingular point. Define \( \omega := \pi^* \omega_{\Gamma} \), and recall [6, 12.8.2–3] there is a canonical section \( a \in H^0(I_{\Gamma}, \omega) \) which vanishes to order 1 at each supersingular point and on which \( d \in (\mathbb{Z}/p\mathbb{Z})^{\times} \) acts (via its action on \( I_{\Gamma} \)) through \( \chi^{-1} \), for \( \chi : (\mathbb{Z}/p\mathbb{Z})^{\times} = F_p^{\times} \to \mathbb{F}_p \), the mod \( p \) Teichmüller character. The following is a straightforward generalization of a theorem of Serre; see [6, Section 12.8] and cf. Propositions 5.7–5.10 of [4].

Proposition 3.1. Fix an integer \( k \geq 2 \). For any integer \( k_0 \leq k \) with \( 2 \leq k_0 \leq p + 1 \), the map \( f \mapsto \pi^* f / a^{k_0-2} \) induces a natural isomorphism of \( \kappa \)-vector spaces

\[
M_k(\mathcal{P}_{\Gamma}/\kappa) \cong H^0(I_{\Gamma}, \Omega^1_{I_{\Gamma}/\kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k-k_0})(\chi^{k_0-2}),
\]

where \( \delta_{k_0} = 1 \) when \( k_0 = p + 1 \) and is zero otherwise; here, ss and cusps are the reduced supersingular and cuspidal divisors, respectively.

\(^3\)Here, we follow the notation of [6, Section 9.4]: By definition \( \mathbb{Z}[\zeta_N] \) is the finite free \( \mathbb{Z} \)-algebra \( \mathbb{Z}[X]/\Phi_N(X) \), where \( \Phi_N \) is the \( N \)-th cyclotomic polynomial and \( \zeta_N \) corresponds to \( X \), equipped with its natural Galois action of \( (\mathbb{Z}/N\mathbb{Z})^{\times} \).
Proof. The proof is a straightforward adaptation of Propositions 5.7–5.10 of [4]; for the convenience of the reader, we sketch the argument. Thanks to [6, 10.13.11], the Kodaira–Spencer map [6, 10.13.10] provides an isomorphism $\omega^2 \simeq \Omega^{1}_{X_{\Gamma}/\kappa}(\text{cusps})$ of line bundles on $X_{\Gamma}$ which, after pullback along $\pi$, gives an isomorphism

$$\omega^2 \simeq \Omega^{1}_{I_{\Gamma}/\kappa}(-(p-2)\text{ss} + \text{cusps})$$

of line bundles on $I_{\Gamma}$ as $\pi$ is étale outside ss and totally (tamely) ramified at each supersingular point.

Since $a \in H^0(I_{\Gamma}, \omega)$ has simple zeroes along ss, via (3.2) any global section $f$ of $\omega^k_{I_{\Gamma}}$ induces a global section $\pi^* f/a^{k_0-2}$ of $\Omega^1_{I_{\Gamma}/\kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k-k_0}$ on which $(\mathbb{Z}/p\mathbb{Z})^\times$ acts through $\chi^{k_0-2}$; thus the map (3.1) is well-defined. As $\pi : I_{\Gamma} \to X_{\Gamma}$ is a degree $p - 1$ generically étale branched cover, the canonical trace mapping $\pi_* \mathcal{O}_{I_{\Gamma}} \to \mathcal{O}_{X_{\Gamma}}$ of locally free $\mathcal{O}_{X_{\Gamma}}$-modules induces a trace mapping $\pi_* : H^0(I_{\Gamma}, \omega^k) \to H^0(X_{\Gamma}, \omega^k_{I_{\Gamma}})$ which satisfies $\pi_* \pi^* = \deg \pi = p - 1$; it follows easily that (3.1) is injective. To prove surjectivity, observe that by (3.2), a global section of $\Omega^1_{I_{\Gamma}/\kappa}(\text{cusps} + \delta_{k_0} \cdot \text{ss}) \otimes \omega^{k-k_0}$ gives a meromorphic section $h$ of $\omega^{k-k_0+2}$ satisfying $\text{ord}_x(h) \geq -(p-1)$ at each supersingular point $x$, with equality possible only when $k_0 = p + 1$. If $h$ lies in the $(k_0 - 2)$-eigenspace of the action of $(\mathbb{Z}/p\mathbb{Z})^\times$, then $f := a^{k_0-2} h$ descends to a meromorphic section of $\omega^k_{I_{\Gamma}}$ over $X_{\Gamma}$ satisfying

$$(p-1) \text{ord}_x(f) = \text{ord}_x(h) + k_0 - 2 \geq k_0 - p - 1$$

at each supersingular point $x \in X_{\Gamma}(\overline{\kappa})$, with equality possible only when $k_0 = p + 1$. Since the left side is a multiple of $p - 1$ and $k_0 \geq 2$, we must have $\text{ord}_x(f) \geq 0$ in all cases, and $f$ is a global (holomorphic) section of $\omega^k_{I_{\Gamma}}$ over $X_{\Gamma}$ with $\pi^* f/a^{k_0-2} = h$. □

Using Proposition 3.1, the Cartier operator $F_\sigma$ on meromorphic differentials induces, by “transport of structure”, a $\sigma^{-1}$-linear map $U : M_k(\mathcal{O}_{\Gamma}/\kappa) \to M_k(\mathcal{O}_{\Gamma}/\kappa)$. If $G$ is any group of automorphisms of $X_{\Gamma}$, then the action of $G$ commutes with $F_\sigma$ (ultimately because the $p$-power map in characteristic $p$ commutes with all ring homomorphisms), and we likewise obtain a $\sigma^{-1}$-linear endomorphism $U$ of $M_k(\mathcal{O}_{\Gamma}/\kappa)^G$. This allows us to define $U$ even when $\mathcal{O}_{\Gamma}/\kappa$ is not representable as follows. Choose a prime $\ell > 3N$, and let $\Gamma'$ be the unique subgroup of GL$_2(\mathbb{Z}/\ell\mathbb{Z})$ projecting to the trivial subgroup of GL$_2(\mathbb{Z}/N\mathbb{Z})$ and to $\Gamma$ in GL$_2(\mathbb{Z}/N\mathbb{Z})$. Then for any perfect field $\kappa'$ of characteristic $p$ admitting a map from $R_{\Gamma'}$, the moduli problem $\mathcal{O}_{\Gamma'}/\kappa'$ is representable, there is a natural action of $G := \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$ on $M_k(\mathcal{O}_{\Gamma'}/\kappa')$, and one has $M_k(\mathcal{O}_{\Gamma}/\kappa') = M_k(\mathcal{O}_{\Gamma'}/\kappa')^G$ (cf. [2, VII.3.3] and [5, Section 1.2]). Via the canonical base-change isomorphism $M_k(\mathcal{O}_{\Gamma}/\kappa) \otimes_{\kappa} \kappa' \simeq M_k(\mathcal{O}_{\Gamma'}/\kappa')$, we obtain the desired endomorphism $U$ of $M_k(\mathcal{O}_{\Gamma}/\kappa)$ by descent, and it is straightforward to check that it is independent of our initial choices of $\ell$ and $\kappa'$. By post-composition with the $\sigma$-linear isomorphism$^4$ $M_k(\mathcal{O}_{\Gamma}/\kappa) \simeq M_k(\mathcal{O}_{\Gamma}/\kappa^\sigma)$ induced by the “exotic isomorphism” of moduli problems $\mathcal{O}_{\Gamma}/\kappa \simeq \mathcal{O}_{\Gamma}^{\sigma^{-1}}/\kappa$ [6, 12.10.1] we obtain a $\kappa$-linear map $U^\# : M_k(\mathcal{O}_{\Gamma}/\kappa) \to M_k(\mathcal{O}_{\Gamma}/\kappa^\sigma)$. When $\mathcal{O}_{\Gamma}$ is defined over$^5 \mathbb{F}_p$ in

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$^4$Explicitly, this isomorphism sends $f \in M_k(\mathcal{O}_{\Gamma}/\kappa)$ to the modular form $f^\sigma$ defined by $f^\sigma(E, \alpha) := f(E^\sigma, \alpha^\sigma)$.

$^5$A sufficient condition for this to happen is that $\det(\Gamma)$ contain the residue class of $p$ mod $N$. 
the sense that \( R_\Gamma \) admits a (necessarily unique) surjection to \( F_p \), one has canonically \( \mathcal{P}_t/F_p = \mathcal{P}_t^{\ell^e}/F_p \) as problems on \((\text{Ell}/F_p)\), and \( U^\# \) is an endomorphism of \( M_k(\mathcal{P}_t/F_p) \). The maps \( U \) and \( U^\# \) are natural generalizations of Atkin's \( U_p \)-operator:

**Proposition 3.2.** Suppose that \( \mathcal{P}_t/\kappa \) is representable and let \( c \) be any cusp of \( X(\Gamma) \) defined over \( \kappa \). Then \( q^{1/e} \) is a uniformizing parameter at \( c \) for some divisor \( e \) of \( N \), and for \( f \in M_k(\mathcal{P}_t/\kappa) \), the formal expansions of \( Uf \) at \( c \) and of \( U^#f \) at \( \sigma^{-1} \) are:

\[
Uf = \sum_{n \geq 0} \sigma^{-1}(a_{np})q^{n/e}, \quad \text{and} \quad U^#f = \sum_{n \geq 0} a_{np}q^{n/e},
\]

*Proof.* Using the well-known local description of the Cartier operator on meromorphic differentials (e.g., [7, Section 10, Proposition 8]), the result follows easily from the arguments of Propositions 2.8 and 5.7 of [4]; see also (the proof of) [4, Proposition 5.9]. \( \square \)

**Corollary 3.3.** Suppose that \( \tilde{\Gamma} = \Gamma_\star(N) \) for \( \star = 0, 1 \). Then \( R_\Gamma = \mathbb{Z} \) and the resulting endomorphisms \( U \) and \( U^\# \) of \( M_k(\mathcal{P}_t/F_p) \) coincide with the Atkin operator \( U_p \), whether or not \( \mathcal{P}_t/F_p \) is representable.

*Proof.* That \( R_\Gamma = \mathbb{Z} \) is clear, as \( \text{det}(\Gamma) = (\mathbb{Z}/N\mathbb{Z})^\times \). By the discussion above, we may reduce to the representable case, and the result then follows from Proposition 3.2 and the \( q \)-expansion principle. \( \square \)

### 4. Proof of Theorem 1.1

We now prove Theorem 1.1. Fix \( k \) and let \( k_0 \) and \( k' \) be as in the statement of Theorem 1.1. First suppose that \( \mathcal{P}_t \otimes_{R_\kappa} \kappa \) is representable and that \(-1\) acts without fixed points on the cusp-labels of \( \Gamma \). Using (3.2) and the fact that \( a \) has simple zeroes along \( ss \) we compute (cf. [6, 12.9.4])

\[
\deg \omega = \frac{2g - 2}{p} + \frac{1}{p} \deg(\text{cusps}) \geq \left\lfloor \frac{2g - 2}{p} \right\rfloor \geq n(I_\Gamma),
\]

where \( g \) is the genus of \( I_\Gamma \). Applying Proposition 2.1 with \( X = I_\Gamma \), \( S = \text{cusps} + \delta_{k_0} \cdot ss \), and \( \mathcal{L} = \omega \), we conclude from (2.3) and the relation \( k - k_0 = p(k' - k_0) \) that the Cartier operator

\[
F_\star : H^0(I_\Gamma, \Omega^1_{I_\Gamma/k}(S) \otimes \omega^{k-k_0}) \rightarrow H^0(I_\Gamma, \Omega^1_{I_\Gamma/k}(S) \otimes \omega^{k'-k_0})
\]

is surjective whenever \( k - k_0 \geq p \). Passing to \( \chi^{k_0-2} \)-eigenspaces for \((\mathbb{Z}/p\mathbb{Z})^\times\) and appealing to Proposition 3.1 then completes the proof in this case.

Now when \( p \leq 3 \), the hypotheses \( N > 4 \) and \( \tilde{\Gamma} \subseteq \Gamma_1(N) \) of Theorem 1.1 ensure that \( \mathcal{P}_t \otimes_{R_\kappa} \kappa \) is representable (as it maps to the moduli problem \([\Gamma_1(N)]\), which is representable for \( N \geq 4 \) by [6, 10.9.6]) and that \(-1\) acts without fixed points on the cusp-labels of \( \Gamma \) [6, 10.7.4]. If \( p \geq 5 \), we may choose a prime \( \ell > 3N \) with \( \ell \neq 0, \pm 1 \text{ mod } p \), so that \( p \nmid |\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})| \). Then for \( N' := N\ell \) and \( \Gamma' \) the subgroup \( 1 \times \Gamma \) of \( \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \text{SL}_2(\mathbb{Z}/N\ell\mathbb{Z}) \), we have (after passing to an appropriate extension \( k' \) of \( \kappa \)) that \( \mathcal{P}_t \otimes_{R_{k'}} \kappa' \) is representable with \(-1\) acting freely on the cusp-labels of \( \Gamma' \) [6, 10.7.1, 10.7.3]. We conclude that the \( U \)-operator induces a surjection of \( \kappa'[\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})]-\text{modules} \ M_k(\mathcal{P}_t/\kappa') \rightarrow M_k(\mathcal{P}_t/\kappa') \). Our choice of \( \ell \)
ensures that the ring \( \kappa'[\text{SL}_2(\mathbb{Z}/\ell\mathbb{Z})] \) is semisimple, so passing to \( \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \)-invariants is exact. As the space of \( \text{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \)-invariant weight \( k \) modular forms for \( \Gamma' \) coincides with \( M_k(\mathcal{P}/\kappa') \) (cf. the definition of \( U \) in Section 3), passing to invariants and descending from \( \kappa' \) to \( \kappa \) then completes the proof of Theorem 1.1 in the general case.

**Acknowledgments**

During the writing of this paper, the author was partially supported by an NSA Young Investigator grant (H98230-12-1-0238). We are very grateful to David Zureick-Brown for many helpful conversations.

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