HILBERT–SAMUEL MULTIPLICITIES OF CERTAIN DEFORMATION RINGS

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Abstract. We compute presentations of crystalline framed deformation rings of a two-dimensional representation \( \bar{\rho} \) of the absolute Galois group of \( \mathbb{Q}_p \), when \( \bar{\rho} \) has scalar semi-simplification, the Hodge–Tate weights are small and \( p > 2 \). In the non-trivial cases, we show that the special fibre is geometrically irreducible, generically reduced and the Hilbert–Samuel multiplicity is either 1, 2 or 4 depending on \( \bar{\rho} \). We show that in the last two cases the deformation ring is not Cohen–Macaulay.

1. Introduction

Let \( p > 2 \) be a prime. Let \( k \) be a finite field of characteristic \( p \), \( E \) be a finite totally ramified extension of \( W(k)[\frac{1}{p}] \) with ring of integers \( \mathcal{O} \) and uniformizer \( \pi \). For a given continuous representation \( \bar{\rho}: G_{\mathbb{Q}_p} \to \text{GL}_2(k) \) we consider the universal framed deformation ring \( R_{\bar{\rho}}^\square \) and the universal framed deformation \( \rho_{\text{univ}}: G_{\mathbb{Q}_p} \to \text{GL}_2(R_{\bar{\rho}}^\square) \).

For all \( \mathfrak{p} \in m\text{-Spec}(R_{\bar{\rho}}^\square[\frac{1}{p}]) \), the set of maximal ideals of \( R_{\bar{\rho}}^\square[\frac{1}{p}] \), we can specialize the universal representation at \( \mathfrak{p} \) to obtain the representation \( \rho_{\mathfrak{p}}: G_{\mathbb{Q}_p} \to \text{GL}_2(R_{\bar{\rho}}^\square[\frac{1}{p}]/\mathfrak{p}) \), where \( R_{\bar{\rho}}^\square[\frac{1}{p}]/\mathfrak{p} \) is a finite extension of \( \mathbb{Q}_p \).

Let \( \tau: I_{G_{\mathbb{Q}_p}} \to \text{GL}_2(E) \) be a representation with an open kernel, where \( I_{G_{\mathbb{Q}_p}} \) is the inertia subgroup of \( G_{\mathbb{Q}_p} \). We also fix integers \( a, b \) with \( b \geq 0 \) and a continuous character \( \psi: G_{\mathbb{Q}_p} \to \mathcal{O}^\times \) such that \( \bar{\psi}\epsilon = \det(\bar{\rho}) \), where \( \epsilon \) is the cyclotomic character. Kisin showed in [10] that there exist unique reduced \( \mathcal{O} \)-torsion free quotients \( R_{\bar{\rho}}^\square,\psi(a, b, \tau) \) and \( R_{\bar{\rho},\text{cris}}^\square(a, b, \tau) \) of \( R_{\bar{\rho}}^\square \) with the property that \( \rho_{\mathfrak{p}} \) factors through \( R_{\bar{\rho}}^\square,\psi(a, b, \tau) \) resp. \( R_{\bar{\rho},\text{cris}}^\square(a, b, \tau) \) if and only if \( \rho_{\mathfrak{p}} \) is potentially semi-stable resp. potentially crystalline with Hodge–Tate weights \( (a, a + b + 1) \) and determinant \( \psi\epsilon \) and inertial type \( \tau \). If \( \tau \) is trivial then \( R_{\bar{\rho},\text{cris}}^\square(a, b, \tau) := R_{\bar{\rho},\text{cris}}^\square(a, b, 1 \oplus 1) \) parametrizes all the crystalline lifts of \( \bar{\rho} \) with Hodge–Tate weights \( (a, a + b + 1) \) and determinant \( \psi\epsilon \). The Breuil–Mézard conjecture, proved by Kisin for almost all \( \bar{\rho} \), see also [2,3,7,8,14], says that the Hilbert–Samuel multiplicity of the ring \( R_{\bar{\rho}}^\square,\psi(a, b, \tau)/\pi \) can be determined by computing certain automorphic multiplicities, which do not depend on \( \bar{\rho} \), and the Hilbert–Samuel multiplicities of \( R_{\bar{\rho},\text{cris}}^\square(a, b) \) in low weights for \( 0 \leq a \leq p - 2, \ 0 \leq b \leq p - 1 \). For most \( \bar{\rho} \), the Hilbert–Samuel multiplicities of \( R_{\bar{\rho},\text{cris}}^\square(a, b) \) have already been determined. Our goal in this paper is to compute the...
Hilbert–Samuel multiplicity of the ring $R_{\bar{\rho},\text{cris}}(a, b)$ with $0 \leq a \leq p - 2$, $0 \leq b \leq p - 1$ when

$$\bar{\rho}: G_{\mathbb{Q}_p} \to \text{GL}_2(k), \quad g \mapsto \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}.$$  

One may show that $R_{\bar{\rho},\text{cris}}(a, b)$ is zero if either $b \neq p - 2$ or the restriction of $\chi$ to $I_{\mathbb{Q}_p}$ is not equal to $\epsilon^a$ modulo $\pi$.

**Theorem 1.** Let $a$ be an integer with $0 \leq a \leq p - 2$ such that $\chi|_{I_{\mathbb{Q}_p}} \equiv \epsilon^a \pmod{\pi}$. Then $R_{\bar{\rho},\text{cris}}(a, p - 2)/\pi$ is geometrically irreducible, generically reduced and

$$e(R_{\bar{\rho},\text{cris}}(a, p - 2)/\pi) = \begin{cases} 1 & \text{if } \bar{\rho} \otimes \chi^{-1} \text{ is ramified,} \\ 2 & \text{if } \bar{\rho} \otimes \chi^{-1} \text{ is unramified, indecomposable,} \\ 4 & \text{if } \bar{\rho} \otimes \chi^{-1} \text{ is split.} \end{cases}$$

In the last two cases, $R_{\bar{\rho},\text{cris}}(a, p - 2)$ is not Cohen–Macaulay.

The multiplicity 4 does not seem to have been anticipated in the literature, see for example [11, 1.1.6]. Our method is elementary in the sense that we do not use any integral $p$-adic Hodge theory. The only $p$-adic Hodge theoretic input is that if $\rho$ is a crystalline lift of $\bar{\rho}$ with Hodge–Tate weights $(0, p - 1)$, then we have an exact sequence

$$0 \to \epsilon^{p-1}\chi_1 \to \rho \to \chi_2 \to 0,$$

where $\chi_1, \chi_2: G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are unramified characters. This allows us to convert the problem into a linear algebra problem, which we solve in Lemma 2. This gives us an explicit presentation of the ring $R_{\bar{\rho},\text{cris}}(a, p - 2)$, using which we compute the multiplicities in Section 4. Our argument gives a proof of the existence of $R_{\bar{\rho},\text{cris}}(a, p - 2)$ independent of [10]. After writing this note we discovered that the idea to convert the problem into linear algebra already appears in [15].

### 2. The universal ring

After twisting we may assume that $\chi = 1$ and $a = 0$ so that

$$\bar{\rho}(g) = \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.$$  

Since the image of $\bar{\rho}$ in $\text{GL}_2(k)$ is a $p$-group, the universal representation factors through the maximal pro-$p$ quotient of $G_{\mathbb{Q}_p}$, which we denote by $G$. We have the following commuting diagram:

$$\begin{array}{ccc}
G_{\mathbb{Q}_p} & \longrightarrow & G \\
\downarrow & & \downarrow \\
G_{\mathbb{Q}_p}^{\text{ab}} & \longrightarrow & G_{\mathbb{Q}_p}^{\text{ab}}(p) \cong G^{\text{ab}},
\end{array}$$
Lemma 1. Let $\eta: G_{Q_p} \to \mathbb{Z}_p^\times$ be a continuous character such that $\eta \equiv 1(p)$. Then $\eta = \epsilon^k \chi$ for an unramified character $\chi$ if and only if $\eta(\gamma) = \epsilon(\gamma)^k$ and $p - 1 | k$.

Proof. “$\Rightarrow$.” Since $\gamma$ maps to identity in $Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$, we clearly have $\chi(\gamma) = 1$ for every unramified character $\chi$. Hence $\epsilon(\gamma)^k \equiv 1(p)$, which implies $p - 1 | k$.

“$\Leftarrow$.” From $\eta \epsilon^{-k}(\gamma) = 1$ and the fact that $\delta$ maps to the image of identity in the maximal pro-$p$ quotient of $Gal(\mathbb{Q}_p(\mu_{p\infty})/\mathbb{Q}_p)$, we see that $\eta \epsilon^{-k} = \chi$ for an unramified character $\chi$. \square

Since $G$ is a free pro-$p$ group generated by $\gamma$ and $\delta$, to give a framed deformation of $\rho$ to $(A, m_A)$ is equivalent to give two matrices in $GL_2(A)$ which reduce to $\bar{\rho}(\gamma)$ and $\bar{\rho}(\delta)$ modulo $m_A$. Thus

$$R^\Box_{\bar{\rho}} = \mathcal{O}[[x_{11}, x_{12}, x_{21}, t_\gamma, y_{11}, y_{12}, y_{21}, t_\delta]]$$

and the universal framed deformation is given by

$$\rho^{univ}: G \to GL_2(R^\Box_{\bar{\rho}}),$$

$$\gamma \mapsto \left( \begin{array}{cc} 1 + t_\gamma + x_{11} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} \end{array} \right),$$

$$\delta \mapsto \left( \begin{array}{cc} 1 + t_\delta + y_{11} & y_{12} \\ y_{21} & 1 + t_\delta - y_{11} \end{array} \right),$$

where $x_{12} := \hat{x}_{12} + [\phi(\gamma)]$, $y_{12} := \hat{y}_{12} + [\phi(\delta)]$ where $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$. 

where $G^ab_{Q_p} := Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ is the maximal abelian quotient of $G_{Q_p}$ and can be described by the exact sequence

$$1 \rightarrow Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{ur}) \rightarrow G^ab_{Q_p} \rightarrow G^F_{Q_p} \rightarrow 1$$

where $\mathbb{Q}_p^{ur}$ is the maximal unramified extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$. Local class field theory implies that the natural map

$$G^ab_{Q_p} \rightarrow Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times Gal(\mathbb{Q}_p(\mu_{p\infty})/\mathbb{Q}_p)$$

is an isomorphism, where $\mu_{p\infty}$ is the group of $p$-power order roots of unity in $\bar{\mathbb{Q}}_p$. The cyclotomic character $\epsilon$ induces an isomorphism

$$Gal(\mathbb{Q}_p(\mu_{p\infty})/\mathbb{Q}_p) \xrightarrow{\epsilon} \mathbb{Z}_p^\times$$

and $Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \cong \hat{\mathbb{Z}}$, hence

$$G^ab \cong (1 + p\mathbb{Z}_p) \times \mathbb{Z}_p,$$

where the map onto the first factor is given by $e^{p-1}$. We choose a pair of generators $\gamma, \delta$ of $G^ab$ such that $\gamma \mapsto (1 + p, 0)$ and $\delta \mapsto (1, 1)$. With [1, Lemma 3.2] we obtain that $G$ is a free pro-$p$ group in two letters $\gamma, \delta$ which project to $\bar{\gamma}, \bar{\delta}$. The way we choose these generators will be of importance in the following.
Remark 1. We note that there are essentially three different cases:

1. \( \bar{\rho} \) is ramified \( \iff \phi(\gamma) \neq 0 \iff x_{12} \in (R_{\bar{\rho}}^\square)^\times \);
2. \( \bar{\rho} \) is unramified, non-split \( \iff \phi(\gamma) = 0, \phi(\delta) \neq 0 \iff x_{12} \in m_{R_{\bar{\rho}}^\square}, y_{12} \in (R_{\bar{\rho}}^\square)^\times \);
3. \( \bar{\rho} \) is split \( \iff \phi(\gamma) = 0, \phi(\delta) = 0 \iff x_{12}, y_{12} \in m_{R_{\bar{\rho}}} \).

Let \( \psi: G_{\mathbb{Q}_p} \to \mathcal{O}^\times \) be a continuous character, such that \( \det(\bar{\rho}) = \overline{\psi \epsilon} \), and let \( R_{\bar{\rho}}^{\square, \psi} \) be the quotient of \( R_{\bar{\rho}}^{\square} \) which parametrizes lifts of \( \bar{\rho} \) with determinant \( \psi \epsilon \). Since \( \gamma, \delta \) generate \( G \) as a group, we obtain

\[
R_{\bar{\rho}}^{\square, \psi} \cong R_{\bar{\rho}}^{\square}/(\det(\rho_{\text{univ}}(\gamma) - \psi(\gamma)), \det(\rho_{\text{univ}}(\delta) - \psi(\delta)))
\]

\[
\cong \mathcal{O}[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}],
\]

because we can eliminate the parameters \( t_\gamma, t_\delta \) due to the relations \((1 + t_\gamma)^2 = \psi(\gamma) + x_{11}^2 + x_{12}x_{21}, t_\gamma \equiv 0(\mathfrak{m}), (1 + t_\delta)^2 = \psi(\delta) + y_{11}^2 + y_{12}y_{21}, t_\delta \equiv 0(\mathfrak{m}) \). We let \( v := \frac{1 - e^{p-1}(\gamma)}{2} \) and define four polynomials

\[
(I_1) \quad I_1 := (v + x_{11})(v - x_{11}) - x_{12}x_{21},
\]

\[
(II) \quad I_2 := (v + x_{11})^2 y_{12} - 2(v + x_{11})x_{12}y_{11} - x_{12}^2 y_{21},
\]

\[
(III) \quad I_3 := x_{21} y_{12} - 2x_{21}(v - x_{11})y_{11} - (v - x_{11})^2 y_{21},
\]

\[
(IV) \quad I_4 := (v + x_{11})x_{21} y_{12} - 2x_{12}x_{21}y_{11} - x_{12}(v - x_{11})y_{21}.
\]

Since for every representation with Hodge–Tate weights \((0, p - 1)\) the determinant is a character of Hodge–Tate weight \( p - 1 \) and \( R_{\bar{\rho}, \text{cris}}^{\square}(0, p - 2) \) parametrizes all lifts \( \rho_{\bar{\rho}} \) with determinant \( \psi \epsilon \), we let from now on \( \psi \) have Hodge–Tate weight \( p - 2 \), as otherwise \( R_{\bar{\rho}, \text{cris}}^{\square}(0, p - 2) \) would be trivial.

Definition 1. We set

\[
R := R_{\bar{\rho}}^{\square, \psi}/(I_1, I_2, I_3, I_4).
\]

Our goal is to show that \( R_{\bar{\rho}, \text{cris}}^{\square}(0, p - 2) \) is isomorphic to \( R \).

Lemma 2. If \( \mathfrak{p} \in \text{m-Spec}(R_{\bar{\rho}}^{\square, \psi}[\frac{1}{p}]) \), then \( \mathfrak{p} \in \text{m-Spec}(R[\frac{1}{p}]) \) if and only if \( \rho_{\mathfrak{p}}(\gamma) \) acts on the \( G \)-invariant subspace with eigenvalue \( e^{p-1}(\gamma) \).

Proof. Let \( \mathfrak{p} \in \text{m-Spec}(R_{\bar{\rho}}^{\square, \psi}[\frac{1}{p}]) \), such that \( \rho_{\mathfrak{p}} \) is reducible and \( \rho_{\mathfrak{p}}(\gamma) \) acts on the \( G \)-invariant subspace with eigenvalue \( e^{p-1}(\gamma) \). Since \( \det(\rho_{\mathfrak{p}}(\gamma)) = \psi(\gamma) = e(\gamma)^{p-1} \) and \( e(\gamma)^{p-1} \) is an eigenvalue of \( \rho_{\mathfrak{p}}(\gamma) \), the other eigenvalue must be 1. Therefore we can write \( 1 + t_{\gamma} = \frac{e(\gamma)^{p-1} + 1}{2} \) and obtain

\[
0 = \det \left( \begin{array}{cc}
1 + t_{\gamma} + x_{11} - e(\gamma)^{p-1} & x_{12} \\
x_{21} & 1 + t_{\gamma} - x_{11} - e(\gamma)^{p-1}
\end{array} \right)
\]

\[
= (v + x_{11})(v - x_{11}) - x_{12}x_{21}.
\]
If we now take \( p \) as above but with \( I_1 := (v + x_{11})(v - x_{11}) - x_{12}x_{21} \in p \), it is easy to see that the vectors \( v_1 = \left( \frac{-x_{12}}{v + x_{11}} \right) \) and \( v_2 = \left( \frac{v - x_{11}}{-x_{21}} \right) \) are eigenvectors for \( \rho_p(\gamma) \) with eigenvalue \( \epsilon(\gamma)^{p-1} \) if they are non-zero. But at least one of them is non-zero because otherwise we obtain \( v = 0 \) and thus \( \epsilon(\gamma)^{p-1} = 1 \), which is a contradiction to the definition of \( \gamma \). So \( \rho_p \) is reducible with an invariant subspace on which \( \rho_p(\gamma) \) acts by \( \epsilon(\gamma)^{p-1} \) if and only if the vectors \( v_1, v_2, \rho^{\text{univ}}(\delta)v_1, \rho^{\text{univ}}(\delta)v_2 \) are pairwise linear dependent. It is easy to check that this is equivalent to the satisfaction of the equations \( I_1 = I_2 = I_3 = I_4 = 0 \). □

Lemma 3.

\[
\text{m-Spec}\left( R \left[ \frac{1}{p} \right] \right) = \text{m-Spec}\left( R_p^{\square, \psi}(0, p - 2) \left[ \frac{1}{p} \right] \right).
\]

Proof. From Khare and Wintenberger [9, Proposition 3.5(i)] we know that every crystalline lift \( \rho_p \) of a reducible two-dimensional representation \( \bar{\rho} \), such that \( \rho_p \) has Hodge–Tate-weights \( (0, p - 1) \), is reducible itself. Moreover, Brinon and Conrad [4, Theorem 8.3.5] say that if \( \rho \) is a reducible two-dimensional crystalline representation, then we have an exact sequence

\[
0 \longrightarrow \epsilon^{p-1} \chi_1 \longrightarrow \rho \longrightarrow \chi_2 \longrightarrow 0.
\]

Thus \( \rho_p(\gamma) \) acts on the invariant subspace as \( \epsilon(\gamma)^{p-1} \) and hence from Lemma 2 it is clear that

\[
\text{m-Spec}\left( R \left[ \frac{1}{p} \right] \right) \supset \text{m-Spec}\left( R_p^{\square, \psi}(0, p - 2) \left[ \frac{1}{p} \right] \right).
\]

For the other inclusion we note that it is also clear from Lemma 2 that any maximal ideal \( p \in \text{m-Spec}(R[\frac{1}{p}]) \) gives rise to a reducible representation \( \rho_p \) such that \( \rho_p(\gamma) \) acts on the invariant subspace as \( \epsilon(\gamma)^{p-1} \) and that the other eigenvalue of \( \rho_p(\gamma) \) is 1. So we obtain with Lemma 1 that \( \rho_p \) is an extension of two crystalline characters

\[
0 \rightarrow \eta_1 \rightarrow * \rightarrow \eta_2 \rightarrow 0,
\]

where the Hodge–Tate weight of \( \eta_1 \) is equal to \( p - 1 \) and the weight of \( \eta_2 \) is equal to 0. Then we can conclude from [13, Proposition 128] that it is semi-stable and from [4, Theorem 8.3.5, Proposition 8.38] that it is crystalline and hence \( p \in \text{m-Spec}(R_p^{\square, \psi}(0, p - 2)\left[ \frac{1}{p} \right]) \). □

Remark 2. We have the following identities mod \( I_1 \):

\[
\begin{align*}
(5) & \quad x_{21}I_1 = (v + x_{11})I_4, \\
(6) & \quad (v - x_{11})I_2 = x_{12}I_4, \\
(7) & \quad x_{21}I_4 = (v + x_{11})I_3, \\
(8) & \quad (v - x_{11})I_4 = x_{12}I_3.
\end{align*}
\]
3. Reducedness

In order to show that $R^{\square,\psi}_{\overline{\rho}}(0, p - 2)$ is equal to $R$, it is enough to show that $R$ is reduced and $\mathcal{O}$-torsion free, since then the assertion follows from Lemma 3, as $R[\frac{1}{p}]$ is Jacobson because $R$ is a quotient of a formal power series ring over a complete discrete valuation ring.

**Lemma 4.** If $\mathcal{O} = W(k)$, then $R$ is an $W(k)$-torsion-free integral domain.

**Proof.** We distinguish two cases.

If $\overline{\rho}$ is ramified, i.e., $x_{12}$ is invertible, we consider the fact that for every complete local ring $A$ with $a \in \mathfrak{m}_A, u \in A^\times$, there is a canonical isomorphism $A[[z]]/(uz - a) \cong A$. Using this we see from (1), (2), (6) and (8) that

$$R = \mathcal{O}[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]/(I_1, I_2) \cong \mathcal{O}[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}],$$

which shows the claim.

In the second case, where $\overline{\rho}$ is unramified, i.e., $x_{12} \notin R^\times$, we consider the ideal $I := (\pi, x_{11}, x_{12}, x_{21})$ and have

$$\text{gr}_I R^{\square,\psi}_{\overline{\rho}} \cong k[[y_{11}, \hat{y}_{12}, y_{21}]]_{[\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{21}]}.$$

Since $\mathcal{O} = W(k)$ we have $v \in I \setminus I^2$ and hence the elements $I_1, I_2, I_3, I_4$ are homogeneous of degree 2, so that

$$\text{gr}_I R \cong k[[y_{11}, \hat{y}_{12}, y_{21}]]_{[\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{21}]}/(I_1, I_2, I_3, I_4),$$

see [6, Example 5.3]. Because $R$ is noetherian it follows from [6, Corollary 5.5] that it is enough to show that $\text{gr}_I R$ is an integral domain.

We define

$$A := k[[y_{11}, \hat{y}_{12}, y_{21}]]_{[\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{21}, \hat{x}_{21}]}/(I_1)$$

and look at the map

$$\phi: A \rightarrow A[x_{12}^{-1}]/(I_2).$$

The latter ring is isomorphic to $(k[[y_{11}, \hat{y}_{12}, y_{21}]]_{[\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{12}^{-1}, \hat{x}_{11}]}/(I_2))$ and since $I_2$ is irreducible it is an integral domain. So we would be done by showing that $\text{ker}(\phi) = (I_2, I_3, I_4)$. The inclusion $(I_2, I_3, I_4) \subset \text{ker}(\phi)$ is clear from (6) and (8). For the other one we consider the fact that

$$\text{ker}(\phi) = \{a \in A : \exists n \in \mathbb{N} \cup \{0\}, b, c, d \in A : x_{12}^n a = bI_2 + cI_3 + dI_4\}.$$

To show that $\text{ker}(\phi) \subset (I_2, I_3, I_4)$, we let $a \in A$ and $n$ be minimal with the property that there exist $b, c, d \in A$ such that

$$x_{12}^n a = bI_2 + cI_3 + dI_4.$$
If \( n = 0 \) there is nothing to show. Now we assume that \( n > 0 \) and consider the prime ideal \( p := (\bar{x}_{12}, \bar{v} - \bar{x}_{11}) \subset A \) and see that

\[
A/p \cong k[[y_{11}, y_{12}, y_{21}]][\bar{x}_{11}, \bar{x}_{12}]
\]

is a unique factorization domain. We also observe that

\[
\begin{align*}
I_2 & \equiv y_{12}(\bar{v} + \bar{x}_{11})^2 \mod p, \\
I_3 & \equiv y_{12}\bar{x}_{21}^2 \mod p, \\
I_4 & \equiv y_{12}(\bar{v} + \bar{x}_{11})\bar{x}_{21} \mod p.
\end{align*}
\]

Modulo \( p \) (9) becomes

\[
0 \equiv y_{12}b(\bar{v} + \bar{x}_{11})^2 + y_{12}c\bar{x}_{21}^2 + y_{12}d(\bar{v} + \bar{x}_{11})\bar{x}_{21}.
\]

Since \( A/p \) is a unique factorization domain there are \( b_1, c_1 \in A \) such that

\[
\begin{align*}
y_{12}b & \equiv b_1\bar{x}_{21} \mod p, \\
y_{12}c & \equiv c_1(\bar{v} + \bar{x}_{11}) \mod p
\end{align*}
\]

and we see that

\[
d \equiv -\frac{b_1\bar{x}_{21} + c_1(\bar{v} + \bar{x}_{11})}{2} \mod p.
\]

Hence we can find \( b_2, b_3, c_2, c_3, d_1, d_2 \in A \) such that

\[
\begin{align*}
b & = b_1\bar{x}_{21} + b_2\bar{x}_{12} + b_3(\bar{v} - \bar{x}_{11}), \\
c & = c_1(\bar{v} + \bar{x}_{11}) + c_2\bar{x}_{12} + c_3(\bar{v} - \bar{x}_{11}), \\
d & = -\frac{b_1\bar{x}_{21} + c_1(\bar{v} + \bar{x}_{11})}{2} + d_1\bar{x}_{12} + d_2(\bar{v} - \bar{x}_{11}).
\end{align*}
\]

Substituting this in (9) we get

\[
\begin{align*}
\bar{x}_{12}^n a & = b\bar{I}_2 + c\bar{I}_3 + d\bar{I}_4 \\
& = \bar{x}_{12}(b_2I_2 + b_3I_4 + c_2I_3 + d_1I_4 + d_2I_3) \\
& \quad + \frac{1}{2}(b_1(\bar{v} + \bar{x}_{11}) + c_1\bar{x}_{21})I_4 + (\bar{v} - \bar{x}_{11})c_3I_3.
\end{align*}
\]

Modulo \( p \) we have \( b_1(\bar{v} + \bar{x}_{11}) + c_1\bar{x}_{21} \equiv 0 \) and hence there are \( b_4, b_5, b_6, c_4, c_5, c_6 \) with

\[
\begin{align*}
b_1 & = b_2b_4 + \bar{x}_{12}b_5 + (\bar{v} - \bar{x}_{11})b_6, \\
c_1 & = (\bar{v} + \bar{x}_{11})c_4 + \bar{x}_{12}c_5 + (\bar{v} - \bar{x}_{11})c_6.
\end{align*}
\]

Hence we can rewrite (18) to

\[
\bar{x}_{12}^n a = \bar{x}_{12}z + \frac{1}{2}(b_4 + c_4)(\bar{v} + \bar{x}_{11})^2I_3 + (\bar{v} - \bar{x}_{11})c_3I_3
\]
for a certain \( z \in (I_2, I_3, I_4) \). So with (21) we see that \( b_4 + c_4 \equiv 0 \mod p \) and \( c_3 \equiv 0 \mod p \) modulo the prime ideal \( p' := (\bar{x}_{12}, \bar{v} + \bar{x}_{11}) \). Therefore we can find some \( c_7, c_8, e_1, e_2 \in A \) with

\[
\begin{align*}
  c_3 &= c_7 \bar{x}_{12} + c_8 (\bar{v} + \bar{x}_{11}), \\
  b_4 + c_4 &= e_1 \bar{x}_{12} + e_2 (\bar{v} - \bar{x}_{11}).
\end{align*}
\]

But since we have \((v + x_{11})(v - x_{11}) = x_{12}x_{21}\) in \( A \) we can finally transform (21) to

\[
\bar{x}_{12}^n a = \bar{x}_{12}z'
\]

for some \( z' \in (I_2, I_3, I_4) \) which shows that \( \bar{x}_{12}^{n-1} a \in (I_2, I_3, I_4) \), since \( A \) is an integral domain. But this is a contradiction to the minimality of \( n \). \( \square \)

**Proposition 1.** \( R \) is reduced and \( \mathcal{O} \)-torsion free for any choice of \( \mathcal{O} \).

**Proof.** Since \( \mathcal{O} \) is flat over \( W(k) \) and we have seen in Lemma 3 that

\[
S := W(k)[[x_{11}, \bar{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(I_1, I_2, I_3, I_4)
\]

is an integral domain, we get an injection

\[
\mathcal{O} \otimes_{W(k)} S \rightarrow \mathcal{O} \otimes_{W(k)} \text{Quot}(S).
\]

As \( S \) is \( W(k) \)-torsion free by Lemma 3, we obtain an isomorphism

\[
\mathcal{O} \otimes_{W(k)} \text{Quot}(S) \cong \mathcal{O} \left[ \frac{1}{p} \right] \otimes_{W(k) \left[ \frac{1}{p} \right]} \text{Quot}(S).
\]

Since \( \mathcal{O} \left[ \frac{1}{p} \right] \) is a separable field extension of \( W(k) \left[ \frac{1}{p} \right] \), we deduce that \( \mathcal{O} \left[ \frac{1}{p} \right] \otimes_{W(k) \left[ \frac{1}{p} \right]} \text{Quot}(S) \) is reduced and \( \mathcal{O} \)-torsion free. \( \square \)

### 4. The multiplicity

We want to compute the Hilbert–Samuel multiplicity of the ring \( R/\pi \) for the given representation

\[
\bar{\rho}: G_{Q_p} \rightarrow \text{GL}_2(k), \quad g \mapsto \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.
\]

We denote the maximal ideal of \( R/\pi \) by \( m \).

**Theorem 2.**

\[
e(R/\pi) = \begin{cases} 
1 & \text{if } \bar{\rho} \text{ is ramified,} \\
2 & \text{if } \bar{\rho} \text{ is unramified, indecomposable,} \\
4 & \text{if } \bar{\rho} \text{ is split.}
\end{cases}
\]

**Proof.** If we set \( J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21} \) we obtain modulo \( \pi \) the relations

\[
\begin{align*}
(I_2 & \equiv -x_{12}J, \\
(I_3 & \equiv x_{21}J, \\
(I_4 & \equiv x_{11}J.
\end{align*}
\]
We split the proof into three cases as in Remark 1. If $\bar{\rho}$ is ramified, i.e., $x_{12}$ is invertible, we see as in the proof of Lemma 4 that
\[
R/\pi \cong k[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J)
\cong k[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]].
\]
Hence it is a regular local ring and therefore $e(R/\pi) = 1$.

Let us assume in the following that $\bar{\rho}$ is unramified, i.e., $x_{12} = \hat{x}_{12} \in m_R$, and we can consider the exact sequence
\[
0 \to (R/\pi)/\Ann_{R/\pi}(J) \to R/\pi \to R/(\pi, J) \to 0.
\]
Since $x_{11}, x_{12}, x_{21} \in \Ann_{R/\pi}(J)$, see (22)–(24), we have $\dim((R/\pi)/\Ann_{R/\pi}(J)) \leq 3$. But $\dim R/\pi = 4$ so that (25) gives us $e(R/\pi) = e(R/(\pi, J))$, see [12, Theorem 14.6]. We obtain that
\[
R/(\pi, J) \cong k[[x_{11}, x_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J)
\cong (k[[x_{11}, x_{12}, x_{21}]]/(x_{11}^2 + x_{12}x_{21}))[y_{11}, \hat{y}_{12}, y_{21}]/(J)
\]
is a complete intersection of dimension 4. So if $q \subseteq R/(\pi, J)$ is an ideal generated by four elements, such that $R/(\pi, J, q)$ has finite length as a $R/(\pi, J)$-module, then these elements form a regular sequence in $R/(\pi, J)$ and $e_q(R/(\pi, J)) = l(R/(\pi, J, q))$, see [12, Theorem 17.11]. Besides, if there exists an integer $n$ such that $qm^n = m^{n+1}$, then $e(R/(\pi, J)) = e_q(R/(\pi, J))$, see [12, Theorem 14.13]. So to finish the proof it would suffice to find such an ideal $q$.

If $\bar{\rho}$ is indecomposable, i.e., $\phi(\delta)$ is non-zero and therefore $y_{12}$ is a unit in $R$, we can write the equation $J = 0$ as
\[
x_{21} = -y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})
\]
and $I_1 = 0$ as
\[
x_{11}^2 = x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})
\]
so that
\[
R/(\pi, J) \cong k[[x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})).
\]
Hence it is clear that $x_{12}, x_{21}, y_{11}, \hat{y}_{12}$ is a system of parameters for $R/(\pi, J)$ that generates an ideal $q$ with $qm = m^2$. So we obtain
\[
e_q(R/(\pi, J)) = l(R/(\pi, J, q)) = l(k[[x_{11}]]/(x_{11}^2)) = 2
\]
and hence $e(R/\pi) = 2$.

If $\bar{\rho}$ is split, which is equivalent to $x_{12}, y_{12} \notin R^\times$, we take $q := (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11})$ and claim that $qm^2 = m^4$. If we write $m = (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11}, x_{11}, x_{12})$ we just have to check that $x_{11}^3, x_{11}^2 x_{12}, x_{11} x_{12}^2, x_{12}^3 \in qm^2$. Therefore it is enough to see that
\[
x_{11}^2 = x_{11}y_{11} - \frac{1}{2}(x_{12} - y_{12})x_{21} - \frac{1}{2}(x_{21} - y_{21})x_{12} \in mq,
\]
\[
x_{12}^2 = -x_{11}^2 + x_{12}(x_{12} - x_{21}) \in mq.
\]
Hence  
\[ e(R/\pi) = l(R/(\pi, J, q)) = l(k[[x_{11}, x_{12}]]/(x_{11}^2, x_{12}^2)) = 4. \]

\[ \square \]

**Corollary 1.** If \( \bar{\rho} \) is unramified, then the ring \( R \) is not Cohen–Macaulay.

**Proof.** Since \( R \) is \( O \)-torsion free, \( \pi \) is \( R \)-regular and hence \( R \) is CM if and only if \( R/\pi \) is CM. In (25) we have constructed a non-zero submodule of \( R/\pi \) of dimension strictly less than the dimension of \( R/\pi \). It follows from [5, Theorem 2.1.2(a)] that \( R/\pi \) cannot be CM. \( \square \)

**Proposition 2.** Spec\((R/\pi)\) is geometrically irreducible and generically reduced.

To prove the proposition we need the following lemma. As in the proof of Theorem 2 we define \( J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21} \).

**Lemma 5.** \( R/(\pi, J) \) is an integral domain.

**Proof.** We again distinguish between three cases as in Remark 1. If \( \bar{\rho} \) is ramified, i.e., \( x_{12} \) is invertible, we have already seen in the proof of Theorem 2 that
\[ R/(\pi, J) \cong k[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J) \]
\[ \cong k[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]]. \]
If \( \bar{\rho} \) is unramified and indecomposable, i.e., \( x_{12} = \hat{x}_{12} \in m_R, y_{12} \in R^\times \) we saw that
\[ R/(\pi, J) \cong k[[x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})) \]
which is easily checked to be an integral domain. If \( \bar{\rho} \) is unramified and split, i.e., \( x_{12}, y_{12} \in m_R \), let \( n \) denote the maximal ideal of \( R/(\pi, J) \). It is enough to show that the graded ring \( \text{gr}_n R/(\pi, J) \) is a domain. Since \( J \) is homogeneous we have
\[ \text{gr}_n R/(\pi, J) \cong k[[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J). \]
We set \( A := k[[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}) \) and have to prove that \( (J) \subset A \) is a prime ideal. We look at the localization map \( A \xrightarrow{\hat{\iota}} A[y_{21}^{-1}] \), which is an inclusion because \( y_{21} \) is regular in \( A \). This gives us a map \( A \xrightarrow{\hat{\iota}} A[y_{21}^{-1}]/(J) \). Since
\[ A[y_{21}^{-1}]/(J) \cong k[[x_{11}, x_{21}, y_{11}, y_{12}, y_{21}]]/(x_{11}^2 - x_{21}y_{21}^{-1}(2x_{11}y_{11} + x_{21}y_{12})) \]
is a domain, we would be done by showing that \( \ker(\hat{\iota}) = (J) \). We have
\[ \ker(\hat{\iota}) = \{ a \in A : y_{21}^{-1}a = bJ \text{ for some } i \in \mathbb{Z}_{\geq 0}, b \in A : y_{21} \nmid b \}. \]
But since \( (y_{21}) \subset A \) is a prime ideal and \( y_{21} \) does not divide \( J \), we see that \( i = 0 \) in all these equations and hence \( \ker(\hat{\iota}) = (J) \). \( \square \)

**Proof of Proposition 2.** Let \( p \) be a minimal prime ideal of \( S := R/\pi \). It follows from (22)–(24) that \( J^2 = 0 \) and thus \( J \in \text{rad}(S) = \bigcap_p \text{ minimal } p \). So Lemma 5 gives us that \( JS \) is the only minimal prime ideal of \( S \), hence Spec\((S)\) is irreducible. If we replace the field \( k \) by an extension \( k' \), we obtain the irreducibility of Spec\((S \otimes_k k')\) analogously, thus Spec\((S)\) is geometrically irreducible.
Spec(S) is called generically reduced if \( S_p \) is reduced for any minimal prime ideal \( p \).
We have already seen that there is only one minimal prime ideal \( p = JS \).
By localizing (25) we obtain \( S_p \cong R/(\pi, J) \).
Lemma 5 implies that \( S_p \) is reduced. \( \square \)

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References


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