A note on weak convergence of singular integrals in metric spaces

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We prove that in any metric space \((X, d)\) the singular integral operators

\[ T^k_{\mu, \epsilon}(f)(x) = \int_{X \setminus B(x, \epsilon)} k(x, y) f(y) \, d\mu(y) \]

converge weakly in some dense subspaces of \(L^2(\mu)\) under minimal regularity assumptions for the measures and the kernels.

1. Introduction

A Radon measure on a metric space \((X, d)\) has \(s\)-growth if there exists some constant \(c_\mu\) such that \(\mu(B(x, r)) \leq c_\mu r^s\) for all \(x \in X, \, r > 0\).

We say that \(k(\cdot, \cdot) : X \times X \setminus \{(x, y) \in X \times X : x = y\} \to \mathbb{R}\) is an \(s\)-dimensional kernel if there exists a constant \(c > 0\) such that for all \(x, y \in X, \, x \neq y:\)

\[ |k(x, y)| \leq c d(x, y)^{-s}. \]

The kernel \(k\) is antisymmetric if \(k(x, y) = -k(y, x)\) for all distinct \(x, y \in X\).

Given a positive Radon measure \(\nu\) on \(X\) and an \(s\)-dimensional kernel \(k\), we define

\[ T^k \nu(x) := \int k(x, y) \, d\nu(y), \quad x \in X \setminus \text{spt } \nu. \]

This integral may not converge when \(x \in \text{spt } \nu\). For this reason, we consider the following \(\epsilon\)-truncated operators \(T^k_\epsilon, \, \epsilon > 0:\)

\[ T^k_\epsilon \nu(x) := \int_{d(x, y) > \epsilon} k(x, y) \, d\nu(y), \quad x \in X. \]

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Given a fixed positive Radon measure $\mu$ on $X$ and $f \in L^1_{\text{loc}}(\mu)$, we write

$$T^k_{\mu} f(x) := T^k(f \mu)(x), \quad x \in X \setminus \text{spt}(f \mu)$$

and

$$T^k_{\mu, \varepsilon} f(x) := T^k_{\varepsilon}(f \mu)(x).$$

Concerning the limit properties of the operators $T^k_{\mu, \varepsilon}$ one can ask if the limit, the so-called principal value of $T$,

$$\lim_{\varepsilon \to 0} T^k_{\mu, \varepsilon}(f)(x),$$

exists $\mu$ almost everywhere. When $\mu$ is the Lebesgue measure in $\mathbb{R}^d$, and $k$ is a standard Calderón–Zygmund kernel, due to cancellations and the denseness of smooth functions in $L^1$, the principal values exist almost everywhere for $L^1$-functions. For more general measures, the question is more complicated. Let $n$ be an integer, $0 < n < d$, and consider the coordinate Riesz kernels

$$R^n_i(x) = \frac{x_i}{|x|^{n+1}}, \quad \text{for } i = 1, \ldots, d.$$

Tolsa proved in [17] that if $E \subset \mathbb{R}^d$ has finite $n$-dimensional Hausdorff measure $\mathcal{H}^n$ the principal values

$$\lim_{\varepsilon \to 0} \int_{E \setminus B(x, \varepsilon)} \frac{x_i - y_i}{|x - y|^{m+1}} d\mathcal{H}^n(y)$$

exist $\mathcal{H}^n$ almost everywhere in $E$ if and only if the set $E$ is $n$-rectifiable i.e., if there exist $n$-dimensional Lipschitz surfaces $M_i$, $i \in \mathbb{N}$, such that

$$\mathcal{H}^n \left( E \setminus \bigcup_{i=1}^{\infty} M_i \right) = 0.$$

Mattila and Preiss had obtained the same result earlier, in [12] under some stronger assumptions for the set $E$. It becomes obvious that the existence of principal values is deeply related to the geometry of the set $E$.

Assuming $L^2(\mu)$-boundedness for the operators $T^k_{\mu}$ one could have expected that more could be deduced about the structure of $\mu$ and the existence of principal values, but this is a hard and, in a large extent, open problem. Dating from 1991 the David–Semmes conjecture, see [5], asks if the $L^2(\mu)$-boundedness of the operators associated with the $n$-dimensional Riesz kernels suffices to imply $n$-uniform rectifiabilty, which can be thought
as a quantitative version of rectifiability. In the very recent deep work [14], Nazarov et al. resolved the conjecture in the codimension 1 case, that is for 

\[ n = d - 1. \]

Mattila et al. in [11], using a special symmetrization property of the Cauchy kernel, had earlier proved the conjecture in the case of one-dimensional Riesz kernels. For all other dimensions and for other kernels few things are known. In fact, there are several examples of kernels whose boundedness does not imply rectifiability, see [1, 4, 9]. For some recent positive results involving other kernels see [2].

Let \( \mu \) be a finite Radon measure and let \( k \) be an antisymmetric kernel in a complete metric space \((X, d)\) where the Vitali covering theorem holds for \( \mu \) and the family of closed balls defined by \( d \). Mattila and Verdera in [13] showed that in this case the \( L^2(\mu) \)-boundedness of the operators \( T_{\mu, \varepsilon}^k \) forces them to converge weakly in \( L^2(\mu) \). This means that there exists a bounded linear operator \( T_{\mu}^k : L^2(\mu) \to L^2(\mu) \) such that for all \( f, g \in L^2(\mu) \),

\[
\lim_{\varepsilon \to 0} \int T_{\mu, \varepsilon}^k(f)(x)g(x) \, d\mu(x) = \int T_{\mu}^k(f)(x)g(x) \, d\mu(x).
\]

Furthermore notions of weak convergence have been recently used by Nazarov et al. [14].

Motivated by these developments it is natural to ask whether limits of this type might exist if we remove the very strong \( L^2 \)-boundedness assumption. We prove that the operators \( T_{\mu, \varepsilon}^k \) converge weakly in dense subspaces of \( L^2(\mu) \) under minimal assumptions for the measures and the kernels in general metric spaces.

The following standard definition is from [7, 2.8.16].

**Definition 1.1.** Let \( \mu \) be a measure on a metric space \((X, d)\). Let \( \mathcal{V} \) be a family of Borel sets in \( X \) and for \( x \in X \) and \( F \subset \mathcal{V} \) denote \( \mathcal{F}_x = \{ S \in \mathcal{F} : x \in S \} \). We say that \( \mathcal{V} \) is a \( \mu \)-**Vitali relation** if for any \( F \subset \mathcal{V} \) and any \( A \subset X \) such that for any \( x \in A \),

\[
\inf \{ \text{diam}(S) : S \in \mathcal{F}_x \} = 0,
\]

then the family \( \{ S \in \mathcal{F}_x : x \in A \} \) has a countable disjointed subfamily \( \mathcal{C} \) such that

\[
\mu(A \setminus \bigcup_{S \in \mathcal{C}} S) = 0.
\]

When \( X = \mathbb{R}^d \) the closed balls defined by various metrics (including the standard \( d_p \) metrics for \( 1 \leq p \leq \infty \)) are Vitali relations for any Radon measure \( \mu \) as a consequence of Besicovitch’s covering theorem, see [10, Theorem 2.8].
Furthermore, the same is true in any doubling metric measure space, that is when there exists some constant $C$ such that for all closed balls $B$, $\mu(2B) \leq C\mu(B)$, see [8, Theorem 1.6]. These facts are frequently referred to as Vitali Covering Theorems. We would also like to point out that there exist metric measure spaces where the closed balls are not Vitali relations. In [6], Davies constructed an example of a compact metric space equipped with a probability measure such that the family of closed balls is not a Vitali relation. Moreover, Preiss [15] provided an example of a Gaussian measure on a separable Hilbert space such that the family of closed balls is not a Vitali relation.

Denote by $\mathcal{X}_B$ the space of finite linear combinations of characteristic functions of balls in $(X,d)$,

$$\mathcal{X}_B = \left\{ \sum_{i=1}^{n} a_i \chi_B(z_i,r_i) : n \in \mathbb{N}, a_i \in \mathbb{R}, z_i \in X, r_i > 0 \right\}.$$ 

We remark that whenever the family of closed balls is a $\mu$-Vitali relation the space $\mathcal{X}_B$ is dense in $L^2(\mu)$.

**Theorem 1.2.** Let $\mu$ be a finite Radon measure with $s$-growth and $k$ an antisymmetric $s$-dimensional kernel on a metric space $(X,d)$. If the family of closed balls in $(X,d)$ is a $\mu$-Vitali relation then there exists subsets $\mathcal{X}_B' \subset \mathcal{X}_B$ which are dense in $L^2(\mu)$ and the weak limits

$$\lim_{\varepsilon \to 0} \int T_{\mu,\varepsilon}^k f(x) g(x) d\mu(x)$$

exist for all $f, g \in \mathcal{X}_B'$.

Until now Theorem 1.2 was only known for measures with $(d-1)$-growth in $\mathbb{R}^d$ under some smoothness assumptions for the kernels, see [3]. We thus extend the result from [3] to measures with $s$-growth for arbitrary $s$ in metric spaces where the family of closed balls is a Vitali relation without requiring any smoothness for the kernels. Our proof follows a completely different strategy using an “exponential growth” lemma for probability measures on intervals and is self-contained (unlike the proof from [3] which depends on several $L^2(\nu)$ to $L^2(\mu)$ boundedness results for separated measures $\nu$ and $\mu$).

Recall that if $k$ is the $(d-1)$-dimensional Riesz kernel in $\mathbb{R}^d$ and $\mu$ has $(d-1)$-growth and is $(d-1)$ purely unrectifiable, that is $\mu(E) = 0$ for all $(d-1)$-rectifiable sets $E$, the principal values diverge $\mu$ almost everywhere and the weak convergence in $L^2(\mu)$ fails. On the other hand, it is of interest
that weak convergence in the sense of Theorem 1.2 holds as it holds for any $s$-dimensional antisymmetric kernel and any finite measure with $s$-growth.

2. Proof of Theorem 1.2

We first prove the following lemma about exponential growth of probability measures on compact intervals. It is motivated by a similar result proved in [16]. Here Leb stands for the Lebesgue measure on the real line and $|I|$ denotes the length of an interval $I \subset \mathbb{R}$.

**Lemma 2.1.** For every integer $\lambda > 2$ the following holds. Let $\nu$ be a finite Borel measure on a compact interval $\Delta \subset \mathbb{R}$. Then for every interval $I \subset \Delta$ there exists a subset $I'(\lambda) \subset I$ such that $\text{Leb}(I'(\lambda)) > |I|(1 - 3(\lambda^{-1} + \lambda^{-2} + \cdots))$ and for every $t \in I'(\lambda)$,

$$\nu([t - \lambda^{-3n}|I|, t + \lambda^{-3n}|I|]) < \lambda^{-n}\nu(I)$$

for all integers $n \geq 1$.

**Proof.** Without loss of generality we can assume that $\nu(\Delta) \leq 1$. Let us partition the interval $I$ into $\lambda^2$ subintervals $J$ of length $|I|\lambda^{-2}$. Let $B_1$ be the family of all intervals $J$ from this partition for which $\nu(J) < \lambda^{-1}$. Obviously, there are at most $\lambda$ intervals in $B_1$. Thus

$$\#B_1 > \lambda^2 - \lambda = \lambda^2 \left(1 - \frac{\lambda}{\lambda^2}\right)$$

and

$$\text{Leb}\left(\bigcup\{J : J \in B_1\}\right) \geq |I|\left(1 - \frac{\lambda}{\lambda^2}\right) = |I|\left(1 - \frac{1}{\lambda}\right).$$

Next, each interval in $B_1$ is divided into $\lambda^2$ subintervals with disjoint interiors and of length $|I|\lambda^{-4}$, and we remove those subintervals for which $\nu(J) \geq \lambda^{-2}$. Denoting by $B_2$ the family of remaining intervals, we see that

$$\#B_2 \geq (\lambda^2)^2 \left(1 - \frac{\lambda}{\lambda^2}\right) - \lambda^2 = (\lambda^2)^2 \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

and

$$\text{Leb}\left(\bigcup\{J : J \in B_2\}\right) \geq |I|\left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right).$$

Proceeding inductively, we partition the interval $I$ into disjoint intervals of length $|I|\lambda^{-2n}$. Next, we define in the same way the family $B_n$. It is formed
by the intervals $J$ of this partition of $n$th generation, which are contained in some interval of the family $B_{n-1}$ and for which $\nu(J) < \lambda^{-n}$. Then

$$\text{Leb}\left(\bigcup\{J : J \in B_n\}\right) \geq \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2} - \cdots - \frac{1}{\lambda^n}\right)|I|.$$ 

For any $t \in I$ let $J_n = J_n(t)$ be the interval of the $n$th partition such that $t \in J_n$. Thus, for every $t \in \cap_{n=1}^{\infty} \bigcup_{J \in B_n} J$, we have that $J_n(t) \in B_n$. Consequently, for all $t \in \cap_{n=1}^{\infty} \bigcup_{J \in B_n} J$, it holds that $\nu(J_n(t)) < \lambda^{-n}$ for all $n \geq 1$. Let now

$$C_n = \{t \in I : [t - |I|\lambda^{-3n}, t + |I|\lambda^{-3n}] \subset J_n(t)\}.$$ 

It is easy to see that $\text{Leb}(C_n^c) < 2|I|\lambda^{-n}$, and, therefore,

$$\text{Leb}\left(\bigcap_{n=1}^{\infty} C_n\right) > |I|\left(1 - 2\left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots\right)\right).$$

Finally, setting

$$I' := \left(\bigcap_{n=1}^{\infty} C_n\right) \cap \left(\bigcap_{i=1}^{\infty} \bigcup_{J \in B_i} J\right)$$

completes the proof.

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**Proof of Theorem 1.2.** We can assume that $\mu(X) \leq 1$. We define finite Borel measures on the unit interval for all $z \in \text{spt } \mu$ by

$$\mu_z(F) = \mu\{x \in X : d(x, z) \in F\}, \quad F \subset [0, 1].$$

Let $G_z = \cup_{\lambda > 2} I'_z(\lambda)$ where $I'_z(\lambda)$ are the sets we obtain after we apply Lemma 2.1 to the measures $\mu_z$. Then Lemma 2.1 implies that $\text{Leb}(G_z) = \text{Leb}([0, 1])$. Let

$$\mathcal{X}_B' := \left\{\sum_{i=1}^{n} a_i \chi_{B(z_i, r_i)} : n \in \mathbb{N}, a_i \in \mathbb{R}, z_i \in \text{spt } \mu, r_i \in G_{z_i}\right\}.$$
Let \( f, g \in \mathcal{X}'_B \) such that
\[
f = \sum_{i=1}^{n} a_i \chi_{B_i} \quad \text{and} \quad g = \sum_{j=1}^{m} b_j \chi_{S_j},
\]
where \( a_i, b_j \in \mathbb{R} \) and \( B_i, S_j \) are closed balls. Then for \( 0 < \delta < \varepsilon \),
\[
\int T_{\mu, \varepsilon}^k f(x)g(x) \, d\mu(x) - \int T_{\mu, \delta}^k f(x)g(x) \, d\mu(x)
= \sum_{j=1}^{m} \sum_{i=1}^{n} a_i b_j \int_{S_j \cap B_i} k(x, y) \, d\mu(y) \, d\mu(x)
\]
Furthermore,
\[
\left| \int_{S_j \cap B_i} k(x, y) \, d\mu(y) \, d\mu(x) \right|
\leq \int_{S_j \cap B_i \cap S_j} k(x, y) \, d\mu(y) \, d\mu(x) + \int_{S_j \cap B_i \setminus S_j} k(x, y) \, d\mu(y) \, d\mu(x)
+ \int_{S_j \setminus B_i \cap S_j} k(x, y) \, d\mu(y) \, d\mu(x) + \int_{S_j \setminus B_i \setminus S_j} k(x, y) \, d\mu(y) \, d\mu(x)
\leq \int_{B_i \cap S_j} |k(x, y)| \, d\mu(y) \, d\mu(x) + 2 \int_{S_j \setminus B_i} |k(x, y)| \, d\mu(y) \, d\mu(x)
\]
The last inequality follows because by antisymmetry and Fubini’s theorem
\[
\int_{S_j \cap B_i \cap S_j} k(x, y) \, d\mu(y) \, d\mu(x) = 0.
\]
Therefore it is enough to show that for any “good” ball \( B = B(z, r) \) with \( z \in \text{spt} \mu \) and \( r \in G_z \)
\[
\lim_{0 < \delta < \varepsilon} \lim_{\varepsilon \to 0} \int_{B \setminus B^c} |k(x, y)| \, d\mu(y) \, d\mu(x) = 0,
\]
which will follow by the monotone convergence theorem if we show that

\[(2.1) \quad \int_{B} \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) < \infty.\]

Since \(B = B(z, r)\) and \(r \in G_z\) Lemma 2.1 implies that \(\mu(\partial B) = 0\) hence it is enough to show that

\[
\int_{B^o} \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) < \infty,
\]

where \(B^o\) stands for the interior of \(B\). For any \(x \in B^o\) let \(n(x) > 0\) such that

\[2^{n(x)} d(x, \partial B) = 3\]

and \(N(x) = \text{integer part of } n(x) + 1\). Therefore, since \(\text{diam}(B) \leq 2\),

\[
B(x, 2) \setminus B \subset \bigcup_{i=1}^{N(x)} B(x, 2^i d(x, \partial B)) \setminus B(x, 2^{i-1} d(x, \partial B)).
\]

Hence for all \(x \in B^o\)

\[
\int_{B(x, 2) \setminus B} |k(x,y)| d\mu(y) \leq \int_{B(x, 2) \setminus B} d(x,y)^{-s} d\mu(y)
\]

\[= \sum_{i=1}^{N(x)} \int_{B(x, 2^i d(x, \partial B)) \setminus B(x, 2^{i-1} d(x, \partial B))} d(x,y)^{-s} d\mu(y)
\]

\[\leq \sum_{i=1}^{N(x)} \mu(B(x, 2^i d(x, \partial B)) \setminus B(x, 2^{i-1} d(x, \partial B)))^{s} d\mu(y)
\]

\[\lesssim N(x) \lesssim |\log d(x, \partial B)|
\]

and

\[
\int_{B^c} |k(x,y)| d\mu(y) \lesssim \int_{B(x, 2)^c} d(x,y)^{-s} d\mu(y) + |\log d(x, \partial B)|
\]

\[\lesssim 1 + |\log d(x, \partial B)|.
\]
Since \( r \in G_z \) there exists some \( \lambda \in \mathbb{N} \) such that \( r \in I_z'(\lambda) \). We write

\[
\int_{B(z,r)^o} |\log d(x, \partial B)| d\mu(x) = \int_{B(z,r-\lambda^{-3})^o} |\log d(x, \partial B)| d\mu(x) \\
+ \sum_{n=1}^{\infty} \int_{\{x : r-\lambda^{-3n} \leq d(z,x) < r-\lambda^{-3(n+1)}\}} |\log d(x, \partial B)| d\mu(x).
\]

Notice that by Lemma 2.1

\[
\mu(\{x : r - \lambda^{-3n} \leq d(z,x) < r - \lambda^{-3(n+1)}\}) = \mu_z([r - \lambda^{-3n}, r - \lambda^{-3(n+1)}) \\
\leq \mu_z([r - \lambda^{-3n}, r + \lambda^{-3n}]) \leq \lambda^{-n}.
\]

Therefore,

\[
\int_{B(z,r)^o} |\log d(x, \partial B)| d\mu(x) \lesssim 3 \log(\lambda)(r - \lambda^{-3})^s \\
\quad + \sum_{i=1}^{n} \lambda^{-n} |\log(\lambda^{-3(n+1)})| < \infty,
\]

and this completes the proof of Theorem 1.2. \( \square \)

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