Boundary regularity of the solution to the complex Monge-Ampère equation on pseudoconvex domains of infinite type

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Let $\Omega$ be a $C^2$-smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ satisfying the “$f$-property”. The $f$-property is a consequence of the geometric “type” of the boundary. All pseudoconvex domains of finite type satisfy the $f$-property as well as many classes of domains of infinite type. In this paper, we prove the existence, uniqueness, and “weak” Hölder-regularity up to the boundary of the solution to the Dirichlet problem for the complex Monge-Ampère equation

$$\begin{cases}
\det \left[ \frac{\partial^2 (u)}{\partial z_i \partial \bar{z}_j} \right] = h \geq 0 \text{ in } \Omega, \\
u = \phi \text{ on } b\Omega.
\end{cases}$$

The idea of our proof goes back to Bedford and Taylor [1]. However, the basic geometrical ingredient is based on a recent result by Khanh [12].

1. Introduction

Let $\Omega$ be a bounded, weakly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$-smooth boundary $b\Omega$. For given functions $h \geq 0$ defined in $\Omega$ and $\phi$ defined on $b\Omega$, the Dirichlet problem for the complex Monge-Ampère equation consists in finding a continuous, plurisubharmonic function $u$ on $\Omega$ such that

$$\begin{cases}
\det [u_{ij}] = h \text{ in } \Omega, \\
u = \phi \text{ on } b\Omega,
\end{cases}$$

(1.1)

where $u_{ij} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ is the $(i, j)^{th}$-entry of $n \times n$-matrix $[u_{ij}]$. When $u$ is not $C^2(\Omega)$, the first Equation in (1.1) means that $(dd^c u)^n = h dV$ in the sense of Bedford-Taylor [1] (where $dV$ is the Lebesgue measure on $\mathbb{C}^n$).
When $\Omega$ is a smooth, bounded, strongly pseudoconvex domain in $\mathbb{C}^n$, a great deal of work has been done about the existence, uniqueness, and regularity of the solution to the complex Monge-Ampère problem. The most general related results are those obtained in [1] and [3].

- In [1], Bedford and Taylor establish the classical solvability of the Dirichlet problem (1.1). Via pluripotential theory [15], the right hand side is developed in the sense of positive currents when $u$ is continuous and plurisubharmonic. The authors prove that if $\Omega$ is a strongly pseudoconvex, bounded domain in $\mathbb{C}^n$ with $C^2$ boundary, and if $\phi \in \text{Lip}_{2\alpha}(b\Omega)$, $0 \leq h^{\frac{1}{n}} \in \text{Lip}_{\alpha}(\overline{\Omega})$, where $0 < \alpha \leq 1$, then there is a unique solution $u \in \text{Lip}_{\alpha}(\overline{\Omega})$ of (1.1). This result is sharp.

- In [3], the smoothness of the solution of (1.1) is also established. In particular, on a bounded strongly pseudoconvex domain with smooth boundary, if $\phi \in \mathcal{C}^\infty(b\Omega)$, then there exists a unique solution $u \in \mathcal{C}^\infty(\overline{\Omega})$ when $h$ is smooth and strictly positive on $\Omega$. The approach of [3] follows the continuity method applied to the real Monge-Ampère equations [9].

When $\Omega$ is not strongly pseudoconvex, there are some known results for the existence and regularity for this problem due to Blocki [2], Coman [7], and Li [18].

- In [2], Blocki also considers the Dirichlet problem (1.1) on a hyperconvex domain. He proves that when the datum $\phi \in \mathcal{C}(b\Omega)$ can be continuously extended to a plurisubharmonic function on $\Omega$ and the right hand is nonnegative and continuous, then the plurisubharmonic solution exists uniquely and is continuous. However, the Hölder continuity for the solution on these domains is still unknown.

- In [7], Coman shows how to connect some geometrical conditions on a domain in $\mathbb{C}^2$ to the existence of a plurisubharmonic upper envelope in Hölder spaces. In particular, the weak pseudoconvexity of finite type $m$ in $\mathbb{C}^2$ and the fact that the Perron-Bremermann function belongs to $\text{Lip}_{m}$ with corresponding data in $\text{Lip}_{\alpha}$ are equivalent. Again, this means that the finite type condition plays a critical role in the Hölder regularity of the solution to the complex Monge-Ampère equation.

- Li [18] studies the problem on a domain admitting a non-smooth, uniformly and strictly plurisubharmonic defining function. In particular, if $\Omega$ admits a uniformly and strictly plurisubharmonic defining function in $\text{Lip}_{\frac{2}{m}}(\overline{\Omega})$ when $0 < \alpha \leq \frac{2}{m}$, and $\phi \in \text{Lip}_{m\alpha}(b\Omega)$ and if $h^{\frac{1}{n}} \in \text{Lip}_{\alpha}(\overline{\Omega})$,
then the solution \( u \in \text{Lip}^\alpha(\Omega) \) of (1.1) exists uniquely. Based on results by Catlin [6] and by Fornaess-Sibony [8], there exists a plurisubharmonic defining function in \( \text{Lip}^{\frac{2}{m}}(\Omega) \) on pseudoconvex domains of finite type \( m \) in \( \mathbb{C}^2 \) or convex domains of finite type \( m \) in \( \mathbb{C}^n \).

The main purpose in this paper is to generalize the above results to a pseudoconvex domain, not necessarily of finite type, but admitting an \( f \)-property. The \( f \)-property assumes the existence of a bounded family of weights in the spirit of [5] and it is sufficient for an \( f \)-estimate for the \( \bar{\partial} \)-Neumann problem [5, 13]. We also notice that when \( \lim_{t \to \infty} \frac{f(t)}{\log t} = \infty \) the solution of the \( \bar{\partial} \)-Neumann problem is regular [14, 16].

**Definition 1.1.** For a smooth, monotonic, increasing function \( f : [1, +\infty) \to [1, +\infty) \) with \( \frac{f(t)}{t^{1/2}} \) decreasing, we say that \( \Omega \) has an \( f \)-property if there exist a neighborhood \( U \) of \( b\Omega \) and a family of functions \( \{\phi_\delta\} \) such that

(i) the functions \( \phi_\delta \) are plurisubharmonic, \( C^2 \) on \( U \), and satisfy \( -1 \leq \phi_\delta \leq 0 \), and

(ii) \( i\partial\bar{\partial}\phi_\delta \gtrsim f(\delta^{-1})^2 \text{Id} \) and \( |D\phi_\delta| \lesssim \delta^{-1} \) for any \( z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\} \), where \( r \) is a \( C^2 \)-defining function of \( \Omega \).

Here and in what follows, \( \lesssim \) and \( \gtrsim \) denote inequalities up to a positive constant. Moreover, we will use \( \approx \) for the combination of \( \lesssim \) and \( \gtrsim \).

**Remark 1.2.** For a pseudoconvex domain, the \( f \)-property is a consequence of the geometric finite type. In [4, 5], Catlin proves that every smooth, pseudoconvex domain \( \Omega \) of finite type \( m \) in \( \mathbb{C}^n \) has the \( f \)-property for \( f(t) = t^\epsilon \) with \( \epsilon = m - n^2 m^{-2} \). Additionally, there are several cases when \( \Omega \) is known to have the \( f \)-property with \( f(t) = t^{1/m} \) where \( m \) is the type: strongly pseudoconvex, pseudoconvex of finite type in \( \mathbb{C}^2 \), decoupled or convex in \( \mathbb{C}^n \) (cf. [6, 10, 19, 20]).

**Remark 1.3.** Khanh and Zampieri study the relationship of the general type (both finite and infinite type) and the \( f \)-property [10, 14]. They prove that if \( P_1, \ldots, P_n : \mathbb{C} \to \mathbb{R}^+ \) are functions such that \( \Delta P_j(z_j) \gtrsim \frac{F(|x_j|)}{x_j} \) or \( \frac{F(|y_j|)}{y_j} \) for any \( j = 1, \ldots, n \), then the pseudoconvex ellipsoid

\[
C = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n P_j(z_j) \leq 1\right\}
\]
has the $f$-property for $f(t) = (F^*(t^{-1}))^{-1}$. Here we denote $F^*$ is the inverse function to $F$.

In this paper, using the $f$-property we prove the “weak” Hölder regularity for the solution of the Dirichlet problem of complex Monge-Ampère equation. For this purpose we recall the definition of the $f$-Hölder spaces in [11].

**Definition 1.4.** Let $f$ be an increasing function such that $\lim_{t \to +\infty} f(t) = +\infty$, $f(t) \lesssim t$. For a subset $A$ of $\mathbb{C}^n$, define the $f$-Hölder space on $A$ by

$$\Lambda^f(A) = \{ u : \| u \|_{L^\infty(A)} + \sup_{z, w \in A, z \neq w} f(|z - w|^{-1}) \cdot |u(z) - u(w)| < \infty \}$$

and set

$$\| u \|_{\Lambda^f(A)} = \| u \|_{L^\infty(A)} + \sup_{z, w \in A, z \neq w} f(|z - w|^{-1}) \cdot |u(z) - u(w)|.$$  

Note that the notion of the $f$-Hölder space includes the standard Hölder space $\Lambda_\alpha$ by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha \leq 1$. When $1 < \alpha \leq 2$, we also define $\Lambda^{t^\alpha}(A) := \Lambda_\alpha(A)$ where

$$\Lambda_\alpha(A) = \{ u : \| u \|_{\Lambda^{t^\alpha}(A)} := \| Du \|_{\Lambda^{t^\alpha-1}(A)} < \infty \}.$$  

The main result in this paper consists in the following:

**Theorem 1.5.** Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain admitting the $f$-property. Suppose that the function $g : [1, \infty) \to [1, \infty)$ defined by

$$g(t)^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty.$$  

If $0 < \alpha \leq 2$, $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and $h \geq 0$ on $\Omega$ with $h^{1/\alpha} \in \Lambda^{\alpha/\alpha}(\Omega)$, then the Dirichlet problem for the complex Monge-Ampère equation

$$(1.2) \begin{cases} 
\det(u_{ij}) = h & \text{in } \Omega, \\
u = \phi & \text{on } b\Omega,
\end{cases}$$

has a unique plurisubharmonic solution $u \in \Lambda^{\alpha/\alpha}(\overline{\Omega})$.

By Remarks 1.2 and 1.3, we immediately have the following.
Corollary 1.6. 1) Let $\Omega$ be a bounded, $C^2$-boundary, pseudoconvex domain of finite type $m$ in $\mathbb{C}^n$ satisfying at least one of the following conditions: strongly pseudoconvex, convex, in $C^2$, or decoupled. If $0 < \alpha \leq 2$, $\phi \in \text{Lip}^\alpha(b\Omega)$, and $h \geq 0$ on $\Omega$ with $h^\frac{1}{n} \in \text{Lip}^\frac{n}{2}(\Omega)$, then $(1.2)$ has a unique plurisubharmonic solution $u \in \text{Lip}^\frac{n}{2}(\Omega)$. If $\Omega$ has finite type $m$, but does not satisfy any one of the above additional conditions, then it is still true that $u \in \text{Lip}^\frac{m}{n}(\Omega)$.

2) Let $\Omega$ be a complex ellipsoid defined by

$$
\Omega = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n \exp\left( -\frac{1}{|z_j|^{s_j}} \right) < e^{-1} \right\}.
$$

If $s := \max_{j=1,\ldots,n} \{s_j\} < 1$, then under the assumption of $\phi$, $h$ and $u$ in Theorem 1.5, we have $u \in \Lambda^{g}(\overline{\Omega})$ where $0 < \alpha \leq 2$ and $g(t) = \log^{\frac{1}{s}} - 1 t$.

We organize the paper as follows. In Section 2, we construct a weak Hölder, uniformly and strictly plurisubharmonic defining function via the work of the second author on peak functions [12]. This particular defining function is the crucial point in the establishing the existence of the solution to the complex Monge-Ampère equation. Following the work by Bedford-Taylor [1], we prove Theorem 1.5 in Section 3.

2. The $f$-property

In this section, under the $f$-property assumption we construct a uniformly and strictly plurisubharmonic defining function with $g^2$-Hölder, where $g$ defined in the following theorem.

Theorem 2.1. Let $f$ be as in Definition 1.1 with $g(t)^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty$. Assume that $\Omega$ is a bounded, pseudoconvex domain admitting the $f$-property. Then there exists a strictly plurisubharmonic defining function of $\Omega$ which belongs to $g^2$-Hölder space of $\overline{\Omega}$, that means, there is a plurisubharmonic function $\rho$ such that

1. $z \in \Omega$ if and only if $\rho(z) < 0$, $b\Omega = \{ z \in \mathbb{C}^n : \rho(z) = 0 \}$;
2. $i\partial \overline{\partial} \rho(X, \overline{X}) \geq |X|^2$ on $\Omega$ in the distribution sense, for any $X \in T^{1,0}\mathbb{C}^n$; and
3. $\rho$ is in the $g^2$-Hölder space of $\overline{\Omega}$, that is, $|\rho(z) - \rho(z')| \lesssim g(|z - z'|^{-1})^{-2}$ for any $z, z' \in \overline{\Omega}$.
Remark 2.2. We note that if $\Omega$ is strongly pseudoconvex then $f(t) \approx t^{1/2}$ and hence $g(t) \approx t^{1/2}$. In this case, it is easy to choose a defining function satisfying this theorem. So in the following we only consider that $\Omega$ is not strongly pseudoconvex, in this case, we can assume that there exists an $\epsilon > 0$ so that $\frac{f(t)}{t^{1/2}}$ is decreasing on $(1, +\infty)$.

The proof of Theorem 2.1 is based on the following result about the existence of a family of plurisubharmonic peak functions which was recently proven by Khanh [12].

Theorem 2.3. Under the assumptions of Theorem 2.1, for any $\zeta \in \partial \Omega$, there exists a $C^2$ plurisubharmonic function $\psi_\zeta$ on $\Omega$ which is continuous on $\overline{\Omega}$ and peaks at $\zeta$ (that means, $\psi_\zeta(z) < 0$ for all $z \in \overline{\Omega} \setminus \{\zeta\}$ and $\psi_\zeta(\zeta) = 0$). Moreover, there are some positive constants $c_1, c_2$ and $c_3$ such that the following hold for any constant $0 < \eta < 1$:

1. $|\psi_\zeta(z) - \psi_\zeta(z')| \leq c_1|z - z'|^\eta$ for any $z, z' \in \overline{\Omega}$; and
2. $g\left(( -\psi_\zeta(z))^{-1/\eta}\right) \leq c_2|z - \zeta|^{-1}$ for any $z \in \overline{\Omega} \setminus \{\zeta\}$.

Before giving the proof of Theorem 2.1, we need the following technical lemma.

Lemma 2.4. Let $g$ and $\eta$ be in Theorem 2.3. For $\delta \in (0, 1)$, let $\omega(\delta) := g(\delta^{-\frac{1}{\eta}})^{-2}$. Then we have

(i) $\omega$ is increasing function on $(0, 1)$ and $\lim_{\delta \to 0^+} \omega(\delta) = 0$;

(ii) for a suitable choice of $\eta > 0$, $\omega$ is concave downward on $(0, 1)$, i.e., $\dot{\omega}(\delta) \leq 0$ for $\delta \in (0, 1)$;

(iii) the inequality $|\omega(\delta) - \omega(\delta')| \leq \omega(|\delta - \delta'|)$ holds for any $\delta, \delta' \in (0, 1)$; and

(iv) for a constant $c > 0$, there is $c' > 0$ such that $\omega(c\delta) \leq c'\omega(\delta)$ for $\delta \in (0, 1)$.

Proof. We first investigate the function $g$. By the definition of $g$, i.e., $\frac{1}{g(t)} := \int_t^\infty \frac{da}{a f(a)} < \infty$, we have

\[ \frac{\dot{g}(t)}{g(t)} = \frac{g(t)}{tf(t)}. \]
and
\begin{equation}
\frac{\dot{g}(t)}{g(t)} = \frac{2\dot{g}(t)}{g(t)} - \frac{1}{t} - \frac{\dot{f}(t)}{f(t)}.
\end{equation}

By Remark 2.2, there exists an \( \epsilon > 0 \) so that \( \frac{f(t)}{t^{2-\epsilon}} \) is decreasing on \((1, \infty)\), we obtain \( \frac{t\dot{f}(t)}{f(t)} \leq \frac{1}{2} - \epsilon \) for any \( t \in (0, \infty) \). We also have
\[
\frac{f(t)}{g(t)} = f(t) \int_t^\infty \frac{da}{af(a)} = f(t) \int_t^\infty \frac{a^{1/2}}{f(a)} \cdot \frac{da}{a^{3/2}} \geq f(t) \int_t^\infty \frac{da}{a^{2}} = 2,
\]
for any \( t \in (1, \infty) \). Then
\begin{equation}
\frac{g(t)}{f(t)} + \frac{t\dot{f}(t)}{f(t)} \leq 1 - \epsilon, \quad \text{for any } t \in (1, \infty).
\end{equation}

Now we prove this lemma. The proof of (i) immediately follows by the
the first derivative of \( \omega \)
\[
\dot{\omega}(\delta) = \frac{2}{\eta} \delta^{-\frac{1}{\eta}} g(\delta^{-\frac{1}{\eta}}) \delta^{-3} g^{-3}(\delta^{-\frac{1}{\eta}}) \geq 0.
\]
For (ii), we have
\begin{equation}
\ddot{\omega}(\delta) = - \left( \frac{2}{\eta^2} \delta^{-\frac{1}{\eta}} - 2 \dot{g}(\delta^{-\frac{1}{\eta}}) \delta^{-3} g^{-3}(\delta^{-\frac{1}{\eta}}) \right) \left[ \eta + 1 + \frac{\delta^{-\frac{1}{\eta}} g(\delta^{-\frac{1}{\eta}})}{\dot{g}(\delta^{-\frac{1}{\eta}})} - 3 \frac{\delta^{-\frac{1}{\eta}} g(\delta^{-\frac{1}{\eta}})}{g(\delta^{-\frac{1}{\eta}})} \right]
\end{equation}
where the second equality follows from (2.2). From (2.3) there is a constant \( \eta < 1 \) such that the bracket term \([\ldots]\) in the last line of (2.4) is positive, particularly we choose \( \eta = \frac{1}{2} \left( 1 + \sup_{t \in (1, \infty)} \left( \frac{g(t)}{f(t)} + \frac{t\dot{f}(t)}{f(t)} \right) \right) \leq 1 - \frac{\xi}{2} < 1 \). Therefore, \( \ddot{\omega} \leq 0 \), i.e., \( \omega \) is concave downward.

Now we prove that \(|\omega(t) - \omega(s)| \leq \omega(|t - s|)\) for any \( t, s \in (0, \delta) \). Assume \( t \geq s \) for some fixed \( s \in [0, \delta] \) and set \( k(t) := \omega(t) - \omega(s) - \omega(t - s) \). Since \( \omega \) is concave downward, \( k(t) = \ddot{\omega}(t) - \omega(t - s) \leq 0 \). That means \( k \) is decreasing, so we obtain \( k(t) \leq k(s) = 0 \). This completes the proof of (iii).

For the inequality (iv), we notice that if \( c \leq 1 \) then \( \omega(c\delta) \leq \omega(\delta) \) since \( \omega \) is increasing. Otherwise, if \( c > 1 \) we use the fact that \( \frac{g(t)}{t^{2\eta}} \) is decreasing for
large $t$ (this is obtained from $\frac{t\dot{g}(t)}{g(t)} = \frac{g(t)}{f(t)} \leq \frac{1}{2}$). This implies $\left(\delta^{\frac{1}{2n}} g(\delta^{-\frac{1}{n}})\right)^{-1}$ is decreasing for small $\delta$, and hence

$$\omega(c\delta) = \left(g((c\delta)^{-\frac{b}{a}})\right)^{-2} = (c\delta)^{\frac{b}{a}} \left((c\delta)^{\frac{1}{2n}} g((c\delta)^{-\frac{1}{n}})\right)^{-2} \leq (c\delta)^{\frac{b}{a}} \left(\delta^{\frac{1}{2n}} g(\delta^{-\frac{1}{n}})\right)^{-2} = \frac{c}{\delta} \omega(\delta).$$

This completes the proof of Lemma 2.4. \hfill \Box

Now, we will prove the aim of this section. Proof of Theorem 2.1. Fix $\zeta \in b\Omega$, we define

$$\rho_\zeta(z) := -2c_2^2 \omega(-\psi_\zeta(z)) + |z - \zeta|^2,$$

where $\psi_\zeta(.)$ and $c_2$ are as in Theorem 2.3. We will show that the function $\rho_\zeta(z)$ satisfies the following properties:

(1) $\rho_\zeta(z) < 0$, for $z \in \Omega$, $\rho_\zeta(\zeta) = 0$;
(2) $\rho_\zeta \in C^2(\Omega)$, $i\partial\bar{\partial}\rho_\zeta(X, \bar{X}) \geq |X|^2$ on $\Omega$, and $X \in T^{1,0} \mathbb{C}^n$; and
(3) $\rho_\zeta$ is in the $g^2$-Hölder space of $\Omega$.

Proof of (1). From the definition of $\omega$ and (2) in Theorem 2.3, we have

$$\omega(-\psi_\zeta(z)) = g \left((-\psi_\zeta(z))^{-\frac{1}{2}}\right)^{-2} \geq \frac{1}{c_2^2} |z - \zeta|^2.$$

Hence,

$$\rho_\zeta(z) = -2c_2^2 \omega(-\psi_\zeta(z)) + |z - \zeta|^2 \leq -2|z - \zeta|^2 + |z - \zeta|^2 < 0,$$

where $\zeta \in b\Omega$, and $z \in \Omega$. Moreover, since $\psi_\zeta(\zeta) = 0$ and $\omega(0) = 0$, it follows that $\rho_\zeta(\zeta) = 0$ for any $\zeta \in b\Omega$.

Proof of (2). Fix $\zeta \in b\Omega$, the Levi form of $\omega(-\psi_\zeta)$ on $\Omega$ is

$$i\partial\bar{\partial}\omega(-\psi_\zeta)(X, \bar{X}) = \omega_i\partial\bar{\partial}\psi_\zeta(X, \bar{X}) - \tilde{\omega} |X\psi_\zeta|^2 \geq 0,$$

where the inequality follows from Lemma 2.4(i) and (ii).
Proof of (3). From Lemma 2.4(iii), we have
\begin{align}
\left| \omega(-\psi_\zeta(z)) - \omega(-\psi_\zeta(z')) \right| &\leq \omega(\left| \psi_\zeta(z) - \psi_\zeta(z') \right|) \\
&\leq \omega(c|z - z'|^\eta) \\
&\leq c'\omega(|z - z'|^{-1})^{-2}.
\end{align}

Here the inequalities are obtained from Theorem 2.3(1) and Lemma 2.4(iii)-(iv).

On the other hand, since \( \Omega \) is bounded and \( g(t) \lesssim t^{\frac{3}{2}} \), we can show that
\begin{align}
||z - \zeta|^2 - |z' - \zeta|^2| &\lesssim |z - z'| \lesssim g(|z - z'|^{-1})^{-2}.
\end{align}

The inequalities (2.7) and (2.8) verify that \( \rho_\zeta(z) \in \Lambda^{g^2}(\overline{\Omega}) \) for uniformly in \( \zeta \in \partial\Omega \).

Now, we are ready to prove Theorem 2.1. We define
\[ \rho(z) = \sup_{\zeta \in \partial\Omega} \rho_\zeta(z). \]

The properties of \( \rho_\zeta \) imply that the function \( \rho \) satisfies (1) of the conclusion and is plurisubharmonic in \( \Omega \) as a consequence of well-known result by Lelong [17]. Moreover, since \( g(0) = 0 \) and \( g: [0, \infty] \to [0, \infty] \), \( \rho \) is also \( g^2 \)-Hölder continuous in \( \overline{\Omega} \) – this follows from the theory of modulus of continuity, the superior envelope of \( g^2 \)-Hölder continuous is \( g^2 \)-Hölder continuous. Finally, the second property of each \( \rho_w \) shows that, in the distribution sense, we have
\begin{align}
\tag{2.9}
i\partial\overline{\partial}\rho(X, \overline{X}) &\geq |X|^2, \quad \text{for any } X \in T^{1,0}\mathbb{C}^n.
\end{align}

This completes the proof of Theorem 2.1. \( \square \)

3. Proof of Theorem 1.5

The proof of Theorem 1.5 follows immediately from Theorems 2.1 and 3.1.

Theorem 3.1. Let \( \Omega \) be a bounded, pseudoconvex domain. Assume that there is a uniformly and strictly plurisubharmonic defining function \( \rho \) of \( \Omega \) such that \( \rho \in \Lambda^{g^2}(\overline{\Omega}) \). If \( 0 < \alpha \leq 2 \), \( \phi \in \Lambda^{\alpha}(\partial\Omega) \), and \( h \geq 0 \) on \( \Omega \) with \( h^{\frac{1}{n}} \in \Lambda^{\alpha}(\Omega) \), then the Dirichlet problem for the complex Monge-Ampère equation (1.2) has a unique plurisubharmonic solution \( u \in \Lambda^{\alpha}(\overline{\Omega}) \).
Let $\Omega$ be a bounded open set in $\mathbb{C}^n$ and $\mathcal{P}(\Omega)$ denote the space of plurisubharmonic functions on $\Omega$. The proof of Theorem 3.1 is adapted from the argument given by Bedford and Taylor [1, Theorem 6.2] for weakly pseudoconvex domains. Based on the approach in [1], we need the following proposition.

**Proposition 3.2.** Let $\Omega$ be a bounded, pseudoconvex domain. Assume that there is a strictly plurisubharmonic defining function $\rho$ of $\Omega$ such that $\rho \in \Lambda^{g^2}(\overline{\Omega})$. Let $0 < \alpha \leq 2$, and $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and let $h \geq 0$ with $h^{1/n} \in \Lambda^{g^\alpha}(\Omega)$. Then, for each $\zeta \in b\Omega$, there exists $v_\zeta \in \Lambda^{g^\alpha}(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ such that

(i) $v_\zeta(z) \leq \phi(z)$ for all $z \in b\Omega$, and $v_\zeta(\zeta) = \phi(\zeta)$,

(ii) $\|v_\zeta\|_{\Lambda^{\alpha t}(\overline{\Omega})} \leq C_0$,

(iii) $\det(H(v_\zeta)(z)) \geq h(z)$,

where $C_0$ is a positive constant depending only on $\Omega$ and $\|\phi\|_{\Lambda^{t^{\alpha}}(b\Omega)}$.

**Proof.** For each $\zeta \in b\Omega$, we may choose the family $\{v_\zeta\}$ by two different ways regarding the value of $\alpha$:

**Case 1:** if $0 < \alpha \leq 1$ then we choose

$$v_\zeta(z) = \phi(\zeta) - K[-2\rho(z) + |z - \zeta|^2]^\frac{\alpha}{2}, \quad z \in \overline{\Omega};$$

**Case 2:** if $1 < \alpha \leq 2$ then we choose

$$v_\zeta(z) = \phi(\zeta) - \sum_{j=1}^{n} 2\text{Re}\left(\frac{\partial \phi(\zeta)}{\partial \zeta_j}(z_j - \zeta_j) - K[-2\rho(z) + |z - \zeta|^2]^\frac{\alpha}{2}\right), \quad z \in \overline{\Omega};$$

where $\rho$ is defined by Theorem 2.1, and $K$ will be chosen step by step later.

It is easy to see that $v_\zeta(\zeta) = \phi(\zeta)$ for any $\zeta \in b\Omega$ in both cases. Moreover, choosing $K$ such that $K \geq \|\phi\|_{\Lambda^{t^\alpha}}$, for all $z \in b\Omega$ we have in Case 1:

$$v_\zeta(z) \leq \phi(\zeta) - \|\phi\|_{\Lambda^{t^\alpha}} |z - \zeta|^\alpha \leq \phi(z);$$

and in Case 2:
\begin{align*}
v_\zeta(z) & \leq \phi(\zeta) - \sum_{j=1}^{n} 2\text{Re} \frac{\partial \phi(\zeta)}{\partial \zeta_j} (z_j - \zeta_j) - \|\phi\|_{A^\alpha} |z - \zeta|^\alpha \\
& \leq \phi(z) + \sum_{j=1}^{n} 2\text{Re} \frac{\partial \phi(\tau \zeta + (1 - \tau)z)}{\partial \zeta_j} (z_j - \zeta_j) \\
& \quad - \sum_{j=1}^{n} 2\text{Re} \frac{\partial \phi(\zeta)}{\partial \zeta_j} (z_j - \zeta_j) - \|\phi\|_{A^\alpha} |z - \zeta|^\alpha \\
& \leq \phi(z) + \|D\phi\|_{A^\alpha} |z - \zeta|^\alpha - |z - \zeta| - \|\phi\|_{A^\alpha} |z - \zeta|^\alpha \\
& \leq \phi(z).
\end{align*}

This proves (i).

For the proof of (ii), in both cases we have the following estimates

\begin{align}
|v_\zeta(z) - v_\zeta(z')| & \leq K \left[ -2\rho(z) + |z - \zeta|^2 \right]^{\frac{\alpha}{2}} - \left[ -2\rho(z') + |z' - \zeta'|^2 \right]^{\frac{\alpha}{2}} + K |z - z'| \\
& \leq K \left[ -2\rho(z) + |z - \zeta|^2 + 2\rho(z') - |z' - \zeta'|^2 \right]^{\frac{\alpha}{2}} + K |z - z'| \\
& \leq K \left[ 2|\rho(z) - \rho(z')| + ||z - \zeta|^2 - |z' - \zeta'|^2| \right]^{\frac{\alpha}{2}} + |z - z'| \\
& \lesssim g^{-\alpha}(|z - z'|^{-1})
\end{align}

Here, the first inequality follows by the fact that $|\delta^\alpha - \eta^\alpha| \leq |\delta - \eta|\frac{\alpha}{2}$ for all $\delta, \eta$ small and $0 < \alpha \leq 2$; the last inequality follows by Theorem 2.1, (2.8) and $g(t) \leq t^{1/2} \leq t^{1/\alpha}$ for large $t$. This implies $v_\zeta \in A^{g^\alpha}(\Omega)$ for all $\zeta \in b\Omega$. Moreover $\|v_\zeta\|_{A^{g^\alpha}(\Omega)}$ is independent on $\zeta$.

To establish (iii), we compute $(v_\zeta)_{ij}$ on $\Omega$. In both cases,

\begin{align}
(v_\zeta(z))_{ij} & = K \frac{\alpha}{2} (-2\rho(z) + |z - \zeta|^2)^{\frac{\alpha}{2} - 2} \\
& \quad \cdot \left[ (-2\rho(z) + |z - \zeta|^2)(2\rho(z)_{ij} - \delta_{ij}) \\
& \quad + \left(1 - \frac{\alpha}{2}\right) (-2\rho_i + \bar{z}_i - \bar{\zeta}_i)(-2\rho_j + \bar{z}_j - \bar{\zeta}_j) \right].
\end{align}
Hence
\[ i\partial\overline{\partial}v_{\zeta}(X, X) \geq K_2(\alpha) (2|\partial\overline{\partial}\rho(X, X) - |X|^2) \]
\[ \geq K_2(\alpha) (2\rho(z) + |z - \zeta|^2)^{\frac{\alpha}{2}} \]
for any \( X \in T^{1,0}C^n \). Here the last inequality follows from Theorem 2.1(2).
Thus \( v_{\zeta} \) is plurisubharmonic and furthermore we obtain
\[ \det[(v_{\zeta})_{ij}](z) \geq \left[ K_2(\alpha) (2\rho(z) + |z - \zeta|^2)^{\frac{\alpha}{2}} \right]^n. \]  
(3.3)

Now, since \( 0 < \alpha \leq 2 \) we choose
\[ K \geq \max \left\{ \frac{2}{\alpha} \max_{z \in \overline{\Omega}, \zeta \in \partial \Omega} (-2\rho(z) + |z - \zeta|^2)^{1-\frac{\alpha}{2}} \|h^{1/n}\|_{L^\infty(\Omega)}, \|\phi\|_{\Lambda^\alpha} \right\}. \]
Then
\[ \det[(v_{\zeta})_{ij}](z) \geq \|h^{1/n}\|^n_{L^\infty(\Omega)} \geq (h^{1/n}(z))^n = h(z), \]  
for all \( z \in \Omega \), and \( \zeta \in \partial \Omega \). This completes the proof of Proposition 3.2. \( \square \)

Before to give a proof of Theorem 3.1, we recall the existence theorem for the problem (1.1) by Bedford and Taylor [1, Theorem 8.3, page 42].

**Theorem 3.3 (Bedford-Taylor [1]).** Let \( \Omega \) be a bounded open set in \( \mathbb{C}^n \).
Let \( \phi \in C(\partial \Omega) \) and \( 0 \leq h \in C(\Omega) \). If the Perron-Bremerman family denoted by
\[ B(\phi, h) = \left\{ v \in \mathcal{P}(\Omega) \cap C(\Omega) : \det[(v)_{ij}] \geq h, \right\} \]
\[ \limsup_{z \to z_0} v(z) \leq \phi(z_0), \text{ for all } z_0 \in \partial \Omega, \]
is non-empty, and its upper envelope
\[ u = \sup\{v : v \in B(\phi, h)\} \]  
(3.5)
is continuous on \( \overline{\Omega} \) with \( u = \phi \) on \( \partial \Omega \), then \( u \) is a solution to the Dirichlet problem (1.1).

**Proof of Theorem 3.1.** First, we see that the set \( B(\phi, h) \) is non-empty, in particular, it contains the family of \( \{v_{\zeta}\}_{\zeta \in \partial \Omega} \) in Proposition 3.2. The proof
of this theorem will be completed if the upper envelope defined in (3.5) has the properties

(1) \( u(\zeta) = \phi(\zeta) \) for all \( \zeta \in b\Omega \);

(2) \( u \in \Lambda^{g^\alpha}(\Omega) \).

We note that the uniqueness of solution follows from the Minimum Principle (cf. [1, Theorem A]).

Next, we define another upper envelope, for each \( z \in \overline{\Omega} \),

\[
v(z) := \sup_{\zeta \in b\Omega} \{ v_\zeta(z) \}.
\]

By the first property of \( \{ v_\zeta \} \) in Proposition 3.2, we have

\[
(3.6) \quad v(\zeta) \geq v_\zeta(\zeta) = \phi(\zeta), \quad \text{for all } \zeta \in b\Omega,
\]

\[
v(z) \leq \phi(z), \quad \text{for all } z \in b\Omega,
\]

and so \( v = \phi \) on \( b\Omega \).

From the second property in Proposition 3.2, we have

\[
|v_\zeta(z) - v_\zeta(z')| \leq C_0(g^\alpha(|z - z'|^{-1}))^{-1}, \quad \text{for all } z, z' \in \overline{\Omega}.
\]

Notice that \( C_0 \) is independent on \( \zeta \) so taking the supremum in \( \zeta \), the theory of the modulus of continuity again implies that

\[
|v(z) - v(z')| \leq C_0(g^\alpha(|z - z'|^{-1}))^{-1}, \quad \text{for all } z, z' \in \overline{\Omega}.
\]

By Proposition 2.8 in [1], the following inequality holds

\[
\det[(v)_{ij}](z) \geq \inf_{\zeta \in \partial\Omega} \{ \det[(v_\zeta)_{ij}](z) \} \geq h(z), \quad \text{for all } z \in \Omega.
\]

Thus, we conclude that \( v \in \mathcal{B}(\phi, h) \cap \Lambda^{g^\alpha}(\overline{\Omega}) \) and \( v(\zeta) = \phi(\zeta) \) for any \( \zeta \in b\Omega \).

By a similar construction there exists a plurisuperharmonic function \( w \in \Lambda^{g^\alpha}(\Omega) \) such that \( w(\zeta) = \phi(\zeta) \) for any \( \zeta \in b\Omega \). Thus, \( v(z) \leq u(z) \leq w(z) \) for any \( z \in \overline{\Omega} \), and hence \( u(\zeta) = \phi(\zeta) \) for any \( \zeta \in b\Omega \). We also obtain

\[
(3.7) \quad |u(z) - u(\zeta)| \leq \max\{\|v\|_{\Lambda^{g^\alpha}(\Omega)}; \|v\|_{\Lambda^{g^\alpha}(\Omega)}\}(g^\alpha(|z - \zeta|^{-1}))^{-1},
\]

for any \( z \in \Omega, \zeta \in b\Omega \). Here, the inequality follows by the facts that \( w, v \in \Lambda^{g^\alpha}(\overline{\Omega}) \) and \( v(\zeta) = u(\zeta) = w(\zeta) = \phi(\zeta) \) for any \( \zeta \in \partial\Omega \).
Finally, using the method by Walsh in [23], we will show that (3.7) also holds for all \( \zeta \in \Omega \). For any small vector \( \tau \in \mathbb{C}^n \), we define

\[
V(z, \tau) = \begin{cases} 
  u(z), & \text{if } z + \tau \notin \Omega, \ z \in \overline{\Omega}, \\
  \max\{u(z), V_\tau(z)\}, & \text{if } z, z + \tau \in \Omega,
\end{cases}
\]

where

\[
V_\tau(z) = u(z + \tau) + \left( K_1|z|^2 - K_2 - K_3 \right) g^{-\alpha}(|\tau|^{-1})
\]

and here

\[
K_1 \geq \max_{k \in \{1, \ldots, n\}} \left( \frac{n}{k} \right)^{1/k} \|h\|^1 \|\Lambda^{\alpha}(\overline{\Omega})\|,
K_2 \geq K_1|z|^2,
\]

and

\[
K_3 \geq \max\{\|v\|_{\Lambda^{\alpha}(\overline{\Omega})}, \|w\|_{\Lambda^{\alpha}(\overline{\Omega})}\}.
\]

We will show that \( V(z, \tau) \in B(\phi, h) \). Observe that \( V(z, \tau) \in \mathcal{P}(\Omega) \) for all \( z, \tau \). Moreover, for \( z \in \partial \Omega \) and \( z + \tau \in \Omega \), we have

\[
(3.8) \quad V_\tau(z) - u(z) = u(z + \tau) - u(z) + \left( K_1|z|^2 - K_2 - K_3 \right) g^{-\alpha}(|\tau|^{-1})
\]

\[
\leq \max\{\|v\|_{\Lambda^{\alpha}(\overline{\Omega})}, \|v\|_{\Lambda^{\alpha}(\overline{\Omega})}\} g^{-\alpha}(|\tau|^{-1})
\]

\[
+ \left( K_1|z|^2 - K_2 - K_3 \right) g^{-\alpha}(|\tau|^{-1})
\]

\[
\leq 0.
\]

Here the first inequality follows by (3.7) and the second follows by the choices of \( K_2 \) and \( K_3 \). This implies that \( \lim \sup_{z \to \zeta} V(z, \tau) \leq \phi(\zeta) \) for all \( \zeta \in \partial b\Omega \). For the proof of \( \det[V(z, \tau)_{ij}] \geq h(z) \), we need the following lemma.

**Lemma 3.4.** Let \( (\alpha_{ij}) \geq 0 \) and \( \beta \in (0, +\infty) \). Then

\[
\det[\alpha_{ij} + \beta I] \geq \sum_{k=0}^{n} \beta^k \det(\alpha_{ij})^{(n-k)/n}.
\]

**Proof of Lemma 3.4.** Let \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( (\alpha_{ij}) \). We have

\[
(3.9) \quad \det[\alpha_{ij} + \beta] = \prod_{j=1}^{n} (\lambda_j + \beta) \geq \sum_{k=0}^{n} \left( \beta^k \prod_{j=k+1}^{n} \lambda_j \right)
\]

\[
\geq \sum_{k=0}^{n} \left( \beta^k \det(\alpha_{ij})^{(n-k)/n} \right).
\]
Here the last inequality follows by
\[
\det[\alpha_{ij}] = \prod_{j=1}^{n} \lambda_j \leq \left( \prod_{j=k+1}^{n} \lambda_j \right)^{n/(n-k)}.
\]

Continuing the proof of Theorem 1.5, for any \( z, z + \tau \in \Omega \) we have
\[
(3.10) \quad \det[(V_\tau(z))_{ij}] = \det[u_{ij}(z + \tau) + K_1 g^{-\alpha}(|\tau|^{-1}) I]
\geq \det[u_{ij}(z + \tau)] + \sum_{k=1}^{n} K_1^k [g^{\alpha}(|\tau|^{-1})]^{-k} \cdot \det[u_{ij}(z + \tau)] \frac{n-k}{n}
\geq h(z + \tau) + \sum_{k=1}^{n} K_1^k [g^{\alpha}(|\tau|^{-1})]^{-k} \cdot (h(z + \tau)) \frac{n-k}{n},
\]
where the first inequality is derived by Lemma 3.4. Since \( h_1^{\frac{1}{n}} \in \Lambda^{g^{\alpha}}(\Omega) \), we obtain
\[
h_1^{\frac{1}{n}}(z) - h_1^{\frac{1}{n}}(z + \tau) \leq g^{-\alpha}(|\tau|^{-1}) \|h_1^{\frac{1}{n}}\|_{\Lambda^{g^{\alpha}}}, \quad \text{for any } z, z + \tau \in \Omega,
\]
and hence
\[
(3.11) \quad h(z) \leq h(z + \tau) + \sum_{k=1}^{n} \binom{n}{k} h(z + \tau)^{(n-k)/n} \left( g^{-\alpha}(|\tau|^{-1}) \|h_1^{\frac{1}{n}}\|_{\Lambda^{g^{\alpha}}} \right)^k.
\]
Combining (3.10), (3.11) with the choice of \( K_1 \), we get
\[
\det[(V_\tau(z))_{ij}] \geq h(z), \quad \text{for any } z, z + \tau \in \Omega.
\]
We conclude that \( V(z, \tau) \in B(\phi, h) \). It follows that for all \( z \in \Omega \), \( V(z, \tau) \leq u(z) \). If \( z, z + \tau \in \Omega \), this yields
\[
(3.12) \quad u(z + \tau) - u(z) \leq V(\tau, z) - (K_1 |z|^2 - K_2 - K_3) g^{-\alpha}(|\tau|^{-1}) - u(z)
\leq (-K_1 |z|^2 + K_2 + K_3) g^{-\alpha}(|\tau|^{-1})
\leq (K_2 + K_3) g^{-\alpha}(|\tau|^{-1}).
\]
By reversing the role of \( z \) and \( z + \tau \), we assert that \( u \in \Lambda^{g^{\alpha}}(\Omega) \). This completes the proof. \qed
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