On two rationality conjectures for cubic fourfolds

Nicolas Addington

Motivated by the question of rationality of cubic fourfolds, we show that a cubic $X$ has an associated K3 surface in the sense of Hassett if and only if the variety $F$ of lines on $X$ is birational to a moduli space of sheaves on a K3 surface, but that having $F$ birational to $\text{Hilb}^2(K3)$ is more restrictive. We compare the loci in the moduli space of cubics where each condition is satisfied.

It is widely expected that a smooth complex cubic fourfold $X$ is rational if and only if it has an associated K3 surface in the sense of Hassett [8] or Kuznetsov [11]. New work of Galkin and Shinder [7] suggests instead that if $X$ is rational then the variety $F$ of lines on $X$ is birational to the Hilbert scheme of two points on a K3 surface. The purpose of this note is to clarify the relationship between these two conditions. The latter is somewhat stronger.

First let us recall Hassett’s Noether–Lefschetz divisors $C_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds [8, §3.2]. For a very general cubic $X$, the algebraic lattice $H^{2,2}(X,\mathbb{Z}) := H^{2,2}(X) \cap H^4(X,\mathbb{Z})$ is generated by $h^2$, the square of the hyperplane class. A special cubic of discriminant $d$ is one for which there is a primitive sublattice $K \subset H^{2,2}(X,\mathbb{Z})$ of rank 2 and discriminant $d$ that contains $h^2$. Such cubics form an irreducible divisor $C_d \subset \mathcal{C}$, non-empty if and only if

\begin{equation}
(*) \quad d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod{6}.
\end{equation}

Moreover there exists a polarized K3 surface $S$ such that $K^\perp \subset H^4(X,\mathbb{Z})$ is Hodge-isometric to $H^2_{\text{prim}}(S,\mathbb{Z})(-1)$ if and only $d$ satisfies the further condition

\begin{equation}
(**) \quad d \text{ is not divisible by } 4, 9, \text{ or any odd prime } p \equiv 2 \pmod{3}.
\end{equation}
Using the Eisenstein integers one can show that (**) is equivalent to saying that $d$ is the norm of a primitive vector in the lattice $A_2 = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, or that $d$ divides $2n^2 + 2n + 2$ for some integer $n$.

**Theorem 1.** The following are equivalent:

(a) $X \in C_d$ for some $d$ satisfying (**).

(b) The transcendental lattice $T_X \subset H^4(X, \mathbb{Z})$ is Hodge-isometric to $T_S(-1)$ for some K3 surface $S$.

(c) $F$ is birational to a moduli space of stable sheaves on $S$.

By a recent result of Bayer and Macrì [5, Thm. 1.2(c)], this last condition is equivalent to saying that $F$ is isomorphic to a moduli space of Bridgeland-stable objects in the derived category of $S$. Thus Theorem 1 answers [13, Question 1.2] in the untwisted case.

Hassett [8, Prop. 6.1.3] showed that if the generic $X \in C_d$ has $F$ isomorphic to $\text{Hilb}^2(S)$ for some K3 surface $S$ then

$$(***) \quad d \text{ is of the form } \frac{2n^2 + 2n + 2}{a^2} \text{ for some } n, a \in \mathbb{Z},$$

and proved a partial converse [8, Thm. 6.1.4]. Thanks to the global Torelli theorem for hyperkähler manifolds [10, 15, 19] we can now prove a more complete result:

**Theorem 2.** The following are equivalent:

(a) $X \in C_d$ for some $d$ satisfying (**).

(b) $F$ is birational to $\text{Hilb}^2(S)$ for some K3 surface $S$.

In contrast to (**), it is hard to tell at a glance whether a number $d$ satisfies (***) if and only if there is an integral solution to the Pell-type equation $m^2 - 2da^2 = -3$; just substitute $m = 2n + 1$. If such an equation has any solution then it has one with $a$ below an explicit bound [2, Thm. 4.2.7]. It is then straightforward to have a computer search for solutions up to this bound. Table 1 lists all $d$ up to 200 that satisfy (*) and (***) in terms of the $A_2$ lattice.
Table 1: Comparison of numerical conditions.

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Outline

In §1 we review Markman’s Mukai lattice for a variety $Y$ of $K3^{[n]}$-type, which governs the global Torelli theorem for such varieties. We give criteria in terms of this lattice for $Y$ to be birational to a moduli space of sheaves or Hilbert scheme of $n$ points on a K3 surface.

In §2 we review Kuznetsov’s K3 category $\mathcal{A}$ associated to $X$, the special classes $\lambda_1, \lambda_2 \in K_{\text{num}}(\mathcal{A})$, and the Mukai lattice $K_{\text{top}}(\mathcal{A})$ introduced in [1]. We prove that

\begin{equation}
H^2(F, \mathbb{Z})(1) \cong \lambda_1^\perp \subset K_{\text{top}}(\mathcal{A}).
\end{equation}
This extends Beauville and Donagi’s result [6, Prop. 6] that \( H^2_{\text{prim}}(F, \mathbb{Z})(1) \cong H^4_{\text{prim}}(X, \mathbb{Z})(2) \), since the latter is Hodge-isometric to \( \langle \lambda_1, \lambda_2 \rangle_{\perp} \subset K_{\text{top}}(A) \).

From (1) we deduce that \( K_{\text{top}}(A)(-1) \) is the Markman–Mukai lattice of \( F \).

All this is consistent with Kuznetsov and Markushevich’s result [12, §5] that \( F \) is a moduli space of objects in the numerical class \( \lambda_1 \in K_{\text{num}}(A) \).

With this lattice theory in hand, we prove Theorems 1 and 2 in §3.

\section*{Convention}

Since we are speaking about transcendental lattices and moduli spaces of sheaves, we will take all K3 surfaces to be projective unless otherwise stated.

\section{1. The Markman–Mukai lattice of a variety of K3\(^[n]\)-type}

A \textit{variety of K3\(^[n]\)-type} is a smooth projective variety \( Y \) deformation-equivalent to the Hilbert scheme of \( n \) points of a K3 surface, \( n \geq 2 \). The second cohomology group \( H^2(Y, \mathbb{Z}) \) carries a quadratic form \( q \), the \textit{Beauville–Bogomolov–Fujiki form}, under which it is a lattice of discriminant \(-2n + 2\) and signature \((3, 20)\). Markman [15, §9] has described an extension of lattices and weight-2 Hodge structures \( H^2(Y, \mathbb{Z}) \subset \tilde{\Lambda} \) with the following properties:

\begin{theorem}[Markman\(^1\)]
\begin{enumerate}
\item As a lattice, \( \tilde{\Lambda} \) is isomorphic to \( U^4 \oplus (-E_8)^2 \).
\item The orthogonal \( H^2(Y, \mathbb{Z})_{\perp} \subset \tilde{\Lambda} \) is generated by a primitive vector of square \( 2n - 2 \).
\item If \( Y \) is a moduli space of sheaves on a K3 surface \( S \) with Mukai vector \( v \in H^*(S, \mathbb{Z}) \) then the extension \( H^2(Y, \mathbb{Z}) \subset \tilde{\Lambda} \) is naturally identified with \( v_{\perp} \subset H^*(S, \mathbb{Z}) \).
\item \( Y_1 \) and \( Y_2 \) are birational if and only if there is a Hodge isometry \( \tilde{\Lambda}_1 \to \tilde{\Lambda}_2 \) taking \( H^2(Y_1, \mathbb{Z}) \) isomorphically to \( H^2(Y_2, \mathbb{Z}) \).
\end{enumerate}
\end{theorem}

Let \( \tilde{\Lambda}_{\text{alg}} \supset H^{1,1}(Y, \mathbb{Z}) \) denote the algebraic part of \( \tilde{\Lambda} \), that is, the integral classes of type \((1, 1)\).

\(^1\)This summary is borrowed from [4, §1].
Proposition 4. Let $Y$ be a variety of $K3^{[n]}$-type, $n \geq 2$. Then the following are equivalent.\footnote{Mongardi and Wandel have proved a similar result independently in \cite[Prop. 2.3]{MongardiWandel}.}

(a) $\tilde{\Lambda}_{\text{alg}}$ contains a copy of the hyperbolic plane $U = (0 \ 1 \\ 1 \ 0)$.

(b) The transcendental lattice $T_Y \subset H^2(Y, \mathbb{Z})$ is Hodge-isometric to $T_S$ for some $K3$ surface $S$.

(c) $Y$ is birational to a moduli space of stable sheaves on $S$.

Proof. (c) $\Rightarrow$ (a): This is immediate from Theorem 3, since the algebraic part of $H^*(S, \mathbb{Z})$ contains a copy of $U$ spanned by $H^0$ and $H^4$.

(a) $\Rightarrow$ (b): Let $L = U^\perp \subset \tilde{\Lambda}$. As a lattice this is isomorphic to $U^3 \oplus (-E_8)^2$, so by the global Torelli theorem it is Hodge-isometric to $H^2(S, \mathbb{Z})$ for some analytic $K3$ surface $S$. In fact $S$ is projective, as follows. By Huybrechts’ projectivity criterion \cite[Thm. 3.11]{Huybrechts} there is a $c \in H^{1,1}(Y, \mathbb{Z})$ with $q(c) > 0$. Let $v$ be a primitive generator of $H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda}$; then $q(v) = 2n - 2 > 0$. Thus $c$ and $v$ span a positive definite sublattice of $\tilde{\Lambda}$. This cannot be contained in $U$, which is indefinite, so $\langle c, v \rangle \cap L$ contains a class of positive square, so $S$ is projective by Huybrechts’ criterion.

Now $T_S$ is the transcendental part of $L$, which is the transcendental part of $\Lambda$, which is $T_Y$.

(b) $\Rightarrow$ (c): We have a Hodge isometry $\varphi: T_Y \rightarrow T_S$, and primitive embeddings $T_Y \subset \tilde{\Lambda} \cong U^4 \oplus (-E_8)^2$ and $T_S \subset H^*(S, \mathbb{Z}) \cong U^4 \oplus (-E_8)^2$. The orthogonal $T_S^\perp$ contains a copy of $U$, so by \cite[Prop. 3.8]{Mukai} any two primitive embeddings $T_S \hookrightarrow U^4 \oplus (-E_8)^2$ differ by an automorphism of $U^4 \oplus (-E_8)^2$. Thus the lattice isomorphism $\varphi: T_Y \rightarrow T_S$ extends to a lattice isomorphism $\tilde{\varphi}: \tilde{\Lambda} \rightarrow H^*(S, \mathbb{Z})$. Since $\varphi$ is a Hodge isometry, it takes $H^{2,0}(Y)$ to $H^{2,0}(S)$, so the extension $\tilde{\varphi}$ does as well, so $\tilde{\varphi}$ is a Hodge isometry.

Again let $v \in \tilde{\Lambda}$ be a primitive generator of $H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda}$, and write $\tilde{\varphi}(v) = (r, c, s) \in H^*(S, \mathbb{Z})$. I claim that either $r > 0$, or we can modify $v$ and $\tilde{\varphi}$ to make it so. If $r < 0$, replace $v$ with $-v$. If $r = 0$ and $s \neq 0$, compose $\tilde{\varphi}$ with the Mukai reflection through $(1, 0, 1) \in H^*(S, \mathbb{Z})$, so now $\tilde{\varphi}(v) = (-s, c, 0)$ and we are reduced to the previous case. If $r = s = 0$, note that $c^2 = q(v) = 2n - 2 > 0$, and compose $\tilde{\varphi}$ with multiplication by $\exp(c) = (1, c, n - 1)$, so now $\tilde{\varphi}(v) = (0, c, n - 1)$ and we are reduced to the previous case.

Now $\tilde{\varphi}(v)$ is a Mukai vector of positive rank, so for a generic polarization of $S$ the moduli space $M$ of stable sheaves on $S$ with Mukai vector $\tilde{\varphi}(v)$ is
smooth and non-empty \[17\]. By construction \( \tilde{\varphi} \) is a Hodge isometry from \( \tilde{\Lambda} \) to \( H^*(S, \mathbb{Z}) \) taking \( H^2(Y, \mathbb{Z}) \) isomorphically to \( \tilde{\varphi}(v)^\perp \), so \( Y \) is birational to \( M \) by Theorem 3.

\[\Box\]

**Proposition 5.** Let \( Y \) be a variety of \( K3^\text{[n]} \)-type, \( n \geq 2 \), and let \( v \) be a primitive generator of \( H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda} \). Then the following are equivalent:

(a) There is a vector \( w \in \tilde{\Lambda}_\text{alg} \) such that \( v.w = -1 \) and \( w^2 = 0 \).

(b) \( Y \) is birational to \( \text{Hilb}^n(S) \) for some K3 surface \( S \).

**Proof.** (b) ⇒ (a): This is immediate from Theorem 3, since \( \text{Hilb}^n(S) \) is the moduli space of sheaves with Mukai vector \( v = (1, 0, 1-n) \in H^*(S, \mathbb{Z}) \); take \( w = (0, 0, 1) \).

(a) ⇒ (b): Observe that \( e := v + (n-1)w \) and \( f := -w \) satisfy \( e^2 = f^2 = 0 \) and \( e.f = 1 \), so they span a copy of \( U \) in \( \tilde{\Lambda}_\text{alg} \). Let \( L = U^\perp = \langle v, w \rangle^\perp \subset \tilde{\Lambda} \). As in the proof of Proposition 4, there is a projective K3 surface \( S \) such that \( H^2(S, \mathbb{Z}) \cong L \). Thus we can produce a Hodge isometry from \( \tilde{\Lambda} = U \oplus L \) to \( H^*(S, \mathbb{Z}) \) that takes \( v \) to \( (1, 0, 1-n) \), so \( Y \) is birational to \( \text{Hilb}^n(S) \) by Theorem 3. \[\Box\]

2. The Markman–Mukai lattice of \( F \)

Recall that \( X \) is a smooth cubic fourfold and \( F \) is the variety of lines on \( X \). Kuznetsov has observed that the triangulated category

\[\mathcal{A} := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^\perp \subset D^b(\text{Coh}(X)) = \{ E \in D^b(\text{Coh}(X)) : \text{Ext}^i(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2 \}\]

is like the derived category of a K3 surface in that it has the same Serre functor and Hochschild homology and cohomology, and has conjectured that \( X \) is rational if and only if \( \mathcal{A} \) is equivalent to the derived category of an actual K3 surface \[11\]. By \[1\], this is essentially equivalent to having \( X \in \mathcal{C}_d \) for some \( d \) satisfying (**).

Let \( K_{\text{num}}(\mathcal{A}) \) be the numerical Grothendieck group of \( \mathcal{A} \), that is, \( K(\mathcal{A}) \) modulo the kernel of the Euler pairing. Let \( \lambda_1, \lambda_2 \in K_{\text{num}}(\mathcal{A}) \) be the classes of the projections of \( \mathcal{O}_L(1) \) and \( \mathcal{O}_L(2) \) into \( \mathcal{A} \), where \( L \) is any line on \( X \). The Euler pairing on the sublattice \( \langle \lambda_1, \lambda_2 \rangle \) is \( -A_2 = \left( \begin{array}{cc} -2 & 1 \\ 1 & -2 \end{array} \right) \).
A Mukai lattice for $\mathcal{A}$ was introduced in [1, Def. 2.2]:

$$K_{\text{top}}(\mathcal{A}) := \{ \kappa \in K_{\text{top}}(X) : \chi([\mathcal{O}_X(i)], \kappa) = 0 \text{ for } i = 0, 1, 2 \}.$$ 

Here $K_{\text{top}}(X)$ is the Grothendieck group of topological vector bundles and $\chi$ is the Euler pairing, which is integer-valued and extends the Euler pairing on $K_{\text{num}}(X)$. It has a Hodge structure of K3 type pulled back via the Chern character or the Mukai vector

$$K_{\text{top}}(\mathcal{A}) \otimes \mathbb{C} \hookrightarrow \bigoplus H^{2i}(X, \mathbb{C})(i).$$

In [1] this was called a weight-two Hodge structure, but it should really be called weight-zero. We will need the following properties:

**Theorem 6 (Addington, Thomas [1, §§2.3–2.4]).**

(a) As a lattice, $K_{\text{top}}(\mathcal{A})$ is isomorphic to $U^4 \oplus E_8^2$.

(b) The algebraic part of $K_{\text{top}}(\mathcal{A})$ is isomorphic to $K_{\text{num}}(\mathcal{A})$.

(c) $\langle \lambda_1, \lambda_2 \rangle^\perp \subset K_{\text{top}}(\mathcal{A})$ is Hodge-isometric to $H^4_{\text{prim}}(X, \mathbb{Z})(2)$.

(d) $X \in C_d$ if and only if there is a primitive sublattice $M \subset K_{\text{num}}(\mathcal{A})$ of rank 3 and discriminant $d$ that contains $\lambda_1$ and $\lambda_2$.

**Proposition 7.** Let $P \subset F \times X$ be the universal line and $p: P \to F$ and $q: P \to X$ the two projections. Then the map $\varphi$ from $\lambda_1^\perp \subset K_{\text{top}}(\mathcal{A})$ to $H^2(F, \mathbb{Z})(1)$ defined by $\varphi(\kappa) = c_1(p_*q^*(\kappa))$ is a Hodge isometry.

**Proof.** Both $\lambda_1^\perp$ and $H^2(F, \mathbb{Z})(1)$ are lattices of rank 23 and discriminant 2. It is enough to show that $\varphi$ is a Hodge isometry when tensored with $\mathbb{Q}$; a priori this only implies that $\varphi$ embeds $\lambda_1^\perp$ as a finite-index sublattice of $H^2(F, \mathbb{Z})(1)$, but since they have the same discriminant the index must in fact be 1.

By the Riemann–Roch formula [3, §3], $\varphi(\kappa)$ is the degree-2 part of

$$(2) \quad p_*(q^*(\text{ch}(\kappa)) \cup \text{td}(T_p)),$$

where $T_p$ is the relative tangent bundle of the $\mathbb{P}^1$-bundle $p: P \to F$. First we calculate $\text{td}(T_p)$. Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class. Let $S$ be the restriction to $F$ of the tautological sub-bundle on $\text{Gr}(2, 6)$. Then $g := -c_1(S) \in H^2(F, \mathbb{Z})$ is the hyperplane class in the Plücker embedding. The
universal line $P$ is the projectivization $\mathbb{P}S$, and $\mathcal{O}_{\mathbb{P}S}(1) = q^*\mathcal{O}_X(1)$. Since $T_p$ is line bundle, we can take determinants in the Euler sequence
\[
0 \to \mathcal{O}_{\mathbb{P}S} \to \mathcal{O}_{\mathbb{P}S}(1) \otimes p^*S \to T_p \to 0
\]
to get $T_p = \mathcal{O}_{\mathbb{P}S}(2) \otimes p^*\det S$. Thus
\[
(3) \quad \text{td}(T_p) = 1 + \frac{1}{2}(2q^*h - p^*g) + \frac{1}{12}(2q^*h - p^*g)^2 + \cdots.
\]

The orthogonal to $\lambda_1$ in $\langle \lambda_1, \lambda_2 \rangle$ is generated by $\lambda_1 + 2\lambda_2$. Since we are tensoring with $\mathbb{Q}$, we have orthogonal direct sums
\[
\begin{align*}
(4) & \quad \lambda_1^\perp = \langle \lambda_1 + 2\lambda_2 \rangle \oplus \langle \lambda_1, \lambda_2 \rangle^\perp \\
(5) & \quad H^2(F, \mathbb{Q}) = \langle g \rangle \oplus H^2_{\text{prim}}(F, \mathbb{Q}).
\end{align*}
\]

By [1, Prop. 2.3], the Chern character\(^3\) gives a Hodge isometry from the second summand of (4) to $H^4_{\text{prim}}(X, \mathbb{Q})(2)$. By [6, Prop. 6], $p_*q^*$ gives a Hodge isometry from this to the second summand of (5). Since the degree-0 part of $\text{td}(T_p)$ is 1, we see that for $\alpha \in H^4(X, \mathbb{Q})$, the degree-2 part of $p_*(q^*\alpha \cup \text{td}(T_p))$ is just $p_*q^*\alpha$. Thus $\varphi$ gives a Hodge isometry from the second summand of (4) to the second summand of (5).

For the first summands of (4) and (5), observe that the Euler square of $\lambda_1 + 2\lambda_2$ is $-6$, and by [8, §2.1] we have $q(g) = -6$ as well (the minus sign comes because we have twisted down to weight zero). Thus it is enough to show that
\[
(6) \quad \varphi(\lambda_1 + 2\lambda_2) = g.
\]

To calculate $\text{ch}(\lambda_1 + 2\lambda_2)$, recall that $\lambda_i$ is the class of the left mutation of $\mathcal{O}_L(i)$ past $\mathcal{O}_X(2)$, $\mathcal{O}_X(1)$, and $\mathcal{O}_X$, where $L$ is any line on $X$, so a straightforward calculation gives
\[
\lambda_1 = [\mathcal{O}_L(1)] - [\mathcal{O}_X(1)] + 4[\mathcal{O}_X] \\
\lambda_2 = [\mathcal{O}_L(2)] - [\mathcal{O}_X(2)] + 4[\mathcal{O}_X(1)] - 6[\mathcal{O}_X]
\]
and thus
\[
\text{ch}(\lambda_1 + 2\lambda_2) = -3 + 3h - \frac{1}{2}h^2 + \cdots.
\]

\(^3\)In fact [1, Prop. 2.3] says that the Mukai vector gives such a Hodge isometry, but since $\text{td}(X)$ is a polynomial in $h$, multiplying by $\sqrt{\text{td}(X)}$ does not affect $H^4_{\text{prim}}(X, \mathbb{Q})$. 

By [8, §2.1] we have $p_*q^*h^2 = g$. We also have $p_*q^*h = 1$: to see this, take a smooth hyperplane section $X \cap H$ and take its preimage under $q$; this is the blow-up of $F$ along the surface of lines contained in the cubic threefold $X \cap H$, hence is generically 1-to-1 over $F$. Combining these facts with (2) and (3) we get (6).

\[ \square \]

Corollary 8. The embedding $H^2(F, \mathbb{Z}) \subset K_{\text{top}}(\mathcal{A})(-1)$ given by the previous proposition can be identified with Markman’s embedding $H^2(F, \mathbb{Z}) \subset \tilde{\Lambda}$ discussed in §1.

Proof. If $n = 2$ or if $n - 1$ is a prime power then for any $Y$ of $K3^{[n]}$-type, any two primitive embeddings of $H^2(Y, \mathbb{Z})$ into $U^4 \oplus (-E_8)^2$ differ by an automorphism of the latter [14, §4.1].

\[ \square \]

3. Proofs of Theorems 1 and 2

Theorem 1. The following are equivalent:

(a) $X \in C_d$ for some $d$ satisfying (**).

(b) The transcendental lattice $T_X \subset H^4(X, \mathbb{Z})$ is Hodge-isometric to $T_S(-1)$ for some K3 surface $S$.

(c) $F$ is birational to a moduli space of stable sheaves on $S$.

Proof. By [1, Thm. 3.1], condition (a) holds if and only if $K_{\text{num}}(\mathcal{A})$ contains a copy of $U \cong -U$. Moreover we have $T_X \cong T_F(-1)$. Thus the theorem follows from Corollary 8 and Proposition 4.

\[ \square \]

To prove Theorem 2 we will have to work in a basis:

Lemma 9. If $X \in C_d$ then there is a $\tau \in K_{\text{num}}(\mathcal{A})$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice of discriminant $d$ with Euler pairing

$$
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
0 & 0 & 2k
\end{pmatrix}
\quad \text{when } d = 6k, \text{ or }
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & 2k
\end{pmatrix}
\quad \text{when } d = 6k + 2.
$$
Proof. By Theorem 6(d), we can choose a \( \tau \in K_{\text{num}}(A) \) such that \( \langle \lambda_1, \lambda_2, \tau \rangle \) is a primitive sublattice of discriminant \( d \). Write the Euler pairing as

\[
\begin{pmatrix}
-2 & 1 & a \\
1 & -2 & * \\
a & * & *
\end{pmatrix}
\]

for some \( a \in \mathbb{Z} \). Replace \( \tau \) with \( \tau - a\lambda_2 \); then the Euler pairing becomes

\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 3b + c \\
0 & 3b + c & *
\end{pmatrix}
\]

for some \( b \) and some \(-1 \leq c \leq 1 \). Replace \( \tau \) with \( \tau + b(\lambda_1 + 2\lambda_2) \); then the Euler pairing becomes

\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & c \\
0 & c & 2k
\end{pmatrix}
\]

for some \( k \), since \( K_{\text{num}}(A) \) is an even lattice. If \( c = 0 \) this has determinant \( 6k \). If \( c = 1 \) this has determinant \( 6k + 2 \). If \( c = -1 \), replace \( \tau \) with \(-\tau\) to get back to the previous case. \( \square \)

Theorem 2. The following are equivalent:

(a) \( X \in C_d \) for some \( d \) satisfying (***)

(b) \( F \) is birational to \( \text{Hilb}^2(S) \) for some K3 surface \( S \).

Proof. We will show that condition (a) holds if and only if there is a \( w \in K_{\text{num}}(A) \) such that

\[
\chi(\lambda_1, w) = 1 \quad \text{and} \quad \chi(w, w) = 0.
\]

Then the theorem follows from Corollary 8 and Proposition 5.

If there is such a \( w \), let \( L = \langle \lambda_1, \lambda_2, w \rangle \subset K_{\text{num}}(A) \). By hypothesis, the Euler pairing on \( L \) is

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & n \\
1 & n & 0
\end{pmatrix}
\]

for some \( n \in \mathbb{Z} \), so \( \text{disc}(L) = 2n^2 + 2n + 2 \). Let \( M \) be the saturation of \( L \), let \( a \) be the index of \( L \) in \( M \), and let \( d = \text{disc}(M) \). Then \( \text{disc}(L) = a^2d \), and \( X \in C_d \) by Theorem 6(d).
Conversely, suppose $X \in C_d$ for some $d$ satisfying (**). Choose integers $n$ and $a$ such that
\[ da^2 = 2n^2 + 2n + 2. \]
Recall that $d$ is even. Since $2n^2 + 2n + 2$ satisfies (**) we see that $a$ is a product of primes $p \equiv 1 \pmod{3}$, and in particular $a \equiv 1 \pmod{3}$. We consider three cases.

Case 1: $n \equiv 1 \pmod{3}$. In this case we find that $d \equiv 0 \pmod{6}$. Write $d = 6k$. By Lemma 9 there is a $\tau \in K_{\text{num}}(A)$ such that the Euler pairing on $\langle \lambda_1, \lambda_2, \tau \rangle$ is
\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
0 & 0 & 2k
\end{pmatrix}.
\]
Let $m = (n - 1)/3$, which is an integer; then we find that
\[
w := m\lambda_1 + (2m + 1)\lambda_2 + a\tau
\]
satisfies (7).

Case 2: $n \equiv 2 \pmod{3}$. In this case we find that $d \equiv 2 \pmod{6}$. Write $d = 6k + 2$. By Lemma 9 there is a $\tau \in K_{\text{num}}(A)$ such that the Euler pairing on $\langle \lambda_1, \lambda_2, \tau \rangle$ is
\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & 2k
\end{pmatrix}.
\]
Let $m = (a - n - 2)/3$, which is an integer; then we find that
\[
w := m\lambda_1 + (2m + 1)\lambda_2 + a\tau
\]
satisfies (7).

Case 3: $n \equiv 0 \pmod{3}$. Again we find that $d \equiv 2 \pmod{6}$. Argue as in the previous case but with $m = (a + n - 1)/3$.

\[ \Box \]

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