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Two dimensional Monge-Ampère equations under incomplete Hölder assumptions

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In this paper, we study the regularity of solutions of the Monge-Ampère equation in two dimensions when the inhomogeneous term is Hölder continuous only in one variable or piecewise Hölder continuous.

1. Introduction

The aim of this paper is to study the Monge-Ampère equation

$$\det D^2u(x) = f(x) \tag{1.1}$$

in a bounded convex domain Ω in $\mathbb{R}^2$. We are concerned with the anisotropic regularity when f is merely partially or piecewise Hölder continuous.

The Monge-Ampere equation has many applications in differential geometry and optimal transportation. The regularity problem for solutions of (1.1) has received a great deal of attention. In [1] Caffarelli established the interior $C^{2,\alpha}$ estimates for the solution u when f is Hölder continuous, and later Jian and Wang [12] obtained the continuity estimates for D^2u for Dini continuous f. In the two dimensional case, the interior $C^{2,\alpha}$ estimates were established by Heinz, and these estimates were used for his solution of Weyl's embedding problem with the aid of a continuity method. In particular, following Lewy's work [17], Heinz developed a characteristic theory for the Monge-Ampère equations in two dimensions [9–11]. See Schulz's book [23] for an excellent exposition to this characteristic theory. The main ingredient of the theory is the so-called partial Legendre transform, which was later used by many authors in their study of the local regularity of certain

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degenerate Monge-Ampère equations, see e.g. [4, 7]. For more recent applications and detailed discussions of the partial Legendre transform, we refer the reader to the paper [8].

The regularity theory mentioned above is isotropic, namely solutions are regular in all variables. An interesting and quite natural question is whether a solution of (1.1) is regular in one variable if the inhomogeneous term is regular in this variable only. For some PDEs from fluid mechanics or physical problems, the anisotropic regularity of the solutions is a significant problem, see e.g. [3, 16] and the references therein. It is known that, for uniformly elliptic equations in second order, if the coefficients and inhomogeneous term satisfy an anisotropic Hölder continuity condition, then the second derivatives of the solution satisfy a related anisotropic Hölder continuity estimates [5, 13, 24]. By [12], equation (1.1) is uniformly elliptic for strictly convex solutions if f is Dini continuous in all variables and

$$\lambda \leq f \leq \Lambda \tag{1.2}$$

for two positive constants λ, Λ. However, if f is not Dini continuous in all variables, the uniform ellipticity of (1.1) is unknown even in two dimensions.

Let φ be a function defined in $\Omega \subset \mathbb{R}^2$, and ξ be a fixed unit vector in $\mathbb{R}^2$. We denote

$$\omega_{\varphi,\xi}(r) = \sup\{|\varphi(x+t\xi) - \varphi(x)| : x+t\xi, x \in \Omega, |t| < r\}.$$

We say φ is Dini continuous in x_1 if

$$\int_0^1 \frac{\omega_{\varphi,e_1}(r)}{r} dr < \infty,$$

and denote this by $\varphi \in C^{\mathrm{Dini}}_{x_1}(\Omega)$. We say φ is Hölder continuous in x_1 with exponent α if $\omega_{\varphi,e_1}(t) \in C^\alpha$, and denote $\varphi \in C^\alpha_{x_1}(\Omega)$, with the norm

$$\|\varphi\|_{C^\alpha_{x_1}(\Omega)} = \sup_\Omega |\varphi| + [\varphi]_{C^\alpha_{x_1}(\Omega)},$$

where

$$[\varphi]_{C^\alpha_{x_1}(\Omega)} = \sup_{r>0} \frac{\omega_{\varphi,e_1}(r)}{r^\alpha}.$$

The first main result of this paper is the following anisotropic Schauder type estimate.

Theorem 1.1. *Let u be a smooth, strictly convex solution of* (1.1) *in a bounded convex domain $\Omega \subset \mathbb{R}^2$. Assume that f satisfies* (1.2). *Then $\forall x, \bar{x} \in \Omega_\delta$ and $\forall \bar{\alpha} \in (0,1)$,*

$$(1.3)\quad |\partial_x\partial_{x_1}u(\bar{x}) - \partial_x\partial_{x_1}u(x)| \le Cd^{\bar{\alpha}}\left(1 + \int_d^1 \frac{\omega_{f,e_1}(r)}{r^{1+\bar{\alpha}}}\right) + C\int_0^d \frac{\omega_{f,e_1}(r)}{r},$$

where $d = |x - \bar{x}|$, $C \ge 0$ depends only on $\bar{\alpha}$, δ, λ, Λ and the modulus of convexity of u. It follows that, for $0 < \alpha < 1$,

$$(1.4)\qquad \|\partial_x\partial_{x_1}u\|_{C^\alpha(\Omega_\delta)} \le C\Big[1 + [f]_{C^\alpha_{x_1}(\Omega)}\Big]$$

for some C only depending on α, δ, λ, Λ, and the modulus of convexity of u. In particular (1.3) *also yields a continuity estimate of the second derivatives $\partial_x\partial_{x_1}u$ in terms of the x_1-direction Dini continuity of f.*

In the above theorem we have used the notion of modulus of convexity of u, which is given by

$$\begin{aligned} m_{u,\Omega}(t) &= \inf\{m_z(t) : z \in \Omega\},\\ m_z(t) &= \inf_{x\in\Omega}\{u(x) - \ell_z(x) : |x - z| > t\}, \end{aligned}$$

where $t > 0$, ℓ_z a support function of u at z. Clearly when u is strictly convex, $m_{u,\Omega}(t)$ is a positive function of $t > 0$.

Assume that f is k-times differentiable in variable x_1. We denote by $f \in C^{k,\alpha}_{x_1}(\Omega)$ (resp. $f \in C^{k,\alpha}_{x_1,\mathrm{loc}}(\Omega)$) if the derivative $\partial^k_{x_1}f$ is Hölder continuous (resp. locally Hölder continuous) in x_1 with exponent α. As a consequence of the previous estimates we have the following theorem.

Theorem 1.2. *Let u be the convex Aleksandrov solution of* (1.1) *in a bounded convex domain $\Omega \subset \mathbb{R}^2$, and $u = 0$ on $\partial\Omega$. Assume that f satisfies* (1.2). *If $f \in C^{Dini}_{x_1}(\Omega)$, then $u \in C^{1,1}_{loc}(\Omega)$. If $f \in C^{k,\alpha}_{x_1,loc}(\Omega)$, for some nonnegative integer k and $0 < \alpha < 1$, then $D^2u \in C^{k,\alpha'}_{x_1,loc}(\Omega)$ and $\partial^{k+1}_{x_1}u \in C^{1,\alpha'}_{loc}(\Omega)$ for any $0 < \alpha' < \alpha$.*

The highlight of our assumptions in the above theorems are the following: no assumptions on the regularity of f in x_2-direction as far as it is uniformly bounded from zero and infinity. Since f is merely measurable in x_2, the conclusions of [12] do not imply that (1.1) is uniformly elliptic, and the higher order regularity does not follow from Evans-Krylov's interior $C^{2,\alpha}$

estimate and the standard Schauder theory for linear elliptic equations. A main work of this paper is to show that (1.1) is indeed uniformly elliptic in two dimensions, under the assumption that f is Dini continuous in only one variable.

Our idea is to show that the solution of (1.1) is very close to the solution of the Monge-Ampère equation with certain inhomogeneous term that only varies in one direction. The regularity of the latter equation shall be studied in Section 2 by making use of the partial Legendre transform. By a perturbation method similar to [12, 26], Theorem 1.1 will be proved in Section 3. As an application of our a priori estimates, we prove Theorem 1.2 also in Section 3.

Our method can be also used to study (1.1) with piecewise Hölder continuous inhomogeneous term. Suppose $\{\Sigma_i\}_{i=1}^N$ is a family of disjoint simple curves in $\mathbb{R}^2$. Assume Σ_i divide Ω into subdomains $\{\Omega_i\}_{i=1}^{N+1}$. Suppose that f is Hölder continuous in each Ω_i, but could be discontinuous across the boundaries of Ω_i. We obtain the following result.

Theorem 1.3. *Let u be the convex Aleksandrov solution of* (1.1) *in a bounded convex domain $\Omega \subset \mathbb{R}^2$, and $u = 0$ on $\partial\Omega$. Assume that f satisfies* (1.2)*, $f \in C^\alpha(\Omega_i)$, and $\partial\Omega_i$ is of class $C^{1,\beta}$, for all $i = 1, \dots, N+1$. Then $u \in C^{1,1}_{loc}(\Omega)$. Moreover, for $x_0 \in \partial\Omega_i \cap \Omega$, if ξ_0 is a unit vector tangential to $\partial\Omega_i$ at x_0, then $\partial_{\xi_0} u \in C^{1,\alpha'}_{loc}(\mathcal{N}_{x_0})$ for $\alpha' < \frac{1}{2}\min\{\alpha, \beta\}$, and some neighborhood $\mathcal{N}_{x_0}$ of x_0.*

Linear elliptic equations with piecewise Hölder continuous coefficients have been studied extensively. This problem is related to the study of composite media, see e.g. [18, 19, 27] and the references therein. Theorem 1.3 deals with an analogous problem for the Monge-Ampère equation. The proof of Theorem 1.3 follows from the same techniques used in previous sections, and will be sketched in Section 4. In the appendix, we give a Pogorelov type estimate which is needed in Section 2. This estimate was known in [22]. For completeness, we also include a proof in the appendix.

2. Equations with inhomogeneous term independent of one variable

In this section, we study the convex solutions of

$$\det D^2 u(x) = f(x_2), \tag{2.1}$$

in a bounded convex domain $U \subset \mathbb{R}^2$. The feature of (2.1) is that the inhomogeneous term f varies only in one direction. We assume that f satisfies (1.2), but does not satisfy any smoothness. Namely f is merely measurable in x_2.

Our motivation is as follows. The $C^{2,\alpha}$ estimate in [1, 12] is based on showing that under the regularity assumption of f, the solution u of (1.1) can be approximated by convex functions which solve the Monge-Ampère equations with constant inhomogeneous term, say 1, in appropriate scaling. Since the latter solution has interior a priori estimates, one can prove by perturbation that C^2 norm of u remains bounded. However, in our case, the inhomogeneous term of (1.1) is not regular in all variables. This forces us to consider (2.1) instead of equations with constant inhomogeneous term.

The main goal of this section is to obtain the interior $C^{1,1}$ estimate for solutions of (2.1). One cannot apply Pogorelov's estimate to get this as the inhomogeneous term is merely measurable in x_2. However, when f is independent of x_1, a partial Pogorelov type estimate has been shown in [22], by which we conclude that if convex function $u \in C^4(U) \cap C(\bar{U})$ solves (2.1) with $u = 0$ on ∂U, then $\sup_{U_\delta} u_{x_1x_1} \le C_\delta$ for any $U_\delta \subset\subset U$. For reader's convenience, we include a proof of this partial Pogorelov type estimate in the appendix, by a slightly different computation. To bound the remaining second derivatives of u, our idea is to reduce the two dimensional Monge-Ampère equation (2.1) to a linear elliptic equation, and then apply the regularity theory of linear equations. This shall be achieved by the partial Legendre transform.

Let us define an injective mapping $\mathcal{P} : U \to \mathbb{R}^2$ by

$$x \mapsto y = \mathcal{P}(x) = (u_{x_1}, x_2). \tag{2.2}$$

Recall that the partial Legendre transform of u is given by

$$u^*(y) = x_1 y_1 - u. \tag{2.3}$$

It is easy to check that

$$u^*_{y_1} = x_1,\ u^*_{y_2} = -u_{x_2}, \tag{2.4}$$

and

$$u^*_{y_1y_1} = \frac{1}{u_{x_1x_1}}, \quad u^*_{y_1y_2} = -\frac{u_{x_1x_2}}{u_{x_1x_1}}, \quad u^*_{y_2y_2} = -\frac{f}{u_{x_1x_1}}. \tag{2.5}$$

Therefore u^* satisfies

$$f(y_2)u^*_{y_1y_1} + u^*_{y_2y_2} = 0 \text{ in } \mathcal{P}(U). \tag{2.6}$$

As we assume that f satisfies (1.2), the equation above is uniformly elliptic. Morrey [20] and Nirenberg [21] proved a remarkable theorem (see also [6, Chapter 12]), which gives Hölder estimates for the first derivatives of solutions of linear uniformly elliptic equations in two variables. The significant feature of their result is that the estimate depends only on bounds on the coefficients and not on any regularity properties. By repeatedly differentiating (2.6) along y_1-direction, we obtain

Proposition 2.1. *Let u^* be bounded smooth solution of* (2.6). *Assume that f satisfies* (1.2). *There exist β and C only depending on λ and Λ, such that for any $U' \subseteq U$, and any $p \neq q$ in $\mathcal{P}(U')$ with $d = \min\{dist(p, \partial\mathcal{P}(U')), dist(q, \partial\mathcal{P}(U'))\} > 0$,*

$$\frac{|\partial_y \partial^k_{y_1} u^*(p) - \partial_y \partial^k_{y_1} u^*(q)|}{|p-q|^\beta} \leq \frac{C}{d^{1+\beta}} \|\partial^k_{y_1} u^*\|_{L^\infty(\mathcal{P}(U'))},$$

where k is any nonnegative integer.

By the proposition above, we are able to prove

Proposition 2.2. *Assume that convex function $u \in C^4(U) \cap C(\bar{U})$ solves* (2.1), *$u = 0$ on ∂U, $B_{1/2} \subseteq U \subseteq B_1$ is normalised, and f satisfies* (1.2). *Then*

$$\|u\|_{C^{1,1}(U_\delta)}, \ \|D^2 u\|_{C^{0,1}_{x_1}(U_\delta)} \leq C_\delta,$$

where $U_\delta = \{x \in U : dist(x, \partial U) > \delta\}$, and C_δ only depends on δ, λ and Λ.

Proof. By the uniform estimate [25],

$$C^{-1} \leq \sup_U |u| \leq C$$

for some universal constant C. (Throughout this proof we say a constant is universal if it only depends on λ and Λ.) It follows that

$$\sup_{U_{\delta/4}} |Du| \leq 4\delta^{-1} \sup_U |u| \leq C_\delta.$$

By Proposition 5.1 in the appendix, we have $\sup_{U_\delta} u_{x_1x_1} \leq C_\delta$.

To prove $\sup_{U_\delta} u_{x_1x_1} \geq C_\delta$, by the partial Legendre transform, it suffices to show $\sup_{\mathcal{P}(U_\delta)} u^*_{y_1y_1} \leq C_\delta$. We shall apply Proposition 2.1 to achieve this.

We first prove

$$\operatorname{dist}\big(\mathcal{P}(U_\delta), \partial\mathcal{P}(U_{\delta/2})\big) \geq C_\delta. \tag{2.7}$$

Let $m_{U_{\delta/4}}(t)$ denote the modulus of convexity of u on $U_{\delta/4}$. It is known that $m_{U_{\delta/4}}(t)$ depends only on δ, λ and Λ. Indeed we have $m_{U_{\delta/4}}(t) \geq C_\delta t^{1+\beta}$ for some universal constant β, see e.g. [25]. For any $z \in U_\delta$, by subtracting an affine function, we assume that $u(z) = 0$ and $Du(z) = 0$. Let $\mathcal{C}_{z,\varepsilon} = \{x \in \mathbb{R}^2 : |\overrightarrow{xz} \cdot e_2| < \varepsilon|\overrightarrow{xz}|\}$, where ε is a constant to be determined. For any $\tilde{z} \in \partial U_{\delta/2}$, let $p = (p_1, p_2) = Du(\tilde{z})$ and $t_0 = |\overrightarrow{\tilde{z}z}|$. To prove (2.7), it suffices to consider two cases: (i) $\tilde{z} \in \mathcal{C}_{z,\varepsilon} \cap \partial U_{\delta/2}$; and (ii) $\tilde{z} \in \partial U_{\delta/2} \setminus \mathcal{C}_{z,\varepsilon}$. For case (i), by the convexity of u,

$$p \cdot \overrightarrow{\tilde{z}z} \geq m_{U_{\delta/4}}(t_0).$$

Consequently

$$t_0\big(|p_1| + |p|\varepsilon\big) \geq p_1\overrightarrow{\tilde{z}z} \cdot e_1 + p_2\overrightarrow{\tilde{z}z} \cdot e_2 \geq C_\delta t_0^{1+\beta}.$$

This implies

$$|p_1| \geq C_\delta t_0^\beta - |p|\varepsilon \geq C_\delta(\delta/2)^\beta - \varepsilon \sup_{U_{\delta/4}} |Du|.$$

Hence by choosing $\varepsilon = \varepsilon_\delta$ small, only depending on δ, λ and Λ, we have

$$\operatorname{dist}\big(\mathcal{P}(z), \mathcal{P}(\tilde{z})\big) \geq |u_{x_1}(\tilde{z}) - u_{x_1}(z)| = |p_1| \geq C_\delta.$$

Now the constant ε is fixed. For case (ii), one gets

$$\operatorname{dist}\big(\mathcal{P}(z), \mathcal{P}(\tilde{z})\big) \geq |\overrightarrow{\tilde{z}z} \cdot e_2| \geq \varepsilon|\overrightarrow{\tilde{z}z}| \geq C_\delta.$$

Hence (2.7) has been proved.

It is easy to check by (2.2), (2.3) and (2.4)

$$\|u^*\|_{L^\infty(\mathcal{P}(U_{\delta/2}))} \leq \operatorname{diam}(U) \cdot \|Du\|_{L^\infty(U_{\delta/2})} + \|u\|_{L^\infty(U)} \leq C_\delta,$$

and

$$\|Du^*\|_{L^\infty(\mathcal{P}(U_{\delta/2}))} \leq \operatorname{diam}(U) + \|Du\|_{L^\infty(U_{\delta/2})} \leq C_\delta.$$

By Proposition 2.1 (for $k = 1$) and the estimate (2.7), we get

$$\|\partial_y\partial_{y_1} u^*\|_{C^\beta(\mathcal{P}(U_\delta))} \leq C_\delta.$$

Hence

$$\|\partial_y \partial_{y_1} u^*\|_{L^\infty(\mathcal{P}(U_\delta))} \le C_\delta. \tag{2.8}$$

It follows from (2.5) that

$$\sup_{U_\delta} u_{x_1x_1} \ge 1/C_\delta \quad \text{and} \quad \sup_{U_\delta} |u_{x_1x_2}| \le C_\delta.$$

Noting that $u_{x_2x_2} = (f + u_{x_1x_2}^2)/u_{x_1x_1}$, we get $\|u\|_{C^{1,1}(U_\delta)} \le C_\delta$.

The argument above shows (2.2) is a Lipschitz mapping. By Proposition 2.1 (for $k = 2$) again, (2.8) implies $\|u^*_{y_1y_1y}\|_{L^\infty(\mathcal{P}(U_\delta))} \le C_\delta$. Differentiating (2.6) along y_1, we get

$$\|u^*_{y_1yy}\|_{L^\infty(\mathcal{P}(U_\delta))} \le C_\delta.$$

It is not hard to see $(u^*)^* = u$. Hence

$$u_{x_1x_1} = \frac{1}{u^*_{y_1y_1}}, \quad u_{x_1x_2} = -\frac{u^*_{y_1y_2}}{u^*_{y_1y_1}}, \quad u_{x_2x_2} = -\frac{f}{u^*_{y_1y_1}}.$$

It follows that $\|D^2u\|_{C^{0,1}_{x_1}(U_\delta)} \le C_\delta$. □

Before closing this section, we prove two lemmas which will be used later.

Kim and Krylov [14, 15] established the interior $W^{2,p}$ estimate for linear uniformly elliptic equations with coefficients that are measurable in one variable and VMO in others. This enables us to prove the following

Lemma 2.1. *Let u_i, $i = 1, 2$, be two convex solutions of (2.1) in B_1. Assume that f satisfies (1.2). Suppose $\|u_i\|_{C^{1,1}(B_1)}$ and $\|D^2u_i\|_{C^{0,1}_{x_1}(B_1)}$ are bounded by C_0. Then if $|u_1 - u_2| \le \nu$ in B_1 for some $\nu > 0$, we have for any $\bar\alpha \in (0, 1)$,*

$$\|u_1 - u_2\|_{C^{1,\bar\alpha}(B_{1/2})} \le C_{\bar\alpha}\nu,$$

and

$$\|\partial_{x_1}u_1 - \partial_{x_1}u_2\|_{C^{1,\bar\alpha}(B_{1/2})} \le C_{\bar\alpha}\nu,$$

where $C_{\bar\alpha}$ only depends on $\bar\alpha$, λ, Λ and C_0.

Proof. We have

$$
\begin{aligned}
0 &= \int_0^1 \frac{d}{dt} \log\det(tD^2u_2 + (1-t)D^2u_1)dt \\
&= \sum_{i,j} a_{ij}\partial^2_{ij}(u_2 - u_1),
\end{aligned} \tag{2.9}
$$

where $a_{ij} = \int_0^1 [tD^2_{ij}u_2 + (1-t)D^2_{ij}u_1]^{-1}dt$. By our assumptions, the operator $L = \sum_{i,j} a_{ij}\partial^2_{ij}$ is uniformly elliptic, and $\|a_{ij}\|_{C^{0,1}_{x_1}(B_1)} \le C$ (only depending on λ, Λ and C_0). Denote $w = u_2 - u_1$. Since a_{ij} is Lipschitz continuous in x_1 (hence VMO), by applying the interior $W^{2,p}$ estimate in [14, 15] to (2.9), we obtain

$$
\|w\|_{W^{2,p}(B_{4/5})} \le C_p\nu, \quad \forall p > 2,
$$

where C_p is a constant only depending on p, λ, Λ and C_0.

Differentiating (2.9) in x_1 variable, we have

$$
\sum_{i,j} a_{ij}\partial^2_{ij}w_{x_1} = -\sum \partial_{x_1}a_{ij}\partial^2_{ij}w \in L^p(B_{4/5}).
$$

By the interior $W^{2,p}$ estimate again, we obtain $\|w_{x_1}\|_{W^{2,p}(B_{3/4})} \le C_p\nu$. We complete the proof by Sobolev inequality immediately. □

The following lemma is a consequence of Proposition 2.2. It is almost identical to a lemma in [12]. Let us first recall some notions. By John's lemma, for bounded convex domain $U \in \mathbb{R}^2$, there is a minimum ellipsoid containing U. We say U is normalised if its minimum ellipsoid is a ball. When U is normalised, one has $B_{r/2} \subset U \subset B_r$ for two concentric balls $B_{r/2}$ and B_r. There is a unique unimodular affine transform T (i.e., $\det T = 1$) such that $T(U)$ is normalised. Choose an appropriate coordinates system such that T is determined by the matrix $\mathrm{diag}\{\lambda_1, \lambda_2\}$, with $\lambda_1 \le \lambda_2$. Following [12], we say U has a *good shape* if

$$
\lambda_2 \le c^*\lambda_1
$$

for some constant c^* under control.

Lemma 2.2. *Let u be a convex solution of* (2.1) *in U which vanishes on ∂U. Assume that f satisfies* (1.2). *Suppose u attains its minimum at the origin and $D^2u(0)$ is uniformly bounded, then the domain U is of good shape.*

Proof. Let T be the unimodular affine transform such that $B_{1/2} \subseteq T(U) \subseteq B_1$, and $\lambda_2 \geq \lambda_1$ be the eigenvalues of T. If U does not have a good shape, the ration λ_2/λ_1 is very large. Let $v(y) = u(T^{-1}(y))$. Then $\det v(y) = f(ay_1 + by_2 + c)$, where a, b, c are determined by the transform T. Notice that f is constant in the direction $\|T_*(e_1)\|^{-1}T_*(e_1)$. By Proposition 2.2, $D_y^2 v(0)$ is uniformly bounded (independent of a, b, c). Hence $D_x^2 u(0) = T' D_y^2 v(0) T$ cannot be uniformly bounded, a contradiction. $\square$

3. Equations with inhomogeneous term regular in one direction

This section is devoted to the proofs of Theorems 1.1 and Theorem 1.2. We prove them by a perturbation argument. Since f is merely measurable in x_2, our idea is to show that the solution u of (1.1) is very close to the solution of certain equation of the form (2.1) (namely, the right hand side term is constant in one direction). By the interior a priori estimates established in the previous section, we can prove by interpolation that the $C^{1,1}$ norm of u remains bounded. Hence (1.1) is uniformly elliptic. As f could be very rough in x_2 variable, one cannot apply Evans-Krylov's interior $C^{2,\alpha}$ estimate. We shall obtain the higher order anisotropic regularity of u by the partial Schauder theory for linear equations [5, 13, 24].

Proof of Theorem 1.1. Our argument follows [12] but some adjustments are needed. The main ingredient of the proof is to show that (1.1) is uniformly elliptic.

For $x_0 \in \Omega_\delta$, by a translation and subtracting an affine function, we suppose that $x_0 = 0$, $u(0) = 0$ and $Du(0) = 0$, so that the origin is the minimum point of u. We consider the solution u in the level set

$$S_h^0 = S_{h,u}^0 = \{x \in \Omega : u(x) < h\}.$$

Choose $h > 0$ small such that S_h^0 is compactly contained in Ω. There is a unimodular affine transform T_h such that $T_h(S_h^0)$ is normalised. By making the change $x \mapsto T_h x/\sqrt{h}$ and $u \mapsto u/h$, we may suppose $h = 1$, S_1^0 is normalised, and

$$\int_0^1 \frac{\omega_{f,\xi}(r)}{r} \leq \varepsilon, \tag{3.1}$$

where $\xi = \|(T_h)_*(e_1)\|^{-1}[(T_h)_*(e_1)]$. By a further rotation if necessary, we may still assume that $\xi = e_1$. Note that ε can be as small as we want, provided h sufficiently small. The Monge-Ampère equation is invariant under the above transform.

Starting with the function u satisfying the above conditions, we define a sequence of convex functions $\{u_k\}_{k=0}^{\infty}$ as the solutions of

$$\begin{aligned} \det D^2 u_k &= \bar{f}(x_2) := f(0, x_2) \text{ in } S^0_{4^{-k},u}, \\ u_k &= u = 4^{-k} \text{ on } \partial S^0_{4^{-k},u}. \end{aligned}$$

The feature of the above equations is that the inhomogeneous terms only depend on x_2. The interior regularity of such equations has been studied in the previous section. Denote

$$\begin{aligned} \nu(t) &= \sup_{z \in B_1} \{|f(y) - f(x)| : x, y \in S^0_{t^2,u}(z), y - x \parallel e_1\}, \\ \nu_k &= \nu(2^{-k}), \end{aligned}$$

which is invariant under unimodular affine transform of x. Here $S^0_{h,u}(z)$ is the level set of u at z, namely

$$S^0_{h,u}(z) = \{x \in \Omega : u(x) < \ell_z(x) + h\},$$

where ℓ_z is the support function of u at z. If $S^0_{t^2,u}$ has a good shape, then $\nu(t) \leq \omega_{f,e_1}(Ct) \leq C\omega_{f,e_1}(t)$.

Since $S^0_{1,u}$ has a good shape, by Lemma 2.2, $\|u_0\|_{C^{1,1}(S^0_{3/4,u})} \leq C$. The key point in our argument below is to understand that $S^0_{t^2,u}$ has a good shape for all $t > 0$ small. Note that

$$\det D^2(1 - C\nu_0)u \leq \det D^2 u_0 \leq \det D^2(1 + C\nu_0)u \text{ in } S^0_{1,u},$$

and $u = u_0 = 1$ on the boundary. By the comparison principle, we have

$$|u - u_0| \leq C\nu_0 \text{ in } S^0_{1,u}.$$

Similarly we have $|u - u_1| \leq C\nu_1$. Hence we obtain $|u_1 - u_0| \leq C\nu_0$. Since $S^0_{1,u}$ has a good shape, so does $S^0_{4^{-1},u}$. It follows by Proposition 2.2 that $|u_1|_{C^{1,1}(S^0_{3/16,u_1})} \leq C$ and $\|D^2 u_1\|_{C^{0,1}_{x_1}(S^0_{3/16,u_1})} \leq C$. By Lemma 2.1, we obtain

$$\|\partial_{x_1} u_1 - \partial_{x_1} u_0\|_{C^{1,\bar{\alpha}}(S^0_{4^{-2},u_1})} \leq C_{\bar{\alpha}} \nu_0. \tag{3.2}$$

In particular, this estimate implies $D^2 u_1(0)$ is uniformly bounded. It then follows from Lemma 2.2 that $S^0_{4^{-2},u_1}$ has a good shape.

By induction we assume that $S^0_{4^{-k-1},u}$ has a good shape, with the shape constant c^*, independent of k. Hence $\nu_k \le C\omega_{f,e_1}(2^{-k})$ for some C (depending on c^* but independent of k). Applying Proposition 2.2 and Lemma 2.1 to $\hat{u}_0 := 4^k u_k(\frac{x}{2^k})$ and $\hat{u}_1 := 4^k u_{k+1}(\frac{x}{2^k})$, we obtain, for $x \in S^0_{4^{-k-2},u_{k+1}}$,

$$\left|Du_k(x) - Du_{k+1}(x)\right| \le C2^{-k}\nu_k,$$

and

$$\left|\partial_x \partial_{x_1} u_k(x) - \partial_x \partial_{x_1} u_{k+1}(x)\right| \le C\nu_k,$$

Note that C is independent of k. Since

$$(u_k)_{x_2x_2} = \frac{\bar{f} + (u_k)^2_{x_1x_2}}{(u_k)_{x_1x_1}},$$

the second inequality above implies

$$|D^2u_k(x) - D^2u_{k+1}(x)| \le C\nu_k. \tag{3.3}$$

Hence for $x \in S^0_{4^{-k-2},u_{k+1}}$

$$\begin{aligned} \left|D^2u_0(x) - D^2u_{k+1}(x)\right| &\le \sum_{i=0}^{k} \left|D^2u_i - D^2u_{i+1}\right| \\ &\le C\int_{2^{-k}}^{1} \frac{\omega_{f,e_1}(r)}{r} \le C\varepsilon. \end{aligned} \tag{3.4}$$

Estimate (3.4) and assumption (3.1) imply that $D^2u_{k+1}(0)$ is uniformly bounded. Hence $S^0_{4^{-k-2},u_{k+1}}$ has a good shape, with the shape constant c^* (independent of k). Consider

$$\hat{u} = 4^{k+1}u\left(\frac{x}{2^{k+1}}\right), \quad \hat{u}_{k+1} = 4^{k+1}u_{k+1}\left(\frac{x}{2^{k+1}}\right).$$

Then $\hat{u}$ and $\hat{u}_{k+1}$ satisfy the equations $\det D^2\hat{u} = f(\frac{x}{2^{k+1}})$ and $\det D^2\hat{u}_{k+1} = f(0, \frac{x_2}{2^{k+1}})$ respectively. By the comparison principle

$$|\hat{u} - \hat{u}_{k+1}| \le C\nu_{k+1}.$$

Hence $S^0_{4^{-k-2},u}$ has a good shape.

From above argument, we obtain by letting $k \to \infty$,

$$\left|D^2 u(x) - D^2 u_0(x)\right| \le C \int_0^1 \frac{\omega_{f,e_1}(r)}{r}.$$

Hence equation (1.1) is uniformly elliptic if f is Dini continuous in x_1-direction. However one cannot apply Evans-Krylov's $C^{2,\alpha}$ estimate to complete the proof, since our estimate (1.3) does not depend on the regularity of f in x_2 variable. For fully nonlinear uniformly elliptic equations, a partial Schauder type estimate has been proved in [24].

For any given $\bar{x}$ near the origin,

$$\begin{aligned}\left|\partial_x\partial_{x_1} u(\bar{x}) - \partial_x\partial_{x_1} u(0)\right| &\le I_1 + I_2 + I_3 \\ &=: \left|\partial_x\partial_{x_1} u_k(\bar{x}) - \partial_x\partial_{x_1} u_k(0)\right| \\ &\quad + \left|\partial_x\partial_{x_1} u_k(0) - \partial_x\partial_{x_1} u(0)\right| \\ &\quad + \left|\partial_x\partial_{x_1} u_k(\bar{x}) - \partial_x\partial_{x_1} u(\bar{x})\right|.\end{aligned} \tag{3.5}$$

Suppose $4^{-k-4} < u(\bar{x}) \le 4^{-k-3}$. Then by (3.3),

$$I_2 \le \sum_{i=k}^{\infty} \left|\partial_x\partial_{x_1} u_i(0) - \partial_x\partial_{x_1} u_{i+1}(0)\right| \le C \int_0^{|\bar{x}|} \frac{\omega_{f,e_1}(r)}{r}. \tag{3.6}$$

Let $u_{\bar{x},j}$ be the solution of

$$\begin{aligned}\det D^2 u_{\bar{x},j} &= \bar{f}_{\bar{x}} := f(\bar{x}_1, x_2) \ \text{ in } S^0_{4^{-j},u}(\bar{x}), \\ u_{\bar{x},j} &= u \quad \text{on } \partial S^0_{4^{-j},u}(\bar{x}),\end{aligned}$$

where $\bar{x}_1$ is the first coordinate of $\bar{x}$. Clearly $\bar{f}_{\bar{x}}$ only varies in one direction. Let $j_k = \inf\{j : S^0_{4^{-j},u}(\bar{x}) \subset S^0_{4^{-k},u}\}$. Obviously $j_k \ge k$. By Caffarelli's strict convexity [2], there exists l_0 (independent of k) such that $j_k \le k + l_0$. Note that $\left|u_k - u_{\bar{x},k+l_0}\right| \le C\nu_k$. By applying Lemma 2.1 to u_k and $u_{\bar{x},k+l_0}$ in $S^0_{4^{-k-l_0},u}(\bar{x})$, it follows that

$$\left|\partial_x\partial_{x_1} u_k(\bar{x}) - \partial_x\partial_{x_1} u_{\bar{x},k+l_0}(\bar{x})\right| \le C\nu_k.$$

Similarly to (3.6) we have

$$\left|\partial_x\partial_{x_1} u(\bar{x}) - \partial_x\partial_{x_1} u_{\bar{x},k+l_0}(\bar{x})\right| \le C \sum_{i=k+l_0}^{\infty} \nu_i \le C \int_0^{|\bar{x}|} \frac{\omega_{f,e_1}(r)}{r}.$$

We obtain the desired estimate for I_3 by the above two inequalities.

Denote $h_i = u_i - u_{i-1}$. By scaling argument and Lemma 2.1, we have

$$\begin{aligned}\left|\partial_x\partial_{x_1}h_i(\bar{x}) - \partial_x\partial_{x_1}h_i(0)\right| &\le \|\partial_x\partial_{x_1}h_i\|_{C^{\bar{\alpha}}(S^0_{4^{-i-1},u_i})} \cdot |\bar{x}|^{\bar{\alpha}} \\ &\le C_{\bar{\alpha}}2^{\bar{\alpha}i}\nu_i|\bar{x}|^{\bar{\alpha}}.\end{aligned}$$

Hence

$$\begin{aligned}I_1 &\le \left|\partial_x\partial_{x_1}u_0(\bar{x}) - \partial_x\partial_{x_1}u_0(0)\right| + \sum_{i=1}^{k}\left|\partial_x\partial_{x_1}h_i(\bar{x}) - \partial_x\partial_{x_1}h_i(0)\right| \\ &\le C_{\bar{\alpha}}|\bar{x}|^{\bar{\alpha}}\left(1 + \int_{|\bar{x}|}^{1}\frac{\omega_{f,e_1}(r)}{r^{1+\bar{\alpha}}}\right).\end{aligned}$$

This completes the proof. □

We introduce a family of smooth functions f_k to approximate the function f which can be very rough in one variable. Let $\varrho(t)$ be the standard mollifier on $\mathbb{R}$. Namely ϱ is a non-negative function in $C^\infty(\mathbb{R})$ vanishing outside the unit interval $(-1,1)$ and satisfying $\int \varrho dt = 1$. Let us define, for $\varepsilon_k \to 0$,

$$\begin{aligned}f_k(x) &= \frac{1}{\varepsilon_k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,s)\varrho\left(\frac{x_1-t}{\varepsilon_k}\right)\varrho\left(\frac{x_2-s}{\varepsilon_k}\right)dtds \\ &= \int_{|s|\le 1}\int_{|t|\le 1} f(x_1-\varepsilon_k t, x_2-\varepsilon_k s)\varrho(t)\varrho(s)dtds.\end{aligned}$$

Choose ε_k small enough such that $\Omega' \subset B_{\varepsilon_k}(\Omega') = \{x : d(x,\Omega') < \varepsilon_k\} \subset \Omega$. It is not hard to see

(i) f_k converges to f strongly in $L^1(\Omega')$;

(ii) If $f \in C^{\mathrm{Dini}}_{x_1}(\Omega)$, then $f_k \in C^{\mathrm{Dini}}_{x_1}(\Omega')$, and $\|f_k\|_{C^{\mathrm{Dini}}_{x_1}(\Omega')} \le \|f\|_{C^{\mathrm{Dini}}_{x_1}(\Omega)}$;

(iii) If $f \in C^{m,\alpha}_{x_1}(\Omega)$, then $f_k \in C^{m,\alpha}_{x_1}(\Omega')$ $\|f_k\|_{C^{m,\alpha}_{x_1}(\Omega')} \le \|f\|_{C^{m,\alpha}_{x_1}(\Omega)}$, where m is nonnegative integer and $0 \le \alpha \le 1$.

In (iii) we use the norm $\|\varphi\|_{C^{m,\alpha}_{x_1}(\Omega)} := \sum_{l=0}^{m}\sup_\Omega|\partial^l_{x_1}\varphi| + [\partial^m_{x_1}\varphi]_{C^\alpha_{x_1}(\Omega)}$.

Proof of Theorem 1.2. We define a family of smooth functions f_k as above. Let u_k be the convex solution of $\det D^2u_k = f_k$ in Ω, vanishing on $\partial\Omega$. It is known that u_k is smooth in Ω, see e.g. [25]. Since f_k converges to f strongly in $L^1(\Omega)$, u_k uniformly converges to u. By the proof of Theorem 1.1, for $\Omega' \subset\subset \Omega$, if $f \in C^{Dini}_{x_1}(\Omega)$, then $\|u_k\|_{C^{1,1}(\Omega')}$ is uniformly bounded, hence $u \in$

$C^{1,1}_{\mathrm{loc}}(\Omega)$. By the estimate (1.4), we have $\|\partial_{x_1}u_k\|_{C^{1,\alpha}(\Omega')} \le C\big[1+[f_k]_{C^{\alpha}_{x_1}(\Omega)}\big]$. Using the equation (1.1), we conclude

$$\|D^2u_k\|_{C^{\alpha}_{x_1}(\Omega')} \le C\big[1+[f_k]_{C^{\alpha}_{x_1}(\Omega)}\big]. \tag{3.7}$$

This implies $\partial_{x_1}u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ and $D^2u \in C^{\alpha}_{x_1,\mathrm{loc}}(\Omega)$ provided $f \in C^{\alpha}_{x_1,\mathrm{loc}}(\Omega)$.

For higher order regularity, we differentiate (1.1) along x_1 to get

$$\sum_{i,j} a_{ij}\partial^2_{ij}u^{(k)} = f^{(k)} - \sum_{l=1}^{k-1}\begin{pmatrix} k-1 \\ l \end{pmatrix}\sum_{i,j} a^{(l)}_{ij}\partial^2_{ij}u^{(k-l)} =: \Psi, \tag{3.8}$$

where a_{ij} is the cofactor of $\{u_{ij}\}$, $u^{(k)} = \partial^k_{x_1}u$, and $a^{(k)}_{ij}$, $f^{(k)}$ are defined similarly. We claim that, if u is a smooth solution of (1.1) in Ω, vanishing on $\partial\Omega$, then for any $0 < \alpha' < \alpha$ and $\delta > 0$, there exists a constant $C_{\alpha',\alpha,\delta,m}$, only depending on α', α, δ, λ, Λ and $\|f\|_{C^{m,\alpha}_{x_1}(\Omega)}$, such that

$$\|\partial^{m+1}_{x_1}u\|_{C^{1,\alpha'}(\Omega_\delta)} \le C_{\alpha',\alpha,\delta,m} \ \text{ and } \ \|D^2u\|_{C^{m,\alpha'}_{x_1}(\Omega_\delta)} \le C_{\alpha',\alpha,\delta,m}. \tag{3.9}$$

By applying the a priori estimate above to u_k, and using the approximation argument, the proof is completed.

We prove the claim by induction. For $m = 0$, we already know that the claim is true. By (3.7), we also have

$$\|a_{ij}\|_{C^{\alpha}_{x_1}(\Omega_{\delta/2})} \le C\big[1+[f]_{C^{\alpha}_{x_1}(\Omega)}\big]. \tag{3.10}$$

Assume (3.9) holds true for $m = 0, \dots, k-1$. Then the inhomogeneous term of (3.8) satisfies, for any $\alpha' < \alpha'' < \alpha$,

$$\|\Psi\|_{C^{\alpha''}_{x_1}(\Omega_{\delta/2})} \le C_{\alpha'',\alpha,\delta,k}. \tag{3.11}$$

Let $v = u^{(k)}$ which solves (3.8). In [13] the authors established a partial Schauder estimate for linear uniformly elliptic equations with coefficients and inhomogeneous term that are Hölder continuous in one direction. They do not assume the continuity of the coefficients in two dimensions. When applying to (3.8), their result reads as, for any $p > \frac{2}{\alpha''}$ and $0 < \beta < \alpha'' - \frac{2}{p}$,

$$\begin{aligned}\big|\partial_x\partial_{x_1}v(\bar{x}) - \partial_x\partial_{x_1}v(x)\big| \le Cd^\beta &\left[\sup_{\Omega_{\delta/2}}|v| + \|\Psi\|_{L^p(\Omega_{\delta/2})} + \frac{\|\Psi\|_{C^{\alpha''}_{x_1}(\Omega_{\delta/2})}}{\alpha''-\beta}\right] \\ &+ C\|a_{ij}\|_{C^{\alpha''}_{x_1}(\Omega_{\delta/2})}\left(\sup_{\Omega_{\delta/2}}|v| + \|\Psi\|_{L^p(\Omega_{\delta/2})}\right)d^{\alpha''-\frac{2}{p}},\end{aligned}$$

where $x, \bar{x} \in \Omega_\delta$ and $d = |x - \bar{x}|$. The constant C only depends on α'', β, p, δ and the elliptic constants of (3.8). Set $\beta = \alpha'$ and $\alpha'' = \frac{1}{2}(\alpha' + \alpha)$, and choose p large (only depending on α',α) such that $\frac{2}{p} < \alpha'' - \alpha'$. Then, by the above estimate, and (3.10) (3.11),

$$\|\partial_x \partial_{x_1} u^{(k)}\|_{C^{\alpha'}(\Omega_\delta)} \leq C_{\alpha',\alpha,\delta,k}.$$

By the equation (3.8), $u^{(k)}_{x_2x_2} = a_{22}^{-1}\big(\Psi - a_{11}u^{(k)}_{x_1x_1} - 2a_{12}u^{(k)}_{x_1x_2}\big)$. Hence (3.9) follows. □

4. Equations with piecewise Hölder continuous inhomogeneous term

We need the following approximation lemma. Its proof can be found in [12].

Lemma 4.1. *Let u_i be the convex Aleksandrov solution of the Monge-Ampère equation* $\det D^2 u_i = f_i \geq 0$ *in convex domain U, $i = 1, 2$. Assume $u_1 = u_2$ on $\partial U \subset \mathbb{R}^n$. Then*

$$\sup_U |u_1 - u_2| \leq C\|f_1 - f_2\|^{1/n}_{L^1(U)},$$

where C depends only on n and diam(U).

To prove Theorem 1.3, it suffices to show the regularity at $x_0 \in \Sigma_{n_0}$. By a translation and subtracting a support function, we assume $x_0 = 0$, and the origin is the minimum point of u. Choose h small so that S^0_h is compactly contained in Ω and does not intersect with Σ_j if $j \neq n_0$. Make the change $x \mapsto T_h x/\sqrt{h}$, $u \mapsto u/h$, where T_h is a unimodular affine transform such that $T_h(S^0_h)$ is normalised. By a further rotation, we assume $\Sigma_{n_0} = \{(x_1, g(x_1))\}$ for some function $g \in C^{1,\beta}$ satisfying $g(0) = g'(0) = 0$. Assume Σ_{n_0} divides B_1 into two parts: $D^+ = \{x \in B_1 : x_2 > g\}$ and $D^- = \{x \in B_1 : x_2 < g\}$. We define locally

$$\bar{f}(x) = \begin{cases} \lim\limits_{y\in D^+, y\to 0} f(y) & \text{in } B_1^+ = B_1 \cap \{x_2 > 0\}, \\ \lim\limits_{y\in D^-, y\to 0} f(y) & \text{in } B_1^- = B_1 \cap \{x_2 < 0\}, \\ f(0) & \text{on } B_1 \cap \{x_2 = 0\}. \end{cases}$$

Clearly $\bar{f}$ only depends on x_2. We denote

$$\tilde{\omega}(r) = \frac{1}{r}\|f - \bar{f}\|^{1/2}_{L^1(B_r(0))}.$$

Note that

$$
\begin{aligned}
\int_{B_r} |f-\bar f| &= \int_{B_r^+\cap D^+} + \int_{B_r^+\setminus D^+} + \int_{B_r^-\cap D^-} + \int_{B_r^-\setminus D^-} \\
&\le C\int_{B_r} |x|^\alpha + 2\Lambda|B_r^+\setminus D^+| + 2\Lambda|B_r^-\setminus D^-| \\
&\le Cr^{2+\gamma},
\end{aligned}
\tag{4.1}
$$

where $\gamma=\min\{\alpha,\beta\}$, and C only depends on λ, Λ, $\|f\|_{C^\alpha(\Omega_i)}$, $\|\Sigma_i\|_{C^{1,\beta}}$. Hence

$$
\int_0^1 \frac{\tilde\omega(r)}{r} \le \varepsilon
$$

for any small ε, if h is sufficiently small.

Again, let convex functions u_k, $k=0,1,\dots$, be the solution of

$$
\begin{aligned}
\det D^2 u_k &= \bar f(0,x_2) \ \text{ in } S^0_{4^{-k},u}, \\
u_k &= u = 4^{-k} \ \text{ on } \partial S^0_{4^{-k},u}.
\end{aligned}
$$

Denote

$$
\tilde\nu(t) = \frac{1}{t}\left(\int_{S^0_{t^2,u}} |f-\bar f|\right)^{1/2} \quad \text{and} \quad \tilde\nu_k = \tilde\nu(2^{-k}).
$$

For each k, by scaling, we consider $\hat u = 4^k u(\frac{x}{2^k})$ and $\hat u_k = 4^k u_k(\frac{x}{2^k})$. Then $\hat u$ and $\hat u_k$ satisfy the equation $\det D^2\hat u = f(2^{-k}x)$ and $\det D^2 \hat u_k = \bar f(2^{-k}x)$ respectively. By Lemma 4.1,

$$
|\hat u - \hat u_k| \le C\tilde\nu_k.
$$

Hence, by the proof of Theorem 1.1 (replacing ω_{f,e_1}, ν by $\tilde\omega$, $\tilde\nu$ respectively), we obtain that $\|u\|_{C^{1,1}(\Omega_\delta)} \le C_\delta$, and for any z close to 0,

$$
|\partial_x\partial_{\xi_0}u(z) - \partial_x\partial_{\xi_0}u(0)| \le C|z|^{\bar\alpha}\left(1+\int_{|z|}^1 \frac{\tilde\omega(r)}{r^{1+\bar\alpha}}\right) + C\int_0^{|z|}\frac{\tilde\omega(r)}{r},
$$

where C depends only on $\bar\alpha$, λ and Λ.

Similar to the proof of Theorem 1.2, Theorem 1.3 follows from (4.1), the above a priori estimates and the standard approximation argument.

5. Appendix

In this appendix, we prove a Pogorelov type estimate for smooth solutions of equation

$$\det D^2 u = f(x') \ \text{ in } U, \tag{5.1}$$

where U is a bounded convex domain in $\mathbb{R}^n$ and $x' = (x_2, ..., x_n)$. This Pogorelov type estimate was known to Savin [22]. Here we include a proof for completeness by a slightly different computation.

Proposition 5.1. *Assume that convex function $u \in C^4(U) \cap C(\bar{U})$ solves (2.1), $u = 0$ on ∂U, and $f > 0$. Then*

$$[-u(x)]u_{11} \leq C_n \left(\sup_U u_1^2 + 1 \right).$$

where C_n only depends on the dimension.

Proof. Consider the auxiliary function

$$H(x) = \rho(x)\eta\left(\frac{1}{2}u_1^2\right)u_{11}.$$

The function $\eta(t)$ is to be determined and ρ is a cut-off function. Assume H attains its maximum at $0 \in U$. By performing the affine transform

$$\tilde{u}(x) = u\left(x_1 - \frac{1}{u_{11}(0)}\sum_{i\geq 2} u_{1i}(0)x_i, x'\right),$$

which does not affect the equation or the maximum, one can also assume that $D^2u(0)$ is diagonal. Then the inverse matrix $\{u^{ij}\} = \mathrm{diag}\{u_{11}^{-1}, \ldots, u_{nn}^{-1}\}$.

We have, at 0,

$$\begin{aligned} 0 &= D_i \log H = \frac{\rho_i}{\rho} + \frac{\eta_i}{\eta} + \frac{u_{11i}}{u_{11}}, \\ 0 &\geq D_{ij}^2 \log H = \frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2} + \frac{\eta_{ii}}{\eta} - \frac{\eta_i\eta_j}{\eta^2} + \frac{u_{11ij}}{u_{11}} - \frac{u_{11i}u_{11j}}{u_{11}^2}, \end{aligned} \tag{5.2}$$

consequently

$$0 \geq \sum u^{ii}\left(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}\right) + \sum u^{ii}\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) + \sum u^{ii}\left(\frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}\right). \tag{5.3}$$

Differentiating (5.1) w.r.t. x_1 up to twice, we get

$$\sum_i u^{ii}u_{ii1} = 0, \tag{5.4}$$

$$\sum u^{ii}u_{ii11} = \sum u^{ii}u^{jj}u_{ij1}^2. \tag{5.5}$$

Plugging (5.5) into (5.3), we find

$$\begin{aligned} 0 \ge &\sum u^{ii}\left(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}\right) + \sum u^{ii}\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) \\ &+ \frac{1}{u_{11}}\sum u^{ii}u^{jj}u_{ij1}^2 - \frac{1}{u_{11}^2}\sum u^{ii}u_{11i}^2 \\ = &\sum u^{ii}\left(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}\right) + \sum u^{ii}\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) \\ &+ \frac{1}{u_{11}}\sum_{i\ge 1, j\ge 2} u^{ii}u^{jj}u_{ij1}^2. \end{aligned} \tag{5.6}$$

Setting $\rho = -u$, we compute

$$\begin{aligned} \sum u^{ii}\left(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}\right) &= -\frac{n}{|u|} - \frac{1}{|u|^2}\sum u^{ii}\rho_i^2 \\ &= -\frac{n}{|u|} - \frac{u_1^2}{|u|^2 u_{11}} - \sum_{i\ge 2} u^{ii}\left(\frac{\eta_i}{\eta} + \frac{u_{11i}}{u_{11}}\right)^2 \\ &= -\frac{n}{|u|} - \frac{u_1^2}{|u|^2 u_{11}} + \sum_{i\ge 2} u^{ii}\left(\frac{\eta_i^2}{\eta^2} + 2\frac{\rho_i\eta_i}{\rho\eta} - \frac{u_{11i}^2}{u_{11}^2}\right) \\ &= -\frac{n}{|u|} - \frac{u_1^2}{|u|^2 u_{11}} - \frac{1}{u_{11}^2}\sum_{i\ge 2} u^{ii}u_{11i}^2, \end{aligned}$$

where we have used $\eta_2 = 0$. We may assume $\frac{u_1^2}{|u|u_{11}} \le C$, otherwise we are through. Hence

$$\sum u^{ii}\left(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}\right) \ge -\frac{C}{|u|} - \frac{1}{u_{11}^2}\sum_{i\ge 2} u^{ii}u_{11i}^2.$$

Plugging this into (5.6), one gets

$$
\begin{aligned}
0 &\geq -\frac{C}{|u|} + \sum u^{ii}\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) + \frac{1}{u_{11}}\sum_{i,j\geq 2} u^{ii}u^{jj}u_{ij1}^2 \\
&\geq -\frac{C}{|u|} + \sum u^{ii}\left[\left(\frac{\eta''}{\eta} - \frac{\eta'^2}{\eta^2}\right)u_1^2u_{1i}^2 + \frac{\eta'}{\eta}u_{1i}^2 + \frac{\eta'}{\eta}u_1u_{1ii}\right] \\
&= -\frac{C}{|u|} + \left[\left(\frac{\eta''}{\eta} - \frac{\eta'^2}{\eta^2}\right)u_1^2 + \frac{\eta'}{\eta}\right]u_{11}.
\end{aligned}
$$

Let $\eta(t) = \left(1 - \frac{t}{2\sup_\Omega u_1^2+1}\right)^{-1/8}$. Clearly η is non-decreasing and $\eta \geq 1$. It is direct to compute

$$
\eta' = \frac{\eta^9}{16\sup_\Omega u_1^2 + 8}, \quad \eta'' = \frac{9\eta^{17}}{(16\sup_\Omega u_1^2 + 8)^2},
$$

and therefore

$$
\frac{\eta''}{\eta} - \frac{\eta'^2}{\eta^2} = \frac{\eta^{16}}{8(2\sup_\Omega u_1^2 + 1)^2}.
$$

Consequently, we have

$$
\begin{aligned}
0 &\geq -\frac{C}{|u|} + \frac{(2\eta^8 - 1)\eta^8}{8(2\sup_\Omega u_1^2 + 1)}u_{11} \\
&\geq -\frac{C}{|u|} + \frac{C}{\sup_\Omega u_1^2 + 1}u_{11},
\end{aligned}
$$

thus finishing the proof. □

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