A characterization of Clifford hypersurfaces among embedded constant mean curvature hypersurfaces in a unit sphere

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Let $\Sigma$ be an $n(\geq 3)$-dimensional compact embedded hypersurface in a unit sphere with constant mean curvature $H \geq 0$ and with two distinct principal curvatures $\lambda$ and $\mu$ of multiplicity $n - 1$ and 1, respectively. It is known that if $\lambda > \mu$, there exist many compact embedded constant mean curvature hypersurfaces [26]. In this paper, we prove that if $\mu > \lambda$, then $\Sigma$ is congruent to a Clifford hypersurface. The proof is based on the arguments used by Brendle [10].
1. Introduction

Let $\Sigma$ be an $n$-dimensional compact embedded hypersurface in an $(n + 1)$-dimensional unit sphere $S^{n+1}$ with constant mean curvature $H$. In case of minimal surfaces in $S^3$ (i.e., $n = 2$ and $H = 0$), Brendle [10] ingeniously proved the famous Lawson conjecture which states that the only embedded minimal torus in $S^3$ is the Clifford torus from a sharp estimate for a two-point function by using the maximum principle. It was observed that the embeddedness condition can be replaced by the weaker assumption that the minimal torus is Alexandrov-immersed in $S^3$ [9]. The technique using the maximum principle for a two-point function was also used by Andrews-Li [5], who gave a complete classification of embedded constant mean curvature tori in $S^3$. More generally, the proof of Lawson conjecture was extended to a class of embedded Weingarten tori in $S^3$ [11]. Hauswirth-Kilian-Schmidt [15] obtained that every mean-convex Alexandrov embedded constant mean curvature torus in $S^3$ is rotationally symmetric by using integrable systems.

It is interesting to find the higher-dimensional analogues of these results. One possible approach to the higher-dimensional problem is to characterize a Clifford hypersurface among embedded constant mean curvature hypersurfaces in $S^{n+1}$. Unfortunately, even when $H = 0$, it is well-known that there exist infinitely many mutually noncongruent embedded minimal hypersurfaces in $S^{n+1}$ which are homeomorphic to the Clifford hypersurface [17]. Recall that an $n$-dimensional Clifford hypersurface in $S^{n+1}$ with constant mean curvature $H$ has two distinct principal curvatures $\lambda$ and $\mu$ of multiplicity $n - k$ and $k$, respectively. Moreover it is given by

$$S^{n-k} \left( \frac{1}{\sqrt{1+\lambda^2}} \right) \times S^k \left( \frac{1}{\sqrt{1+\mu^2}} \right),$$

where $\lambda$ and $\mu$ satisfy $nH = (n - k)\lambda + k\mu$ and $\lambda\mu + 1 = 0$.

In view of this observation, we restrict ourselves to consider compact embedded constant mean curvature hypersurfaces in a unit sphere with two distinct principal curvatures. Otsuki [22] proved that if the multiplicities of two distinct principal curvatures are greater than 1, then the minimal hypersurface is locally congruent to a Clifford minimal hypersurface. Later, by studying an ordinary differential equation derived from the two distinct principal curvature condition, Otsuki [23, 24] also proved that a compact embedded minimal hypersurface in $S^{n+1}$ with two distinct principal curvatures of multiplicity $n - 1$ and 1, respectively, is congruent to a Clifford minimal
hypersurface (see also [12]). Therefore he gave the following characterization of Clifford minimal hypersurfaces in $\mathbb{S}^{n+1}$.

**Theorem 1.1 ([22–24]).** Let $\Sigma$ be an $n(\geq 3)$-dimensional compact embedded minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures of multiplicity $n-k$ and $k$ for $1 \leq k \leq n-1$. Then $\Sigma$ is congruent to a Clifford minimal hypersurface $\mathbb{S}^{n-k} \left( \sqrt{\frac{n-k}{n}} \right) \times \mathbb{S}^{k} \left( \frac{1}{\sqrt{1+\frac{k}{n}}} \right)$.

In case of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, Wei [31] obtained the analogue of Otsuki’s result, provided the multiplicities of two principal curvatures are at least 2, applying a similar argument as in [22].

**Theorem 1.2 ([31]).** Let $\Sigma$ be an $n(\geq 3)$-dimensional hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H$ and with two distinct principal curvatures $\lambda$ and $\mu$ of multiplicities $n-k$ and $k$, respectively, for $2 \leq k \leq n-2$. Then $\Sigma$ is isometric to a Clifford hypersurface $\mathbb{S}^{n-k} \left( \frac{1}{\sqrt{1+\lambda^2}} \right) \times \mathbb{S}^{k} \left( \frac{1}{\sqrt{1+\mu^2}} \right)$, where $\lambda$ and $\mu$ satisfy $nH = (n-k)\lambda + k\mu$ and $\lambda\mu + 1 = 0$.

Therefore it suffices to consider constant mean curvature hypersurfaces with two distinct principal curvatures $\lambda$ and $\mu$, $\mu$ being simple (i.e., multiplicity 1). Perdomo [26] obtained the existence of compact embedded constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ other than the totally geodesic $n$-spheres and Clifford hypersurfaces (see also [12, 27, 33]). Indeed, he constructed such examples by analyzing an ordinary differential equation arising from the two distinct principal curvatures $\lambda$ and $\mu$ satisfying that $\lambda > \mu$.

**Theorem 1.3 ([26]).** For any integer $m \geq 2$ and $H$ between $\cot \frac{\pi}{m}$ and $\frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$, there exists a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H$ other than the totally geodesic $n$-spheres and Clifford hypersurfaces.

On the other hand, in the study of $n$-dimensional constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures of multiplicity $n-1$ and 1, it mostly requires an additional assumption to obtain a characterization of Clifford hypersurfaces. For instance, Perdomo [25] and Wang [30] independently obtained a curvature integral inequality for minimal hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, which
characterizes a Clifford minimal hypersurface. Later, Wei [32] showed that the similar curvature integral inequality holds for hypersurfaces with the vanishing $m$-th order mean curvature (i.e., $H_m \equiv 0$). More precisely, they proved

**Theorem ([25, 30, 32]).** Let $M$ be an $n(\geq 3)$-dimensional closed hypersurface in $S^{n+1}$ with $H_m \equiv 0$ ($1 \leq m < n$) and with two distinct principal curvatures, one of them being simple. Then

$$\int_M |A|^2 \leq \frac{n(m^2 - 2m + n)}{m(n - m)} \text{Vol}(M),$$

where equality holds if and only if $M$ is isometric to a Clifford hypersurface $S^{n-1}(\sqrt{n-m/n}) \times S^1(\sqrt{m/n})$.

In [3], Andrews-Huang-Li obtained a uniqueness of Clifford hypersurface among compact embedded Weingarten hypersurfaces in the unit sphere with two distinct principal curvatures satisfying a linear relation between them. Very recently, the authors [21] obtained a more general sharp curvature integral inequality for hypersurfaces in $S^{n+1}$ with constant $m$-th order mean curvature and with two distinct principal curvatures, which generalizes Simons’ integral inequality and gives a characterization of Clifford hypersurfaces in $S^{n+1}$.

In contrast to the 2-dimensional problem for embedded constant mean curvature tori, we consider embedded constant mean curvature hypersurfaces with two distinct principal curvatures of multiplicity $n - 1$ and 1 without assuming any topological restriction. In this paper, we give the following characterization theorem (Theorem 5.3) of Clifford hypersurfaces:

**Theorem.** Let $\Sigma$ be an $n(\geq 3)$-dimensional compact embedded hypersurface in $S^{n+1}$ with constant mean curvature $H \geq 0$ and with two distinct principal curvatures $\lambda$ and $\mu$, $\mu$ being simple. If $\mu > \lambda$, then $\Sigma$ is congruent to a Clifford hypersurface $S^{n-1}(\sqrt{1+\lambda^2}) \times S^1(\sqrt{\lambda/1+\lambda^2})$, where $\lambda = \frac{nH - \sqrt{n^2H^2+4(n-1)}}{2(n-1)}$.

The key ingredients in the proof of our theorem are the following: We first define a suitable two-point function on an embedded constant mean curvature hypersurface based on the non-collapsing argument and compute the first and second order derivatives of the two-point function. This technique was pioneered by Huisken [18] and was developed by Andrews [2].
Secondly, we obtain a Simons-type identity for constant mean curvature hypersurfaces with two distinct principal curvatures. Indeed, this provides a sufficient condition for constant mean curvature hypersurfaces to attain the equality in Kato’s inequality. Combining with Simons-type identity and adapting the arguments by Brendle [10] with a slight modification finally gives a characterization of Clifford hypersurfaces.

We remark that every constant mean curvature torus in $S^3$ has two distinct principal curvatures which implies that there is no umbilic point. (See [20] for minimal tori and [14, 16] for constant mean curvature tori in $S^3$.) Moreover, constant mean curvature tori in $S^3$ automatically satisfy the condition that $\mu > \lambda$. Hence our main theorem can be regarded as an extension of the results by Brendle [10] and Andrews-Li [5] to higher-dimensional cases.

2. Preliminaries

Let $F : \Sigma^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ be a compact embedded constant mean curvature hypersurface in $S^{n+1}$ with two distinct principal curvatures, one of them being simple. Let $\nu(x)$ be the unit normal vector at $x \in \Sigma$ in $S^{n+1}$. Let $h$ and $A$ be the second fundamental form and the shape operator of $\Sigma$, respectively. Note that $A$ is a self-adjoint endomorphism of the tangent space at each point $x$ in $\Sigma$ such that $\langle A(X), Y \rangle = h(X, Y)$ for all $X, Y \in T_x \Sigma$. Since $\Sigma$ has two distinct principal curvatures and one of them is simple, we may assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$, where each $\lambda_i$ denotes the principal curvature on $\Sigma$ for $1 \leq i \leq n$. The normalized mean curvature $H$ is defined by

$$H = \frac{1}{n} \text{tr}(h) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{n-1}{n} \lambda + \frac{1}{n} \mu.$$
Define the two-point function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by
\begin{equation}
Z(x, y) := \Psi(x)(1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle.
\end{equation}
It is easy to check that for any $y \in \Sigma$,
\begin{align*}
Z(x, y) > 0 & \text{ if } F(y) \in \text{int}B_T(x, \frac{1}{\Psi(x)}), \\
Z(x, y) = 0 & \text{ if } F(y) \in \partial B_T(x, \frac{1}{\Psi(x)}), \\
Z(x, y) < 0 & \text{ if } F(y) \not\in B_T(x, \frac{1}{\Psi(x)}),
\end{align*}

since
\[ \frac{2}{\Psi(x)}Z(x, y) = |F(y) - p(x)|^2 - \left( \frac{1}{\Psi(x)} \right)^2. \]

We recall the definition of the interior ball curvature at $x \in \Sigma$, which was originally given by Andrews-Langford-McCoy [4] (see also [5]).

**Definition 2.1.** The interior ball curvature $k$ is a positive function on $\Sigma$ defined by
\[ k(x) := \inf \left\{ \frac{1}{r} : B_T(x, r) \cap \Sigma = \{ x \}, \ r > 0 \right\}. \]

Because $\Sigma$ is compact and embedded in $S^{n+1}$, one can see that the function $k$ is a well-defined positive function on $\Sigma$. From the definition of $k(x)$ for every point $x \in \Sigma$, it follows that
\[ k(x)(1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle \geq 0 \]
for all $y \in \Sigma$. Let $\Phi(x) := \max\{\lambda(x), \mu(x)\}$ be the maximum value of the principal curvatures of $\Sigma$ in $S^{n+1}$ at $F(x)$. Note that the two distinct principal curvature condition guarantees that $\Sigma$ has no umbilic point and hence $\Phi(x) - H > 0$.

Motivated by the works of Brendle [10] and Andrews-Li [5], we introduce the constant $\kappa$ as follows:
\[ \kappa := \sup_{x \in \Sigma} \frac{k(x) - H}{\Phi(x) - H}. \]

For convenience, we will write $\varphi(x) := \Phi(x) - H$. 

Proposition 2.2 (Uniform boundedness of $\kappa$). Let $\Sigma$ be a compact embedded constant mean curvature hypersurface with two distinct principal curvatures in $\mathbb{S}^{n+1}$. Then there exists a constant $K > 0$ satisfying

$$1 \leq \kappa < K.$$ 

Proof. By definition, one sees that $\varphi > 0$. Because $\Sigma$ is compact, $\varphi$ is uniformly bounded and $k$ is uniformly bounded above. From the definition of $k$, it immediately follows that $k(x) \geq \Phi(x)$ for all $x \in \Sigma$, which gives the conclusion. 

Define a positive function $\Psi(x) := \kappa \varphi(x) + H = \kappa(\Phi(x) - H) + H$ on $\Sigma$. Then $\Psi(x) \geq k(x)$. It follows that

$$Z(x, y) = \Psi(x)(1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle \geq 0$$

for all $(x, y) \in \Sigma \times \Sigma$. Therefore if there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x}, \overline{y}) = 0$, then

$$\frac{\partial Z}{\partial x_i}(\overline{x}, \overline{y}) = \frac{\partial Z}{\partial y_i}(\overline{x}, \overline{y}) = 0,$$

since the function $Z$ attains its global minimum at $(\overline{x}, \overline{y})$. Note that the global minimum of the function $Z$ is attained at $(x, x) \in \Sigma \times \Sigma$ for all $x \in \Sigma$. Furthermore, one can see that there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x}, \overline{y}) = 0$ and $\overline{x} \neq \overline{y}$ by making use of the compactness of $\Sigma$ and the property of interior ball curvature derived from the embeddedness of $\Sigma$ (see Lemma 5.1).

3. Simons-type identity for constant mean curvature hypersurfaces

Let $\Sigma$ be an $n(\geq 3)$-dimensional compact embedded constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. The traceless part of the second fundamental form $h$ is defined to be a differential 2-form $\eta$ on $\Sigma$ with the coefficient function $\eta_{ij}$ in local coordinates as follows:

$$\eta_{ij} := h_{ij} - \delta_{ij}H,$$

where $\delta_{ij}$ is the Kronecker delta. The corresponding traceless shape operator $\hat{A}$ is defined by

$$\langle \hat{A}(X), Y \rangle = \eta(X, Y)$$
for all $X, Y \in T_x \Sigma$, where $T_x \Sigma$ denotes the tangent space of $\Sigma$ at $x \in \Sigma$.

In 1970, Otsuki [22] observed the following interesting property of the eigenspace of principal curvatures.

**Theorem 3.1 ([22]).** Let $\Sigma$ be a hypersurface immersed in an $(n + 1)$-dimensional Riemannian manifold of constant curvature such that the multiplicities of principal curvatures are all constant. Then we have the following:

- The distribution of the space of principal vectors corresponding to each principal curvature is completely integrable.
- If the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.

Let $(x_1, \ldots, x_n)$ be the geodesic normal coordinates at $x \in \Sigma$ (i.e. the metric tensor is given by $g_{ij} = \delta_{ij}$ and the Christoffel symbol $\Gamma^k_{ij}(x)$ at $x$ vanishes). We may assume that $h_{ij} = \lambda_i \delta_{ij}$ with $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. We will denote the coefficient function of the covariant derivative $\nabla^\Sigma h$ by $h_{ijk}$. Then

$$h_{ijk}(x) = \frac{\partial h_{ij}(x)}{\partial x_k}$$

at $x \in \Sigma$. As a consequence of Theorem 3.1, one can compute $\eta_{ijk}$ for $1 \leq i, j, k \leq n$.

**Lemma 3.2.** Let $\Sigma$ be a constant mean curvature hypersurface in $S^{n+1}$ with two distinct principal curvatures $\lambda$ and $\mu$, $\mu$ being simple: $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. Then for all $1 \leq i, j, k \leq n$, we have

$$\eta_{ijk} = h_{ijk}$$

and

$$\begin{cases} 
\eta_{ijk} = 0 & \text{if } i, j, k \text{ are all distinct}, \\
\eta_{iik} = 0 & \text{if } k \neq n, \\
\eta_{nnn} = -(n-1)\eta_{iin} & \text{for } i = 1, \ldots, n-1.
\end{cases}$$

**Proof.** One can easily see that $\eta_{ijk} = h_{ijk}$ for $1 \leq i, j, k \leq n$ on $\Sigma$. If $i, j, k$ are all distinct, then at least two of them are contained in the set $\{1, \ldots, n-1\}$. Using the Codazzi equations, we may assume that $i$ and $j$ are in the set
{1, \ldots, n - 1}. Since \( h_{ijk} = \frac{\partial h_{ij}}{\partial x_k} \) at \( x \), the first part of Theorem 3.1 implies

\[
\eta_{ijk} = \frac{\partial h_{ij}}{\partial x_k} = \frac{\partial \lambda_i}{\partial x_k} \delta_{ij} = 0
\]

at \( x \in \Sigma \). To check the last two equalities, we let \( i, j \in \{1, \ldots, n - 1\} \). Then \( h_{iik} = h_{jjk} \) for any \( k \in \{1, \ldots, n\} \), and \( h_{iij} = 0 \) are direct consequences of the second part of Theorem 3.1. The constant mean curvature assumption implies that \( h_{nnk} = -\sum_{i=1}^{n-1} h_{iik} \). Hence the conclusion immediately follows. □

When a constant mean curvature hypersurface has two distinct principal curvatures, we first prove the following useful identity.

**Proposition 3.3.** Let \( \Sigma \) be a constant mean curvature hypersurface in \( \mathbb{S}^{n+1} \) with two distinct principal curvatures \( \lambda \) and \( \mu \), \( \mu \) being simple. Then \( |\hat{A}| \) is strictly positive and

\[
|\nabla^\Sigma \hat{A}|^2 = \frac{n+2}{n}|\nabla^\Sigma |\hat{A}||^2.
\]

**Remark 3.4.** It is well-known that a constant mean curvature hypersurface \( \Sigma \) in space forms satisfies

\[
|\nabla^\Sigma \hat{A}|^2 - |\nabla^\Sigma |\hat{A}||^2 \geq \frac{2}{n}|\nabla^\Sigma |\hat{A}||^2,
\]

which is so-called Kato’s inequality \([6, 19, 28, 34]\). It would be interesting to characterize the equality case. Proposition 3.3 gives a sufficient condition for Kato’s inequality (4) to attain the equality.

**Proof of Proposition 3.3.** Since \( \Sigma \) has two distinct principal curvatures \( \lambda \) and \( \mu \), the functions \( \lambda - H \) and \( \mu - H \) never vanish. Thus the function \( |\hat{A}| \) is strictly positive. For \( x \in \Sigma \), we choose the geodesic normal coordinates at \( x \) as above. Then we have

\[
|\nabla^\Sigma \hat{A}|^2 = \sum_{i,j,k=1}^{n} \eta^2_{ijk} = \sum_{i=1}^{n} \eta^2_{ii} + 3 \sum_{i,k=1, i \neq k}^{n} \eta^2_{iik} + \sum_{i,j,k=1, i,j,k \text{ distinct}}^{n} \eta^2_{ijk}
\]

\[
= \eta^2_{nnn} + 3(n-1)\eta^2_{11n}
\]

\[
= (n-1)^2 \eta^2_{11n} + 3(n-1)\eta^2_{11n}
\]

\[
= (n-1)(n+2)\eta^2_{11n}.
\]
where we used the relations \( \eta_{nnn} = - \sum_{i=1}^{n-1} \eta_{ni} = -(n-1) \eta_{11n} \) in the second and third equality. Since \( 2 |\hat{A}| \nabla |\hat{A}| = \nabla |\hat{A}|^2 \),

\[
| \nabla^\Sigma |\hat{A}|^2 |^2 = \frac{1}{4|\hat{A}|^2} | \nabla^\Sigma |\hat{A}|^2 |^2
= \frac{1}{|\hat{A}|^2} \sum_{i,j,k=1}^{n} \eta_{ii} \eta_{ik} \eta_{jj} \eta_{jk}
= \frac{1}{|\hat{A}|^2} \sum_{i,j,k=1}^{n} \eta_{ii} \eta_{ik} \eta_{jj} \eta_{jk} + \frac{1}{|\hat{A}|^2} \sum_{i,j=1}^{n} \eta_{ii} \eta_{in} \eta_{jj} \eta_{jn}.
\]

Using Lemma 3.2, one sees that the first term of the right hand side of the identity (6) vanishes. Moreover

\[
\sum_{i=1}^{n} \eta_{ki} = (n-1) \eta_{11} + \eta_{nn} = 0.
\]

Therefore the second term of the right hand side of the identity (6) can be written as

\[
\frac{1}{|\hat{A}|^2} \sum_{i,j=1}^{n} \eta_{ii} \eta_{in} \eta_{jj} \eta_{jn}
= \frac{1}{|\hat{A}|^2} \sum_{i,j=1}^{n-1} \eta_{ii} \eta_{in} \eta_{jj} \eta_{jn} + \frac{2}{|\hat{A}|^2} \sum_{i=1}^{n-1} \eta_{ii} \eta_{in} \eta_{nn} \eta_{nnn} + \frac{1}{|\hat{A}|^2} \eta_{nn} \eta_{nnn} \eta_{nn} \eta_{nnn}
= \frac{1}{|\hat{A}|^2} \left( (n-1)^2 \eta_{11}^2 \eta_{11}^2 + 2(n-1)^3 \eta_{11}^2 \eta_{11}^2 + (n-1)^4 \eta_{11}^2 \eta_{11}^2 \right)
= \frac{n^2(n-1)^2}{|\hat{A}|^2} \eta_{11}^2 \eta_{11}^2.
\]

From the fact that

\[
|\hat{A}|^2 = (n-1) \eta_{11}^2 + (n-1)^2 \eta_{11}^2 = n(n-1) \eta_{11}^2 > 0,
\]

we finally obtain

(7) \[
| \nabla^\Sigma |\hat{A}|^2 |^2 = n(n-1) \eta_{11n}^2.
\]
Hence combining the equations (5) and (7),

$$|\nabla^\Sigma A|^2 = (n - 1)(n + 2)\eta_1^2 = \frac{n + 2}{n}|\nabla^\Sigma |A||^2,$$

which completes the proof.

The following second order partial differential equation of the second fundamental form of a minimal hypersurface in $S^{n+1}$ was established by Simons [29].

$$\Delta^\Sigma |A|^2 - 2|\nabla^\Sigma A|^2 + 2(|A|^2 - n)|A|^2 = 0.$$ 

More generally one can obtain the analogue of the above equation by Simons for a constant mean curvature hypersurface $\Sigma$ in a Riemannian manifold. The Gauss equations and the Ricci formulas state that

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

$$h_{ijkl} = h_{ijlk} + \sum_{r=1}^{n} h_{rj}R_{rkl} + \sum_{r=1}^{n} h_{ir}R_{rjk},$$

where $R_{ijkl}$ denotes the components of the Riemann curvature tensor of $\Sigma$. The Laplacian of $h$ can be computed by making use of the Codazzi equations as follows:

$$\Delta^\Sigma h_{ij} = \sum_{k=1}^{n} h_{ijkk} = \sum_{k=1}^{n} h_{kijk}$$

$$= \sum_{k=1}^{n} h_{kikj} + \sum_{k,r=1}^{n} h_{ri}R_{rjk} + \sum_{k,r=1}^{n} h_{kr}R_{rjk}$$

$$= \sum_{k=1}^{n} h_{kikj} + \sum_{k,r=1}^{n} h_{ri}(\delta_{rj}\delta_{kk} - \delta_{rk}\delta_{kj} + h_{rj}h_{kk} - h_{rk}h_{kj})$$

$$+ \sum_{k,r=1}^{n} h_{kr}(\delta_{rj}\delta_{ik} - \delta_{rk}\delta_{ij} + h_{rj}h_{ik} - h_{rk}h_{ij})$$

$$= \sum_{k=1}^{n} h_{kkij} + n h_{ij} - h_{ij} + nH \sum_{r=1}^{n} h_{ri}h_{rj}$$

$$- \sum_{k,r=1}^{n} h_{ir}h_{rk}h_{kj} + h_{ij} - nH\delta_{ij} + \sum_{k,r=1}^{n} h_{ik}h_{kr}h_{rj} - |A|^2 h_{ij}$$

$$= \sum_{k=1}^{n} h_{kkij} + (n - |A|^2) h_{ij} + nH \sum_{r=1}^{n} h_{ri}h_{rj} - nH\delta_{ij}.$$
Therefore we have

\[ (8) \quad \Delta \Sigma h_{ij} = (n - |A|^2) h_{ij} + nH h_{ij} - nH \delta_{ij} + nH \sum_{k=1}^{n} h_{ik} h_{kj}. \]

Note that the above equation (8) holds for any hypersurface \( \Sigma \) in \( S^{n+1} \).

In the following we have second-order elliptic partial differential equation on the trace-less second fundamental form, which was obtained by Alías-de Almeida-Brasil [1]. For completeness we give the proof which is slightly different from their proof.

**Proposition 3.5 ([1]).** Let \( \Sigma \) be a constant mean curvature hypersurface in \( S^{n+1} \) with two distinct principal curvatures \( \lambda \) and \( \mu \), \( \mu \) being simple. Then

\[ \Delta \Sigma |\hat{A}| - \frac{2}{n} \frac{|\nabla^{\Sigma} |\hat{A}|^2}{|\hat{A}|^2} + (|A|^2 - n)|\hat{A}| \]

\[ - 2nH^2 |\hat{A}| + \text{sgn}(\lambda - \mu) \frac{n(n-2)}{\sqrt{n(n-1)}} H |\hat{A}|^2 = 0. \]

**Proof.** Using the equation (8), we get

\[ \sum_{i,j=1}^{n} \eta_{ij} \Delta \Sigma \eta_{ij} = \sum_{i,j=1}^{n} \eta_{ij} \Delta \Sigma h_{ij} \]

\[ = \sum_{i,j=1}^{n} (n - |A|^2) \eta_{ij}^2 + nH \sum_{i,j,k=1}^{n} \eta_{ij} (\eta_{ik} \eta_{kj} + H \eta_{ik} \delta_{kj} + H \delta_{ik} \eta_{kj}) \]

\[ = (n - |A|^2) \sum_{i=1}^{n} \eta_{ii}^2 + nH \sum_{i=1}^{n} \eta_{ii}^3 + 2nH^2 \sum_{i=1}^{n} \eta_{ii}^2. \]

The left hand side is equal to \( \frac{1}{2} \Delta \Sigma |\hat{A}|^2 - |\nabla^{\Sigma} \hat{A}|^2 \), and the right hand side is equal to \( (n - |A|^2)|\hat{A}|^2 + 2nH^2 |\hat{A}|^2 + nH \sum_{i=1}^{n} \eta_{ii}^3 \). We also see that

\[ \sum_{i=1}^{n} \eta_{ii}^3 = (n-1) \eta_{11}^3 - (n-1)^3 \eta_{11}^3 \]

\[ = -n(n-1)(n-2) \eta_{11}^3 = -\text{sgn}(\lambda - \mu) \frac{n-2}{\sqrt{n(n-1)}} |\hat{A}|^3, \]
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\[ |\hat{A}|^3 = (|\hat{A}|^2)^{\frac{3}{2}} = \text{sgn}(\lambda - \mu) \sqrt{n(n - 1)^3} \eta^3_{11}. \]

Therefore we have the following Simons-type identity:

\[ \Delta_\Sigma |\hat{A}|^2 - 2|\nabla^\Sigma \hat{A}|^2 + 2(|A|^2 - n)|\hat{A}|^2 \]
\[ - 4nH^2|\hat{A}|^2 + \text{sgn}(\lambda - \mu) \frac{2n(n - 2)}{\sqrt{n(n - 1)}} H|\hat{A}|^3 = 0. \]

Since \( \Delta_\Sigma |\hat{A}|^2 = 2|\hat{A}| \Delta_\Sigma |\hat{A}| + 2|\nabla^\Sigma |\hat{A}|^2, \)

\[
\Delta_\Sigma |\hat{A}| + \frac{|\nabla^\Sigma |\hat{A}|^2}{|\hat{A}|} - \frac{|\nabla^\Sigma \hat{A}|^2}{|\hat{A}|} 
\] 
\[ + (|A|^2 - n)|\hat{A}| - 2nH^2|\hat{A}| + \text{sgn}(\lambda - \mu) \frac{n(n - 2)}{\sqrt{n(n - 1)}} H|\hat{A}|^2 = 0. \]

Therefore applying the equation (3) gives the conclusion. \( \square \)

Applying Proposition 3.5 to the function \( \varphi = \Phi - H, \) where \( \Phi \) is the maximum value of the principal curvatures, we get the following:

**Corollary 3.6.** Let \( \Sigma \) be a constant mean curvature hypersurface in \( \mathbb{S}^{n+1} \) with two distinct principal curvatures \( \lambda \) and \( \mu, \mu \) being simple. Then

\[ \Delta_\Sigma \varphi - \frac{2}{n} \frac{|\nabla^\Sigma \varphi|^2}{\varphi} + (|A|^2 - n)\varphi - 2nH^2\varphi + \text{sgn}(\lambda - \mu)nf(n)H\varphi^2 = 0, \]

where the function \( f(n) \) is defined by

\[ f(n) := \begin{cases} 
\frac{n-2}{n-1} & \text{if } \Phi = \mu, \\
n - 2 & \text{if } \Phi = \lambda.
\end{cases} \]

**Proof.** Note that if \( \Phi = \mu, \) then \( |\hat{A}|^2 = \frac{n}{n-1}\varphi^2 \) and if \( \Phi = \lambda, \) then \( |\hat{A}|^2 = n(n - 1)\varphi^2. \) The conclusion follows from Proposition 3.5 and the linearity of \( \Delta_\Sigma \) and \( \nabla^\Sigma. \) \( \square \)

For later use, we define a constant \( g(n) \) depending on the dimension \( n \) as follows:

\[ g(n) = \begin{cases} 
\frac{1}{n-1} & \text{if } \Phi = \mu, \\
n - 1 & \text{if } \Phi = \lambda.
\end{cases} \]

Then one can write \( |\hat{A}|^2 = ng(n)\varphi^2. \)
4. First and second order derivatives of the two-point function

Let $\Sigma$ be an $n(\geq 3)$-dimensional compact embedded constant mean curvature hypersurface in $S^{n+1}$ with two distinct principal curvatures $\lambda$ and $\mu$ of the multiplicity $n - 1$ and 1, respectively. Consider a pair of points $(\bar{x}, \bar{y}) \in \Sigma \times \Sigma$ such that $Z(\bar{x}, \bar{y}) = 0$. Then by the equation (2)

$$\frac{\partial Z}{\partial x_i}(\bar{x}, \bar{y}) = \frac{\partial Z}{\partial y_i}(\bar{x}, \bar{y}) = 0.$$ 

Let us choose geodesic normal coordinates $(x_1, \ldots, x_n)$ at $x$ in $\Sigma$ satisfying that

$$h_{ij} = \lambda_i \delta_{ij}$$

with $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$ and geodesic normal coordinates $(y_1, \ldots, y_n)$ at $y$ in $\Sigma$. Therefore the first order derivatives of the function $Z(x, y)$ are given by

$$0 = \frac{\partial Z}{\partial x_i}(\bar{x}, \bar{y}) = \frac{\partial \Psi(x)}{\partial x_i} \left(1 - \langle F(x), F(y) \rangle - \Psi(x) \frac{\partial F(x)}{\partial x_i}, F(y) \right)$$

$$+ \sum_{k=1}^{n} h_{ik}(x) \left( \frac{\partial F(x)}{\partial x_k}, F(y) \right),$$

and

$$0 = \frac{\partial Z}{\partial y_i}(\bar{x}, \bar{y}) = -\Psi(x) \left( F(x), \frac{\partial F(y)}{\partial x_i} \right) + \left( \nu(x), \frac{\partial F(y)}{\partial x_i} \right).$$

In this section, using these relations in geodesic normal coordinates as above, we are able to compute the second order derivatives of the function $Z$.

**Proposition 4.1.** At the point $(\bar{x}, \bar{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\bar{x}, \bar{y}) = 0$ and $x \neq y$, we have

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) = \left( \Delta_{\Sigma} \Psi(x) - 2 \sum_{i=1}^{n} \frac{|\partial \Psi(x)|^2}{\Psi(x) - \lambda_i(x)} \right)$$

$$+ \left( |A(x)|^2 - n \right) \Psi(x) (1 - \langle F(x), F(y) \rangle)$$

$$+ n \Psi(x) + n H \Psi(x) \langle \nu(x), F(y) \rangle - n H \langle F(x), F(y) \rangle.$$
Proof. Differentiating the equation (9) in the direction $\frac{\partial}{\partial x_i}$ gives

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) = \Delta_{\Sigma} \Psi(\bar{x}) (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) - 2 \sum_{i=1}^{n} \frac{\partial \Psi(\bar{x})}{\partial x_i} \left\langle \frac{\partial F(\bar{x})}{\partial x_i}, F(\bar{y}) \right\rangle$$

$$- \Psi(\bar{x}) \langle \Delta_{\Sigma} F(\bar{x}), F(\bar{y}) \rangle + \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{\partial h_i^k}{\partial x_i} \left\langle \frac{\partial F(\bar{x})}{\partial x_k}, F(\bar{y}) \right\rangle$$

$$+ \sum_{i,k=1}^{n} h_i^k(\bar{x}) \left( - h_{ik}(\bar{x}) \nu(\bar{x}) - \delta_{ik} F(\bar{x}), F(\bar{y}) \right).$$

Since $F : \Sigma \to S^{n+1} \subset \mathbb{R}^{n+2}$ is a constant mean curvature hypersurface,

$$\Delta_{\Sigma} F(x) + nF(x) = -nH \nu(x).$$

By using the Codazzi equations,

$$\sum_{i=1}^{n} \frac{\partial h_i^k}{\partial x_i} = \sum_{i=1}^{n} \nabla_{\Sigma} h_i^k(\bar{x}) = \sum_{i=1}^{n} h_{iki}(\bar{x}) = \sum_{i=1}^{n} h_{iik}(\bar{x}) = 0$$

at $\bar{x}$. Thus

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) = \Delta_{\Sigma} \Psi(\bar{x}) (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) - 2 \sum_{i=1}^{n} \frac{\partial \Psi(\bar{x})}{\partial x_i} \left\langle \frac{\partial F(\bar{x})}{\partial x_i}, F(\bar{y}) \right\rangle$$

$$+ n\Psi(\bar{x}) \langle F(\bar{x}), F(\bar{y}) \rangle + n\Psi(\bar{x})H \langle \nu(\bar{x}), F(\bar{y}) \rangle$$

$$- |A(\bar{x})|^2 \langle \nu(\bar{x}), F(\bar{y}) \rangle - nH \langle F(\bar{x}), F(\bar{y}) \rangle.$$

Rearranging the above formula by using the equation (1) with $Z(\bar{x}, \bar{y}) = 0$ yields

$$(11) \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) = (\Delta_{\Sigma} \Psi(\bar{x}) + (|A(\bar{x})|^2 - n) \Psi(\bar{x})) (1 - \langle F(\bar{x}), F(\bar{y}) \rangle)$$

$$+ n\Psi(\bar{x}) - 2 \sum_{i=1}^{n} \frac{\partial \Psi(\bar{x})}{\partial x_i} \left\langle \frac{\partial F(\bar{x})}{\partial x_i}, F(\bar{y}) \right\rangle$$

$$+ nH \Psi(\bar{x}) \langle \nu(\bar{x}), F(\bar{y}) \rangle - nH \langle F(\bar{x}), F(\bar{y}) \rangle.$$

Using the formula (9) with $\frac{\partial Z}{\partial x_i}(\bar{x}, \bar{y}) = 0$, we have

$$(12) \left\langle \frac{\partial F(\bar{x})}{\partial x_i}, F(\bar{y}) \right\rangle = (1 - \langle F(\bar{x}), F(\bar{y}) \rangle) \frac{\partial \Psi(\bar{x})}{\partial x_i},$$
Putting the equation (12) in the equation (11), we get the conclusion. □

Let \( w_i(\vec{x}, \vec{y}) \) be the reflection of the vector \( \frac{\partial F(x)}{\partial x_i} \) in \( \mathbb{R}^{n+2} \) with respect to the hyperplane orthogonal to \( F(\vec{x}) - F(\vec{y}) \) and passing through the origin. The vector \( w_i(\vec{x}, \vec{y}) \) is given by

\[
w_i(\vec{x}, \vec{y}) = \frac{\partial F(\vec{x})}{\partial x_i} - 2 \left\langle \frac{\partial F(\vec{x})}{\partial x_i}, \frac{F(\vec{x}) - F(\vec{y})}{|F(\vec{x}) - F(\vec{y})|} \right\rangle \frac{F(\vec{x}) - F(\vec{y})}{|F(\vec{x}) - F(\vec{y})|}.
\]

We remark that \( \{w_1(\vec{x}, \vec{y}), \ldots, w_n(\vec{x}, \vec{y})\} \) is the set of mutually orthogonal unit tangent vectors in \( T_F(\vec{y})S^{n+1} \). On the other hand, the following three properties hold at \( (\vec{x}, \vec{y}) \) for \( 1 \leq i \leq n \).

1. \[
\left\langle \frac{\partial F(\vec{y})}{\partial y_i}, \Psi(\vec{x})F(\vec{x}) - \nu(\vec{x}) \right\rangle = -\frac{\partial Z}{\partial y_i}(\vec{x}, \vec{y}) = 0,
\]
2. \[
\left\langle w_i(\vec{x}, \vec{y}), \Psi(\vec{x})F(\vec{x}) - \nu(\vec{x}) \right\rangle = \frac{\left\langle \frac{\partial F(\vec{y})}{\partial x_i}, F(\vec{y}) \right\rangle}{1 - \langle F(\vec{x}), F(\vec{y}) \rangle} Z(\vec{x}, \vec{y}) = 0,
\]
3. \[
|F(\vec{y})|^2 |\Psi(\vec{x})F(\vec{x}) - \nu(\vec{x})|^2 - \langle F(\vec{y}), \Psi(\vec{x})F(\vec{x}) - \nu(\vec{x}) \rangle^2 = (1 + \Psi(\vec{x})^2) - \Psi(\vec{x})^2 = 1 \neq 0.
\]

Thus one sees that \( \text{Span} \left( \frac{\partial F(\vec{y})}{\partial y_1}, \ldots, \frac{\partial F(\vec{y})}{\partial y_n} \right) = \text{Span} \left( w_1(\vec{x}, \vec{y}), \ldots, w_n(\vec{x}, \vec{y}) \right) \). Moreover, if we choose the coordinates at \( \vec{y} \) satisfying that for \( 1 \leq i \neq j \leq n \)

\[
\left\langle w_i(\vec{x}, \vec{y}), \frac{\partial F(\vec{y})}{\partial y_i} \right\rangle \geq 0 \quad \text{and} \quad \left\langle w_i(\vec{x}, \vec{y}), \frac{\partial F(\vec{y})}{\partial y_j} \right\rangle = 0,
\]

then the above three properties implies that

(13) \[
w_i(\vec{x}, \vec{y}) = \frac{\partial F(\vec{y})}{\partial y_i}.
\]

Equipped with the local coordinates chosen as above, we are able to get the following second order derivatives at the global minimum points of the two-point function \( Z \).

**Proposition 4.2.** At the point \( (\vec{x}, \vec{y}) \in \Sigma \times \Sigma \) satisfying that \( Z(\vec{x}, \vec{y}) = 0 \) and \( \vec{x} \neq \vec{y} \), we have

\[
\frac{\partial^2 Z}{\partial x_i \partial y_i}(\vec{x}, \vec{y}) = \lambda_i(\vec{x}) - \Psi(\vec{x}).
\]
Proof. Differentiating the equation (9) in the direction \( \frac{\partial}{\partial y_i} \) gives
\[
\frac{\partial^2 Z}{\partial x_i \partial y_i}(\mathbf{x}, y) = -\frac{\partial \Psi(\mathbf{x})}{\partial x_i} \left\langle F(\mathbf{x}), \frac{\partial F(y)}{\partial y_i} \right\rangle - \Psi(\mathbf{x}) \left\langle \frac{\partial F(\mathbf{x})}{\partial x_i}, \frac{\partial F(y)}{\partial y_i} \right\rangle
+ \sum_{k=1}^{n} h_{i}^{k}(\mathbf{x}) \left\langle \frac{\partial F(\mathbf{x})}{\partial x_k}, \frac{\partial F(y)}{\partial y_i} \right\rangle.
\]

Here the second equality follows from the equation (9) with \( \frac{\partial Z}{\partial x_i}(\mathbf{x}, y) = 0 \).

Moreover it can be expressed in terms of \( w_i(\mathbf{x}, y) \) as follows:
\[
\frac{\partial^2 Z}{\partial x_i \partial y_i}(\mathbf{x}, y) = -\frac{1}{1 - \langle F(\mathbf{x}), F(\mathbf{y}) \rangle} \left( (\lambda_i(\mathbf{x}) - \Psi(\mathbf{x})) \left\langle \frac{\partial F(\mathbf{x})}{\partial x_i}, F(\mathbf{y}) \right\rangle \right)
\times \left( \langle \lambda_i(\mathbf{x}) - \Psi(\mathbf{x}) \rangle \left\langle \frac{\partial F(\mathbf{x})}{\partial x_i}, F(\mathbf{y}) \right\rangle \right)
+ (\lambda_i(\mathbf{x}) - \Psi(\mathbf{x})) \left\langle \frac{\partial F(\mathbf{x})}{\partial x_i}, \frac{\partial F(\mathbf{y})}{\partial y_i} \right\rangle.
\]

Plugging the equation (13) into the above identity gives the conclusion. \( \square \)

**Proposition 4.3.** At the point \((\mathbf{x}, y) \in \Sigma \times \Sigma\) satisfying that \( Z(\mathbf{x}, y) = 0 \) and \( \mathbf{x} \neq \mathbf{y} \), we have
\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2}(\mathbf{x}, y) = n\Psi(\mathbf{x}) + nH \Psi(\mathbf{x}) \langle F(\mathbf{x}), \nu(\mathbf{y}) \rangle - nH \langle \nu(\mathbf{x}), \nu(\mathbf{y}) \rangle.
\]

Proof. Differentiating the equation (10) in the direction \( \frac{\partial}{\partial y_i} \) gives
\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2}(\mathbf{x}, y) = -\Psi(\mathbf{x}) \left\langle F(\mathbf{x}), \Delta \Sigma F(\mathbf{y}) \right\rangle + \langle \nu(\mathbf{x}), \Delta \Sigma F(\mathbf{y}) \rangle
= n\Psi(\mathbf{x}) \left\langle F(\mathbf{x}), F(\mathbf{y}) \right\rangle + nH \Psi(\mathbf{x}) \langle F(\mathbf{x}), \nu(\mathbf{y}) \rangle
- n \langle \nu(\mathbf{x}), F(\mathbf{y}) \rangle - nH \langle \nu(\mathbf{x}), \nu(\mathbf{y}) \rangle
= n\Psi(\mathbf{x}) + nH \Psi(\mathbf{x}) \langle F(\mathbf{x}), \nu(\mathbf{y}) \rangle - nH \langle \nu(\mathbf{x}), \nu(\mathbf{y}) \rangle. \ \square
Proposition 4.4. For \((x, y) \in \Sigma \times \Sigma\) satisfying that \(Z(x, y) = 0\) and \(x \neq y\),

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} = (1 - \langle F(x), F(y) \rangle) \times \left( \Delta_{\Sigma} \Psi(x) - 2 \sum_{i=1}^{n} \frac{\left| \frac{\partial \Psi(x)}{\partial x_i} \right|^2}{\Psi(x) - \lambda_i(x)} + \left( |A(x)|^2 - n \right) \Psi(x) - nH \Psi(x)^2 + nH \right).
\]

Proof. Applying Proposition 4.1, Proposition 4.2, and Proposition 4.3, we have

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} = (1 - \langle F(x), F(y) \rangle) \times \left( \Delta_{\Sigma} \Psi(x) - 2 \sum_{i=1}^{n} \frac{\left| \frac{\partial \Psi(x)}{\partial x_i} \right|^2}{\Psi(x) - \lambda_i(x)} + \left( |A(x)|^2 - n \right) \Psi(x) - nH \Psi(x)^2 + nH \right)
\]

In order to get the conclusion, we need the following computations:

- \(\langle \nu(x), F(y) \rangle = -\Psi(x) (1 - \langle F(x), F(y) \rangle)\),

Since \(B_T(x, \frac{1}{\Psi(x)})\) touches \(\Sigma\) at \(F(x)\) and \(F(y)\) simultaneously, the center \(p(x)\) of the geodesic ball \(B_T(x, \frac{1}{\Psi(x)})\) is given by

\[
p(x) = F(x) - \frac{1}{\Psi(x)} \nu(x) = F(y) - \frac{1}{\Psi(x)} \nu(y),
\]

which gives \(\nu(y) = \nu(x) + \Psi(x)(F(y) - F(x))\). Thus

- \(\langle F(x), \nu(y) \rangle = \langle F(x), \nu(x) + \Psi(x)(F(y) - F(x)) \rangle = -\Psi(x) (1 - \langle F(x), F(y) \rangle)\).

Moreover

- \(\langle \nu(x), \nu(y) \rangle = \langle \nu(x), \nu(x) + \Psi(x)(F(y) - F(x)) \rangle = 1 + \Psi(x) \langle \nu(x), F(y) \rangle = 1 - \Psi(x)^2 (1 - \langle F(x), F(y) \rangle)\),

- \(\langle F(x), F(y) \rangle = 1 - (1 - \langle F(x), F(y) \rangle)\).
Combining these computations with the equation (14), we get the conclusion.

Since \( \Phi(x) \leq k(x) \leq \Psi(x) \), one sees that for \( 1 \leq i \leq n \)

\[
\Psi(x) - \lambda_i = \Psi(x) - \left( nH - \sum_{j \neq i} \lambda_j \right)
= \Psi(x) + \sum_{j \neq i} \lambda_j - nH \leq n(\Psi(x) - H).
\]

We remark that \( \Psi(x) - \lambda_j < n(\Psi(x) - H) \) for some \( 1 \leq j \leq n \) because \( F(\Sigma) \) has two distinct principal curvatures. As a consequence of Proposition 4.4, we have the following:

**Corollary 4.5.** For \((x, y) \in \Sigma \times \Sigma\) satisfying that \( Z(x, y) = 0 \) and \( x \neq y \),

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} 
\leq (1 - \langle F(x), F(y) \rangle) \times \left( \Delta_\Sigma \Psi(x) - \frac{|\nabla_\Sigma \Psi(x)|^2}{\Psi(x) - H} + (|A(x)|^2 - n) \Psi(x) - nH \Psi(x)^2 + nH \right).
\]

Moreover, if \( \kappa > 1 \), equality holds only when \( \nabla_\Sigma \Psi(x) = 0 \).

## 5. Proof of Main Theorem

We begin with showing the existence of a global minimum point \((x, y) \in \Sigma \times \Sigma\) of the function \( Z \) which is not contained in the diagonal \( D = \{(x, x) : x \in \Sigma\} \subset \Sigma \times \Sigma \) when \( \kappa > 1 \).

**Lemma 5.1.** If \( \kappa > 1 \), then there exists a point \((x, y) \in \Sigma \times \Sigma \setminus D\) such that \( Z(x, y) = 0 \), where \( D \) is the diagonal.

**Proof.** Since \( \Sigma \) is compact, \( \kappa \) is attained at some point \( x \in \Sigma \). Thus

\[
(15) \quad \Psi(x) = \kappa \varphi(x) + H = k(x).
\]

By the definition of the interior ball curvature \( k(x) \) at \( x \in \Sigma \), there exists a point \( y \in \Sigma \) satisfying that \( y \in B_T(x, \frac{1}{k(x)}) \cap \Sigma \setminus \{x\} \). This is equivalent to that there exists a point \( y \in \Sigma \setminus \{x\} \) such that \( Z(x, y) = 0 \), which follows from the definition of the function \( Z \) and the equation (15). \( \square \)
By definition of the interior ball curvature \( k(x) \), it holds \( \Phi(x) \leq k(x) \) for every \( x \in \Sigma \), in general. However the following proposition shows that if \( \Sigma \) has a constant mean curvature \( H > 0 \) and two distinct principal curvatures of multiplicity \( n - 1 \) and \( 1 \), then \( k(x) = \Phi(x) \) for every \( x \in \Sigma \).

**Proposition 5.2.** Let \( \Sigma \) be an \( n (\geq 3) \)-dimensional compact embedded hypersurface in \( S^{n+1} \) with constant mean curvature \( H \) with two distinct principal curvatures, one of them being simple. If \( H > 0 \). Then the interior ball curvature \( k(x) \) is the same as the maximum principal curvature \( \Phi(x) \) for all \( x \in \Sigma \).

**Proof.** Suppose that \( \kappa > 1 \). By Lemma 5.1, there exists a point \((\bar{x}, \bar{y}) \in \Sigma \times \Sigma \) with \( \bar{x} \neq \bar{y} \) satisfying that \( Z(\bar{x}, \bar{y}) = 0 \). Using Corollary 3.6 and Corollary 4.5 together with \( \Psi(\bar{x}) = \kappa \phi(\bar{x}) + H \), we get

\[
\frac{1}{1 - \langle F(\bar{x}), F(\bar{y}) \rangle} \left( \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} \right) \\
\leq \kappa \Delta_{\Sigma} \phi(\bar{x}) - \frac{2\kappa}{n} \frac{|\nabla_{\Sigma} \phi(\bar{x})|^2}{\phi(\bar{x})} + (|A(\bar{x})|^2 - n) (\kappa \phi(\bar{x}) + H) \\
- nH(\kappa \phi(\bar{x}) + H)^2 + nH \\
= H|A(\bar{x})|^2 - \kappa^2 nH \phi(\bar{x})^2 - nH^3 - \text{sgn}(\lambda - \mu) \kappa n f(n) H \phi(\bar{x})^2,
\]

where \( f(n) = \frac{n-2}{n-1} \) if \( \Phi = \mu \), and \( f(n) = n - 2 \) if \( \Phi = \lambda \). Using the relation

\[
|\hat{A}|^2 = |A|^2 - nH^2 = ng(n) \phi^2,
\]

we get

\[
\frac{1}{1 - \langle F(\bar{x}), F(\bar{y}) \rangle} \left( \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} \right) \\
\leq -nH \phi(\bar{x})^2 (\kappa^2 + \text{sgn}(\lambda - \mu) f(n) \kappa - g(n)) \\
< -nH \phi(\bar{x})^2 (1 + \text{sgn}(\lambda - \mu) f(n) - g(n)) \\
\leq 0
\]

where we used the identity \( 1 + \text{sgn}(\lambda - \mu) f(n) - g(n) = 0 \).

However, since the point \((\bar{x}, \bar{y}) \in \Sigma \times \Sigma \setminus D \) is a global minimum point of the function \( Z \), we see

\[
0 \leq \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2},
\]
which is a contradiction. From Proposition 2.2, it follows that
\[ k(x) = \Phi(x) = \Psi(x) \]
for all \( x \in \Sigma \).

We are now ready to prove our main theorem.

**Theorem 5.3.** Let \( \Sigma \) be an \( n(\geq 3) \)-dimensional compact embedded hypersurface in \( S^{n+1} \) with constant mean curvature \( H \geq 0 \) and with two distinct principal curvatures \( \lambda \) and \( \mu \), \( \mu \) being simple. If \( \mu > \lambda \), then \( \Sigma \) is congruent to a Clifford hypersurface \( S^{n-1} \left( \frac{1}{\sqrt{1+\lambda^2}} \right) \times S^1 \left( \frac{|\lambda|}{\sqrt{1+\lambda^2}} \right) \), where

\[ \lambda = \frac{nH - \sqrt{n^2H^2 + 4(n-1)^2}}{2(n-1)} \]

**Proof.** If \( H = 0 \), then \( \Sigma \) is congruent to a Clifford minimal hypersurfaces from Theorem 1.1 by Otsuki. Thus it suffices to consider the case of \( H > 0 \). Since \( \mu > \lambda \), we have \( \Phi = \mu \). From Proposition 5.2, we have
\[ \Phi(x)(1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle \geq 0, \]
for all \( x, y \in \Sigma \).

Fix \( x \in \Sigma \) and choose an orthonormal frame \( \{ e_1, \ldots, e_n \} \) in a neighborhood of \( x \) such that \( h(e_n, e_n) = \Phi \). Let \( \gamma(t) \) be a geodesic on \( \Sigma \) such that \( \gamma(0) = F(x) \) and \( \gamma'(0) = e_n \). For simplicity, let us identify the hypersurface \( \Sigma \) with its image under the embedding \( F \), so that \( F(x) = x \). Define a function \( f : \mathbb{R} \to \mathbb{R} \) by
\[ f(t) := Z(F(x), \gamma(t)) = \Phi(x)(1 - \langle F(x), \gamma(t) \rangle) + \langle \nu(x), \gamma(t) \rangle. \]

Then, by definition, \( f(t) \geq 0 \) and \( f(0) = 0 \). A simple computation shows
\[
\begin{align*}
f'(t) &= -\langle \Phi(x)F(x) - \nu(x), \gamma'(t) \rangle, \\
f''(t) &= \langle \Phi(x)F(x) - \nu(x), \gamma(t) + h(\gamma'(t), \gamma'(t))\nu(\gamma(t)) \rangle, \\
f'''(t) &= \langle \Phi(x)F(x) - \nu(x), \gamma'(t) + (\nabla_{\gamma'(t)}h)\gamma'(t)\nu(\gamma(t)) \\
&\quad + h(\gamma'(t), \gamma'(t))\nabla_{\gamma'(t)}\nu(\gamma(t)) \rangle,
\end{align*}
\]
where \( \nabla \) is the covariant derivative of \( \mathbb{R}^{n+2} \). In particular, it follows that
\[ f(0) = f'(0) = 0, \]
\[ f''(0) = \langle \Phi(x) F(x) - \nu(x), F(x) + \Phi(x) \nu(x) \rangle = 0. \]

Moreover the fact that \( f \geq 0 \) implies that \( f'''(0) = 0 \). Hence

\[ 0 = f'''(0) = \langle \Phi(x) F(x) - \nu(x), e_n + h_{nnn}(x) \nu(x) \rangle = -h_{nnn}(x), \]

since \( \nabla\gamma'(t) \nu(\gamma(t)) \) is tangent to \( \Sigma \). Therefore we get \( e_n \lambda = h_{11n} = -\frac{1}{n-1} h_{nnn} = 0 \). Combining this with Lemma 3.2, one sees that \( \lambda \) and \( \mu \) are constant on \( \Sigma \), which implies that \( \Sigma \) is an isoparametric hypersurface in \( S^{n+1} \) with two distinct principal curvatures. From the classification of isoparametric hypersurfaces with two principal curvatures due to Cartan [13], it follows that \( \Sigma \) is congruent to the Riemannian product \( S^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times S^1\left(\frac{1}{\sqrt{1+\mu^2}}\right) \), where \( \lambda \) and \( \mu \) satisfy \( nH = (n-1)\lambda + \mu \).

We now claim that \( \lambda \mu + 1 = 0 \) on \( \Sigma \). To see this, let \( \{e_1, \ldots, e_n, e_{n+1}\} \) be a local orthonormal frame around \( p \in \Sigma \) such that \( e_{n+1} \) is normal to \( \Sigma \) and \( h_{ij} = \lambda_i \delta_{ij} \) at \( p \). Let \( \{\omega_1, \ldots, \omega_n, \omega_{n+1}\} \) be a dual coframe. We use the following convention on the ranges of indices:

\[ 1 \leq A, B, C, \ldots \leq n + 1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n. \]

Then the structure equations of a unit sphere \( S^{n+1} \) are given by

\begin{align}
\omega_{AB} &= -\sum_{B=1}^{n+1} \omega_{AB} \land \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\
\omega_{AB} &= -\sum_{C=1}^{n+1} \omega_{AC} \land \omega_{CB} + \Omega_{AB}, \\
\Omega_{AB} &= \frac{1}{2} \sum_{C,D=1}^{n+1} K_{ABCD} \omega_C \land \omega_D, \\
K_{ABCD} &= \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}.
\end{align}

We restrict these forms to \( \Sigma \). Then we have \( \omega_{n+1} = 0 \) on \( \Sigma \). Moreover

\[ 0 = d\omega_{n+1} = -\sum_{i=1}^{n} \omega_{n+1,i} \land \omega_i \quad \text{and} \quad \omega_{n+1,i} = \sum_{j=1}^{n} h_{ij} \omega_j = \sum_{i=1}^{n} \lambda_i \omega_i. \]

Recall that \( h_{ijk} \) is defined by
\[
\sum_{k=1}^{n} h_{ijk} \omega_k = dh_{ij} - \sum_{k=1}^{n} h_{ik} \omega_{kj} - \sum_{k=1}^{n} h_{kj} \omega_{ki}.
\]

Let \( \theta_{ij} := (\lambda_i - \lambda_j) \omega_{ij} = \theta_{ji}. \) Then we have

\[
\sum_{k=1}^{n} h_{ijk} \omega_k = \delta_{ij} d\lambda_j - (\lambda_i - \lambda_j) \omega_{ij} = \delta_{ij} d\lambda_j - \theta_{ij}.
\]

Since each \( \lambda_i \) is constant on \( \Sigma \), Lemma 3.2 shows that

\[
\theta_{in} = \delta_{in} d\lambda_n - \sum_{k=1}^{n} h_{ink} \omega_k = - \sum_{k=1}^{n-1} h_{ink} \omega_k - h_{inn} \omega_n = 0
\]

for \( 1 \leq i \leq n - 1 \). This implies that \( \omega_{in} = \frac{\theta_{in}}{\lambda - \mu} = 0 \). Therefore, for \( 1 \leq i \leq n - 1 \), using the equation (17) gives

\[
0 = d\omega_{in} = - \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kn} - \omega_{i,n+1} \wedge \omega_{n+1,n} + \omega_i \wedge \omega_n = (\lambda \mu + 1) \omega_i \wedge \omega_n,
\]

which shows that \( \lambda \mu + 1 = 0 \) on \( \Sigma \). Therefore \( \Sigma \) is congruent to a Clifford hypersurface \( S^{n-1} \left( \frac{1}{\sqrt{1+\lambda^2}} \right) \times S^1 \left( \frac{|\lambda|}{\sqrt{1+\lambda^2}} \right) \), where \( \lambda = \frac{nH-\sqrt{n^2H^2+4(n-1)^2}}{2(n-1)} \) since \( \mu > \lambda \).

6. Appendix: The case of \( H = 0 \)

Our proof of Theorem 5.3 still works for the case of \( H = 0 \). Although Otsuki gave a classification theorem for embedded minimal hypersurfaces with two distinct principal curvatures in Theorem 1.1, we here give another proof of Theorem 1.1. If \( H = 0 \), then \( \mu = -(n-1)\lambda \). Therefore, by choosing a suitable orientation, we have \( \mu(x) > \lambda(x) \) for every \( x \in \Sigma \). The proof is divided into two cases: \( \kappa = 1 \) and \( \kappa > 1 \). The proof in case of \( \kappa = 1 \) is similar to that of Theorem 5.3 with \( \Phi = \mu \). For this reason, it suffices to consider the case of \( \kappa > 1 \). The proof uses basically Brendle’s argument [10].

**Proposition 6.1.** Let \((\bar{x}, \bar{y}) \in \Sigma \times \Sigma \setminus D\) such that \( Z(\bar{x}, \bar{y}) = 0 \). Then \( \nabla_{\Sigma} \Phi(\bar{x}) = 0 \).
Proof. Since the function $Z$ attains its global minimum at $(\overline{x}, \overline{y})$, it follows from the inequality (16) and $H = 0$ that

$$0 \leq \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} = H|A(\overline{x})|^2 - \kappa^2 nH\varphi(\overline{x})^2 - nH^3 - \text{sgn}(\lambda - \mu)\kappa n f(n) H \varphi(\overline{x})^2 = 0.$$ 

Therefore equality holds in Corollary 4.5, which implies that $\nabla^\Sigma \Psi(\overline{x}) = 0$. Hence we obtain that $\nabla^\Sigma \Phi(\overline{x}) = 0$. □

For a point $(\overline{x}, \overline{y})$ such that $Z(\overline{x}, \overline{y}) = 0$, we choose open neighborhoods $U_1$ and $U_2$ of $\overline{x}$ and $\overline{y}$, respectively, such that $U_1 \times U_2 \cap D = \emptyset$. Then there exist a constant $\Lambda_1 > 0$ depending on $\overline{x}$ and $\overline{y}$ such that

$$\sup_{U_1 \times U_2} \{|\nabla^\Sigma \Psi|, |\nabla^\Sigma F|, |A|^2\} < \Lambda_1,$$

and

$$\inf_{U_1 \times U_2} \{\Psi - \lambda, \Psi - \mu, 1 - \langle F(x), F(y) \rangle\} > \frac{1}{\Lambda_1}.$$

For a sufficiently small $\varepsilon > 0$, there exist open neighborhoods $N_1 \subset U_1$ and $N_2 \subset U_2$ of $\overline{x}$ and $\overline{y}$ such that

$$|Z(x, y)| < \varepsilon \quad \text{and} \quad |dZ(x, y)| < \varepsilon$$

for $(x, y) \in N_1 \times N_2$. Obviously, the neighborhood $N_1 \times N_2$ is disjoint from $D$. In order to compute the second order derivatives for an arbitrary point $(x, y) \in N_1 \times N_2$, let us choose geodesic normal coordinates $(x_1, \ldots, x_n)$ at $x$ satisfying that $h_{ij} = \lambda_i \delta_{ij}$ at $x$.

We recall that the vector $w_i(x, y)$ is defined by the reflection of $\frac{\partial F(x)}{\partial x_i}$ in $\mathbb{R}^{n+2}$ with respect to the hyperplane orthogonal to $F(x) - F(y)$ and passing through the origin. Thus the vector $w_i(x, y)$ is given by

$$w_i(x, y) = \frac{\partial F(x)}{\partial x_i} - 2 \left\langle \frac{\partial F(x)}{\partial x_i}, \frac{F(x) - F(y)}{|F(x) - F(y)|} \right\rangle \frac{F(x) - F(y)}{|F(x) - F(y)|}.$$ 

For any point $(x, y) \in N_1 \times N_2$, we obtain the following estimates:

$$\left| \left\langle \frac{\partial F(y)}{\partial y_i}, \Psi(x) F(x) - \nu(x) \right\rangle \right| = \left| -\frac{\partial Z}{\partial y_i}(x, y) \right| < \varepsilon,$$ 

(20)
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\[ \langle w_i(x, y), \Psi(x)F(x) - \nu(x) \rangle = \left| \frac{\langle \frac{\partial F}{\partial x_i}(x), F(y) \rangle}{1 - \langle F(x), F(y) \rangle} \right| Z(x, y) < \Lambda_1^2 \varepsilon, \]

and

\[ |F(y)|^2 |\Psi(x)F(x) - \nu(x)|^2 - \langle F(y), \Psi(x)F(x) - \nu(x) \rangle^2 = (1 + \Psi(x)^2) - (\Psi(x) - Z(x, y))^2 \]
\[ = 1 + 2\Psi(x)Z(x, y) - Z(x, y)^2 \]
\[ > 1 - \varepsilon^2 \neq 0. \]

Let \( V(x, y) \) be the orthogonal projection of \( \Psi(x)F(x) - \nu(x) \) onto \( T_{F(y)}S^{n+1} \), which is spanned by \( \{ \frac{\partial F}{\partial y_i}(y) \}_{i=1}^n \) and \( \nu(y) \). For a suitably chosen small \( \varepsilon \), we can conclude that

- \( F(y) \) and \( \Psi(x)F(x) - \nu(x) \) are linearly independent,
- \( |V(x, y)|^2 = 1 + 2\Psi(x)Z(x, y) - Z(x, y)^2 \)

from the formula (22). The inequality (20) implies that

- The set \( \{ \frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_n}, V \} \) is a basis of \( T_{F(y)}S^{n+1} \). Moreover, we can make the angle between \( V \) and \( \nu(y) \) arbitrarily close to 0 in \( N_1 \times N_2 \) for a sufficiently small \( \varepsilon > 0 \).

Note that the vectors \( w_i(x, y) \) and \( w_j(x, y) \) for \( 1 \leq i, j \leq n \) and \( i \neq j \) are mutually orthogonal unit vectors in \( T_{F(y)}S^{n+1} \). Finally the inequality (21) implies that

- The set \( \{ w_1, \ldots, w_n, V \} \) is also a basis of \( T_{F(y)}S^{n+1} \). Moreover, we can make the angle between \( w_i \) (\( 1 \leq i \leq n \)) and \( V \) arbitrarily close to \( \frac{\pi}{2} \) in \( N_1 \times N_2 \) for a sufficiently small \( \varepsilon > 0 \).

Therefore we choose geodesic normal coordinates at \( y \) satisfying that for \( 1 \leq i < j \leq n \)

- the angle between \( \frac{\partial F}{\partial y_i}(y) \) and \( w_i(x, y) \) is sufficiently small, which depends on \( \varepsilon > 0 \),
- \( \langle w_i(x, y), \frac{\partial F}{\partial y_i}(y) \rangle \geq 0 \) and \( \langle w_i(x, y), \frac{\partial F}{\partial y_j}(y) \rangle = 0. \)

Moreover, the magnitude of the difference vector between \( w_i(x, y) \) and \( \frac{\partial F}{\partial y_i}(y) \) can be controlled by \( Z(x, y) \) and \( |dZ(x, y)| \) as follows:
Lemma 6.2. Let \((x, y) \in \Sigma \times \Sigma \setminus D\) such that \(Z(x, y) = 0\). Then there exist open neighborhoods \(N_1\) of \(x\) and \(N_2\) of \(y\) in \(\Sigma\) satisfying that \((N_1 \times N_2) \cap D = \emptyset\) and there exists a constant \(\Lambda_2 > 0\) depending only on \(x, y\) such that

\[
\left| w_i(x, y) - \frac{\partial F}{\partial y_i}(y) \right|^2 \leq \Lambda_2 \left( Z(x, y) + |dZ(x, y)| \right)
\]

for any point \((x, y) \in N_1 \times N_2\).

**Proof.** Let us choose the open neighborhoods \(N_1\) and \(N_2\) of \(x\) and \(y\) as above. Furthermore, for an arbitrary point \((x, y) \in N_1 \times N_2\), we choose the geodesic normal coordinates at \(x\) and \(y\) as above. If \(w_i(x, y) = \frac{\partial F}{\partial y_i}(y)\), then it is trivial. Thus we may assume that \(w_i(x, y) \neq \frac{\partial F}{\partial y_i}(y)\). Since we can make the angle between \(\frac{\partial F}{\partial y_i}(y)\) and \(w_i(x, y)\) sufficiently small, we may assume that the angle between the vectors \(w_i(x, y) - \frac{\partial F}{\partial y_i}(y)\) and \(V\) is less than \(\frac{\pi}{4}\). It is easy to see that

\[
\left| w_i(x, y) - \frac{\partial F}{\partial y_i}(y) \right|^2 < 2 \left\langle w_i(x, y) - \frac{\partial F}{\partial y_i}(y),\frac{V}{|V|} \right\rangle \right|^2.
\]

Therefore

\[
\left| w_i(x, y) - \frac{\partial F}{\partial y_i}(y) \right|^2 < \frac{2}{|V|^2} \left\langle w_i(x, y) - \frac{\partial F}{\partial y_i}(y),\frac{V}{|V|} \right\rangle \right|^2
\]

\[
= \frac{2}{|V|^2} \left\langle w_i(x, y) - \frac{\partial F}{\partial y_i}(y),\Psi(x)F(x) - \nu(x) \right\rangle \right|^2
\]

\[
= 4 \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle Z(x, y) + \frac{\partial Z}{\partial y_i}(x, y) \right|^2
\]

\[
\leq 4\Lambda_1^4 \varepsilon Z(x, y) + 4\varepsilon (2\Lambda_1^2 + 1) |dZ(x, y)|
\]

\[
\leq \Lambda_2 \left( Z(x, y) + |dZ(x, y)| \right),
\]

for any point \((x, y) \in N_1 \times N_2\). □

In the proof of Proposition 4.1, Proposition 4.2, and Proposition 4.3, we used the property

\[
Z(x, y) = \frac{\partial Z}{\partial x_i}(x, y) = \frac{\partial Z}{\partial y_i}(x, y) = 0
\]

at a global minimum point \((x, y)\). However it is no longer valid in general. Using the geodesic normal coordinates we picked in the above, we have the
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following second order derivatives in general. For any point \((x, y) \in N_1 \times N_2\),

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2}(x, y) = \left( \Delta \Psi(x) - 2 \sum_{i=1}^{n} \frac{|\partial \Psi(x)|^2}{\Psi(x) - \lambda_i(x)} \right) + \left( |A(x)|^2 - n \right) \Psi(x) \right) \left( 1 - \langle F(x), F(y) \rangle \right) + n \Psi(x) + 2 \sum_{i=1}^{n} \frac{\partial \Psi(x)}{\Psi(x) - \lambda_i(x)} \frac{\partial Z}{\partial x_i} - |A(x)|^2 Z(x, y),
\]

\[
\frac{\partial^2 Z}{\partial x_i \partial y_i}(x, y) = \lambda_i(x) - \Psi(x) - \frac{1}{2} \left| w_i(x, y) - \frac{\partial F(y)}{\partial y_i} \right|^2 - \frac{1}{1 - \langle F(x), F(y) \rangle} \left\langle F(x), \frac{\partial F(y)}{\partial y_i} \right\rangle \frac{\partial Z}{\partial x_i}(x, y),
\]

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2}(x, y) = n \Psi(x) - n Z(x, y).
\]

**Lemma 6.3.** Let \((\bar{x}, \bar{y}) \in \Sigma \times \Sigma \setminus D\) such that \(Z(\bar{x}, \bar{y}) = 0\). Then there exist open neighborhoods \(N_1\) of \(\bar{x}\) and \(N_2\) of \(\bar{y}\) in \(\Sigma\) satisfying that \((N_1 \times N_2) \cap D = \emptyset\) and there exists a constant \(\Lambda > 0\) depending only on \(\bar{x}, \bar{y}\) such that

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2}(x, y) + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i}(x, y) + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2}(x, y) \leq \Lambda \left( Z(x, y) + |dZ(x, y)| \right)
\]

for all \((x, y) \in N_1 \times N_2\).

**Proof.** Applying the estimates (18), (19), the equalities (23), (24), (25), and Lemma 6.2, we have

\[
\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial y_i^2} \leq \left( 1 - \langle F(\bar{x}), F(\bar{y}) \rangle \right) \left( \Delta \Sigma \Psi(\bar{x}) - 2 \frac{|\nabla \Sigma \Psi(\bar{x})|^2}{\Psi(\bar{x})} + \left( |A(\bar{x})|^2 - n \right) \Psi(\bar{x}) \right) + 2n \Lambda^2_1 |dZ| + \Lambda_1 Z + n(\Lambda_2 Z + \Lambda_2 |dZ|) + 2n \Lambda^2_1 |dZ| + n Z \leq \Lambda (Z + |dZ|).
\]

□
We define the set $\Omega$ by

$$\Omega := \{ \bar{x} \in \Sigma : \text{there exists a point } \bar{y} \in \Sigma \setminus \{ \bar{x} \} \text{ such that } Z(\bar{x}, \bar{y}) = 0 \}.$$  

Lemma 5.1 shows that the set $\Omega$ is nonempty. Moreover we can prove the following:

**Theorem 6.4.** The set $\Omega$ is an open subset of $\Sigma$.

We need the following strict maximum principle for a degenerate second order elliptic partial differential equation.

**Theorem 6.5 ([7, 8]).** Let $\Omega$ be an open subset of an $n$-dimensional Riemannian manifold, and let $X_1, \ldots, X_m$ be smooth vector fields on $\Omega$. Assume that $\varphi : \Omega \to \mathbb{R}$ is a nonnegative smooth function satisfying

$$\sum_{j=1}^{m} (D^2 \varphi)(X_j, X_j) \leq -L \inf_{|\xi| \leq 1} (D^2 \varphi)(\xi, \xi) + L|d\varphi| + L\varphi,$$

where $L$ is a positive constant. Let $F = \{ x \in \Omega : \varphi(x) = 0 \}$ be the zero set of the function $\varphi$. Moreover, suppose that $\gamma : [0, 1] \to \Omega$ is a smooth path such that $\gamma(0) \in F$ and $\gamma'(s) = \sum_{j=1}^{m} f_j(s)X_j(\gamma(s))$ for suitable smooth functions $f_1, \ldots, f_m : [0, 1] \to \mathbb{R}$. Then $\gamma(s) \in F$ for all $s \in [0, 1]$.

**Proof of Theorem 6.4.** Take an open subset $\Omega$ of $2n$-dimensional manifold $\Sigma \times \Sigma$ as an open neighborhood $N_1 \times N_2$ of $(\bar{x}, \bar{y})$ in Lemma 6.3 with the usual product topology. Let $X_i = \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}$ for $i = 1, \ldots, n$. Then Lemma 6.3 shows that the condition of Theorem 6.5 is satisfied. Applying Theorem 6.5 with $L = \Lambda$, we have the conclusion. □

We now complete our proof of Theorem 1.1. By Proposition 6.1 and Theorem 6.4, we see that $\Delta_{\Sigma} \Psi(\bar{x}) = 0$ for all $\bar{x} \in \Omega$. Thus Corollary 3.6 implies that $|A(\bar{x})|^2 = n$ for all $\bar{x} \in \Omega$. Furthermore, since $|A|^2 = n(n-1)\lambda^2$ by minimality and two distinct principal curvatures condition, we conclude that $\lambda$ and $\mu$ are constant on $\Omega$, which implies that $\Psi$ is constant on $\Omega$. By analytic continuation for solutions of elliptic partial differential equations, we see that $\Psi$ is constant on $\Sigma$. Hence $\lambda$ and $\mu$ are constant on $\Sigma$, which shows that $\Sigma$ is an isoparametric minimal hypersurface with two distinct principal curvatures. From the Cartan’s classification of isoparametric hypersurfaces and analysis of the structure equations as before, it follows that
\( \Sigma \) is congruent to a Clifford minimal hypersurface. However, this is a contradiction to the fact that \( \kappa = 1 \) on any Clifford minimal hypersurface, which completes the proof.

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