§1. Introduction

The purpose of this paper is to give a natural (unconditional) construction of a family of non-tempered Arthur packets of $G_2$, and to construct the submodule in the space of square-integrable automorphic forms associated to these Arthur packets. A surprising aspect of our definition is that a representation in one of these local Arthur packets can actually be reducible. To the best of our knowledge, this is the first instance of such a phenomenon for split $p$-adic groups.

Our construction is based on an earlier paper of Rallis and Schiffmann [RS] which we recall briefly. In the paper [RS], Rallis and Schiffmann constructed a lifting of cuspidal automorphic forms from the metaplectic group $\widetilde{SL}_2$ to the split exceptional group of type $G_2$ over a number field $F$. This was achieved by exploiting the fact that $SL_2 \times G_2$ is a subgroup of $SL_2 \times O_7$, which is the classical dual pair in $Sp_{14}$. The lifting is then defined using the theta kernel furnished by the Weil representation $\omega_\psi^{(7)}$ of $\widetilde{Sp}_{14}$ (which depends on the choice of an additive character $\psi$).

The surprising discovery of Rallis-Schiffmann is that, despite restricting from $O_7$ to the smaller group $G_2$, one still obtains a correspondence of representations. More precisely, if $\sigma$ is an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$, let $V(\sigma)$ be the theta lift of $\sigma$; it is a non-zero subspace of the space of automorphic forms on $G_2$. Then the main results of Rallis-Schiffmann are:

- $V(\sigma)$ is contained in the space of cusp forms if and only if the theta lift (associated to $\psi$) of $\sigma$ to $SO_3$ (studied by Waldspurger) is zero.

- The cuspidal representations obtained as lifts from $\widetilde{SL}_2$ are characterized as those having a non-zero period with respect to some quasi-split $SU_3$ (which is a subgroup of $G_2$).
• The local correspondence of unramified representations is precisely determined. In particular, when \( V(\sigma) \) is cuspidal, the local components of each irreducible constituent of \( V(\sigma) \) are determined for almost all places \( v \), in terms of the local components of \( \sigma \).

• As a consequence of the unramified correspondence, the irreducible cuspidal representations contained in \( V(\sigma) \) are non-generic and CAP with respect to the Heisenberg parabolic or the Borel subgroup of \( G_2 \). This gives the first construction of CAP representations of \( G_2 \).

In this paper, we complete the study initiated in [RS] by giving a precise determination of the representation \( V(\sigma) \). The first step in this is the complete determination of the local theta correspondence. Since the archimedean correspondence has to a large extent been determined by Li-Schwermer [LS], we shall only discuss the non-archimedean case here. More precisely, if \( v \) is a \( p \)-adic place of \( F \) and \( \sigma_v \) an irreducible representation of \( \tilde{SL}_2(F_v) \), the maximal \( \sigma_v \)-isotypic quotient of \( \omega_v^{(7)} \) can be expressed as \( \sigma_v \otimes \theta(\sigma_v) \), where \( \theta(\sigma_v) \) is a smooth representation of \( G_2(F_v) \). Let \( \Theta(\sigma_v) \) be the maximal semisimple quotient of \( \theta(\sigma_v) \). Our main local result is:

\[(1.1) \text{ Theorem } \Theta(\sigma_v) \text{ can be completely determined for any } \sigma_v \text{ (to the extent that classification of representations of } G_2(F_v) \text{ is known). It turns out that } \Theta(\sigma_v) \text{ is irreducible except when } \sigma_v = \omega_v^{\pm} \text{ (the even and odd Weil representations of } \tilde{SL}_2(F_v) \text{ associated to } \psi_v). \text{ In these two exceptional cases, } \Theta(\sigma_v) \text{ is the sum of 2 unipotent representations.} \]

A precise statement of the results is given in Theorem 2.16.

We now turn to the global situation. For any cuspidal \( \sigma \) on \( \tilde{SL}_2 \), one can show that \( V(\sigma) \) is contained in the space of square-integrable automorphic forms. Thus \( V(\sigma) \) is semisimple and is a non-zero summand of \( \Theta(\sigma) := \otimes_v \Theta(\sigma_v) \). Our precise local result immediately shows that \( V(\sigma) \cong \Theta(\sigma) \), whenever \( \sigma_v \) is not the even or odd Weil representations associated to \( \psi_v \), for any place \( v \), since \( \Theta(\sigma) \) is irreducible then. However, when \( \Theta(\sigma) \) is reducible, there are more than one possibilities for \( V(\sigma) \). The determination of \( V(\sigma) \) in this case is easily the trickiest part of the paper. In any case, our main global result (Theorem 3.7) says:

\[(1.2) \text{ Theorem If } \sigma \subset \mathcal{A}_{00}, \text{ i.e. } \sigma \text{ is not an irreducible summand of any Weil representation, then } V(\sigma) \cong \Theta(\sigma). \]

In fact, one can define the theta lift of any square-integrable automorphic representation of \( \tilde{SL}_2 \) by a regularization of the theta integral. Thus, one can
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speak of the regularized theta lift of the orthogonal complement of $\mathcal{A}_{00}$, which consists of the Weil representations of $\tilde{SL}_2$. One can show that the space of automorphic forms of $G_2$ thus obtained is precisely equal to that constructed in [GGJ], by restriction of the minimal representation of the various quasi-split $Spin_8$’s.

Let us highlight a corollary of the global theorem above. It pertains to the question of whether there are cuspidal representations of $G_2$ with non-zero $SL_3$-period. Such cuspidal representations should be very scarce, but can be obtained by restriction from the minimal representation of split $Spin_8$ [GGJ]. It wasn’t known previously if other $SL_3$-distinguished cuspidal representations exist. Now, as a consequence of our global theorem, we know that they do. Indeed, when $\sigma \subset \mathcal{A}_{00}$ is such that its theta lift to $SO_3$ is non-zero, we know from [RS] that $V(\sigma)$ is not totally contained in $\mathcal{A}_{usp}(G_2)$. However, this does not exclude the possibility that $V(\sigma) \cap \mathcal{A}_{usp}(G_2)$ is non-zero. In [RS, Pg. 823], Rallis-Schiffmann remarked that they do not know if such a possibility can actually happen. Our global theorem implies that it does, and the cuspidal representations thus obtained are $SL_3$-distinguished.

This paper serves as an announcement of some of the results of the longer paper [GG]. In particular, though we provide a precise statement of the local theorem, we do not give its proof here. We do, however, give the proof of the global theorem, since it provides a justification for our definition of the local Arthur packets, especially in the case when the local packet contains a reducible representation. The details can be found in Section 4. In [GG], we provide further justification of the correctness of our local packets by showing that the spaces we constructed are full near equivalence classes; this last aspect will not be discussed here.

§2. Local Results

In this section, we shall state our precise local results. For this, we need to introduce a number of notations and recall a number of background facts. Throughout this section, $F$ will denote a non-archimedean local field of characteristic zero, and we fix a non-trivial additive character $\psi$ of $F$. For any $a \in F^\times$, we let $\psi_a$ be the character defined by $\psi_a(x) = \psi(ax)$.

(2.1) The group $\tilde{SL}(2)$. The group $\tilde{SL}_2(F)$ is a topological central extension of $SL_2(F)$ by $\{\pm 1\}$. As usual, we shall let $T$ denote the diagonal torus of $SL_2$ and $N$ the group of unipotent upper triangular matrices. Hence $B = TN$ is the usual Borel subgroup. For a subgroup $H$ of $SL_2(F)$, let $\hat{H}$ be its inverse image
in $\widetilde{SL}_2(F)$.

One can define a character $\chi_\psi$ of $\tilde{T}$ by:

$$\chi_\psi(t(a), \epsilon) = \epsilon \cdot \gamma_\psi / \gamma_\psi a$$

where $t(a) = \text{diag}(a, a^{-1})$, $\epsilon = \pm 1$ and $\gamma_\psi$ is the 8th root of unity associated to $\psi$ by Weil. Let us recall the classification of irreducible genuine representations of $\widetilde{SL}_2(F)$.

(2.2) The Weil representations of $\widetilde{SL}_2(F)$. Let $\chi$ be a quadratic character of $F^\times$ (possibly trivial). Then $\chi$ corresponds to an element $a_\chi \in F^\times / F^\times 2$. Associated to $\chi$ is a Weil representation $\omega_\chi$ of $\widetilde{SL}_2(F)$. As a representation of $\widetilde{SL}_2(F)$, $\omega_\chi$ is reducible. In fact, it is the direct sum of two irreducible representations:

$$\omega_\chi = \omega_\chi^+ \oplus \omega_\chi^-,$$

where $\omega_\chi^-$ is supercuspidal and $\omega_\chi^+$ is not.

(2.3) The principal series. The principal series representations of $\widetilde{SL}_2(F)$ can be parametrized by the characters $\mu$ of $F^\times$ (cf. [W2, Pg. 225]). The representation associated to $\mu$ is the induced representation

$$\tilde{\pi}(\mu) = \text{Ind}_{B}^{\tilde{G}} \chi_{\psi} \cdot \delta_B^{1/2} \cdot \mu.$$

Note that this parametrization depends on the additive character $\psi$. Now we have:

(2.4) Proposition (i) $\tilde{\pi}(\mu)$ is irreducible if and only if $\mu^2 \neq -1$, in which case $\tilde{\pi}(\mu) \cong \tilde{\pi}(\mu^{-1})$.

(ii) If $\mu = \chi \cdot | - |^{1/2}$ where $\chi$ is a quadratic character, then we have a short exact sequence:

$$0 \longrightarrow sp_\chi \longrightarrow \tilde{\pi}(\mu) \longrightarrow \omega_\chi^+ \longrightarrow 0.$$ 

We call $sp_\chi$ the special representation associated to $\chi$.

(iii) If $\mu = \chi \cdot | - |^{-1/2}$, then we have a short exact sequence,

$$0 \longrightarrow \omega_\chi^+ \longrightarrow \tilde{\pi}(\mu) \longrightarrow sp_\chi \longrightarrow 0.$$ 

The proposition gives all the non-supercuspidal genuine representations of $\widetilde{SL}_2(F)$. The other irreducible representations of $SL_2(F)$ are all supercuspidal, including the $\omega_\chi^-$’s introduced above.
(2.5) **Whittaker functionals.** For any \( a \in F^\times \), let \( \psi_a \) be the character of \( N \) defined by:

\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \psi(ax).
\]

It is known that

\[
\dim(\sigma_{N,\psi_a}) \leq 1.
\]

We say that \( \sigma \) has a \( \psi_a \)-Whittaker functional if \( \sigma_{N,\psi_a} \neq 0 \). It is also known that any genuine \( \sigma \) has a \( \psi_a \)-Whittaker functional for some \( a \). We let

\[
\hat{F}(\sigma) = \{ a \in F^\times : \sigma \text{ has a } \psi_a\text{-Whittaker functional} \}.
\]

Clearly, \( \hat{F}(\sigma) \) is a non-empty union of square classes.

(2.6) **The Weil representation of \( \widetilde{SL}_2(F) \times SO(V,q) \).** If \( (V,q) \) is a quadratic space, then as is well-known, one can define a Weil representation \( \omega_{\psi,q} \) of the group \( \widetilde{SL}_2(F) \times SO(V,q) \). For example, if \( \chi \) is a quadratic character of \( F^\times \) and \( (V,q) \) is the rank 1 quadratic space \( \langle a_\chi \rangle \), then \( \omega_{\psi,q} \) is simply the representation of \( \widetilde{SL}_2(F) \) denoted by \( \omega_\chi \) in 2.2.

Let \( (V_m,q_m) \) be the \((2m+1)\)-dimensional quadratic space \( \langle 1 \rangle \oplus \mathbb{H}^m \), where \( \mathbb{H} \) is the rank 2 hyperbolic space. We shall write \( \omega_\psi^{(m)} \) for the Weil representation of \( \widetilde{SL}_2(F) \times SO(V_m,q_m) \).

(2.7) **Waldspurger’s lift and packets for \( \widetilde{SL}_2(F) \).** By a detailed study of the representation \( \omega_{\psi,q} \) as \( (V,q) \) ranges over all 3-dimensional quadratic spaces [W1,2], Waldspurger defined a map \( Wd_\psi \) from the set of irreducible representations of \( \widetilde{SL}_2(F) \) which are not equal to \( \omega_\chi^+ \) to the set of infinite dimensional representations of \( PGL_2(F) \). This leads to a partition of the set of such representations of \( \widetilde{SL}_2(F) \) indexed by the infinite dimensional representations of \( PGL_2(F) \). Namely, if \( \tau \) is such a representation of \( PGL_2(F) \), we set

\[
\tilde{A}_\tau = \text{inverse image of } \tau \text{ under } Wd_\psi.
\]

It turns out that

\[
\#\tilde{A}_\tau = \begin{cases} 2 & \text{if } \tau \text{ is discrete series;} \\ 1 & \text{if } \tau \text{ is not.} \end{cases}
\]
In the first case, the set $\tilde{A}_\tau$ has a distinguished element $\sigma^+\tau$, which is characterized by the fact that $\tau \otimes \sigma^+\tau$ is a quotient of $\omega_3^{(3)}$. The other element of $\tilde{A}_\tau$ will be denoted by $\sigma^-\tau$. In the second case, we shall let $\sigma(\tau)^+$ be the unique element in $\tilde{A}_\tau$ and set $\sigma(\tau)^- = 0$.

(2.8) The group $G_2$. Now we come to the split group $G_2$. It is the automorphism group of the octonion algebra $O$. Like the quaternion algebra, the octonion algebra carries a quadratic norm form $N$ and a linear trace form $Tr$ and these are preserved by $G_2$. Let $V$ be the space of trace zero elements in $O$, equipped with the quadratic form $q = -N$. Then $(V, q)$ is isomorphic to $(V_7, q_7)$ and $G_2$ acts as automorphisms of $(V, q)$. This gives us an embedding $G_2 \hookrightarrow SO(V_7, q_7)$.

The group $G_2$ has two conjugacy classes of maximal parabolic subgroups. One of them is the Heisenberg parabolic $P_2 = L_2 \cdot U_2$, with $U_2$ a 5-dimensional Heisenberg group. Denote the other maximal parabolic by $P_1 = L_1 \cdot U_1$. Its unipotent radical $U_1$ is a 3-step nilpotent group. In both cases, the Levi subgroups are isomorphic to $GL_2$ and we fix these isomorphisms.

Now let us recall some facts about representations of $G_2(F)$.

(2.9) Langlands quotients. Let $\tau$ be a tempered representation of $GL_2(F)$ and $s > 0$. In the following, we shall use standard notions for the representations of $PGL_2$. For example, $St$ denotes the Steinberg representation, $St_\chi$ the twist of $St$ by the quadratic character $\chi$ and $\pi(\mu_1, \mu_2)$ for a principal series representation. Now the induced representations

$$I_{P_1}(\tau, s) = Ind_{P_1}^{G_2} \delta^{1/2} \cdot \tau \cdot |det|^s$$

has a unique irreducible quotient $J_{P_1}(\tau, s)$. The reducibility points of these induced representations are known by [M, Thm. 3.1 and Thm. 5.3].

(2.10) Degenerate principal series. Consider now the induced representation

$$I_{P_2}(\mu) = Ind_{P_2}^{G_2} \delta^{1/2} (\mu \circ \det),$$

where $\mu$ is a character of $F^\times$. The following was shown in [M, Thm. 3.1 and Props. 4.1, 4.3, 4.4]:

(2.11) Lemma Assume that $|\mu| = | - |^s$ with $Re(s) \geq 0$. Then $I_{P_1}(\mu)$ is irreducible unless $\mu^2 = | - |$ or $\mu = | - |^{5/2}$. For these exceptional cases, we have the following non-split exact sequences:

(i) If $\mu = \chi | - |^{1/2}$, with $\chi \neq 1$ a quadratic character, then we have:

$$0 \rightarrow J_{P_2}(St_\chi, 1/2) \rightarrow I_{P_1}(\mu) \rightarrow J_{P_1}(\pi(1, \chi), 1) \rightarrow 0$$
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(ii) If $\mu = | - |^{1/2}$, then we have:

\[ 0 \longrightarrow J_{P_1}(St, 1/2) \longrightarrow I_{P_1}(\mu) \longrightarrow J_{P_1}(\pi(1, 1), 1) \oplus J_{P_2}(St, 1/2) \longrightarrow 0 \]

(iii) If $\mu = | - |^{5/2}$,

\[ 0 \longrightarrow J_{P_2}(St, 3/2) \longrightarrow I_{P_1}(\mu) \longrightarrow 1 \longrightarrow 0 \]

(2.12) $U_2$-spectrum. The $L_2(F)$-orbits of characters of $U_2(F)$ can be naturally parametrized by cubic algebras over $F$ (cf. [GGJ]). For a cubic algebra $E$, let us write $\psi_E$ for a character in the corresponding orbit. The $U_2$-spectrum of a smooth representation $\pi$ of $G_2$ is the set of those cubic algebras $E$ such that the corresponding twisted Jacquet module $\pi_{U_2, \psi_E}$ is non-zero. In this paper, we shall only look at the cubic algebras of the form $F \times K$ where $K$ is an étale quadratic algebra. We set

\[ \hat{F}(\pi) = \{ a \in F^\times : K_a = F(\sqrt{a}) \text{ is in the } U_2\text{-spectrum of } \pi \} \]

Clearly, $\hat{F}(\pi)$ is a union of square classes.

(2.13) Some unipotent representations. We recall the results of [HMS] concerning the restriction of the (unique) unitarizable minimal representation $\Pi_K$ of the quasi-split $Spin^K(8)$ to the subgroup $G_2$, where $K$ is an étale quadratic algebra. The representation $\Pi_K$ is trivial on the center of $Spin^K_8$ and can be extended to a representation of $SO^K_8$. Any such extension will be called a minimal representation of $SO^K_8$ and each has the same restriction to $G_2$. Now we have:

(2.14) Proposition

(i) When $K = F \times F$,

\[ \Pi_K = J_{P_1}(\pi(1, 1), 1) \oplus 2 \cdot J_{P_2}(St, 1/2) \oplus \pi_\epsilon \]

where $\pi_\epsilon$ is supercuspidal.

(ii) When $K$ is a quadratic field, with associated quadratic character $\chi$,

\[ \Pi_K = J_{P_1}(\pi(1, \chi), 1) \oplus \pi(\chi) \]

where $\pi(\chi)$ is supercuspidal.

For a given $K$, the irreducible constituents of $\Pi_K$ obtained in the above proposition make up a unipotent Arthur packet, as explained in [GGJ].
(2.15) The Weil representation for $\widetilde{SL}_2(F) \times G_2(F)$. Finally, we are in a position to describe our main local theorem. Since $G_2$ is a subgroup of $SO(V_7, q_7)$, we may restrict the representation $\omega^{(7)}_\psi$ to $\widetilde{SL}_2(F) \times G_2(F)$. Given a representation $\sigma$ of $\widetilde{SL}_2(F)$, we have defined the smooth representation $\theta(\sigma)$ and $\Theta(\sigma)$ in the introduction. The main local theorem is:

(2.16) Theorem The representation $\theta(\sigma)$ is non-zero and admissible. Moreover, we have:

(a) (Principal series) If $\sigma = \tilde{\pi}(\mu)$ is an irreducible principal series (so that $\mu^2 \neq |-|^{\pm 1}$) with $\mu \neq |-|^{5/2}$, then

$$\theta(\sigma) \cong I_{P_1}(\mu^{-1}).$$

In particular, $\theta(\sigma) = \Theta(\sigma)$ is irreducible, unless $\sigma = \tilde{\pi}(|-|^{\pm 5/2})$, in which case $\Theta(\sigma)$ is the trivial representation of $G_2$.

(b) (Special representations) If $\sigma = sp_\chi$, then

$$\Theta(\sigma) \cong \begin{cases} J_{P_2}(St\chi, 1/2) & \text{if } \chi \neq 1; \\ J_{P_1}(St, 1/2) & \text{if } \chi = 1. \end{cases}$$

(c) (Weil representations) If $\sigma = \omega^\chi_+$ where $\chi$ is a quadratic character of $F^\times$, then

$$\theta(\sigma) = \begin{cases} J_{P_1}(\pi(1, \chi), 1) & \text{if } \chi \neq 1; \\ J_{P_1}(\pi(1, 1), 1) \oplus J_{P_2}(St, 1/2) & \text{if } \chi = 1. \end{cases}$$

If $\sigma = \omega^-_\chi$, then

$$\theta(\sigma) = \begin{cases} \pi(\chi) & \text{if } \chi \neq 1; \\ J_{P_2}(St, 1/2) \oplus \pi_\epsilon & \text{if } \chi = 1. \end{cases}$$

Here $\pi(\chi)$ and $\pi_\epsilon$ were defined in Prop. 2.14.

(d) (Supercuspidals) Suppose that $\sigma$ is supercuspidal and $\sigma \neq \omega^-_\chi$ for any $\chi$. Let $\tau = Wd_{\omega}(\sigma)$. If $\sigma = \sigma^+_\tau$, then

$$\theta(\sigma) = J_{P_2}(\tau, 1/2).$$

If $\sigma = \sigma^-_\tau$, then $\theta(\sigma)$ is an irreducible non-generic supercuspidal representation such that

$$\hat{F}(\theta(\sigma)) = \hat{F}(\sigma).$$
Moreover, if $\theta(\sigma) \cong \theta(\sigma')$, then $\sigma \cong \sigma'$.

The main ingredients in the proof of the theorem are the computation of the Jacquet and Fourier-Jacobi functors of $\omega_\psi^{(7)}$ with respect to various unipotent subgroups of $\widehat{SL}_2$ and $G_2$, as well as the the study of the $U_2$-spectrum of $\theta(\sigma)$ in terms of the $N$-spectrum of $\sigma$.

(2.17) Remark Even though we have restricted ourselves to the non-archimedean case in this section, the archimedean correspondence can also be completely determined. To a large extent, this was already done by Li-Schwermer [LS].

§3. Global Results
In this section, we let $F$ be a number field with adele ring $\mathbb{A}$ and fix a non-trivial additive character $\psi$ of $F \setminus \mathbb{A}$. We shall describe our main global results below.

(3.1) Cusp forms of $\widetilde{SL}_2(\mathbb{A})$. Let $\mathcal{A}^2$ denote the space of square-integrable genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Then there is an orthogonal decomposition

$$\mathcal{A}^2 = \mathcal{A}_0 \bigoplus \left( \bigoplus_\chi \mathcal{A}_\chi \right).$$

Here, $\chi$ runs over all quadratic characters of $F^\times \setminus \mathbb{A}^\times$.

Let us describe the space $\mathcal{A}_\chi$ more concretely. If $\omega_\chi = \otimes_v \omega_{\chi_v}$ is the global Weil representation attached to $\chi$, then the formation of theta series gives a map $\theta_\chi : \omega_\chi \to \mathcal{A}^2$, whose image is the space $\mathcal{A}_\chi$. To describe the decomposition of $\mathcal{A}_\chi$, for a finite set $S$ of places of $F$, let us set

$$\omega_{\chi,S} = (\otimes_{v \in S} \omega_{\chi_v}^-) \otimes (\otimes_{v \not\in S} \omega_{\chi_v}^+)$$

so that

$$\omega_\chi = \bigoplus_S \omega_{\chi,S}.$$ 

Then we have

$$\mathcal{A}_\chi \cong \bigoplus_{\# S \text{ even}} \omega_{\chi,S}.$$ 

Moreover, $\omega_{\chi,S}$ is cuspidal if and only if $S$ is non-empty.
(3.2) Near equivalence classes. In a profound piece of work [W2], Waldspurger has described the near equivalence classes of representations in $A_{00}$. Earlier, in [W1], he has shown that $A_{00}$ satisfies multiplicity one. Let us describe his results.

Given a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of $PGL_2$, we define a set of irreducible unitary representations of $\tilde{SL}_2(\mathbb{A})$ as follows. Recall that for each place $v$, we have a local packet $\tilde{A}_{\tau_v} = \{\sigma^+_v, \sigma^-_v\}$ where $\sigma^-_v = 0$ if $\tau_v$ is not discrete series. Now set

$$\tilde{A}_\tau = \{\sigma = \otimes_v \sigma_v^{\epsilon_v} : \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

This is called the global packet associated to $\tau$.

For $\sigma = \otimes_v \sigma_v^{\epsilon_v} \in \tilde{A}_\tau$, let us set

$$\epsilon_\sigma = \prod_v \epsilon_v.$$

Then we have:

$$A_{00} = \bigoplus_{\text{cuspidal } \tau} A(\tau)$$

where each $A(\tau)$ is a near equivalence class of cuspidal representations and

$$A(\tau) = \bigoplus_{\sigma \in \tilde{A}_\tau : \chi = \epsilon(\tau, 1/2)} \sigma.$$

(3.3) Fourier coefficients. For a character $\chi$ of $N(F) \backslash N(\mathbb{A})$, the $\chi$-Fourier coefficient of an automorphic form $f$ of $\tilde{SL}(\mathbb{A})$ is the function defined by

$$f_\chi(h) = \int_{N(F) \backslash N(\mathbb{A})} \chi(n) \cdot f(nh) \, dn.$$

Say that $\sigma$ has missing $\chi$-coefficient if $f_\chi = 0$ for all $f \in \sigma$.

(3.4) Global theta lift. Let $\omega^{(7)}_\psi = \otimes_v \omega^{(7)}_\psi$ be the global Weil representation of $\tilde{Sp}_{14}(\mathbb{A})$ associated to $\psi$. By the formation of theta series, we have a map

$$\theta : \omega^{(7)}_\psi \longrightarrow A(\tilde{Sp}_{14}).$$
Now if $\sigma \subset A^2$ is a cuspidal representation of $\tilde{SL}_2(\mathbb{A})$, then we let $V(\sigma)$ denote the linear span of all functions on $G_2(\mathbb{A})$ of the form

$$\theta(\varphi, f)(g) = \int_{\tilde{SL}_2(F) \backslash \tilde{SL}_2(\mathbb{A})} \theta(\varphi)(gh) \cdot f(h) \, dh,$$

for $\varphi \in \omega_\psi^{(7)}$ and $f \in \sigma$.

The complex conjugate over $f(h)$ is introduced to ensure the compatibility of global and local theta lifts.

From the results of [RS], one deduces:

**3.5 Proposition** The space $V(\sigma)$ is non-zero and is contained in the space of square-integrable automorphic forms on $G_2$, so that $V(\sigma)$ is semisimple and is a non-zero summand of $\Theta(\sigma) = \otimes_v \Theta(\sigma_v)$. It is contained in the space of cusp forms if and only if $\sigma$ has missing $\psi$-coefficient.

**3.6 Regularized theta lift.** It is desirable to extend the definition of the theta lift to all summands of $A^2$, i.e. for the non-cuspidal representations $\omega_{\chi,S}$ ($S$ empty). Let us explain how this is done.

For simplicity, let us take the case when $\chi = 1$ is trivial, so that $\omega_1 = \omega_\psi^{(1)}$. With $V_8 := V_7 \oplus (-V_1) \cong \mathbb{H}^4$, we have the following seesaw diagram:

$$\begin{array}{ccc}
\tilde{SL}_2 \times \tilde{SL}_2 & \xrightarrow{SO(V_8)} & \Delta SL_2 \\
\Delta SL_2 & \xleftarrow{G_2} & \end{array}
$$

As a representation of $G_2(\mathbb{A}) \times \Delta SL_2(\mathbb{A})$, $\omega_\psi^{(7)} \otimes \omega_\psi^{(1)}$ is (a dense subspace of) the restriction to $SO(V_8)(\mathbb{A}) \times \Delta SL_2(\mathbb{A})$ of the Weil representation $W_\psi$ of $\tilde{Sp}_{16}(\mathbb{A})$.

Let $\Theta : W_\psi \to \mathcal{A}(\tilde{Sp}_{16})$ be the usual theta map. In particular, for $\varphi \in \omega_\psi^{(7)}$ and $f \in \omega_\psi^{(1)}$, the function

$$(g, h) \mapsto \theta(\varphi)(gh) \cdot f(h)$$

is the restriction to $G_2(\mathbb{A}) \times \Delta SL_2(\mathbb{A})$ of the element $\Theta(\varphi \otimes f)$. It follows that the absolute convergence of $\theta(\varphi, f)$ is equivalent to the absolute convergence of

$$\int_{\Delta SL_2(F) \backslash \Delta SL_2(\mathbb{A})} \Theta(\varphi \otimes f)(gh) \, dh$$

for all $g \in G_2(\mathbb{A})$. 

Now the convergence of this latter integral is a well-studied problem in the topic of regularized Siegel-Weil formula. In our case, if we realize $W_\psi$ in the Schrodinger model $S(V_8(\mathbb{A}))$, it is easy to see that the integral of $\Theta(\Phi)$ (for $\Phi \in W_\psi$) over $\Delta SL_2$ converges absolutely iff $W_\psi(h)\Phi(0) = 0$ for all $h \in \Delta SL_2(\mathbb{A})$. So if $\varphi \otimes f$ has this property, then the integral $\theta(\varphi, f)$ converges. This is the case, for example, if $f$ lies in $\omega_{\chi,S}$ with $S$ non-empty.

In general, the map sending $\Phi$ to the function $h \mapsto W_\psi(h)\Phi(0)$ gives a $(SO(V_8) \times \Delta SL_2)$-equivariant map

$$T : W_\psi \longrightarrow Ind_{B}^{\Delta SL_2} \delta_B^2 \quad \text{(unnormalized induction)}.$$ 

Now fix an archimedean place $v_0$. It is clear that one can find an element $Z$ in the center of the universal enveloping algebra of $\Delta SL_2(F_{v_0})$ such that

$$Z = \begin{cases} 
1 \text{ on the trivial representation;} \\
0 \text{ on the above principal series.}
\end{cases}$$

Then $W_\psi(Z)\Phi$ lies in $ker(T)$ and this allows us to define the regularized theta lift:

$$\theta^{reg}(\varphi, f)(g) := \int_{\Delta SL_2(F) \backslash \Delta SL_2(\mathbb{A})} \Theta(W_\psi(Z)(\varphi \otimes f))(gh)dh.$$ 

Note that there is a unique equivariant extension of the theta integral from $Ker(T)$ to $W_\psi$. Hence the regularized theta lift defined here is canonical.

Let

$$V_\chi = \text{regularized theta lift of } A_\chi.$$ 

Then a consideration of the above see-saw diagram and [GRS, Theorem 6.9] (which says that the regularized theta lift of the trivial representation of $\Delta SL_2$ is a minimal representation of $SO_8$) gives the first part of our main global theorem:

**Theorem** (i) The space $V_\chi$ is equal to the space of automorphic forms obtained by restricting the automorphic minimal representation of the quasi-split $Spin_8^{\chi}$. The latter space consists of square-integrable automorphic forms and was determined in [GGJ] as an abstract representation.

(ii) If $\sigma \subset A_{00}$, then $V(\sigma) \cong \Theta(\sigma)$. 

(3.7)
(3.8) Sketch proof. We give a sketch of the proof of Theorem 3.7(ii). Clearly, there is nothing to prove if $\Theta(\sigma)$ is irreducible. In general, the proof is achieved by studying the Fourier coefficients of $\theta(\varphi, f)$, as we now explain.

We begin with some generalities which hold for any cuspidal $\sigma$. Recall that, with the character $\psi$ as a base point, the $\tilde{T}$-orbits of Fourier coefficients for $SL_2$ are parametrized by quadratic $F$-algebras. If $\sigma \subset A_0$, then $\sigma$ has at least 2 non-vanishing ($\tilde{T}$-orbits of) Fourier coefficients, whereas the representation $\omega_{\chi, S}$ supports only one Fourier coefficient, namely the one determined by the quadratic character $\chi$. For a quadratic field $K$, we shall let $\psi_K$ denote a character of $N(F) \backslash N(\AA)$ in the orbit indexed by $K$.

As for $G_2$, we shall consider Fourier expansion along $U_2$, in which case the $L_2$-orbits of Fourier coefficients are parametrized by cubic $F$-algebras. For a quadratic field $K$, we shall let $\psi_K$ denote a character of $U_2(F)$, trivial on $U_2(F)$, which lies in the orbit indexed by $F \times K$. For $\theta(\varphi, f) \in V(\sigma)$, we set

$$\theta(\varphi, f)_{\psi_K}(g) = \int_{U_2(F) \backslash U_2(\AA)} \overline{\psi_K(u)} \cdot \theta(\varphi, f)(ug) \, du, \quad g \in G_2(\AA).$$

Let $W(\sigma, \psi_K)$ denote the span of all the functions $\theta(\varphi, f)_{\psi_K}$ with varying $\varphi$ and $f \in \sigma$. Then we have a $G_2(\AA)$-equivariant surjective map from $V(\sigma)$ to $W(\sigma, \psi_K)$, so that $W(\sigma, \psi_K)$ is a semisimple representation of $G_2(\AA)$ and is a summand of $\Theta(\sigma)$.

(3.9) Proposition The space $W(\sigma, \psi_K)$ is non-zero iff $\sigma$ has non-zero $\tilde{\psi}_K$-Fourier coefficient. In this case, if $f = \otimes_v f_v$ and $\varphi = \otimes_v \varphi_v$, then one has an expression:

$$\theta(\varphi, f)_{\psi_K}(g) = \prod_v W_{\psi_{K_v}}(\varphi_v, f_v, g_v)$$

where $W_{\psi_{K_v}}(\varphi_v, f_v, g_v)$ is a local expression depending only on $\varphi_v \in \omega_{\psi_v}^{(7)}$ and $f_v \in \sigma_v$ (and the character $\psi_{K_v}$).

Let $W(\sigma_v, \psi_{K_v})$ denote the span of all the functions $W_{\psi_{K_v}}(\varphi_v, f_v, -)$, with varying $\varphi_v$ and $f_v$. Then as a corollary, we have:

(3.10) Corollary As a representation of $G_2(\AA)$,

$$W(\sigma, \psi_K) \cong \otimes_v W(\sigma_v, \psi_{K_v})$$

and $W(\sigma_v, \psi_{K_v})$ is a non-zero summand of $\Theta(\sigma_v)$. 
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(3.11) The proof. Now suppose that $\sigma \subset A_{00}$. Choose a quadratic field $K$ so that $\sigma$ supports a $\tilde{\psi}_K$-Fourier coefficient. Then to prove the theorem for $\sigma$, it suffices to show that:

$$W(\sigma_v, \psi_{K_v}) \cong \Theta(\sigma_v) \quad \text{for all } v.$$ 

Again this is clear if $\Theta(\sigma_v)$ is irreducible. Hence, we are reduced to showing this for the representation $\omega_{\psi_v}$ with $K_v = F_v \times F_v$. More precisely, we need to show that

$$W(\omega_{\psi_v}, \psi_{K_v}) \cong \Theta(\omega_{\psi_v}) = J_{P_2}(St_v, 1/2) \oplus \pi_{\epsilon_v}.$$ 

It is likely that there is a purely local proof of this statement. However, we shall present a local-global argument which we find rather amusing.

(3.12) A local-global argument. Suppose we want to prove the local statement above for a place $v_0$. Choose a quadratic field $K$ split at $v_0$, and let $\chi_K$ be the quadratic character corresponding to $K$. Let $v_1$ be another place where $K$ is split and set $S_0 = \{v_0, v_1\}$.

Consider the two representations $\pi_1$ and $\pi_2$ of $G_2(\mathbb{A})$ defined as follows:

$$(\pi_1)_v = \begin{cases} \pi_v, & \text{if } v \in S_0; \\ J_{P_1}(\pi(1, \chi_v), 1) & \text{if } v \notin S_0 \end{cases} \quad \text{and} \quad (\pi_2)_v = \begin{cases} \pi_v, & \text{if } v = v_1; \\ J_{P_2}(St_v, 1/2) & \text{if } v = v_0; \\ J_{P_1}(\pi(1, \chi_v), 1) & \text{if } v \notin S_0. \end{cases}$$

By the results of [GGJ], we know that $\pi_1$ and $\pi_2$ occur with multiplicity one in the restriction of the minimal representation of $Spin^K_S$. Hence, by Theorem 3.7(i), $\pi_1$ and $\pi_2$ occur with multiplicity one in $V_{\chi_K}$. So they must occur in $V(\omega_{\chi_K,S})$ for some $S$ of even cardinality. By our local results, one sees that the only possibility for $S$ is $S_0$. Hence we have:

$$\pi_1 \oplus \pi_2 \subset V(\omega_{\chi_K,S_0}).$$

Now since $\omega_{\chi_K,S_0}$ has non-zero $\tilde{\psi}_K$-coefficient, we have a surjective map from $V(\omega_{\chi_K,S_0})$ onto the non-zero space $W(\omega_{\chi_K,S_0}, \psi_K)$. In fact, by results of [GGJ], this map is non-zero when restricted to any irreducible summand of $V(\omega_{\chi_K,S_0})$. Hence, for $i = 1$ or 2, $\pi_i \subset W(\omega_{\chi_K,S_0}, \psi_K)$ and thus $(\pi_i)_{v_0} \subset W(\omega_{\psi_{v_0}}, \psi_{K_{v_0}})$, which is what we desire to prove.

§4. Arthur Packets

In this section, we shall explain how our main results allow us to give a definition of a family of Arthur packets for $G_2$. 

(4.1) **Arthur parameters.** Let $L_F$ be the conjectural Langlands group of $F$. We shall be considering a family of Arthur parameters for $G_2$:

$$\psi : L_F \times SL_2(\mathbb{C}) \rightarrow G_2(\mathbb{C}).$$

To write down the relevant family, let us observe that

$$SL_{2,l} \times_{\mu_2} SL_{2,s} \subset G_2$$

where $(SL_{2,l}, SL_{2,s})$ is a pair of commuting $SL_2$’s corresponding to a pair of mutually orthogonal long and short roots. Now suppose that $\tau$ is a cuspidal representation of $PGL_2$. Conjecturally, $\tau$ corresponds to an irreducible representation

$$\phi_\tau : L_F \rightarrow SL_2(\mathbb{C}).$$

We define an Arthur parameter by

$$\psi_\tau : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau \times id} SL_{2,l}(\mathbb{C}) \times SL_{2,s}(\mathbb{C}) \xrightarrow{i} G_2(\mathbb{C}).$$

If $S_{\psi_\tau}$ is the component group of the centralizer of $\psi_\tau$, then $S_{\psi_\tau} \cong \mathbb{Z}/2\mathbb{Z}$.

The global parameter gives rise to local parameters $\psi_{\tau,v}$ for each place $v$. The local component groups $S_{\psi_{\tau,v}}$ are given by

$$S_{\psi_{\tau,v}} = \begin{cases} 1, & \text{if } \phi_{\tau,v} \text{ is reducible} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \phi_{\tau,v} \text{ is irreducible}. \end{cases}$$

Note the condition $\phi_{\tau,v}$ is irreducible is equivalent to $\tau_v$ being a discrete series representation of $PGL_2(F_v)$.

(4.2) **Local Arthur packets.** Now Arthur’s conjecture (cf. [A1,2]) predicts that for each place $v$, there is a finite set $A_{\tau,v}$ of unitary representations of $G_2(F_v)$ associated to $\psi_{\tau,v}$. The representations should be indexed by the irreducible characters of $S_{\psi_{\tau,v}}$. Hence, in our case, $A_{\tau,v}$ has the form:

$$A_{\tau,v} = \begin{cases} \{\pi_v^+\}, & \text{if } \tau_v \text{ is not discrete series} \\ \{\pi_v^+, \pi_v^-\}, & \text{if } \tau_v \text{ is discrete series}. \end{cases}$$

Here, $\pi_v^+$ is indexed by the trivial character of $S_{\tau,v}$.

The set $A_{\tau,v}$ is called a local A-packet, and should satisfy
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• for almost all $v$, $\pi^{\pm}_{\tau_v}$ is irreducible and unramified with Satake parameter

$$s_{\tau,v} = i \left( t_{\tau,v} \times \begin{pmatrix} q^{1/2}_v & 0 \\ 0 & q^{-1/2}_v \end{pmatrix} \right) \in G_2(\mathbb{C})$$

where $t_{\tau,v} \in SL_2(\mathbb{C})$ is the Satake parameter of $\tau_v$.

• the distribution $\pi^{+}_{\tau_v} - \pi^{-}_{\tau_v}$ is stable.

• certain identities involving transfer of character distributions to endoscopic groups of $G(F_v)$ should hold.

(4.3) Definition of local Arthur packets. Now we can use Theorem 2.16 to give a natural candidate for the packet $A_{\tau,v}$. Recall that $\tau_v$ determines a set $\tilde{\pi}_{\tau_v}$ of representations of $\tilde{SL}_2(F_v)$. This has 2 or 1 elements $\sigma^{\pm}_{\tau_v}$, depending on whether $\tau_v$ is discrete series or not. We set

$$\pi^{\pm}_{\tau_v} = \Theta(\sigma^{\pm}_{\tau_v}).$$

This defines the packet $A_{\tau,v}$.

Why is this a reasonable definition? For one thing, when $\tau_v$ is unramified, then $\Theta(\sigma^{\pm}_{\tau_v})$ is indeed irreducible and unramified with the required Satake parameter $s_{\tau,v}$. As a second justification, we consider the following special case.

(4.4) A special case. We would like to highlight the case when $\tau_v$ is the Steinberg representation $St$. In this case, according to our definition,

$$\begin{cases} 
\pi^{+}_{\tau_v} = \Theta(\omega_{\psi_v}) = J_{P_2}(St, 1/2) + \pi_\epsilon \\
\pi^{-}_{\tau_v} = \Theta(\psi_{\pi_1}) = J_{P_1}(St, 1/2).
\end{cases}$$

For the case of split $p$-adic groups, this is the first instance we know in which the representation in a packet can be reducible, and this is quite surprising at first sight. The initial guess would be to take $\pi^{+}_{\tau_v}$ simply as $J_{P_2}(St, 1/2)$. However, we have:

(4.5) Proposition Assume that the packet of unipotent representations in Prop. 2.14(i) is indeed an Arthur packet, so that $J_{P_1}(\pi(1, 1), 1) + 2J_{P_2}(St, 1/2) + \pi_\epsilon$ is stable. Then $(J_{P_2}(St, 1/2) + \pi_\epsilon) - J_{P_1}(St, 1/2)$ is stable.

The proposition justifies our definition of $\pi^{+}_{\tau_v}$. A more powerful justification is given by our global theorem 3.7, as we explain below.
(4.6) **Global A-packets.** With the local packets $A_{\tau,v}$ at hand, we may define the global A-packet by:

$$A_{\tau} = \{ \pi = \otimes v \pi_v : \pi_v \in A_{\tau,v}, \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v \}.$$  

It is a set of nearly equivalent representations of $G_2(\mathbb{A})$. If $S$ is the set of places $v$ where $\tau_v$ is discrete series, then $\#A_{\tau} = 2\#S$.

To each $\pi \in A_{\tau}$, one can attach a multiplicity $m(\pi)$ as follows. Arthur attached to $\psi_{\tau}$ a quadratic character $\epsilon_{\psi_{\tau}}$ of the component group $S_{\psi_{\tau}}$. In the case at hand, $\epsilon_{\psi_{\tau}}$ is the non-trivial character of $S_{\psi_{\tau}} \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau,1/2) = -1$. Now if $\pi = \otimes v \pi_v \in A_{\tau}$, set

$$m(\pi) = \begin{cases} 
1, & \text{if } \epsilon_{\pi} := \prod v \epsilon_v = \epsilon(\tau,1/2); \\
0, & \text{if } \epsilon_{\pi} = -\epsilon(\tau,1/2). 
\end{cases}$$

If we let

$$V_{\tau} = \bigoplus_{\pi \in A_{\tau}: \epsilon_{\pi} = \epsilon(\pi,1/2)} \pi,$$

then Arthur conjectures that there is a $G_2(\mathbb{A})$-equivariant embedding

$$\iota_{\tau} : V_{\tau} \hookrightarrow L^2_d(G_2(\mathbb{F}) \backslash G_2(\mathbb{A})).$$

Now our global theorem 3.7 says that for the given $\tau$, the global theta correspondence constructs a subspace of $L^2_d(G_2(\mathbb{F}) \backslash G_2(\mathbb{A}))$ isomorphic to

$$\bigoplus_{\sigma \in \tilde{A}_{\tau}: \epsilon_{\sigma} = \epsilon(\tau,1/2)} \Theta(\sigma).$$

This is isomorphic to $V_{\tau}$ with our definition of the local packets $A_{\tau,v}$. This provides compelling global justification for our definition, especially for taking $\pi_{\tau_v}^\pm$ to be reducible when $\tau_v$ is Steinberg.

**References**


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