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Stable Configurations of Linear Subspaces and Quotient Coherent Sheaves

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1 Introduction

In this paper, we begin to study GIT stability of systems of geometric objects, using the Hilbert-Mumford numerical criterion and moment map. Here we focus on linear subspaces and quotient coherent sheaves.

Consider the product $\Pi_{i=1}^m \mathrm{Gr}(k_i, V \otimes W)$ of the Grassmannians of k_i-dimensional subspaces of $V \otimes W$, on which $\mathrm{SL}(V)$ acts diagonally, where V and W are two fixed vector spaces over complex numbers. For a set ω of positive integers, set

$$L_\omega = \otimes_{i=1}^m \pi_i^*(\mathcal{O}_{\mathrm{Gr}(k_i, V \otimes W)}(\omega_i)).$$

L_ω admits a unique $\mathrm{SL}(V)$-linearization. Then, using Hilbert-Mumford numerical criterion, we showed that a system of linear subspaces $\{K_i \subset V \otimes W\}$, as a point of $\Pi_{i=1}^m \mathrm{Gr}(k_i, V \otimes W)$, is semistable (resp. stable) with respect to the $\mathrm{SL}(V)$-linearized invertible sheaf L_ω if and only if, for all nonzero proper subspace $H \subset V$, we have

$$\frac{1}{\dim H} \sum_i \omega_i \dim(K_i \cap (H \otimes W)) \leq \frac{1}{\dim V} \sum_i \omega_i \dim K_i$$

(resp. $<$). This is Theorem 2.2, which generalizes Mumford's Proposition 4.3 of [21], where he treated the case $\mathrm{Gr}(k, V)^m$, and Dolgachev's Theorem 11.1 of [2], where he treated the case of subspaces of V. An equivalent version of

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I thank Igor Dolgachev for informing me of this (after posting the first version) and two examples of quotients he constructed in §11.3. See the remark after Theorem 2.2.

the above criterion is given in Theorem 2.2' in terms of systems of a_i-dimensional quotients of $V \otimes W$, as points in $\Pi_{i=1}^m \mathrm{Gr}(V \otimes W, a_i)$. This alternative formulation is necessary for the later application to quotient coherent sheaves.

To apply moment map, we assume that $\dim W = 1$ and consider the special case of systems of subspaces in V. We showed that a configuration $\{V_i\} \in \prod_i \mathrm{Gr}(k_i, V)$ is polystable if and only if $\{V_i\}$ can be (uniquely) *balanced* with respect to a Hermitian metric on V. Here, a Hermitian metric h on V is said to be a *balance metric* for the weighted configuration of vector subspaces $(\{V_i\}, \omega)$ if the weighted sum of the orthogonal projections from V onto V_i, for $1 \leq i \leq m$, is the scalar operator $\wp_\omega(\{V_i\}) = \frac{1}{\dim V} \sum_i \omega_i k_i$. That is

$$\sum_{i=1}^m \omega_i \pi_{V_i} = \wp_\omega(\{V_i\}) \, \mathrm{Id}_V$$

where $\pi_{V_i} : V \to V_i \hookrightarrow V$ is the orthogonal projection from V to V_i and Id_V is the identity map from V to V. In this case, we also say the weighted configuration $(\{V_i\}, \omega)$ is *balanced* with respect to the metric h. We say $(\{V_i\}, \omega)$ can be (uniquely) balanced if there is a (unique) $u \in \mathrm{SU}(V) \backslash \mathrm{SL}(V)$ such that $(\{u \cdot V_i\}, \omega)$ is balanced.

When the configuration $\{V_i\}$ is a so-called m-filtration, the existence of a balanced metric was proved by Totaro [28] where the term *good metric* was used instead. It was also proved in Klyachko's paper [18]. Totaro's motivation is to use *good metric* to give an elementary proof of G. Faltings and G. Wüstholz's theorem on the stability of tensor product [7]. Indeed, we have hoped that the results obtained here may be used to study some problems on Diophantine approximations. This is actually one of our original motivations to investigate the stability of systems of vector subspaces.

Along the way, we generalize the Gelfand-MacPherson correspondence ([11]) from configurations of *points* to configurations of *linear subspaces*. More precisely, we show that there is a one-to-one correspondence between the set of $\mathrm{GL}(V)$-orbits on the product of the Grassmannians $\Pi_{i=1}^m \mathrm{Gr}(k_i, V)$ and the set of $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m)$-orbits on the Grassmannian $\mathrm{Gr}(n, \mathbb{C}^{k_1+\cdots+k_m})$ where $n = \dim V$ and $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m) \subset \mathrm{GL}(k_1 + \cdots + k_m)$ acts on coordinate subspaces block-wise. Then, following the approach of Kapranov ([17]), we prove that there is a natural one-to-one correspondence between the set of GIT quotients of $\Pi_{i=1}^m \mathrm{Gr}(k_i, V)$ by the diagonal action of $\mathrm{GL}(V)$ and the set of GIT quotients of $\mathrm{Gr}(n, \mathbb{C}^{k_1+\cdots+k_m})$ by the action of $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m)$. It should follow from here that there is also an isomorphism between the Chow quotients of the two actions (cf. Theorem 3.6 of [15]). When $k_1 = \cdots = k_m = 1$, $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m)$

becomes a maximal torus of $\mathrm{GL}(k_1 + \cdots + k_m, \mathbb{C})$. And in this case, the above correspondence becomes the usual Gelfand-MacPherson correspondence. The case of a product of $\mathrm{Gr}(2, \mathbb{C}^4)$ was already treated by P. Foth and G. Lozano in [8]. After posting this paper on ArXiv, Ciprian Borcea e-mailed me that his paper [1] contains a generalization of the Gelfand-MacPherson correspondence, at birational level, to flag configurations.

In addition, by combining the generalized GM correspondence and the isomorphism between $\mathrm{Gr}(n, \mathbb{C}^{k_1+\cdots+k_m})$ and $\mathrm{Gr}(k_1+\cdots+k_m-n, \mathbb{C}^{k_1+\cdots+k_m})$, we obtain a *generalized Gale transform* from configurations of subspaces in $\Pi_{i=1}^m \mathrm{Gr}(k_i, \mathbb{C}^n)$ to configurations of subspaces in $\Pi_{i=1}^m \mathrm{Gr}(k_i, \mathbb{C}^{k_1+\cdots+k_m-n})$. The duality is well-defined up to linear transformations. This seems to be what was suggested by Eisenbud and Popescu in [6]. What is the geometric significance of this duality? This is a question worth pursuing.

To compute the $\mathrm{GL}(V)$-ample cone of $\Pi_{i=1}^m \mathrm{Gr}(k_i, V)$ or equivalently the $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m)$-ample cone of $\mathrm{Gr}(n, \mathbb{C}^{k_1+\cdots+k_m})$, we introduce a new polytope, the *diagonal hypersimplex* or *subhypersimplex*, which generalizes the usual hypersimplex (§5.2). As an interesting observation, we found that some diagonal hypersimplexes provide natural examples of G-ample cones *without* any top chambers. Not many examples of this sort are previously known (cf. the Appendix of [4]).

Finally, as an application, we consider systems of quotient coherent sheaves. Let X be a projective scheme (possibly singular) over the field of complex numbers. Let $\{\mathcal{E}_i\}$ $(1 \le i \le m)$ be a system of (quotient) coherent sheaves over X, realized as a point in the product of certain Quot schemes $\mathrm{Quot}(V \otimes \mathcal{W}, P_i)$ over X, where V is a vector space and $\mathcal{W}$ is a coherent sheaf. The group $\mathrm{SL}(V)$ of special linear transformations acts diagonally on the total product space. On the product space, there is a $\mathrm{SL}(V)$-linearization $\mathcal{L}_\omega$ associated to any given set of positive weights $\omega = \{\omega_i\}$ via the Grothendieck embeddings of the corresponding Quot schemes. We prove that $\{\mathcal{E}_i\}$ is GIT semistable (resp. stable) with respect to the $\mathrm{SL}(V)$-linearized invertible sheaf $\mathcal{L}_\omega$ if and only if for every proper linear subspace H of V,

$$\frac{1}{\dim V} \sum_i \omega_i \chi(\mathcal{E}_i(k)) \le \frac{1}{\dim H} \sum_i \omega_i \chi(\mathcal{F}_i(k))$$

(resp. $<$) where $\mathcal{F}_i$ is the subsheaf of $\mathcal{E}_i$ generated by $H \otimes \mathcal{W}$, and $\chi(\bullet)$ is the Euler characteristic. (See §6 for more details.)

Using the relation between GIT stability and the vanishing of moment map, we proved, in the special case of subbundles of the trivial bundle V, that a configuration $\{\mathcal{E}_i\}$ of vector subbundles in $\Pi_{i=1}^m \operatorname{Quot}(V, P_i)$ is polystable if and only if $\{\mathcal{E}_i\}$ can be (uniquely) balanced. Here we say that the configuration $\{\mathcal{E}_i\}$ of vector subbundles in $\Pi_{i=1}^m \operatorname{Quot}(V, P_i)$ is balanced if

$$\sum_{i=1}^{m} \omega_i \int_X A_i(x) A_i^*(x) dV = \wp_\omega(\{\mathcal{E}_i\}) \operatorname{Vol}(X) I$$

where $A_i(x)$ is a matrix representation of $(\mathcal{E}_i)_x \subset \mathbb{C}^N$ whose columns form an orthonormal basis for $(\mathcal{E}_i)_x$ $(1 \leq i \leq m)$, I is the identity matrix, $\operatorname{Vol}(X)$ is the volume of X, and $\wp_\omega(\{\mathcal{E}_i\}) = \sum_i \omega_i \frac{\operatorname{rank}(\mathcal{E}_i)}{N}$. We say $\{\mathcal{E}_i\}$ can be (uniquely) balanced if there is a (unique) element $u \in \mathrm{SU}(N) \backslash SL(N)$ such that $\{u \cdot \mathcal{E}_i\}$ is balanced.

When the system consists of a single vector bundle (i.e., $m = 1$) over a smooth projective variety, the above becomes a differential geometric criterion for the Gieseker-Simpson stability, which is originally due to Wang ([30]) and Phong-Sturm ([22]). Similar circle of ideas appeared earlier in the papers of Zhang ([32]) and Luo ([20]).

The outcome of this paper relies on the ideas of many other people in their earlier works, my sole contribution is to generalize them to systems of vector subspaces and coherent sheaves, in the hope that they will be used in future applications and references. The use of balance metrics was inspired by Totaro [28], Klyachko [18], and by Wang ([30]), Phong-Sturm ([22]) and the earlier papers of Zhang ([32]) and Luo ([20]); The GIT constructions of the moduli spaces of stable configurations of coherent sheaves follow very closely the approach of Simpson ([24]); The generalized GM correspondence obviously plainly follows Gelfand-MacPherson ([11]) and Kapranov ([17]); The author benefited from the conversations with P. Foth and W.-P. Li, and from the correspondence with I. Dolgachev and C. Simpson. I thank them all. Financial support and hospitality from Harvard University and Professor S.-T. Yau, from NCTS Taiwan and Professor C.L Wang, and from Hong Kong UST and Professors W.-P. Li and Y. Ruan are gratefully acknowledged. The research is partially supported by NSF and NSA. The paper was finished in early 2003.

2 Configuration of subspaces and quotients of tensor product

Throughout the paper, we will work over the field of complex numbers. Let V and W be two vector spaces. Consider the product of the Grassmannians

$$\Pi_{i=1}^{m} \operatorname{Gr}(k_i, V \otimes W).$$

The group $\mathrm{SL}(V)$ acts diagonally on $\Pi_{i=1}^{m} \operatorname{Gr}(k_i, V \otimes W)$ by operating on the factor V. We will study the GIT of this action.

2.1 Stability Criteria

To proceed we need a lemma.

Lemma 2.1. *Let* $\mathbf{q}$ *be the vector* $(q_1, \ldots, q_n)$ *such that*

$$(\star) \quad q_1 \geq q_2 \ldots \geq q_n, \quad \text{and} \quad q_1 + \ldots + q_n = 0.$$

Let $\mathbf{q}_s$ *be the vector* $(q_1, \ldots, q_n)$ *such that*

$$q_1 = \ldots = q_s = n - s, q_{s+1} = \ldots = q_n = -s, \quad \text{for} \quad s = 1, \ldots, (n-1).$$

Then $\mathbf{q}$ *is a linear combination of* $\mathbf{q}_s$*,* $s = 1, \ldots, (n-1)$*, with nonnegative coefficients.*

Proof. Indeed, one can check that

$$\mathbf{q} = \frac{q_1 - q_2}{n} \mathbf{q}_1 + \ldots + \frac{q_{n-1} - q_n}{n} \mathbf{q}_{n-1}.$$

□

Let $\omega = \{\omega_1, \ldots, \omega_m\}$ be a set of positive integers, and

$$L_\omega = \otimes_{i=1}^{m} \pi_i^* (\mathcal{O}_{\operatorname{Gr}(k_i, V \otimes W)}(\omega_i))$$

be the ample line bundle over $\Pi_{i=1}^{m} \operatorname{Gr}(k_i, V \otimes W)$ associated with ω, where π_i is the projection from the product space to the i-th factor. This line bundle has a unique $\mathrm{SL}(V)$-linearization because $\mathrm{SL}(V)$ is semisimple ([21]).

We refer the reader to consult [21] for the definition of GIT stability and for the Hilbert-Mumford numerical criterion.

Theorem 2.2. *A system of linear subspaces $\{K_i \subset V \otimes W\}$ as a point of $\Pi_{i=1}^m \mathrm{Gr}(k_i, V\otimes W)$ is semistable (resp. stable) with respect to the* $\mathrm{SL}(V)$*-linearized invertible sheaf L_ω if and only if, for all nonzero proper subspace $H \subset V$, we have*

$$\frac{1}{\dim H}\sum_i \omega_i \dim(K_i \cap (H \otimes W)) \leq \frac{1}{\dim V}\sum_i \omega_i \dim K_i$$

(resp. $<$).

Proof. Choose a basis $v_1, \ldots, v_n$ of V such that $H = \mathrm{span}\{v_1, \ldots, v_s\}$. Set $H_i = \mathrm{span}\{v_1, \ldots, v_i\}$. In particular, we have $H_n = V$ and $H_s = H$.

Let $w_1, \ldots, w_d$ be a basis for W. We list the basis of $V \otimes W$ made of $v_i \otimes w_j$ as

$$\{v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_n \otimes w_d\}.$$

Let E_i $(1 \leq i \leq nm)$ be spanned by the first i vectors in the above basis.

Let K be any subspace of $V \otimes W$. Then for any integer $1 \leq j \leq k = \dim K$, there are integers l_j such that

$$\dim K \cap E_{l_j - 1} = j - 1, \ \dim K \cap E_{l_j} = j.$$

Under the basis $\{v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_n \otimes w_d\}$, K can be represented by a matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1l_1} & 0 & \cdots & 0 & 0\cdots 0 \\ a_{21} & \cdots & \cdots & a_{2l_2} & \cdots & 0 & 0\cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & \cdots & \cdots & \cdots & \cdots & a_{kl_k} & 0\cdots 0 \end{pmatrix}$$

In the Plucker embedding, one sees that

$$p_{i_1 \cdots i_k}(K) = 0 \ \text{ if } \ i_j > l_j$$

$$p_{l_1 \cdots l_k}(K) \neq 0.$$

We now apply the above to all K_i $(1 \leq i \leq m)$ and let $l_j^{(i)}$ be the numbers associated to K_i.

Next, consider the one-parameter subgroup $\lambda(t)$ of $\mathrm{SL}(V)$ defined by a vector $\mathbf{q} = (q_1, \ldots, q_n)$ as a diagonal matrix

$$\lambda(t) = \mathrm{diag}(t^{q_1}, \ldots, t^{q_n})$$

with

$$q_1 + \ldots + q_n = 0.$$

By permutation if necessary, we can further assume that

$$q_1 \geq q_2 \ldots \geq q_n.$$

Let each q_i repeat m times, we obtain a new diagonal matrix

$$\lambda'(t) = \operatorname{diag}(t^{q'_1}, \ldots, t^{q'_{mn}}).$$

Under this convention and from the matrix representations of K, we see that

$$p_{i_1 \cdots i_k}(\lambda(t)K) = t^{q'_{i_1} + \ldots q'_{i_k}} p_{i_1 \cdots i_k}(K).$$

Hence by the minimality of the numerical function we obtain

$$\mu^{L_\omega}(\{K_i\}, \lambda) = \sum_{i=1}^{m} \omega_i \sum_{j=1}^{k_i} q'_{l^i_j}.$$

Using the fact that $\dim K_i \cap E_j - \dim K_i \cap E_{j-1}$ equals 0 when $j \neq l_j^{(i)}$ and equals 1 otherwise, we can rewrite

$$\mu^{L_\omega}(\{K_i\}, \lambda) = \sum_{i=1}^{m} \omega_i (\sum_{j=1}^{mn} q'_j (\dim K_i \cap E_j - \dim K_i \cap E_{j-1})).$$

(Note here that $\mu^{L_\omega}(\{K_i\}, \lambda)$ is linear in $(q_1, \ldots, q_n)$. This observation will be useful later.)

Now replace λ by the one-parameter subgroup λ_s defined by $\mathbf{q}_s$ ($1 \leq s \leq (n-1)$, see Lemma 2.1), then we have

$$\mu^{L_\omega}(\{K_i\}, \lambda_s) = \sum_{i=1}^{m} \omega_i (\sum_{j=1}^{sm} (n-s)(\dim K_i \cap E_j - \dim V_i \cap E_{j-1}))$$

$$- \sum_{i=1}^{m} \omega_i (\sum_{j=sm+1}^{nm} s(\dim K_i \cap E_j - \dim V_i \cap E_{j-1})).$$

After cancelation, we obtain

$$\mu^{L_\omega}(\{K_i\}, \lambda_s) = \sum_{i=1}^{m} \omega_i ((n-s) \dim K_i \cap E_{sm} - s(\dim K_i \cap E_{mn} - \dim K_i \cap E_{sm})).$$

That is,

$$\mu^{L_\omega}(\{K_i\}, \lambda_s) = \sum_{i=1}^{m} \omega_i (n \dim K_i \cap E_{sm} - s \dim K_i \cap E_{mn}),$$

Noting that $E_{sm} = H \otimes W$ and $E_{mn} = V \otimes W$, we have

$$\mu^{L_\omega}(\{K_i\}, \lambda_s) = \dim V \sum_{i=1}^{m} \omega_i \dim K_i \cap (H \otimes W) - \dim H \sum_{i=1}^{m} \omega_i \dim K_i.$$

Now if $\{K_i\}$ is L_ω-semisimple (resp. simple), then

$$\mu^{L_\omega}(\{K_i\}, \lambda_s) \leq 0$$

(resp. $<$) which is the same as that

$$\frac{1}{\dim H} \sum_i \omega_i \dim(K_i \cap (H \otimes W)) \leq \frac{1}{\dim V} \sum_i \omega_i \dim K_i$$

(resp. $<$).

Conversely, if the inequality

$$\frac{1}{\dim H} \sum_i \omega_i \dim(K_i \cap (H \otimes W)) \leq \frac{1}{\dim V} \sum_i \omega_i \dim K_i$$

holds for all H, but $\{K_i\}$ is not L_ω-semistable. Then there is one-parameter subgroup λ such that $\mu^{L_\omega}(\{K_i\}, \lambda) > 0$. By conjugation and permutation, we can assume that the vector $\mathbf{q}$ that defines λ satisfies $(\star)$ (see Lemma 2.1). Note that such a vector $\mathbf{q}$ is a linear combination of $\mathbf{q}_s$ $(1 \leq s \leq n-1)$ with non-negative coefficients. Note also from the above that $\mu^{L_\omega}(\{K_i\}, \lambda)$ is linear in $\mathbf{q}$. Hence there must exists s $(1 \leq s \leq n-1)$ such that $\mu^{L_\omega}(\{K_i\}, \lambda_s) > 0$, but this is equivalent to that

$$\frac{1}{\dim H} \sum_i \omega_i \dim(K_i \cap (H \otimes W)) > \frac{1}{\dim V} \sum_i \omega_i \dim K_i$$

for some vector subspace H, a contradiction.

Similarly, if the strict inequality

$$\frac{1}{\dim H} \sum_i \omega_i \dim(K_i \cap (H \otimes W)) < \frac{1}{\dim V} \sum_i \omega_i \dim K_i$$

holds for all H, then by the above $\{K_i\}$ is L_ω-semistable. Assume that it is not stable. Then there is a λ that satisfies $(\star)$ of Lemma 2.1 such that $\mu^{L_\omega}(\{K_i\}, \lambda) = 0$. Then the same arguments as above plus that we already know $\mu^{L_\omega}(\{K_i\}, \lambda_s) \leq 0$ will yield that there is s $(1 \leq s \leq n-1)$ such that $\mu^{L_\omega}(\{K_i\}, \lambda_s) = 0$, but this is equivalent to that

$$\frac{1}{\dim H} \sum_i \omega_i \dim(K_i \cap (H \otimes W)) = \frac{1}{\dim V} \sum_i \omega_i \dim K_i$$

for some vector subspace H, a contradiction.

This completes the proof. □

In the case of systems of linear subspaces of V, Dolgachev in Theorem 11.1, [3] already provided a proof of the stability criterion. More interestingly, §11.3 of [3] contains two nice explicit examples: 4 lines in $\mathbb{P}^3$ where the quotient is $\mathbb{P}^2$, and, 6 lines in $\mathbb{P}^3$ where the quotient is a double cover of a toric space ramified over an explicitly given hypersurface. It seems that these are the only explicitly known nontrivial examples of quotients.

Now let us go back to our setups. The above theorem can also be equivalently stated in terms of quotients. We will use the notation $\mathrm{Gr}(V \otimes W, a)$ for the Grassmannian of quotient linear spaces of $V \otimes W$ of dimension a.

Let ω be a set of positive integers and

$$L'_\omega = \otimes_{i=1}^m \pi_i^*(\mathcal{O}_{\mathrm{Gr}(V \otimes W, a_i)}(\omega_i))$$

be the ample line bundle over $\Pi_{i=1}^m \mathrm{Gr}(V \otimes W, a_i)$ defined by ω where π_i is the projection from the product space to the i-th factor.

Theorem 2.2'. *A configuration $\{V \otimes W \xrightarrow{f_i} U_i \to 0\}$ as a point of $\Pi_{i=1}^m \mathrm{Gr}(V \otimes W, a_i)$ is semistable (resp. stable) with respect to the $\mathrm{SL}(V)$-linearized invertible sheaf L'_ω if and only if, for all nonzero proper subspace $H \subset V$, we have*

$$\frac{1}{\dim V} \sum_{i=1}^m \omega_i \dim U_i \leq \frac{1}{\dim H} \sum_{i=1}^m \omega_i \dim f_i(H \otimes W)$$

(resp. $<$). In particular, $f_i(H \otimes W) \neq 0$ for some i.

We note that when $m = 1$, this is Simpson's Proposition 1.14 of [24], where it is used to construct the moduli space of coherent sheaves.

For the action of $\mathrm{SL}(V)$ on $\mathrm{Gr}(V \otimes W, a)$, if there is a GIT quotient, then it will be *unique* because there is only one ample line bundle over $\mathrm{Gr}(V \otimes W, a)$ up to homothety, and, this line bundle has a unique $\mathrm{SL}(V)$ linearization. There could be none, for example, this will be the case when $\dim W = 1$. From now on, we assume that a GIT quotient exists and we use $\mathfrak{M}$ to denote this unique quotient variety.

Fix a set of positive numbers $\omega = \{\omega_1, \ldots, \omega_m\}$ and let $\mathfrak{M}_\omega$ be the quotient variety of the locus of the L'_ω-semistable configurations.

Proposition 2.3. *Fix an integer $1 \leq i \leq m$. For sufficiently large ω_i (relative to other ω_j), we have*

1. *If a configuration $\{V \otimes W \to U_j \to 0\}$ is L'_ω-semistable, then its i-th component $V \otimes W \to U_i \to 0$ is also semistable.*
2. *If the i-th component of a configuration $\{V \otimes W \to U_j \to 0\}$ is stable, then the configuration $\{V \otimes W \to U_j \to 0\}$ is L'_ω-stable.*
3. *In particular, there is a surjective projective morphism from $\mathfrak{M}_\omega$ to $\mathfrak{M}$*

$$\pi_i : \mathfrak{M}_\omega \to \mathfrak{M}.$$

Similar statements hold in terms of systems of subspaces.

Proof. For any subspace $H \subset V$, the inequality

$$\frac{1}{\dim V} \sum_j \omega_j \dim U_j \leq \frac{1}{\dim H} \sum_j \omega_j \dim f_j(H \otimes W) \text{ (resp } <)$$

holds if and only if

$$\dim H \dim U_i - \dim V \dim f_i(H \otimes W) \leq$$

$$\frac{1}{\omega_i}(\dim H \sum_{j \neq i} \omega_j \dim U_j - \dim V \sum_{j \neq i} \omega_j \dim f_j(H \otimes W)) \text{ (resp } <)$$

holds.

Let R be the right hand term of the last inequality. Then we can choose a sufficiently large ω_i, such that $|R| < 1$. Now if the inequality "$\leq$" holds, the left hand term

$$\dim H \dim U_i - \dim V \dim f_i(H \otimes W)$$

must be nonpositive since it is an integer. This proves (1).

For (2), if $\dim H \dim U_i - \dim V \dim f_i(H \otimes W) < 0$, then

$$\dim H \dim U_i - \dim V \dim f_i(H \otimes W) \leq -1.$$

Since we have $|R| < 1$, we obtain

$$\dim H \dim U_i - \dim V \dim f_i(H \otimes W) < R$$

which implies

$$\frac{1}{\dim V} \sum_j \omega_j \dim U_j < \frac{1}{\dim H} \sum_j \omega_j \dim f_j(H \otimes W).$$

(3). The existence of the morphism $\pi_i : \mathfrak{M}_\omega \to \mathfrak{M}$ follows from (1). The surjectivity follows from (2). $\square$

2.2 Harder-Narasimhan and Jordan-Hölder filtrations

Theorem 2.2 motivates the following definition. Let $\omega = (\omega_1, \ldots, \omega_m)$ be a set of positive numbers, called the weights. The *normalized total weighted dimension* of $\mathcal{K} = \{K_i\} \in \prod_i \mathrm{Gr}(k_i, V \otimes W)$ with respect to ω is defined by

$$\wp_\omega(\mathcal{K}) = \frac{1}{\dim V} \sum_i \omega_i \dim K_i.$$

For any subspace H of V, there is an induced subconfiguration of linear subspaces in $H \otimes W$

$$\mathcal{H} = (K_1 \cap (H \otimes W), \ldots, K_m \cap (H \otimes W))$$

whose *normalized total weighted dimension* with respect to ω is

$$\wp_\omega(\mathcal{H}) = \frac{1}{\dim H} \sum_i \omega_i \dim K_i \cap (H \otimes W).$$

Definition 2.4. The configuration $\mathcal{K}$ is $\wp_\omega$-semistable (resp. stable) with respect to the weights ω if

$$\wp_\omega(\mathcal{H}) \leq \wp_\omega(\mathcal{K}) \ \ (\text{resp.} <)$$

for every subspace H of V.

Then, Theorem 2.2 can be restated as

Theorem 2.5. *The configuration $\mathcal{K}$ is GIT semistable (stable) with respect to the linearized line bundle L_ω if and only if it is $\wp_\omega$-semistable (stable) with respected to the weight set ω.*

Let $f : V \to Q$ be a linear map. Then, the induced map $V \otimes W \to Q \otimes W$, still denoted by f, induces a configuration $\{f(K_i)\}$ of linear subspaces in $Q \otimes W$. A subconfiguration of $\{K_i\}$ is the one induced from an inclusion map $i : H \hookrightarrow V$.

Lemma 2.6. *Let $\{K_i\}$ be a configuration of linear subspaces of $V \otimes W$ and*

$$0 \to F \to V \to Q \to 0$$

be an exact sequence. Let $\mathcal{F}$ and $\mathcal{Q}$ be the inducing configurations. Then

1. *$\wp_\omega(\mathcal{Q}) \geq \wp_\omega(\mathcal{V})$(resp. $>$) if $\wp_\omega(\mathcal{F}) \leq \wp_\omega(\mathcal{V})$(resp. $>$);*
2. *$\wp_\omega(\mathcal{F}) \geq \wp_\omega(\mathcal{Q})$(resp. $>$) if $\wp_\omega(\mathcal{F}) \geq \wp_\omega(\mathcal{V})$(resp. $>$).*

Proof. We prove (2) and leave (1) for the reader.

Let F_i be $K_i \cap (F \otimes W)$ and Q_i be the image of K_i under the map $f : V \otimes W \to Q \otimes W$ for all i. If $\wp_\omega(\mathcal{F}) \geq \wp_\omega(\mathcal{V})$, then

$$\frac{1}{\dim F} \sum_i \omega_i \dim F_i \geq$$

$$\frac{1}{\dim V} \sum_i \omega_i \dim K_i = \frac{1}{\dim V} (\sum_i \omega_i \dim F_i + \sum_i \omega_i \dim Q_i).$$

Hence

$$\frac{\dim F + \dim Q}{\dim F} \sum_i \omega_i \dim F_i \geq (\sum_i \omega_i \dim F_i + \sum_i \omega_i \dim Q_i).$$

Therefore

$$\frac{\dim Q}{\dim F} \sum_i \omega_i \dim F_i \geq \sum_i \omega_i \dim Q_i.$$

That is, $\wp_\omega(\mathcal{F}) \geq \wp_\omega(\mathcal{Q})$.

The strict inequality can be proved similarly. □

Definition 2.7. For any configuration $\{K_i\}$ of linear subspaces of $V \otimes W$, if there is a filtration

$$0 = V^0 \subset V^1 \subset \cdots \subset V^h = V$$

with the inducing subconfigurations

$$\{0\} = \{K_i^{(0)}\} \subset \{K_i^{(1)}\} \subset \cdots \subset \{K_i^{(h)}\} = \{K_i\}, \ \ 1 \leq i \leq m$$

where $K_i^{(l)} = K_i \cap (V^l \otimes W)$ $(1 \leq l \leq h)$ such that the quotient configuration $\{K_i^{(l)}/K_i^{(l-1)}\}_i$ $(1 \leq l \leq h)$ is $\wp_\omega$-semistable and the normalized total weighted dimension

$$\frac{1}{\dim V^l/V^{l-1}} \sum_i \omega_i \dim(K_i^{(l)}/K_i^{(l-1)}), \ \ 1 \leq l \leq h$$

is strictly decreasing, then the filtration

$$0 = V^0 \subset V^1 \subset \cdots \subset V^k = V$$

or rather the filtered configuration

$$\{0\} = \{K_i^{(0)}\} \subset \{K_i^{(1)}\} \subset \cdots \subset \{K_i^{(h)}\} = \{K_i\}, \ \ 1 \leq i \leq m$$

will be called a *Harder-Narasimhan filtration* for $\{K_i\}$.

Proposition 2.8. *For every configuration $\{K_i\}$ of linear subspaces of $V \otimes W$, the Harder-Narasimhan filtration exists and is unique.*

Proof. Let H be a subspace of V such that

$$\wp_\omega(\mathcal{H}) = \frac{1}{\dim H} \sum_i \omega_i \dim K_i \cap (H \otimes W)$$

is the maximal. If $H = V$, then $\mathcal{K}$ is $\wp_\omega$-semistable, we are done. Otherwise, by maximality, $\mathcal{H}$ is $\wp_\omega$-semistable. Now assume H_1 is another linear subspace such that $\wp_\omega(\mathcal{H}_1)$ is maximal, that is, $\wp_\omega(\mathcal{H}_1) = \wp_\omega(\mathcal{H})$. Then $\mathcal{H} \oplus \mathcal{H}_1$ is $\wp_\omega$-semistable of $\wp_\omega(\mathcal{H} \oplus \mathcal{H}_1) = \wp_\omega(\mathcal{H})$. Consider the addition map

$$f : H \oplus H_1 \to V.$$

Since $H \oplus H_1$ is $\wp_\omega$-semistable, we have that the normalized weighted dimension of the kernel $\mathrm{Ker}(f)$ is less than or equal to $\wp_\omega(\mathcal{H})$, therefore the normalized weighted dimension of the image $\mathcal{H} + \mathcal{H}_1$ is greater than or equal to $\wp_\omega(\mathcal{H})$ (by Lemma 2.6 (1)) and hence equal to $\wp_\omega(\mathcal{H})$ by the maximality of $\wp_\omega(\mathcal{H})$. This showed that there is a unique subspace V^1 such that $\wp_\omega(\mathcal{K}^1)$ is largest, where

$\mathcal{K}^1$ is the induced configuration from V^1. This constitutes the first step of the filtration

$$0 \subset V^1 \subset V.$$

Next consider V/V^1. If $\mathcal{K}/cK^1$ is semistable, we are done because Lemma 2.6 (2) implies that $\wp_\omega(\mathcal{K}^1) > \wp_\omega(\mathcal{K}/\mathcal{K}^1)$. If $\mathcal{K}/\mathcal{K}^1$ is not semistable, the above procedure can be applied word for word to produce a unique linear subspace V^2 ($V^1 \subset V^2 \subset V$) with $\mathcal{K}^2/\mathcal{K}^1$ semistable. By Lemma 2.6 (2) again, $\wp_\omega(\mathcal{K}^1) > \wp_\omega(\mathcal{K}^2/\mathcal{K}^1)$ because $\wp_\omega(\mathcal{K}^1) > \wp_\omega(\mathcal{K}^2)$. Hence by induction, we will obtain the desired filtration.

The uniqueness is clear from the proof. □

Definition 2.9. Assume that $\{K_i\}$ is $\wp_\omega$-semistable. If there is a filtration

$$0 = V^0 \subset V^1 \subset \cdots \subset V^k = V$$

with the inducing subconfigurations

$$\{0\} = \{K_i^{(0)}\} \subset \{K_i^{(1)}\} \subset \cdots \subset \{K_i^{(h)}\} = \{K_i\}, \;\; 1 \le i \le m$$

such that the quotient systems $\{K_i^{(l)}/K_i^{(l-1)}\}_i$ are $\wp_\omega$-stable with the same normalized total weighted dimension $\wp_\omega(\mathcal{V})$, then the filtration is called a Jordan-Hölder filtration.

Proposition 2.10. *For any $\wp_\omega$-semistable $\{K_i\}$, a Jordan-Hölder filtration exists.*

Proof. A construction goes as follows. If $\mathcal{K}$ is stable, we are done. Otherwise, let H be a maximal subspace such that $\wp_\omega(\mathcal{H}) = \wp_\omega(\mathcal{K})$. Then $\mathcal{H}$ must also be semistable. Applying Lemma 2.6 (1) and (2), one can check that $\mathcal{K}/\mathcal{H}$ is $\wp_\omega$-stable and $\wp_\omega(\mathcal{K}/\mathcal{H}) = \wp_\omega(\mathcal{K})$. Repeat the same procedure to $\mathcal{H}$, we will obtain a desired filtration. □

From the proof one see that a Jordan-Hölder filtration always exists but depends on a choice of maximal subspaces H, hence it needs not to be unique.

2.3 Splitting and Merging

To conclude §2, we will make some elementary observations for the purpose of future references. Let

$$\mathcal{K} = (K_1, \ldots, K_m) \in \mathrm{Gr}(k_1, V \otimes W) \times \ldots \times \mathrm{Gr}(k_m, V \otimes W)$$

be a configuration of vector subspaces of $V = \mathbb{C}^n$ with weightes $\omega = (\omega_1, \ldots, \omega_m)$. If for every i, $\omega_i = s_i + t_i$ where s_i and t_i are nonnegative integers, then we can split K_i with weight ω_i into K_i with weight s_i and K_i with weight t_i. In this way, we obtain a new configuration $\widetilde{\mathcal{K}}$ with new weights $\tilde{\omega}$. We may call such a process splitting or separation.

Conversely, as opposed to splitting, one may consider "merging". That is, for any configuration of vector subspaces $\widetilde{\mathcal{K}}$ with weights $\widetilde{\omega}$, if $\tilde{K}_i = \tilde{K}_j$ for some $i \neq j$, then we can merge the two as one and count it with new weight $\omega_i = \tilde{\omega}_i + \tilde{\omega}_j$. This way, we obtain a new configuration $\mathcal{K}$ with new weights ω. We may call such a process merging.

Clearly in either splitting or merging, we have that

$$\wp_\omega(\mathcal{K}) = \wp_{\tilde{\omega}}(\widetilde{\mathcal{K}}),$$

and, it can also be easily checked that for any subspace $H \subset K$,

$$\wp_\omega(\mathcal{H}) = \wp_{\tilde{\omega}}(\widetilde{\mathcal{H}}).$$

For any weights ω, if we write every ω_i as the sum $1 + \cdots + 1$ (ω_i many), then we obtain a new weight set $\mathbb{I} = (1, \ldots, 1)$ which we shall call the trivial weight. Now, let M_ω denote the GIT quotient of $X = \mathrm{Gr}(k_1, V \otimes W) \times \ldots \times \mathrm{Gr}(k_m, V \otimes W)$ defined by the $\mathrm{SL}(V)$-linearized line bundle L_ω. Then it follows that

Proposition 2.11. *$\mathcal{K}$ is semistable (stable) with respect to ω if and only if $\widetilde{\mathcal{K}}$ is semistable (stable) with respect to $\tilde{\omega}$. Consequently, this induces a closed embedding from the GIT quotient space M_ω to the corresponding GIT quotient space $M_{\tilde{\omega}}$. In particular, every M_ω can be embedded in $M_{\mathbb{I}}$ as a closed subvariety.*

3 Balance metrics and Stability

3.1 Polystable configurations

In this section, we will focus on the special case when $\dim W = 1$. So, let

$$\mathcal{V} = (V_1, \ldots, V_m) \in \mathrm{Gr}(k_1, V) \times \ldots \times \mathrm{Gr}(k_m, V)$$

be a configuration of vector subspaces of $V = \mathbb{C}^n$ and $\omega = (\omega_1, \ldots, \omega_m)$ be a set of positive numbers.

Proposition 3.1. *If $\mathcal{V}$ is $\wp_\omega$-semistable and is a direct sum $\oplus_{i=1}^{l}\mathcal{H}_i$ of a finite number of subconfigurations, then all $\mathcal{H}_i$ and $\mathcal{V}$ have the same normalized total weighted dimension. In particular, all $\mathcal{H}_i$ are also semistable.*

Proof. We first prove the case when $\mathcal{V} = \mathcal{H}_1 \oplus \mathcal{H}_2$. We have

$$0 \to \mathcal{H}_1 \to \mathcal{V} \to \mathcal{H}_2 \to 0$$

and

$$0 \to \mathcal{H}_2 \to \mathcal{V} \to \mathcal{H}_1 \to 0.$$

Since $\mathcal{V}$ is semistable, $\wp_\omega(\mathcal{H}_1) \leq \wp_\omega(\mathcal{V})$ and $\wp_\omega(\mathcal{H}_2) \leq \wp_\omega(\mathcal{V})$. By Lemma 2.6, $\wp_\omega(\mathcal{H}_2) \geq \wp_\omega(\mathcal{V})$ and $\wp_\omega(\mathcal{H}_1) \geq \wp_\omega(\mathcal{V})$. Hence they are all equal.

In general, write $\mathcal{V} = \mathcal{H}_i \oplus$ (the rest), by the case $l = 2$, $\wp_\omega(\mathcal{H}_i) = \wp_\omega(\mathcal{V})$ for every i. □

Definition 3.2. A semistable configuration $\mathcal{V} = (V_1, \ldots, V_m)$ is called polystable if it is a direct sum of a finite number of stable subconfigurations of the same normalized total weighted dimension.

Proposition 3.3. *$\mathcal{V}$ is polystable if and only if as a point in the product of the Grassmannians its orbit is closed in the semistable locus.*

Proof. Suppose that $\mathcal{V} = \{V_i\}$ is polystable and is the direct sum of stable subconfigurations $\{\mathcal{H}_q\}$ induced from the decomposition $V = \oplus_q H_q$. Let $\mathcal{V}(t)$ be a curve in $G \cdot \mathcal{V}$ for t near t_0. Let $\mathcal{V}(0)$ be the limit of $\mathcal{V}(t)$ in the semistable locus at t_0. Then $\mathcal{V}(0)$ is the direct sum of $\{\mathcal{H}_q(0)\}$ where $\mathcal{H}_q(0)$ is in the closure of $G \cdot \mathcal{H}_q$. By Proposition 3.1, $\mathcal{H}_q(0)$ is semistable. Since $\mathcal{H}_q$ is stable, $\mathcal{H}_q(0)$ in the orbit $G \cdot \mathcal{H}_q$. This means there is a linear isomorphism l_q of V sending H_q to $H_q(0)$ and inducing isomorphisms between $H_q \cap V_i$ and $H_q(0) \cap V_i$ for all i. Since V is the direct sum $\oplus_q H_q$, one can build a linear isomorphism l of V from $l_q|_{H_q}$ (for all q), sending H_q to $H_q(0)$ for all q and inducing isomorphisms between $H_q \cap V_i$ and $H_q(0) \cap V_i$ for all i. Hence $\mathcal{V}(0) = \{H_q(0)\}_q$ is in the orbit $G \cdot \mathcal{V}$. This shows that the orbit $G \cdot \mathcal{V}$ is closed in the semistable locus.

Conversely if $\mathcal{V}$ is a semistable configuration and the orbit $G \cdot \mathcal{V}$ is closed in the semistable locus, we need to show that $\mathcal{V}$ is polystable. Let $F \subset V$ be a subspace such that $\{F \cap V_i\}$ constitute the first step in the Jordan-Hölder filtration of $\mathcal{V}$. Choose a basis for F and extend it to a basis for V. Then under this basis we can represent each V_i as an $(n \times k_i)$ matrix

$$\begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix}$$

where A_i generates $F \cap V_i$. Let d be the dimension of F and $\lambda(t)$ be a one-parameter subgroup of $\mathrm{GL}(V)$ defined by

$$\lambda(t) = \begin{pmatrix} tI_d & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

Then we have

$$\lambda(t)V_i = \begin{pmatrix} tA_i & tB_i \\ 0 & D_i \end{pmatrix}$$

As t tends to zero, this splits the limit as the direct sum

$$F \cap V_i \oplus Q \cap V_i$$

where Q is spanned by the basis element of V that are not in F. By Proposition 3.1, the configuration $\{F \cap V_i \oplus Q \cap V_i\}_i$ is semistable. Since $G \cdot \mathcal{V}$ is closed in the semistable locus, this shows that $\{V_i\}$ and

$$\{F \cap V_i \oplus Q \cap V_i\}$$

are in the same orbit. Repeat the Jordan-Hölder process, this will eventually show that $\mathcal{V}$ is polystable. □

3.2 Balanced metrics and polystable configurations

Definition 3.4. A Hermitian metric h on V is said to be a balance metric for the weighted configuration $(\mathcal{V}, \omega)$ of vector subspaces if the weighted sum of the orthogonal projections from V onto V_i $(1 \leq i \leq m)$ is the scalar operator $\wp_\omega(\mathcal{V}) \operatorname{Id}_V$. That is

$$\sum_{i=1}^{m} \omega_i \pi_{V_i} = \wp_\omega(\mathcal{V}) \operatorname{Id}_V$$

where $\pi_{V_i} : V \to V_i \hookrightarrow V$ is the orthogonal projection from V to V_i and Id_V is the identity map from V to V. In this case, we also say that the weighted configuration $(\mathcal{V}, \omega)$ is balanced with respect to the metric h. We say $(\mathcal{V}, \omega)$ can be (uniquely) balanced if there is a (unique) $g \in \mathrm{SU}(V) \backslash \mathrm{SL}(V)$ such that $(g \cdot \mathcal{V}, \omega)$ is balanced.

Theorem 3.5. *A configuration $\mathcal{V} = (V_1, \ldots, V_m)$ is polystable with respect to a weight set ω if and only if there is a balance metric on V for the configuration.*

Proof. First, it is easy to check that under the linearized line bundle L_ω, the moment map

$$\Phi : \mathrm{Gr}(k_1, V) \times \ldots \times \mathrm{Gr}(k_m, V) \to \sqrt{-1}\,\mathrm{su}(V)$$

of the diagonal action of $\mathrm{SU}(V)$ is given by

$$\Phi(\mathcal{V}) = \sum_i \omega_i A_i A_i^* - \wp_\omega(\mathcal{V}) I_n$$

where $\mathcal{V} = (V_1, \ldots, V_m) \in \mathrm{Gr}(k_1, V) \times \ldots \times \mathrm{Gr}(k_m, V)$ and A_i is a matrix representation of V_i such that its columns form an orthonormal basis for V_i $(1 \le i \le m)$. (Here using an orthonormal basis $\{e_1, \cdots, e_n\}$ of V, we identify $\mathrm{su}(V)$ with $\mathrm{su}(n)$. Also, using the Killing form, we identify $\mathrm{su}(n)^*$ with $\sqrt{-1}\,\mathrm{su}(n)$.)

Assume that $\mathcal{V} = (V_1, \ldots, V_m)$ is polystable with respect to the weighted ω. By Proposition 3.3, its orbit in the semistable locus is closed. Hence by, for example, Theorem 2.2.1 (1) of [4], there is an element $g \in \mathrm{SL}(V)$ such that $\Phi(g \cdot \mathcal{V}) = 0$. If g is the identity, this means that

$$\sum_i \omega_i A_i A_i^* = \wp_\omega(\mathcal{V}) I_n$$

which is equivalent to

$$\sum_{i=1}^m \omega_i \pi_{V_i} = \wp_\omega(\mathcal{V})\,\mathrm{Id}_V$$

because by a direct computation in linear algebra one can verify that the orthogonal projection π_{V_i} can be identified with the matrix $A_i A_i^*$ under the identification between V and $\mathbb{C}^n$ (using the orthonormal basis $\{e_1, \cdots, e_n\}$). That is, the standard hermitian metric h is a balance metric on V for the configuration. Similarly, when g is not the identity, the hermitian metric $gh(\bullet, \bullet) = h(g\bullet, g\bullet)$ is a balance metric on V for the configuration.

Conversely, if there is hermitian metric h' such that it is a balance metric for the configuration $\mathcal{V} = (V_1, \ldots, V_m)$, then by scaling, we may assume that h' and h have the same volume form. Hence there is $g \in \mathrm{SL}(V)$ such that $h' = gh$. This implies that

$$\Phi(g \cdot \mathcal{V}) = 0.$$

Hence (again by, for example, Theorem 2.2.1 (1) of [4]), the orbit through $g \cdot \mathcal{V}$ is closed in the semistable locus. Therefore by Proposition 3.3, $\mathcal{V}$ is polystable with respect to L_ω. $\square$

This theorem was previously known for the so-called m-filtrations with the trivial weights $\mathbb{I} = (1, \ldots, 1)$ and was proved by Klyachko ([18]) and Totaro ([28]).

3.3 Stability of tensor product

Of special interest is the so-called m-filtration. A filtration $V^\bullet$ is a weakly decreasing configuration of subspaces

$$V = V^0 \supset V^1 \supset \ldots \supset \{0\}.$$

By a m-filtration, we mean a collection $V^\bullet(s)$ of filtrations of V, for $1 \leq s \leq m$.

In [7] (cf. also [28]), Faltings and Wüstholz defined a stability for m-filtration. Their definition coincides with our definition when considering the m-filtration as a configuration of vector subspaces with trivial weights $\mathbb{I} = (1, \ldots, 1)$.

Conversely, if we treat each V_i as a (trivial) filtration $V \supset V^i \supset \{0\}$ and use splitting process, then any configuration of vector subspaces with weights ω can also be considered as a m-filtration with trivial weights $\mathbb{I}$, and again the two stabilities coincides.

Given two m-filtrations $V^\bullet(s)$ and $W^\bullet(s)$, $1 \leq s \leq m$. We define the tensor product $(V \otimes W)^\bullet(s)$ as

$$(V \otimes W)^l(s) = \sum_{p+q=l} V^p(s) \otimes W^q(s).$$

If $V^\bullet(s)$ and $W^\bullet(s)$ have attached weights ω and ω', then we will use the splitting and merging method to give $\{(V \otimes W)^l(s)\}$ the induced weights $\tilde{\omega}$.

Proposition 3.6. *If $V^\bullet(s)$ and $W^\bullet(s)$ are $\wp_\omega$-semistable and $\wp_{\omega'}$-semistable, respectively, then $(V \otimes W)^\bullet(s)$ is $\wp_{\tilde{\omega}}$-semistable.*

Proof. This proposition marginally generalizes Theorem 1 of [28]. It also follows from the proof of [28] using the splitting and merging method to relate weighted filtrations with unweighted (or trivially weighted) filtrations. $\square$

A different way to prove this may be done via calculating the moment map of $\mathrm{Gr}(pq, V \otimes W)$ using the moment map of $\mathrm{Gr}(p, V)$ and $\mathrm{Gr}(q, W)$.

4 Generalized Gelfand-MacPherson correspondence

4.1 Correspondence between orbits

Choosing a basis of V, we can identify V with $\mathbb{C}^n$. Then a k-dimensional vector subspace $E \subset V \cong \mathbb{C}^n$ can be represented by a full rank matrix M of size $n \times k$. The group $G = \mathrm{SL}(n, \mathbb{C})$ acts on M from the left. The group $G_k = \mathrm{SL}(k, \mathbb{C})$ acts on M from the right. Two such matrices represent the same vector subspace if and only if they are in the same orbit of G_k. Let $U^0_{n,k}$ be the space of all full rank matrices of size $n \times k$. Then $\mathrm{Gr}(k, \mathbb{C}^n)$ is the orbit space $U^0_{n,k}/G_k$.

Now assume that $n < k_1 + \ldots + k_m$. Given a configuration of vector subspaces

$$(V_1, \ldots, V_m) \in \mathrm{Gr}(k_1, n) \times \ldots \times \mathrm{Gr}(k_m, n),$$

let $(M_1, \ldots, M_m)$ be their corresponding (representative) matrices. Now, think of

$$M = (M_1, \ldots, M_m)$$

as a matrix of size $n \times (k_1 + \ldots + k_m)$ and let $U^0_{n,(k_1,\ldots,k_m)}$ be the space of matrices of size $n \times (k_1 + \ldots + k_m)$ such that M and each of its block matrix M_j is of full rank for $1 \le j \le m$.

There are two group actions on $U^0_{n,(k_1,\ldots,k_m)}$: one is the action of the group $G = \mathrm{SL}(n, \mathbb{C})$ from the left; the other is the action of the product group

$$G_{k_1,\ldots,k_m} = \mathrm{S}(\mathrm{GL}(k_1, \mathbb{C}) \times \ldots \times \mathrm{GL}(k_m, \mathbb{C})) \subset \mathrm{SL}(k_1 + \cdots + k_m, \mathbb{C})$$

with each factor acting on the corresponding block from the right. For simplicity, we sometimes use G' to denote $G_{k_1,\ldots,k_m}$. Quotienting $U^0_{n,(k_1,\ldots,k_m)}$ by the group $G_{k_1,\ldots,k_m}$, we obtain

$$X = \mathrm{Gr}(k_1, n) \times \ldots \times \mathrm{Gr}(k_m, n)$$

with the residual group $G = \mathrm{SL}(n, \mathbb{C})$ acting diagonally as usual; Quotienting $U^0_{n,(k_1,\ldots,k_m)}$ by the group $G = \mathrm{SL}(n, \mathbb{C})$, we obtain

$$Y = \mathrm{Gr}(n, k_1 + \ldots + k_m)$$

with the residual group $G' = G_{k_1,\ldots,k_m}$ acting block-wise.

It follows that

Proposition 4.1. *There is a bijection between G-orbits on X and G'-orbits on Y. Indeed, there is a homeomorphism between the (non-Hausdorff) orbit spaces X/G and Y/G'.*

When $k_1 = k_2 = \ldots = k_m = 1$, X is $(\mathbb{P}^{n-1})^m$ and $G_{1,\ldots,1}$ is a maximal torus of $\mathrm{SL}(m, \mathbb{C})$. In this case, the proposition is the Gelfand-MacPherson correspondence ([11]).

Proof. The correspondence exists because each set of orbits are in one-to-one correspondence with $G \times G'$-orbits on $U^0_{n,(k_1,\ldots,k_m)}$. □

4.2 Quotients in stages

From the previous section one naturally expects that the correspondence between orbits will induce a correspondence between the set of GIT quotients of

$$X = \mathrm{Gr}(k_1, n) \times \ldots \times \mathrm{Gr}(k_m, n)$$

by the group $G = \mathrm{SL}(n, \mathbb{C})$ and the set of GIT quotients of

$$Y = \mathrm{Gr}(n, k_1 + \ldots + k_m)$$

by the group $G' = G_{k_1,\ldots,k_m}$. The detail of this goes as follows.

First, recall that any G-linearized ample line bundle on $X = \mathrm{Gr}(k_1, n) \times \ldots \times \mathrm{Gr}(k_m, n)$ must be of the form L_ω for some weights ω. For the Grassmannian $Y = \mathrm{Gr}(n, k_1+\ldots+k_m)$, there is only one line bundle $L = \mathcal{O}_Y(1)$ up to homothety. But the character group of $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_m, \mathbb{C})$ can be identified with $\mathbb{Z}^m$. That is, each set ω of positive integers defines a character

$$\chi_\omega : \mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m) \to \mathbb{C}^*.$$

Let $L(\chi_\omega)$ be the ample line bundle $\mathcal{O}_Y(1)$ twisted by the character χ_ω. $L(\chi_\omega)$ is linearized for $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_m, \mathbb{C})$ and hence for its subgroup $G' = \mathrm{S}(\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_m, \mathbb{C}))$.

Theorem 4.2. *There is a one-to-one correspondence between the set of GIT quotients of* $X = \mathrm{Gr}(k_1, n) \times \ldots \times \mathrm{Gr}(k_m, n)$ *by the group* $G = \mathrm{SL}(n, \mathbb{C})$ *and the set of GIT quotients of* $Y = \mathrm{Gr}(n, k_1 + \ldots + k_m)$ *by the group* $G' = G_{k_1,\ldots,k_m}$. *More precisely, for any sequence* ω *of positive integers, we have a natural isomorphism between* $X^{ss}(L_\omega)//G$ *and* $Y^{ss}(L(\chi_\omega))//G'$

When $k_1 = \ldots k_m = 1$, the theorem was previously proved by Kapranov using the standard Gelfand-MacPherson correspondence. Here we reproduce his proof in the general case.

Proof. First, recall that the coordinate ring of $\mathrm{Gr}(k, \mathbb{C}^n)$ in the Plüker embedding can be identified with the ring of polynomials f in matrices M of size $n \times k$ such that $f(M \cdot g) = f(M)$ for all $g \in \mathrm{SL}(k, \mathbb{C})$. In particular, we have that the section space $\Gamma(\mathrm{Gr}(k, \mathbb{C}^n), \mathcal{O}_{\mathrm{Gr}(k,\mathbb{C}^n)}(d))$ can be identified with

$$\{f(M) | f(tM) = t^d f(M), f(M \cdot g) = f(M), g \in \mathrm{SL}(k, \mathbb{C})\}$$

for all integers $d > 0$.

Now using the group $\mathrm{GL}(n, \mathbb{C})$ in place of $\mathrm{SL}(k, \mathbb{C})$, the above has an equivalent but more concise expression as follows. Recall that the character group of $\mathrm{GL}(k, \mathbb{C})$ can be naturally identified with the group of integers $\mathbb{Z}$. For any integer $d > 0$, let

$$\chi_d : \mathrm{GL}(k, \mathbb{C}) \to \mathbb{C}^*$$

be the corresponding character of $\mathrm{GL}(k, \mathbb{C})$. Then we have

$$\Gamma(\mathrm{Gr}(k, \mathbb{C}^n), \mathcal{O}_{\mathrm{Gr}(k,\mathbb{C}^n)}(d)) = \{f | f(M \cdot g) = \chi_d(g) f(M), g \in \mathrm{GL}(k, \mathbb{C})\}.$$

This is because the two identities:

$$f(tM) = t^d f(M) \ \text{ and } \ f(M \cdot g) = f(M), g \in \mathrm{SL}(k, \mathbb{C})$$

can be combined together in the single identity

$$f(M \cdot g) = \chi_d(g) f(M), g \in \mathrm{GL}(k, \mathbb{C}).$$

From the above and considering the ring of polynomials in matrices M of size $n \times (k_1 + \ldots + k_m)$ one checks that

$$\Gamma(X, L_\omega^d) = \{f(M) | f(M \cdot g) = \chi_{d\omega}(g) f(M), g \in \mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m)\}$$

and
$$\Gamma(Y, L^d(\chi_\omega)) = \{f(M) | f(g' \cdot M) = \chi_d(g') f(M), g' \in \mathrm{GL}(n, \mathbb{C})\}.$$

Therefore by taking the projective spectrum of the invariants of
$$A = \oplus_d \Gamma(X, L_\omega^d)$$
under the action of the group $\mathrm{GL}(n, \mathbb{C})$ and by taking the projective spectrum of the invariants of
$$B = \oplus_d \Gamma(Y, L^d(\chi_\omega))$$
under the action of the group
$$\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m),$$
we see that the both quotients
$$X^{ss}(L_\omega)//G \text{ and } Y^{ss}(L(\chi_\omega))//G'$$
can be naturally identified with the projective spectrum of the ring
$$R = \oplus_d R_d$$
where
$$R_d = \{f(M) | f(M \cdot g) = \chi_{d\omega}(g) f(M), f(g' \cdot M) = f(M)\}$$
for all $g \in \mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m), g' \in \mathrm{SL}(n, \mathbb{C})$. (Note that here we take $g' \in \mathrm{SL}(n, \mathbb{C})$ instead of $g' \in \mathrm{GL}(n, \mathbb{C})$. This is because the effect of the central part of $\mathrm{GL}(n, \mathbb{C})$ is already reflected by the scalar matrices in $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m)$.)

This has established the desired correspondence. □

5 The Cone of effective linearizations

As the stability depends on ω, so does the moduli. In this section, we study the G-ample cone to pave a way for the study of the variation of the moduli. In particular, we will introduce a family of new polytopes: *diagonal hypersimpleces*.

5.1 Effective linearizations

Given a linearized line bundle L over X, it is called G-effective if $X^{ss}(L) \neq \emptyset$. Not all of L_ω are G-effective. The following should characterize the effective ample ones.

We will always assume that the group $G = \mathrm{SL}(V)$ acts freely on generic configuration of linear subspaces, that is, G acts freely on an open subset of generic points in $\Pi_{i=1}^m \mathrm{Gr}(k_i, V)$. This should be true when $n < k_1 + \cdots + k_m$ and $n^2 \leq \sum_i k_i(n - k_i)$.

Conjecture 5.1. *Under the above (and perhaps some additional natural) conditions, we have*

1. $X^{ss}(L_\omega) \neq \emptyset$ *if and only if* $\omega_i \leq \frac{1}{n}\sum_i k_i\omega_i$ *for all* $1 \leq i \leq m$ *if and only if* $max\{\omega_i\}_i \leq \frac{1}{n}\sum_i k_i\omega_i$;
2. $X^s(L_\omega) \neq \emptyset$ *if and only if* $\omega_i < \frac{1}{n}\sum_i k_i\omega_i$ *for all* $1 \leq i \leq m$ *if and only if* $max\{\omega_i\}_i < \frac{1}{n}\sum_i k_i\omega_i$.

The necessary parts of both (1) and (2) are true.

Proof. (1). The necessary direction is easy. Assume that $X^{ss}(L_\omega) \neq \emptyset$ and let $\mathcal{V} = \{V_i\} \in X^{ss}(L_\omega)$. We have that for all $W \subset V$,

$$\frac{1}{\dim W}\sum_j \omega_j \dim(V_j \cap W) \leq \frac{1}{n}\sum_i k_i\omega_i.$$

Now for any given i, take $W = V_i$, then we obtain

$$\omega_i \leq \frac{1}{\dim W}\sum_j \omega_j \dim(V_j \cap W) \leq \frac{1}{n}\sum_i k_i\omega_i$$

for all i.

The necessary part of (2) can be proved similarly. □

Equivalently,

Conjecture 5.2. *1.* $Y^{ss}(L(\chi_\omega)) \neq \emptyset$ *if and only if* $\omega_i \leq \frac{1}{n}\sum_i k_i\omega_i$ *for all* $1 \leq i \leq m$ *if and only if* $max\{\omega_i\}_i \leq \frac{1}{n}\sum_i k_i\omega_i$;

2. *$Y^s(L(\chi_\omega)) \neq \emptyset$ if and only if $\omega_i < \frac{1}{n}\sum_i k_i\omega_i$ for all $1 \leq i \leq m$ if and only if $max\{\omega_i\}_i < \frac{1}{n}\sum_i k_i\omega_i$.*

The necessary parts of both (1) and (2) are true.

5.2 Diagonal hypersimplex and G-ample cone

The previous conjectures lead to the discovery of the following polytope. Setting $x_i = n\omega_i / \sum_i k_i\omega_i$, then x_i satisfy $0 \leq x_i \leq 1$ and $\sum_i k_i x_i = n$. Hence we introduce the polytope

$$\Delta^m_{n,\{k_i\}} = \{(x_1, \ldots, x_m) | 0 \leq x_i \leq 1, \sum_i k_i x_i = n.\}.$$

Recall the standard hypersimplex Δ^m_n is defined as

$$\Delta^m_n = \{(x_1, \ldots, x_m) | 0 \leq x_i \leq 1, \sum_i x_i = n.\}.$$

Thus $\Delta^m_{n,\{k_i\}}$ is a subpolytope of $\Delta^{k_1+\cdots+k_m}_n$. In fact, let $D_{k_1,\ldots,k_m}$ be the diagonal subspace of $\mathbb{R}^{k_1+\cdots+k_m}$ such that the first k_1 coordinates coincide, the next k_2 coordinate coincide, and so on, then

$$\Delta^m_{n,\{k_i\}} = \Delta^{k_1+\cdots+k_m}_n \cap D_{k_1,\ldots,k_m}.$$

Clearly $\Delta^m_{n,\{k_i\}}$ is the hypersimplex Δ^m_n when all k_i are equal to 1. Hence, it seems reasonable to call $\Delta^m_{n,\{k_i\}}$ a *diagonal hypersimplex* or simply a *generalized hypersimplex*.

Let $G = \mathrm{SL}(V)$ and $G' = S(\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_m, \mathbb{C}))$. Let also $C^G(X)$ and $C^{G'}(Y)$ be the G-ample cone of X and G'-ample cone of Y, respectively. (For the definition and properties of a general G-ample cone, see Definition 3.2.1 and §3 of [4].)

Then, as a corollary of either of the above conjectures, we have

Conjecture 5.3. *Both $C^G(X)$ and $C^{G'}(Y)$ can be naturally identified with the positive cone over the generalized hypersimplex $\Delta^m_{n,\{k_i\}}$.*

5.3 Walls and Chambers

In §3 of [4], a natural *wall and chamber structure* in $C^G(X)$ is introduced. However it can happen that there are no (top) chambers at all in $C^G(X)$. Not many examples of this type are previously known. Here we produce an interesting one.

Consider the product of m-copies of $\mathrm{Gr}(2, \mathbb{C}^4)$,

$$X = \Pi_{i=1}^m \mathrm{Gr}(2, \mathbb{C}^4).$$

Proposition 5.4. *All the above conjectures are true for* $\Pi_{i=1}^m \mathrm{Gr}(2, \mathbb{C}^4)$.

Proof. We only need to prove it for Conjecture 5.1, the rest follow from this. Take any configuration $\{V_i\} \in \Pi_{i=1}^m \mathrm{Gr}(2, \mathbb{C}^4)$ such that $V_i \cap V_j = \{0\}$ $(i \neq j)$. We will check that $\{V_i\}$ is $\wp_\omega$-semistable. First note that $\frac{1}{\dim V} \sum_i \omega_i \dim V_i = \frac{1}{2} \sum_i \omega_i$. Let F be an arbitrary proper subspace of V. We examine it case by case.

$\dim F = 1$. F can intersect non-trivially (i.e., be contained in) only one V_i. Hence we have

$$\frac{1}{\dim F} \sum_i \omega_i \dim(F \cap V_i) \leq \omega_i \leq \frac{1}{2} \sum_i \omega_i.$$

$\dim F = 2$. If $\dim F \cap V_i = 2$, then $F = V_i$, hence

$$\frac{1}{\dim F} \sum_i \omega_i \dim(F \cap V_i) = \omega_i \leq \frac{1}{2} \sum_i \omega_i.$$

Otherwise, $\dim F \cap V_i \leq 1$ for all i, hence

$$\frac{1}{\dim F} \sum_i \omega_i \dim(F \cap V_i) \leq \frac{1}{2} \sum_i \omega_i.$$

$\dim F = 3$. If $\dim F \cap V_i \leq 1$ for all i, then the stability condition is trivially true. Otherwise, $\dim F \cap V_i = 2$ can only be true for only one i. In this case,

$$\begin{aligned} \frac{1}{\dim F} \sum_i \omega_i \dim(F \cap V_i) &\leq \frac{1}{3}(2\omega_i + \sum_{j \neq i} \omega_j) \\ &= \frac{1}{3}(\omega_i + \sum_j \omega_j) \leq \frac{1}{2} \sum_j \omega_j. \end{aligned}$$

□

An equivalent version of the following proposition already appeared in Foth-Lozano's paper [8] in terms of polygons.

Proposition 5.5. *([8]) For every weight set $\omega \in \Delta^m_{4,\{2,\cdots,2\}}$,*

$$X^{ss}(L_\omega) \setminus X^s(L_\omega) \neq \emptyset.$$

In particular, there is not any (top) chamber in the G-ample cone.

Proof. Let F be a 2-dimensional subspace. Take a configuration $\{V_i\} \in \Pi_{i=1}^m \mathrm{Gr}(2,\mathbb{C}^4)$ such that $V_i \cap V_j = \{0\}$ $(i \neq j)$ and $\dim V_i \cap F = 1$ for all i. Then $\{V_i\}$ is semistable for all $\omega \in \Delta^m_{4,\{2,\cdots,2\}}$ by the proof of the previous proposition. Since

$$\frac{1}{\dim F}\sum_i \omega_i \dim(F \cap V_i) = \frac{1}{2}\sum_i \omega_i = \frac{1}{\dim V}\sum_i \omega_i \dim V_i,$$

$\{V_i\} \in X^{ss}(L_\omega) \setminus X^s(L_\omega)$ for all

$$\omega \in \Delta^m_{4,\{2,\cdots,2\}}.$$

□

Remark 5.6. Finally, note that

$$\Delta^m_{4,\{2,\cdots,2\}} = \{(x_1,\ldots,x_m)|0 \leq x_i \leq 1, \sum_{i=1}^m 2x_i = 4\}$$

$$= \{(x_1,\ldots,x_m)|0 \leq x_i \leq 1, \sum_{i=1}^m x_i = 2\}$$

which is just the standard hypersimplex Δ^m_2. Recall we just showed that $\Delta^m_{4,\{2,\cdots,2\}}$ has no chambers as $\mathrm{SL}(4,\mathbb{C})$-ample cone of $\mathrm{Gr}(2,\mathbb{C}^4)^m$. However, Δ^m_2, as $\mathrm{SL}(2,\mathbb{C})$-ample cone of $(\mathbb{P}^1)^m$ has natural wall and chamber structure. It would be interesting to investigate in detail the implications of the above on the problem of variation of GIT quotients by the two distinct, yet related actions. Likewise, one should also study the implication of the identity

$$\Delta^m_{kn,\{n,\ldots,n\}} = \Delta^m_k.$$

6 Stable Configuration of Coherent Sheaves

6.1 Quot scheme and Grothendieck embedding

Let X be a projective scheme over $\mathbb{C}$ (possibly singular) with a very ample invertible sheaf $\mathcal{O}(1)$. The Hilbert polynomial $p(\mathcal{E}, k) = \chi(\mathcal{E}(k))$ is uniquely defined by the condition that

$$p(\mathcal{E}, k) = \dim H^0(X, \mathcal{E}(k)), \quad \text{for} \ \ k >> 1.$$

Let $d = d(\mathcal{E})$ denote the dimension of the support of $\mathcal{E}$. It is equal to the degree of $p(\mathcal{E}, k)$. So,

$$p(\mathcal{E}, k) = \frac{r}{d!}k^d + \frac{a}{(d-1)!}k^{d-1} + \cdots .$$

Here r is the rank of $\mathcal{E}$ and a/r is defined to be the slope of $\mathcal{E}$. We say $\mathcal{E}$ is of pure dimension if for any $0 \neq \mathcal{F} \subset \mathcal{E}$, we have $d(\mathcal{F}) = d(\mathcal{E})$.

Fix a vector space V and a coherent sheaf $\mathcal{W}$ over X. Also fix a (Hilbert) polynomial P. We will consider the Quot scheme

$$\mathrm{Quot}(V \otimes \mathcal{W}, P),$$

parameterizing the coherent quotient sheaves

$$V \otimes \mathcal{W} \to \mathcal{E} \to 0$$

such that $p(\mathcal{E}, k) = P(k)$.

For $k >> 1$, Grothendieck proves that there is an explicit embedding $\mathrm{Quot}(V \otimes \mathcal{W}, P) \to \mathrm{Gr}(V \otimes W, P(k))$ where $W = H^0(\mathcal{W}(k))$. Indeed, let $\mathcal{U}$ be the universal quotient sheaf over

$$\mathrm{Quot}(V \otimes \mathcal{W}, P) \times X,$$

and $\mathcal{L}(k) = \mathrm{Det}(p_*(\mathcal{U} \otimes q^*\mathcal{O}_X(k))$ be the determinant line bundle over $\mathrm{Quot}(V \otimes \mathcal{W}, P)$ where p and q are the natural projections

$$\begin{array}{ccc} \mathrm{Quot}(V \otimes \mathcal{W}, P) \times X & \xrightarrow{q} & X \\ {\scriptstyle p}\big\downarrow & & \\ \mathrm{Quot}(V \otimes \mathcal{W}, P) & & \end{array}$$

Then this is very ample for $k >> 1$ and is the same as the ample line bundle induced from the embedding into the Grassmaniann (see 1.32 of [29]).

6.2 Stability of configurations of coherent sheaves

Consider a configuration of coherent quotient sheaves

$$\{V \otimes \mathcal{W} \to \mathcal{E}_i \to 0\}_i$$

with $p(\mathcal{E}_i, k) = P_i(k)$ where P_i are some fixed Hilbert polynomials. Let $L_{k,i}$ be the linearized ample line bundle on $\mathrm{Quot}(V \otimes \mathcal{W}, P_i)$ induced from the embedding

$$\mathrm{Quot}(V \otimes \mathcal{W}, P_i) \to \mathrm{Gr}(V \otimes W, P_i(k))$$

for sufficiently large k (we choose k so large that it works for all i). For a given set of positive integers $\omega = \{\omega_1, \ldots, \omega_m\}$, let $L_{k,\omega}$ be the linearization on

$$\Pi_i \, \mathrm{Quot}(V \otimes \mathcal{W}, P_i) \subset \Pi_i \, \mathrm{Gr}(V \otimes W, P_i(k))$$

defined by

$$L_{k,\omega} = \otimes_i L_{k,i}^{\omega_i}.$$

We need a simple lemma

Lemma 6.1. *(Theorem 1.19, [21]). Let $i : Z \to \widetilde{Z}$ be a G-invariant closed embedding from a scheme Z to a scheme $\widetilde{Z}$ and L a linearized ample line bundle over $\widetilde{Z}$. Then $Z^{ss}(i^*L) = i^{-1}(\widetilde{Z}^{ss}(L))$ and $Z^s = i^{-1}(\widetilde{Z}^s)$.*

This lemma when applied to the Grothendiek embedding will allow us to work directly on the Grassmannian instead of the Quot scheme.

Theorem 6.2. *There is an integer M such that for $k \geq M$, the following holds. Suppose that $\{V \otimes \mathcal{W} \xrightarrow{f_i} \mathcal{E}_i \to 0\}$ is a point in*

$$\Pi_i \, \mathrm{Quot}(V \otimes \mathcal{W}, P_i),$$

and for any subspace $H \subset V$, let $\mathcal{F}_i$ denote the subsheaf of $\mathcal{E}_i$ generated by $H \otimes \mathcal{W}$. Then $\{\mathcal{E}_i\}$ is semistable (resp. stable) with respect to the $\mathrm{SL}(V)$-linearization $L_{k,\omega}$ if and only if

$$\frac{1}{\dim V} \sum_i \omega_i \chi(\mathcal{E}_i(k)) \leq \frac{1}{\dim H} \sum_i \omega_i \chi(\mathcal{F}_i(k))$$

(resp. $<$). In particular, $\chi(\mathcal{F}_i(k)) > 0$ for some i.

Proof. For $k >> 1$, we have the product of the Grothendieck embeddings

$$\Pi_i \operatorname{Quot}(V \otimes \mathcal{W}, P_i) \to \Pi_i \operatorname{Gr}(V \otimes W, P_i(k))$$

where $W = H^0(\mathcal{W}(k))$. Consider the sequences

$$\{H \otimes \mathcal{W} \stackrel{f_i}{\to} \mathcal{F}_i \to 0\}.$$

Let $\mathcal{K}_i$ be the kernal of f_i. Since all such H runs over a bounded family, so does $\mathcal{F}_i$. Hence $\mathcal{K}_i$ also runs over a bounded family. In particular we may choose M large enough so that when $k \geq M$, $\chi(\mathcal{F}_i(k)) = h^0(\mathcal{F}_i(k))$ and $h^1(\mathcal{K}_i(k)) = 0$ for all such $\mathcal{F}_i$ and $\mathcal{K}_i$. Twist the exact sequence

$$0 \to \mathcal{K}_i \to H \otimes \mathcal{W} \stackrel{f_i}{\to} \mathcal{F}_i \to 0$$

by $\mathcal{O}_X(k)$ and take the long exact sequence of cohomology, we get an exact sequence

$$H \otimes W \stackrel{f_i}{\to} H^0(\mathcal{F}_i(k)) \to H^1(\mathcal{K}_i(k)).$$

The third term vanishes so that this gives

$$\dim f_i(H \otimes W) = \chi(\mathcal{F}_i(k)).$$

Now Theorem 2.2' can be applied to the configuration

$$\{0 \to V \otimes W \stackrel{f_i}{\to} H^0(\mathcal{E}_i(k)) \to 0\}$$

to conclude the proof. □

6.3 Moduli of Semistable Configuration of Coherent Sheaves

Let $\mathfrak{M}_P$ be the moduli space of semistable coherent sheaves over X with the Hilbert polynomial P.

Fix a set of positive numbers $\omega = \{\omega_1, \ldots, \omega_m\}$ and (Hilbert) polynomials $\mathbf{P} = \{P_1, \ldots, P_m\}$. Let $\mathfrak{M}_{\mathbf{P},\omega}$ be the moduli space of semistable configurations of coherent sheaves over X with the Hilbert polynomial P_i and with respect to the weight $\omega = \{\omega_1, \ldots, \omega_m\}$.

Proposition 6.3. *Fix an integer $1 \leq i \leq m$. For sufficient large ω_i (relative to other ω_j), we have*

1. *If a configuration* $\{V \otimes \mathcal{W} \to \mathcal{E}_j \to 0\}$ *of coherent sheaves is* ω*-semistable, then its* i*-th component* $V \otimes \mathcal{W} \to \mathcal{E}_i \to 0$ *is a semistable sheaf;*
2. *If the* i*-th component* $V \otimes \mathcal{W} \to \mathcal{E}_i \to 0$ *of a configuration* $\{V \otimes \mathcal{W} \to \mathcal{E}_j \to 0\}$ *is stable, then the configuration* $\{V \otimes \mathcal{W} \to \mathcal{E}_j \to 0\}$ *is* ω*-stable;*
3. *In particular, there is a porjective morphism from* $\mathfrak{M}_{\mathbf{P},\omega}$ *to* $\mathfrak{M}_P$

$$\pi_i : \mathfrak{M}_{\mathbf{P},\omega} \to \mathfrak{M}_P.$$

Proof. The proof is completely similar to that of Proposition 2.3, thus is omitted. □

Recall that the stability of a coherent sheaf $\mathcal{E}$ is defined as follows. $\mathcal{E}$ is semi stable (resp. stable) if for every proper subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have that

$$\frac{\chi(\mathcal{F}(k))}{\mathrm{rk}(\mathcal{F})} \le \frac{\chi(\mathcal{E}(k))}{\mathrm{rk}(\mathcal{E})}$$

(resp. $<$) for sufficiently large k (e.g., [9], [12], [16], [24]). It would be nice to also have an intrinsic stability criterion (definition) for configurations of coherent sheaves without using Grothendieck's Grassmannian embeddings. Other directions of further research include: to study the properties of the moduli (cf., e.g., [13] and [19]), and to study the dependence of the moduli on the parameters (cf., e.g., [10] and [23], among others).

7 Balanced Configuration and Moment Map

7.1 Quot scheme and $\mathrm{Hom}(X, \mathrm{Gr})$

After tensoring coherent sheaves by $\mathcal{O}_X(N)$ for large enough N, we may assume that they are generated by global sections, hence regard them as quotient sheaves of the trivial sheaf $V = \mathbb{C}^N \times X$

$$V \xrightarrow{f_i} U_i \to 0.$$

We will focus on vector bundles only. This allows us to switch the viewpoint and consider vector subbundles of $\mathbb{C}^N \times X$ instead of quotient bundles. So, let

$$\mathcal{E}_i \subset \mathbb{C}^N \times X$$

be a configuration of vector subbundles of rank r_i over X with the Hilbert polynomial P_i $(1 \leq i \leq m)$.

Each $\mathcal{E}_i$ corresponds to a map

$$g_i : X \to \mathrm{Gr}(r_i, \mathbb{C}^N)$$

where g_i sends $x \in X$ to the fiber $(\mathcal{E}_i)_x \subset \mathbb{C}^N$. Conversely, every morphism

$$g : X \to \mathrm{Gr}(r, \mathbb{C}^N)$$

defines a vector subbundle by pulling back the universal bundle

$$\mathcal{E} = \{(v, x) \in \mathbb{C}^N \times X | v \in g(x)\}.$$

Let

$$\mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N); P_i)$$

be the set of morphisms that correspond to vector subbundles of Hilbert polynomial P_i. Then we have an embedding

$$j : \Pi_{i=1}^m \mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N); P_i) \to \Pi_{i=1}^m \mathrm{Quot}(V, P_i).$$

We will use the pull-back bundle $j^* L_\omega$ as the linearization on

$$\Pi_{i=1}^m \mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N); P_i)$$

where L_ω is $L_{1,\omega}$ as defined in §7.2. Intrinsically, this bundle admits a description similar to $L_{m,\omega}$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N)) \times X & \xrightarrow{ev} & \mathrm{Gr} \\ \pi \downarrow & & \\ \mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N)) & & \end{array}$$

Let $\mathcal{U}_i$ be the universal vector bundle over Gr. Then

$$L_i = \mathrm{Det}(\pi_*(ev_*(\mathcal{U}_i \otimes \mathcal{O}_X(1)))$$

is very ample. For a weight set ω, the tensor product $\otimes_i L_i^{\omega_i}$ of these line bundles on $\Pi_{i=1}^m \mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N); P_i)$ is $j^* L_\omega$.

By Lemma 6.1, a configuration $\{\mathcal{E}_i\}$ of vector subbundles of $\mathbb{C}^N \times X$ is (semi) stable with respect to L_ω if and only if the corresponding configuration of morphisms $g_i : X \to \mathrm{Gr}(r_i, \mathbb{C}^N)$ is (semi) stable with respect to $j^* L_\omega$.

7.2 Moment map for singular varieties

Let Z be any (possibly) singular variety acted upon by a compact group K. Let Ω be a bilinear skew-symmetric form on the Zariski tangent space TZ which restricts to a symplectic form on Z^0, the smooth locus of Z. A continuous equivariant map

$$\Phi : Z \to \mathfrak{k}^*$$

is called a moment map if the resrtiction

$$\Phi_{Z^0} : Z^0 \to \mathfrak{k}^*$$

is a moment map (in the usual sense) for the action of K on Z^0. That is, at a smooth point of Z, we have

$$d\langle \Phi, a\rangle = i_{\xi_a}\omega$$

for every $a \in \mathfrak{k}$ where ξ_a is the vector field generated by a. By continuity, the moment map $\Phi : Z \to \mathfrak{k}^*$ is uniquely determined by the moment map $\Phi_{Z^0} : Z^0 \to \mathfrak{k}^*$. Note also that the moment map, when exists, is unique if the group G is semisimple. For linear actions on projective varieties, a moment map always exists.

If in addition, Z can be equivariantly embedded in a smooth ambient variety $\widetilde{Z}$, then the restriction of a moment map

$$\widetilde{\Phi} : \widetilde{Z} \to \mathfrak{k}^*$$

to Z will be a moment map for the K-action on Z. This situation is the case we will be interested. That is, we will consider the equivariant embeddings of the Quot schemes in the Grassmannians. Lemma 6.1 will allow us to apply some results in the smooth case to the singular case.

7.3 Moment map for $\Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$

Now consider the space

$$\Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i).$$

$\mathrm{SL}(N)$ acts on it diagonally by moving the images. We assume that $\Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$ is generically smooth (hence every component $\operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$ is also generically smooth). The line bundle j^*L_ω induces a symplectic form Ω on the smooth locus of $\Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$ as follows. At any given point $f : X \hookrightarrow \operatorname{Gr}(r_i, \mathbb{C}^N)$, the tangent space $T_f \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N))$ is $H^0(X, f^*T \operatorname{Gr}(r_i, \mathbb{C}^N))$. We can define a skew-symmetric bilinear form Ω_i on $H^0(X, f^*T \operatorname{Gr}(r_i, \mathbb{C}^N))$ by setting

$$(\Omega_i)_f(u, v) = \int_X f^*(\omega_i)_{FS}(u, v)dV$$

where $u, v \in H^0(X, f^*T \operatorname{Gr}(r_i, \mathbb{C}^N))$ and $(\omega_i)_{FS}$ is the symplectic form induced from the Fubini-Study Kähler form on $\operatorname{Gr}(r_i, \mathbb{C}^N)$. The form Ω_i restricts to a symplectic form on the smooth locus of $\operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$. Then the form on $\Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$ is

$$\Omega = \sum_{i=1}^m \omega_i \Omega_i.$$

Let $\operatorname{Vol}(X)$ be the volume of X and I denote the identity matrix in $\operatorname{su}(N)$. Then we have

Proposition 7.1. *Under the symplectic form Ω, the moment map Φ of the action of $\operatorname{SU}(N)$ on $\Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$ is given by*

$$\Phi(\{g_i\}) = \sum_{i=1}^m \omega_i \int_X A_i(x) A_i^*(x) dV - \wp_\omega(\{g_i\}) \operatorname{Vol}(X) I$$

where $\{g_i\} \in \Pi_{i=1}^m \operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$, $A_i(x)$ is a matrix representation of $g_i(x) \subset \mathbb{C}^N$ whose columns is an orthonormal basis for $g_i(x)$ $(1 \le i \le m)$, and $\wp_\omega(\{g_i\}) = \sum_i \omega_i \frac{r_i}{N}$.

Proof. One first checks that for any given i the moment map Φ_i of the action of $\operatorname{SU}(N)$ on $\operatorname{Hom}(X, \operatorname{Gr}(r_i, \mathbb{C}^N); P_i)$ is the integration over X of the moment map ϕ_i of the action of $\operatorname{SU}(N)$ on the Grassmannian $\operatorname{Gr}(r_i, \mathbb{C}^N)$. For any $a \in \operatorname{su}(N)$, it generates a vector field ξ_a on $\operatorname{Gr}(r_i, \mathbb{C}^N)$. At any smooth point $f \in \operatorname{Hom}(X, \operatorname{Gr}(r, \mathbb{C}^N); P)$, we have

$$\begin{aligned} i_{\xi_a}(\Omega_i)_f &= \int_X f^* i_{\xi_a}(\omega_i)_{FS} dV \\ &= \int_X f^* \langle d\phi_i, a\rangle \wedge dV \\ &= \pi_*(ev^*(d\langle \phi_i, a\rangle) \wedge dV) \\ &= d\langle \pi_*(ev^*\phi_i dV), a\rangle \end{aligned}$$

$$= d\langle \int_X \phi_i dV, a\rangle.$$

This implies that $\Phi_i = \int_X \phi_i dV$.

Therefore the moment map Φ of the action of $\mathrm{SU}(N)$ on

$$\Pi_{i=1}^m \mathrm{Hom}(X, \mathrm{Gr}(r_i, \mathbb{C}^N); P_i)$$

is the same as the integration over X of the moment map Φ_0 of the diagoanl action of $\mathrm{SU}(N)$ on the product of the Grassmannians. Since

$$\Phi_0(\{g_i(x)\}) = \sum_i \omega_i (A_i(x)A_i^*(x) - \frac{r_i}{N} I),$$

we have

$$\Phi(\{g_i\}) = \int_X \Phi_0 d\,\mathrm{Vol} = \sum_i \int_X \omega_i (A_i(x)A_i^*(x) - \frac{r_i}{N} I) d\,\mathrm{Vol}\,.$$

That is

$$\Phi(\{g_i\}) = \sum_i \omega_i \int_X A_i(x)A_i^*(x) dV - \wp_\omega(\{g_i\})\,\mathrm{Vol}(X) I.$$

□

7.4 Balanced Configuration and Stability

Definition 7.2. Let $\{g_i : X \to \mathrm{Gr}(r_i, \mathbb{C}^N)\}$ be a configuration of morphism into the Grassmannians. We say that the configuration $\{g_i\}$ is balanced if

$$\sum_{i=1}^m \omega_i \int_X A_i(x)A_i^*(x) dV = \wp_\omega(\{g_i\})\,\mathrm{Vol}(X) I.$$

We say $\{g_i\}$ can be (uniquely) balanced if there is a (unique) element $u \in \mathrm{SU}(N)\backslash SL(N)$ such that $\{u \cdot g_i\}$ is balanced.

The following theorem follows from Proposition 7.1 and Lemma 6.1.

Theorem 7.3. *$\{g_i\}$ is stable if and only if $\{g_i\}$ can be (uniquely) balanced and its stabilizer group is finite.*

Definition 7.4. Let $\{\mathcal{E}_i\}$ be a configuration of vector subbundles in $\Pi_{i=1}^m \operatorname{Quot}(V, P_i)$ where V is the trivial vector bundle $\mathbb{C}^N \times X$. We say the system $\{\mathcal{E}_i\}$ is balanced if

$$\sum_{i=1}^m \omega_i \int_X A_i(x) A_i^*(x) dV = \wp_\omega(\{\mathcal{E}_i\}) \operatorname{Vol}(X) I$$

where $A_i(x)$ is a matrix representation of $(\mathcal{E}_i)_x \subset \mathbb{C}^N$ whose columns is an orthonormal basis for $(\mathcal{E}_i)_x$ $(1 \le i \le m)$, and $\wp_\omega(\{\mathcal{E}_i\}) = \sum_i \omega_i \frac{r_i}{N}$. . We say $\{\mathcal{E}_i\}$ can be (uniquely) balanced if there is a (unique) element $u \in \mathrm{SU}(N) \backslash \mathrm{SL}(N)$ such that $\{u \cdot \mathcal{E}_i\}$ is balanced.

As a consequence of Theorem 7.3, we obtain

Theorem 7.5. *Let $\{\mathcal{E}_i\}$ be a configuration of vector subbundles in $\Pi_{i=1}^m \operatorname{Quot}(V, P_i)$. Then $\{\mathcal{E}_i\}$ is stable if and only if $\{\mathcal{E}_i\}$ can be (uniquely) balanced and its stabilizer group is finite.*

When $m > 1$, the condition that "the stabilizer group of the configuration $\{\mathcal{E}_i\}$ is finite" is a quite weak condition. For example, it will be the case when $\cap_i \operatorname{Stab}(\mathcal{E}_i)$ is finite where $\operatorname{Stab}(\mathcal{E}_i)$ is the stabilizer group of $\mathcal{E}_i$ $(1 \le i \le m)$.

Finally, consider the case when $m = 1$. Let $\mathcal{E}$ be a vector subbundle in $\mathbb{C}^N \times X$. Then we obtain a result of Wang ([30]) and Phong-Sturm ([22])

Theorem 7.6. *$\mathcal{E}$ is Gieseker-Simpson stable if and only if it can be (uniquely) balanced and its automorphism group is finite.*

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