The Projective Tangent Bundles of a Complex Three-Fold

Friedrich Hirzebruch

Dedicated to my friend Armand Borel
in memory of the golden fifties in Princeton

§1. Introduction and statement of results.

When I visited the Institute for Advanced Study for its 75th anniversary in March 2005, I lectured about my joint work with Armand in the fifties [BH] and reported in particular about an example we studied in §13.9 and §24.11. This concerns the 10-dimensional flag manifold

\[X = \mathbb{U}(4)/\mathbb{U}(2) \times \mathbb{U}(1) \times \mathbb{U}(1).\]

The points of \(X\) are the ordered triples of one 2-dimensional and two 1-dimensional linear subspaces of the standard hermitian space \(\mathbb{C}^4\) which are pairwise hermitian orthogonal. The manifold \(X\) carries two homogenous complex structures, namely the 5-dimensional complex manifold \(X_1\) consisting of the flags in \(\mathbb{C}^4\) of type \((0) \subset (1) \subset (3) \subset (4)\), i.e. the origin contained in a one-dimensional linear subspace, contained in a three-dimensional linear subspace, contained in \(\mathbb{C}^4\), and the complex manifold \(X_2\) consisting of flags of type \((0) \subset (1) \subset (2) \subset (4)\). Borel and I prove for the first Chern class \(c_1\) and the Chern number \(c_1^5\)

\[\begin{align*}
(1) & \quad c_1(X_1) \text{ is divisible by } 3 \\
(2) & \quad c_1(X_2) \text{ is not divisible by } 3 \\
& \quad c_1^5[X_1] = 3^5 \cdot 20 = 4860 \\
& \quad c_1^5[X_2] = 4500
\end{align*}\]

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Borel and I did this as an example to our general method to study characteristic classes of homogeneous spaces by using the root systems of Lie groups. As written in [BH], §24.11, "... we get an example of two 5-dimensional algebraic varieties which are $C^\infty$-differentially homeomorphic, but have different Chern numbers".

E. Calabi attended my Princeton talk and mentioned to me afterwards that $X_1$ is the projective covariant tangent bundle of the projective space $\mathbb{P}_3(\mathbb{C})$, and $X_2$ is the projective contravariant tangent bundle of $\mathbb{P}_3(\mathbb{C})$. Indeed, the points of the projective covariant bundle consist of a point $x$ in $\mathbb{P}_3(\mathbb{C})$ and a plane going through $x$ which as a tangent plane in $x$ is annihilated by a covariant tangent vector in $x$. This corresponds to the inclusion $(1) \subset (3)$. Similarly $(1) \subset (2)$ describes the projective contravariant tangent bundle.

Motivated by the remark of Calabi I studied the following problem:

Let $B$ be a compact complex manifold of dimension 3 and $c = 1 + c_1 + c_2 + c_3$ its total Chern class ($c_i \in H^{2i}(B, \mathbb{Z})$). Let $X_1, X_2$ be the projective covariant tangent bundle of $B$ and the projective contravariant tangent bundle respectively. Then $X_1, X_2$ are 5-dimensional compact complex manifolds fibred over $B$ with the projective plane as fibre. After introducing a hermitian metric in $B$ the covariant and contravariant tangent bundles with structural group $U(3)$ are anti-isomorphic by complex conjugation in each fibre. Therefore $X_1$ and $X_2$ are diffeomorphic. The cohomology ring $H^*(B, \mathbb{Z})$ maps isomorphically into the cohomology ring of $X_1$ and $X_2$ respectively and $H^*(B, \mathbb{Z})$ is an extension of $H^*(B, \mathbb{Z})$ by the first Chern class $\eta$ of the tautological line bundle over $X_i$. The total Chern classes of $X_1, X_2$ will be denoted by

\[ d = 1 + d_1 + d_2 + d_3 + d_4 + d_5 \text{ where } d_j \in H^2(X_i, \mathbb{Z}) \]

Each Chern number of $X_1, X_2$ can be calculated as a linear combination of the Chern numbers $c_1^3, c_1 c_2, c_3$ of $B$. I did not carry this out completely. I formulate here only the following result for $d_1$ and the Chern number $d_1^5$.

**Proposition 1.** We have

\[ d_1(X_1) = -3\eta \]
\[ d_1(X_2) = 2c_1 - 3\eta \]

\[ d_1^5[X_1] = 3^5(c_1^3 - 2c_1c_2 + c_3)[B] \]
\[ d_1^5[X_2] = 9(23c_1^3 - 36c_1c_2 - 27c_3)[B] \]

Of course, (1) and (2) follow from (3) and (4). Let $g \in H^2(\mathbb{P}_3(\mathbb{C}), \mathbb{Z})$ be the positive generator of the cohomology ring. The total Chern class of $\mathbb{P}_3(\mathbb{C})$ is
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$(1 + g)^4$, and the Chern numbers are

$$c_1^3 = 64, c_1 c_2 = 24, c_3 = 4,$$

and (4) gives the values 4860 and 4500 respectively.

We now specialize to the case that the complex 3-fold $B$ is a Calabi-Yau manifold, i.e. we assume that the first Chern class of $B$ vanishes. Then we only have the Chern number $c_3[B]$ which is the Euler number of $B$ and all Chern numbers of $X_1$ and $X_2$ are multiples of $c_3[B]$. We have the following table for $X_1, X_2$ and the Cartesian product $B \times \mathbb{P}^2(\mathbb{C})$.

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$B \times \mathbb{P}^2(\mathbb{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_5$</td>
<td>$3c_3$</td>
<td>$3c_3$</td>
<td>$3c_3$</td>
</tr>
<tr>
<td>$d_1d_4$</td>
<td>$9c_3$</td>
<td>$9c_3$</td>
<td>$9c_3$</td>
</tr>
<tr>
<td>$d_3d_2$</td>
<td>$3c_3$</td>
<td>$3c_3$</td>
<td>$3c_3$</td>
</tr>
<tr>
<td>$d_1d_4^2$</td>
<td>$9c_3$</td>
<td>$9c_3$</td>
<td>$9c_3$</td>
</tr>
<tr>
<td>$d_1^2d_3$</td>
<td>$27c_3$</td>
<td>$-27c_3$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_2d_1^3$</td>
<td>$81c_3$</td>
<td>$-81c_3$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_1^3$</td>
<td>$243c_3$</td>
<td>$-243c_3$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

(5)

If the Euler number $c_3[B]$ is not zero, then $X_1, X_2$ have different Chern numbers, therefore they are not bholomorphically equivalent. The complex cobordism class of a compact complex manifold is determined by its Chern numbers [M]. Therefore the complex cobordism classes of $X_1$ and $X_2$ are equal if and only if $c_3[B] = 0$. From table (5) we obtain

**Proposition 2.** *For the projective covariant or contravariant tangent bundles $X_1, X_2$ of the Calabi-Yau manifold $B$ we have in the complex cobordism ring the equation*

(6) \[ X_1 + X_2 = 2B \times \mathbb{P}^2(\mathbb{C}) \]

**Remarks:**

(1) The results (5) and (6) are true if we only assume that the Chern numbers $c_1^3$ and $c_1 c_2$ of $B$ vanish.
(2) The Todd genus is multiplicative in projective bundles ([H1] and [H3]). The Todd genus of $B$ equals $\frac{1}{2} c_1 c_2 |B| = 0$. Hence the Todd genus of $X_1$ and $X_2$ vanishes. For a 5-fold $Y$ with Chern classes $d_i$ the Todd genus is given by ([H3], §1)

$$T(Y) = \frac{1}{1440} (-d_4 d_1 + d_3 d_2 + 3 d_2 d_1 - d_2 d_3) [Y]$$

which indeed vanishes for $X_1$, $X_2$ and $B \times \mathbb{P}_2(\mathbb{C})$, see table (5).

§2. Proofs

Let $E$ be a complex vector bundle (fibre $\mathbb{C}^n$, structural group U(n)) over the compact manifold $B$ with total Chern class

$$1 + a_1 + ... + a_n , \quad a_i \in H^{2i}(B, \mathbb{Z}).$$

Consider the associated projective bundle $X$ over $B$ with fibre $\mathbb{P}_{n-1}(\mathbb{C})$ and projection $\pi : X \rightarrow B$. We have

$$\pi^* E = L \oplus \tilde{E}$$

where $L$ is the tautological line bundle over $X$ and $\tilde{E}$ (fibre $\mathbb{C}^{n-1}$) its hermitian orthogonal complement. Then $\tilde{E} \otimes L^{-1}$ is the tangential vector bundle $T$ along the fibres of $X$ (see [H3], §13) and

$$(1) \quad \pi^* E \otimes L^{-1} = T \oplus 1$$

where 1 denotes the trivial line bundle. We always use tacitly that $\pi^*$ maps the integral cohomology ring of $B$ isomorphically into $H^*(X, \mathbb{Z})$ and usually omit $\pi^*$ in the notation. The ring $H^*(X, \mathbb{Z})$ is an extension of $H^*(B, \mathbb{Z})$ by the first Chern class $\eta \in H^2(X, \mathbb{Z})$ of the tautological line bundle $L$ over $X$. The total Chern class of $\pi^* E$ can be split formally (or by lifting $E$ to the associated bundle with $U(n)/U(1)^n$ as fibre)

$$1 + a_1 + ... + a_n = (1 + x_1) ... (1 + x_n) , \quad x_i \in H^2$$

and (1) implies that the total Chern class of $T$ equals

$$(2) \quad (1 + x_1 - \eta) ... (1 + x_n - \eta) = \sum_{i=0}^{n} (1 - \eta)^{n-i} a_i$$

and, since the $n$-th Chern class of $T$ is zero,
(3) \[ \sum_{i=0}^{n} (-\eta)^{n-i} a_i = 0 \] (formula of Guy Hirsch)

The ring \( H^*(X, \mathbb{Z}) \) is the extension of \( H^*(B, \mathbb{Z}) \) by \( \eta \) with the relation (3). Thus \( H^*(X, \mathbb{Z}) \) is a free module over \( H^*(B, \mathbb{Z}) \) with base 1, \( \eta, \ldots, \eta^{n-1} \).

All this is already mentioned in [H1] and in the following paper [H2] where the total Chern class of the flag manifold \( U(n)/U(1)^n \) is calculated. This is the beginning of the joint work with A. Borel. When I told him in 1953 about the formula for \( U(n)/U(1)^n \) he pointed out that this can be generalized to \( G/T \) where \( G \) is a compact connected Lie group and \( T \) a maximal torus of \( G \) (see [H2], remark at the end of §7).

If \( B \) is a complex manifold of dimension \( m \) with total Chern class

\[ c(B) = 1 + c_1 + c_2 + \ldots c_m \]

then

(4) \[ c(X) = c(B) \cdot \sum_{i=0}^{n} (1 - \eta)^{n-i} a_i \]

To calculate the Chern numbers of \( X \) we use that \( -\eta \) when restricted to the fibre \( \mathbb{P}_{n-1}(\mathbb{C}) \) over \( X \) is the positive generator of \( H^2(\mathbb{P}_{n-1}(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z} \) represented by a hyperplane.

We denote the total Chern class \( c(X) \) by \( d \)

\[ d = 1 + d_1 + \ldots + d_{m+n-1} \]

A Chern number \( d_{r_1}d_{r_2}\ldots d_{r_s} \) \[ r_1 + r_2 + \ldots + r_s = m + n - 1 \] equals by (3) and (4) a linear combination of monomials of dimension \( m \) in the \( a_i \) and \( c_j \) multiplied with \( (-\eta)^{n-1} \). Since \( (-\eta)^{n-1} \) gives 1 when evaluated on the fibre, the Chern number \( d_{r_1}d_{r_2}\ldots d_{r_s} [X] \) equals the above polynomial in \( a_i \) and \( c_j \) of dimension \( m \) evaluated on \( B \).

We now carry this out in the cases we are interested in.

We specialize to \( m = n = 3 \). The Hirsch relation (3) gives

(5) \[ \eta^3 = \eta^2 a_1 - \eta a_2 + a_3 \]
and by repeated applications of (5)

(5') \[ \eta^4 = \eta^2(a_1^2 - a_2) + \eta(-a_1a_2 + a_3) \]

(5'') \[ \eta^5 = \eta^2(a_1^3 - 2a_1a_2 + a_3) \]

We now consider the covariant tangent bundle of the 3-dimensional complex manifold $B$ with Chern classes $c_1, c_2, c_3$. The Chern classes of the covariant tangent bundle of $B$ are $-c_1, c_2, -c_3$. According to (4), the total Chern class of the projective covariant tangent bundle $X_1$ equals

(6) \[ d = (1 + c_1 + c_2 + c_3)((1 - \eta)^3 - (1 - \eta)^2c_1 + (1 - \eta)c_2 - c_3) \]

\[ d_1 = -3\eta \]

We have by (5'')

(7) \[ d_1^5 = -3^5\eta^2(-c_1^3 + 2c_1c_2 - c_3) \]

Thus the first formula in (4) of §1 is proved.

If we take the contravariant tangent bundle of $B$, then by definition its Chern classes are those of $B$. The total Chern class $d$ of the associated projective bundle $X_2$ is now given by

(8) \[ d = (1 + c_1 + c_2 + c_3)((1 - \eta)^3 + (1 - \eta)^2c_1 + (1 - \eta)c_2 + c_3) \]

\[ d_1 = 2c_1 - 3\eta \]

For the calculation of $d_1^5 = (2c_1 - 3\eta)^5$ we have to replace in this expression $\eta^3, \eta^4, \eta^5$ by the quadratic polynomials in $\eta$ given in (5), (5'), (5'') with $a_i = c_i$.

We have

\[ (2c_1 - 3\eta)^5 = 9(-27\eta^5 + 90c_1\eta^4 - 120c_1^2\eta^3 + 80c_1^3\eta^2) \]
and by (5), (5'), (5'')

\[ \eta^5 = (c_1^3 - 2c_1c_2 + c_3)\eta^2 \]
\[ c_1\eta^4 = (c_1^3 - c_1c_2)\eta^2 \]
\[ c_1^2\eta^3 = c_1^3\eta^2 \]

which gives the second formula in (4) of §1.

We now assume that \( B \) is a Calabi-Yau manifold (\( c_1 = 0 \)). The Chern class \( c_2 \) cannot occur in the final result because \( c_1c_2 = 0 \). Therefore, for the calculation we put \( c_2 = 0 \) and have in the covariant case \( \eta^3 = -c_3 \) and for the total Chern class of \( X_1 \)

\[ d(X_1) = (1 + c_3)(1 - 3\eta + 3\eta^2) \tag{9} \]
\[ d_1 = -3\eta, d_2 = 3\eta^2, d_3 = c_3, d_4 = -3c_3\eta, d_5 = 3c_3\eta^2 \tag{10} \]

The values in table (5) of §1 are now easily obtained.

For the projective contravariant bundle \( X_2 \) we have (again mod \( c_2 \)) the same formulas as in (9) and (10), but in this case

\[ \eta^3 = c_3 \]

This checks with table 5 in §1. For the Cartesian product \( B \times \mathbb{P}_2(\mathbb{C}) \) again (9) and (10) hold, but \( \eta^3 = 0 \) and the values in table 5 result.

**Remark:**

Proposition 2 and table (5) of §1 can be generalized to higher dimensions. We assume that \( B \) is a compact almost complex manifold of dimension \( n \) and that all Chern numbers of \( B \) vanish except \( c_n[B] \), the Euler number. Consider the projective covariant tangent bundle \( X_1 \) of \( B \) and the projective contravariant tangent bundle \( X_2 \) of \( B \). Both \( X_1 \) and \( X_2 \) are fibred over \( B \) with \( \mathbb{P}_{n-1}(\mathbb{C}) \) as fibre. The Chern classes of \( X_1, X_2 \) and \( B \times \mathbb{P}_{n-1}(\mathbb{C}) \) are denoted \( d_i \). Using the preceding methods, the following is proved easily.

If \( n \) is even, then \( X_1 \) and \( X_2 \) have equal Chern numbers, hence \( X_1 = X_2 \) in the complex cobordism ring.
If $n$ is odd, the following holds:
If a Chern number is a monomial of dimension $2n-1$ in Chern classes $d_i$ with all $i < n$, then for $X_2$ it is minus the corresponding Chern number of $X_1$, whereas it vanishes for $B \times \mathbb{P}_{n-1}(\mathbb{C})$. The other Chern numbers agree on $X_1, X_2$, and $B \times \mathbb{P}_{n-1}(\mathbb{C})$. Hence we have in the complex cobordism ring

$$X_1 + X_2 = 2B \times \mathbb{P}_{n-1}(\mathbb{C}).$$

References


[H1] F. Hirzebruch, Todd arithmetic genus for almost complex manifolds, mimeographed notes, Princeton University 1953, published as Nr. 4 in Vol. 1 of the Gesammelte Abhandlungen


[H3] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Springer 1956


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