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A Local-Global Compatibility Conjecture in the p-adic Langlands Programme for $\mathrm{GL}_{2/\mathbb{Q}}$

Matthew Emerton

To John Coates, for his 60th birthday

Contents

1. Introduction

1.1. **The local-global compatibility conjecture.** Fix a prime p, as well as a finite extension E of $\mathbb{Q}_p$. If K^p is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}^p)$ (referred to as a "tame level"), then one can define a certain E-Banach space $\widehat{H}^1(K^p)_E$, equipped with an action of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$, by taking the inductive limit of the étale cohomology with coefficients in E of the modular curves of arbitrary p-power level and of tame level K^p, and then completing with respect to the norm induced by the $\mathcal{O}_E$-submodule of integral cohomology classes. Passing to the

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locally convex inductive limit over all tame levels K^p, we obtain a complete locally convex topological E-vector space $\widehat{H}^1_E$ equipped with a representation of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$ that is (so to speak) "smooth in the prime-to-p-directions, but unitary Banach in the p-adic direction". (See Subsection 7.2 for the precise definitions of these various topological vector spaces.) The object of this note is to explain a conjecture on the multiplicities with which certain $G_{\mathbb{Q}}$-representations appear in $\widehat{H}^1_E$. This conjecture is in some sense the most optimistic possible, in light of what is already known, or believed, to be true.

In order to state the conjecture, we must first admit the truth of a "local p-adic Langlands conjecture for GL_2". The idea that such a conjecture should (or even could) exist is due largely to Breuil, and has been extensively developed both by him and others. In what follows, we will take as given the most optimistic version of this conjecture, namely that to any continuous representation of $G_{\mathbb{Q}_p}$ on a two dimensional E-vector space V there is associated in a natural manner an admissible unitary Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. Granting the existence of this local correspondence, we may now state our conjecture.

1.1.1. **Conjecture** (Local-global compatibility)**.** *If V is an arbitrary odd irreducible continuous two dimensional representation of $G_{\mathbb{Q}}$ over E that is unramified outside of a finite set of primes, then the space $\mathrm{Hom}_{G_{\mathbb{Q}}}(V, \widehat{H}^1_E)$ (which is naturally a $\mathrm{GL}_2(\mathbb{A}_f)$-representation) decomposes as a restricted tensor product*

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(V, \widehat{H}^1_E) \xrightarrow{\sim} B(V_{|D_p}) \otimes \bigotimes_{\ell \neq p}{}' \, \pi_\ell^{\mathrm{m}}(V),$$

where $B(V_{|D_p})$ is the admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $V_{|D_p}$ via the local p-adic Langlands correspondence, while for each prime $\ell \neq p$, $\pi_\ell^{\mathrm{m}}(V)$ is the admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ associated to $V_{|D_\ell}$ via the modified classical local Langlands correspondence discussed in 2.1.1 below.[1]

At least for those V that are attached to classical modular forms, this conjecture is due to Breuil [13]. One of our goals will be to provide some justification for extending his conjecture to more general V.

Of course, for Conjecture 1.1.1 to have a precise meaning, one has to make the local p-adic Langlands correspondence precise. Thus a second goal of our paper is to discuss various results and conjectures related to the local correspondence. So far this correspondence has been specified for two dimensional $G_{\mathbb{Q}_p}$-representations that are trianguline in the sense of [27] (with some caveats in the case of reducible representations). This is work primarily of Breuil and Colmez.

[1]The representation $\pi_\ell^{\mathrm{m}}(V)$ coincides with the representation $\pi_\ell(V)$ attached to $V_{|D_\ell}$ by the classical local Langlands correspondence with respect to the Tate normalization, except in those cases in which the latter representation is not generic (i.e. not infinite dimensional).

In Section 6 we will recall the definition of this trianguline correspondence in detail.

Currently, very little seems to have been proved regarding the existence of the local correspondence for those two dimensional $G_{\mathbb{Q}_p}$-representations V that are not trianguline.[2]

Nevertheless, it is possible to formulate a minimal set of requirements that the local p-adic correspondence should satisfy, and we do this in Conjecture 3.3.1 below. Our set of requirements is by no means definitive (and it certainly does not specify the correspondence uniquely; cf. Remark 3.3.6 below). However, it does serve to formalize some of the ideas and expectations about the relationship between p-adic Galois representations and p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that have appeared in the literature in the last several years.[3] Of all the conditions that we incorporate into our conjecture, perhaps the most novel is a condition (namely condition (8) of Conjecture 3.3.1) that relates locally analytic Jacquet modules on the $\mathrm{GL}_2(\mathbb{Q}_p)$ side (as defined in [32]) to a certain notion of refinement on the $G_{\mathbb{Q}_p}$ side (related to the one introduced in [45] for crystalline representations, and also closely related to the theory of two dimensional trianguline representations developed in [27]).

Having formulated Conjecture 3.3.1, we are immediately confronted with the problem of proving that it is satisfied by the correspondence as it has been defined in the trianguline case. A good part of the paper is devoted to discussing this problem either directly or indirectly (although we do not solve it – see Subsection 6.6 for a discussion of what parts of the problem remain open). We focus in particular on describing the relationship between the trianguline correspondence and the theory of Jacquet modules. For this we rely heavily on the constructions and results of [27, 32, 35].

When Conjectures 1.1.1 and 3.3.1 are combined, they provide an appealing framework within which to consider a number of important conjectures relating p-adic Galois representations and modular forms. For example, we will prove the following result.

1.1.2. **Proposition.** *Assume that the local p-adic Langlands correspondence exists, and satisfies Conjecture 3.3.1. Then Conjecture 1.1.1 has the following consequences. (As usual we let V denote an odd irreducible continuous two dimensional representation of $G_{\mathbb{Q}}$ over E.)*

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[2]Breuil and Strauch (in unpublished work) have proposed a candidate for $B(V)$ in (at least some of) those cases when V is potentially semi-stable but not trianguline. However, as far as I am aware, little is known about their proposed $B(V)$. For example, it does not seem to be known whether or not it is non-zero.

[3]A caveat: There is an important connection between the $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ worlds, discovered by Colmez, which we have not attempted to incorporate into our conjecture; see Remark 3.3.7 below.

(1)$_?$ *The representation $V_{|D_p}$ is potentially semi-stable with distinct Hodge-Tate weights if and only if V is the twist by a power of the p-adic cyclotomic character of a Galois representation attached to a classical cuspidal Hecke eigenform of weight $k \geq 2$.*

(2)$_?$ *The representation $V_{|D_p}$ is trianguline if and only if V is a twist of a Galois representation attached to a p-adic overconvergent cuspidal Hecke eigenform of finite slope.*

The ? are to indicate that each assertion is contingent on Conjectures 1.1.1 and 3.3.1. The first statement is a conjecture of Fontaine and Mazur [37, Conj. 3c]. The second statement implies [41, Conj. 11.8], but is in fact stronger; it also implies the equality $X_{\mathrm{fs}}^{\circ} = X_{\mathrm{fs}}$ discussed in note (2) on p. 450 of that reference.

Proposition 1.1.2 is proved in Subsection 7.9 below. (More precisely, it is a reformulation of parts (3)$_?$ and (4)$_?$ of Proposition 7.9.1.) The proof of (2)$_?$ depends on a result which may be of interest in its own right. If K^p is a fixed tame level, then applying the theory of locally analytic Jacquet modules to $\widehat{H}^1(K^p)_E$ one can construct a certain coherent sheaf of algebras on the space $\widehat{\mathrm{T}}$ of characters of $(\mathbb{Q}_p^{\times})^2$, whose relative spectrum $\operatorname{Spec} \mathcal{A}(K^p)$ was shown in [33, §4.4] to contain the reduced eigensurface of tame level K^p. The question of whether or not this inclusion is an equality was left open in that reference. Theorem 7.5.8 below shows that this inclusion is (essentially) an equality. As the proof of this theorem is rather technical, we only sketch it here. The details will appear elsewhere.

We now state some partial results in the direction of Conjecture 1.1.1 related to the case when V is an irreducible two dimensional $G_{\mathbb{Q}}$-representation associated to an overconvergent p-adic eigenform of finite slope. (In this case $V_{|D_p}$ is trianguline – see [41] and also Thm. 7.6.1 below. Thus $B(V_{|D_p})$ is defined, and Conjecture 1.1.1 has a precise meaning.) The following theorem incorporates results of Berger, Breuil, and Colmez, as well as of the author.

1.1.3. **Theorem.** *Let V be an irreducible continuous $G_{\mathbb{Q}}$-representation attached to an overconvergent eigenform f over E of finite slope. Assume furthermore that:*

(a) If $V_{|D_p}$ is irreducible and potentially semi-stable (in which case f is necessarily classical of weight $k \geq 2$, by [41, Thm. 6.6]*) then it is Frobenius semi-simple.*

(b) If $V_{|D_p}$ is reducible then it is potentially crystalline, and f is classical of weight $k \geq 2$.

The following results then hold:

(1) *If* $V_{|D_p}$ *is indecomposable and* f *is classical then there is an isomorphism*

$$\bigotimes_{\ell \neq p}{}' \pi_\ell^{\mathrm{m}}(V) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(B(V_{|D_p}), \mathrm{Hom}_{G_{\mathbb{Q}}}(V, \widehat{H}^1_E)\big).$$

(2) *In general, if* f *has tame level* K^p*, then there is a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant embedding* $B(V_{|D_p}) \to \mathrm{Hom}_{G_{\mathbb{Q}}}(V, \widehat{H}^1_E)^{K^p}$.

The next theorem provides something of a converse to the previous one.

1.1.4. **Theorem.** *Let* V *be as in Conjecture 1.1.1, and suppose furthermore that* V *is not a twist of a representation having finite image. Let* W *be an irreducible trianguline continuous two dimensional representation of* $G_{\mathbb{Q}_p}$*. If there is a non-zero continuous* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant map* $B(W) \to \mathrm{Hom}_{G_{\mathbb{Q}}}(V, \widehat{H}^1_E)$*, then* V *is a twist of a representation attached to an overconvergent eigenform of finite slope, and* $V_{|D_p} \cong W$.

These theorems follow from the results proved in Subsection 7.10 below.

Let us close this introductory discussion by remarking that one could hope for a still stronger conjecture than Conjecture 1.1.1, for while this conjecture predicts the multiplicities with which all odd irreducible two dimensional Galois representations appear inside $\widehat{H}^1_E$, it does not explain how they are "glued together" inside $\widehat{H}^1_E$. By contrast, the classical Langlands correspondence provides a complete description of the parabolic cohomology of modular curves as a representation of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$ (see Theorem 2.5.1 below). In a forthcoming paper we will formulate a strengthening of Conjecture 1.1.1 that gives an analogous description of $\widehat{H}^1_E$ (or more precisely, of the localization of $\widehat{H}^1_E$ at the maximal ideal in an appropriate Hecke algebra corresponding to some fixed absolutely irreducible residual two dimensional $G_{\mathbb{Q}}$-representation) as a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-module.

1.2. Contents and arrangement of the paper.

In Section 2 we briefly recall the classical Langlands correspondence for $\mathrm{GL}_{2/\mathbb{Q}}$ in both the local and global contexts. Given that our ultimate subject is Conjecture 1.1.1, we put a particular emphasis on the realization of the global correspondence in the cohomology of modular curves.

Sections 3, 4, 5, and 6 are devoted to discussing various aspects of the local p-adic Langlands conjecture.

In Section 3, after presenting some initial motivation and recalling the notion of admissible unitary Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, we present our conjecture regarding the local p-adic Langlands correspondence (Conjecture 3.3.1).

In Section 4 we introduce the concept of a refinement of a two dimensional $G_{\mathbb{Q}_p}$-representation, and establish some basic properties on the existence and classification of refinements. (For this we rely heavily on the work of Colmez [27].) A fundamental point is that a two dimensional $G_{\mathbb{Q}_p}$-representation admits a refinement if and only if it is trianguline. Furthermore, as we explain, the language of refinements gives a convenient way to express the classification (due to Colmez) of two dimensional trianguline representations.

In Section 5 we recall what little is known about the classification of topologically irreducible admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. One basic technique for studying such a representation B is to pass to the subspace B_{an} of locally analytic vectors, which is dense in B, and is a strongly admissible locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. Unfortunately, not much is known about the classification of such representations in general. However, one is much closer to having a classification of those B for which either B_{lalg} (the space of locally algebraic vectors in B) or $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$ (the Jacquet module of B_{an} with respect to the Borel subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$) is non-zero, as we explain in some detail. The discussion relies heavily on work of Colmez [26, 27] and of the author [32, 35].

As we remarked above, the local correspondence $V \mapsto B(V)$ has been defined for most trianguline two dimensional $G_{\mathbb{Q}_p}$-representations V. More precisely, it has been defined for irreducible trianguline V by Breuil and Colmez [12, 26], and for Frobenius semi-simple potentially crystalline reducible V by Breuil and the author [6, 14]. In Section 6 we recall the definition of the correspondence in these various cases, and also discuss the expected structure of $B(V)$ in the remaining reducible cases. We discuss the extent to which this explicit correspondence is known to satisfy the conditions of Conjecture 3.3.1. We also state a conjecture on the structure of the space of locally analytic vectors $B(V)_{\mathrm{an}}$ in $B(V)$ (generalizing a conjecture of Breuil in the potentially crystalline case).

With our discussion of the local p-adic Langlands conjecture completed, in Section 7 we return to the subject of Conjecture 1.1.1. We begin by describing in more detail the completions $\widehat{H}^1(K^p)_E$ and $\widehat{H}^1_E$ introduced above. After an aside on various notions and results related to systems of Hecke eigenvalues, we discuss the structure of $(\widehat{H}^1_E)_{\mathrm{lalg}}$, and of the Jacquet module $J_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)$. In particular, we state and sketch the proof of Theorem 7.5.8, which establishes the precise relationship between $J_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)$ and the eigensurface of tame level K^p. After another aside, in which we reformulate some of the results of [41] in the language of refinements, we turn to our main topic: the local-global compatibility conjecture. We state the conjecture in various forms, and deduce some of its consequences (perhaps the most interesting of which are the two stated in Proposition 1.1.2). Finally, we state and prove some results (summarized in Theorems 1.1.3 and 1.1.4 above) that provide some small amount of evidence for the conjecture.

1.3. **Notation, terminology, and conventions.** We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, as well as an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ for each prime ℓ. We also fix embeddings $\imath_\ell : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ for each ℓ, and an embedding $\imath_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. As usual, we write $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $G_{\mathbb{Q}_\ell} := \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. The embedding $\imath_\ell$ induces an embedding $G_{\mathbb{Q}_\ell} \hookrightarrow G_\mathbb{Q}$ for each ℓ; we sometimes denote the image by D_ℓ (the decomposition group at ℓ).

For each prime ℓ, the residue field of the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}}_\ell$ is an algebraic closure $\overline{\mathbb{F}}_\ell$ of the finite field $\mathbb{F}_\ell$. There is a natural surjection $G_{\mathbb{Q}_\ell} \to \mathrm{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$, with kernel equal to the inertia subgroup I_ℓ; the target of this surjection is procyclic, topologically generated by the Frobenius Frob_ℓ. For each prime ℓ we let W_ℓ denote the Weil group at ℓ; i.e. the subgroup of $G_{\mathbb{Q}_\ell}$ consisting of elements which act on $\overline{\mathbb{F}}_\ell$ via an integral power of Frobenius. We topologize W_ℓ in the usual way, by declaring the inertia subgroup (endowed with the topology it inherits as a closed subgroup of $G_{\mathbb{Q}_\ell}$) to be open. The local Artin map induces an isomorphism $\mathbb{Q}_\ell^\times \xrightarrow{\sim} \mathrm{W}_\ell^{\mathrm{ab}}$, which we normalize by identifying the uniformizer ℓ in $\mathbb{Q}_\ell^\times$ with a lift of Frob_ℓ^{-1} (i.e. geometric Frobenius).

We let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$, and as usual we write $\mathbb{A}_f := \mathbb{Q} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$ for the ring of finite adèles over $\mathbb{Q}$. More generally, if Σ is any finite set of primes then we write $\widehat{\mathbb{Z}}^\Sigma$ to denote the prime-to-Σ profinite completion of $\mathbb{Z}$, and $\mathbb{A}_f^\Sigma := \mathbb{Q} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^\Sigma$ to denote the prime-to-Σ finite adèles. If $\Sigma = \{\ell\}$ for a single prime ℓ, we write $\widehat{\mathbb{Z}}^\ell := \widehat{\mathbb{Z}}^{\{\ell\}}$ and $\mathbb{A}_f^\ell := \mathbb{A}_f^{\{\ell\}}$. The embeddings $\mathbb{Q} \to \mathbb{A}_f$ and $\widehat{\mathbb{Z}} \to \mathbb{A}_f$ induce an isomorphism $\mathbb{Q}_{>0} \times \widehat{\mathbb{Z}}^\times \xrightarrow{\sim} \mathbb{A}_f^\times$, and hence an isomorphism

$$\widehat{\mathbb{Z}}^\times \xrightarrow{\sim} \mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times. \tag{1}$$

The global Artin map (which we normalize to be compatible with our choice of normalization of the local Artin maps) induces an isomorphism

$$\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times \xrightarrow{\sim} G_\mathbb{Q}^{\mathrm{ab}}. \tag{2}$$

As usual, we will use the isomorphisms (1) and (2) to identify characters ψ of $G_\mathbb{Q}^{\mathrm{ab}}$ with characters of $\widehat{\mathbb{Z}}^\times$. If ψ is unramified outside of a finite set of primes Σ, then the corresponding character of $\widehat{\mathbb{Z}}^\times$ factors through the projection $\widehat{\mathbb{Z}}^\times \to \widehat{\mathbb{Z}}^\times/(\widehat{\mathbb{Z}}^\Sigma)^\times$, and thus we will regard ψ as a character of $\widehat{\mathbb{Z}}^\times/(\widehat{\mathbb{Z}}^\Sigma)^\times$. Let us point out that if ψ is a character of $G_\mathbb{Q}^{\mathrm{ab}}$ unramified outside of Σ, if ℓ is a prime not in Σ, and if ψ_ℓ denotes the restriction of ψ to $\mathrm{W}_\ell^{\mathrm{ab}}$, identified with a character of $\mathbb{Q}_\ell^\times$ via the local Artin map, then we have the formula $\psi(\ell) = \psi_\ell(\ell^{-1})$ (where on the left hand side ℓ is regarded as an element of $\widehat{\mathbb{Z}}^\times/(\widehat{\mathbb{Z}}^\Sigma)^\times$ and on the right hand side as an element of $\mathbb{Q}_\ell^\times$).

The prime p will be fixed throughout the paper. We let $|\ \ |: \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ denote the p-adic absolute value, normalized by $|\, p\, | = p^{-1}$. (We will occasionally use the same notation to denote the ℓ-adic absolute value on $\mathbb{Q}_\ell^\times$, for some prime $\ell \neq p$, normalized so that $|\ \ell\ |= \ell^{-1}$. We will always make it clear when we are doing this.) For any integer w, we write simply z^w to denote the character $\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ defined by $z \mapsto z^w$.

We let $\varepsilon : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character. We use the same symbol ε to denote its restriction to $G_{\mathbb{Q}_p}$. (It will be clear from the context whether we are referring to the global or local character.) Global class field theory identifies ε with the character of $\widehat{\mathbb{Z}}^\times$ given by projection onto the pth factor: $\widehat{\mathbb{Z}}^\times \to \mathbb{Z}_p^\times$.

We fix a finite extension E of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. We let $\mathcal{O}_E$ denote the ring of integers in E, ϖ the uniformizer of $\mathcal{O}_E$, and $\mathbb{F} := \mathcal{O}_E/\varpi$ the residue field of $\mathcal{O}_E$.

A continuous character $\eta : \mathrm{W}_p \to E^\times$ extends to a continuous character of $G_{\mathbb{Q}_p}$ if and only if η is unitary, i.e. if and only if η takes values in $\mathcal{O}_E^\times$. If this extension exists, it is unique, and we denote it by the same symbol η. If $\alpha \in E^\times$, we let $\mathrm{ur}(\alpha) : \mathbb{Q}_p^\times \to E^\times$ denote the character that maps p to α and is trivial on $\mathbb{Z}_p^\times$. Regarded as a character of W_p, the character $\mathrm{ur}(\alpha)$ is unramified (hence the notation) and maps a geometric Frobenius element to α. This character is unitary, and hence extends to a continuous E-valued character of $G_{\mathbb{Q}_p}$, precisely when $\alpha \in \mathcal{O}_E^\times$. Assuming this condition holds, then according to the convention just signalled, we use the same notation $\mathrm{ur}(\alpha)$ to denote the extended character. The same convention allows us to regard the local cyclotomic character ε as a unitary character of $\mathbb{Q}_p^\times$. We then have the following useful formula: $\varepsilon =|\ \ | z$.

For any commutative ring R with 1 we let $\mathrm{GL}_2(R)$ denote (as usual) the group of invertible 2×2 matrices over R, let $\mathrm{P}(R)$ denote the subgroup of upper triangular matrices, let $\overline{\mathrm{P}}(R)$ denote the subgroup of lower triangular matrices, and let $\mathrm{T}(R) := \mathrm{P}(R) \bigcap \overline{\mathrm{P}}(R)$ denote the subgroup of diagonal matrices. There is a canonical identification $\mathrm{T}(R) = (R^\times)^2$ (given by identifying $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with the ordered pair of units (a, d)).

If B is an E-Banach space equipped with a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$, then we let B_{lalg} (resp. B_{an}) denote the $\mathrm{GL}_2(\mathbb{Q}_p)$-subspace of locally algebraic (resp. locally analytic) vectors in B (as defined in [31]; see also [51] in the case of locally analytic vectors). Recall that the space B_{an} may be equipped in a natural way with an inductive limit topology, making it a barrelled locally convex E-space equipped with a locally analytic action of $\mathrm{GL}_2(\mathbb{Q}_p)$. (See [31], but note that in that reference B_{an} is denoted by B_{la}.) The topology on B_{an} is finer than the topology induced on it as a subspace of B. In [51] another topology on B_{an} is considered, which realizes it as a closed subspace of the space of locally analytic B-valued functions on $\mathrm{GL}_2(\mathbb{Q}_p)$. When B is admissible unitary, which is the main case of interest to us, these two topologies coincide [14, Rem. A.1.1], and [51, Thm. 7.1] shows that B_{an} is a strongly admissible locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (in the sense of [49]), which is dense as a subset of B.

We let $\widehat{\mathrm{T}}$ denote the rigid analytic space over E that classifies the continuous (equivalently, locally analytic) characters of $\mathrm{T}(\mathbb{Q}_p)$. For any finite extension E' of $\mathbb{Q}_p$, the space $\widehat{\mathrm{T}}(E')$ of E'-valued points of $\widehat{\mathrm{T}}$ is thus canonically identified with the space of continuous characters $\mathrm{T}(\mathbb{Q}_p) \to E'^{\times}$. Via the canonical identification $\mathrm{T} = (\mathbb{Q}_p^{\times})^2$ we may regard an element $\chi \in \widehat{\mathrm{T}}(E)$ as an ordered pair (χ_1, χ_2) of characters $\chi_i : \mathbb{Q}_p^{\times} \to E^{\times}$; we will then write $\chi = \chi_1 \otimes \chi_2$. We may equally well identify the pair (χ_1, χ_2) with a representation $\mathbb{Q}_p^{\times} \to \mathrm{T}(E) = (E^{\times})^2$, and hence via local class field theory with a representation $\mathrm{W}_p \to \mathrm{T}(E)$. In summary, we have a natural identification $\widehat{T}(E) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))$.

We will have cause to consider several kinds of parabolically induced representations attached to characters $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$. For any such character we may form the continuous induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ and the locally analytic induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$. Each is defined as a space of functions $f : \mathrm{GL}_2(\mathbb{Q}_p) \to E$ satisfying the condition

$$f(\overline{p}g) = \chi_1(a)\chi_2(d)f(g) \tag{3}$$

for all $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $\overline{p} = \left(\begin{smallmatrix} a & 0 \\ * & d \end{smallmatrix}\right) \in \overline{\mathrm{P}}(\mathbb{Q}_p)$. The $\mathrm{GL}_2(\mathbb{Q}_p)$-action is via right translation. For the continuous (resp. locally analytic) induction, the function f is required to be continuous (resp. locally analytic). Since the projection $\mathrm{GL}_2(\mathbb{Z}_p) \to \overline{\mathrm{P}}(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)$ is surjective, restricting functions to $\mathrm{GL}_2(\mathbb{Z}_p)$ embeds the continuous (resp. locally analytic) induction as a closed subspace of the space of continuous (resp. locally analytic) functions on $\mathrm{GL}_2(\mathbb{Z}_p)$. Thus it is naturally an admissible Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, in the sense of [50] (resp. a strongly admissible locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, in the sense of [49]). The locally analytic induction is naturally identified (as a topological $\mathrm{GL}_2(\mathbb{Q}_p)$-representation) with the space of locally analytic vectors in the continuous induction (as follows from [31, Prop. 3.5.11]).

If $\chi_1\chi_2^{-1}$ has non-negative integral Hodge-Tate weight, then we may also form the locally algebraic induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}}$, defined to consist of those functions f satisfying condition (3), and such that $\big(\chi_2^{-1} \circ \det\big)(g)f(g)$ is a locally algebraic function on $\mathrm{GL}_2(\mathbb{Q}_p)$. If χ_2 itself is of integral Hodge-Tate weight, then the locally algebraic induction is naturally identified with the subspace of locally algebraic vectors in the continuous induction. In general (i.e. if χ_2 is not of integral Hodge-Tate weight), then it consists of the subspace of locally SL_2-algebraic vectors in the continuous induction. (Hopefully this slight inconsistency in our use of the terminology "locally algebraic" will not cause confusion.)

Finally, if χ_1 and χ_2 are smooth characters of $\mathrm{T}(\mathbb{Q}_\ell)$ for some prime ℓ, with values in any field, then we will write $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_\ell)}^{\mathrm{GL}_2(Q_\ell)} \chi_1 \otimes \chi_2\big)_{\mathrm{sm}}$ to denote the corresponding smooth induction. (In the case when $\ell = p$ and $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$ with both χ_1 and χ_2 being smooth characters, smooth induction is a special case of locally algebraic induction.)

If we are given a family π_ℓ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ vector spaces indexed by the primes ℓ lying outside some finite set of primes Σ, such that for almost all ℓ the representation π_ℓ contains a non-zero $\mathrm{GL}_2(\mathbb{Z}_\ell)$-invariant vector e_ℓ, unique up to multiplication by a scalar, then we will denote by $\bigotimes'_{\ell \notin \Sigma} \pi_\ell$ the restricted tensor product of the representations π_ℓ with respect to the vectors e_ℓ (as discussed in [40, §9], for example, where the notation is $\bigotimes_{e_\ell} \pi_\ell$.). The restricted tensor product is naturally a $\mathrm{GL}_2(\mathbb{A}_f^\Sigma)$-representation, which up to isomorphism is independent of the choice of the e_ℓ.

We let $\underline{1}$ denote the trivial character (of any group; in any particular usage, the group in question will be clear from the context).

If M is a $\mathbb{Z}_p$-module, or a sheaf of $\mathbb{Z}_p$-modules, and A is a $\mathbb{Z}_p$-algebra, then we sometimes write $M_A := A \otimes_{\mathbb{Z}_p} M$.

We assume that the reader is familiar with the basic aspects of the p-adic Hodge theory of $G_{\mathbb{Q}_p}$-representations, including Hodge-Sen-Tate weights, the period rings of Fontaine and the related conditions on a representation of being Hodge-Tate, de Rham, (potentially) semi-stable, or (potentially) crystalline. We use the standard notation for these period rings, and for the associated Dieudonné modules. (We always form Dieudonné modules in the covariant sense – i.e. by tensoring with the corresponding period ring and then passing to $G_{\mathbb{Q}_p}$-invariants.) A useful survey of this material is given in [4]. (We caution the reader that in this reference, Hodge-Sen-Tate weights are referred to as generalized Hodge-Tate weights.)

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2. Classical Langlands for $\mathrm{GL}_{2/\mathbb{Q}}$

In this section we briefly describe the classical Langlands correspondence for GL_2 over $\mathbb{Q}$.

2.1. **The classical local Langlands correspondence.** For any prime ℓ, the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_\ell)$ (whose existence was established by Tunnell [61], building on earlier results of Jacquet-Langlands [40] and Langlands [43]) establishes a certain bijection between the set of isomorphism classes of irreducible admissible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ on $\overline{\mathbb{Q}}_p$-vector spaces, and the set of isomorphism classes of two dimensional Frobenius semi-simple Weil-Deligne representations $\mathrm{WD}_\ell \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ over $\overline{\mathbb{Q}}_p$ (as defined in [29, §8] or [58, §4]).

In fact there are various choices of correspondence, depending on the desired normalization. The so-called unitary correspondence is uniquely determined by the requirement that the local L- and ε-factors attached to a pair of corresponding isomorphism classes should coincide. However, in our discussion we will always employ the so-called Tate normalization (as described in [29]). If σ is a two dimensional Frobenius semi-simple representation of WD_ℓ over $\overline{\mathbb{Q}}_p$, then we let $\pi_\ell(\sigma)$ denote the irreducible admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ associated to σ via the Tate normalized local Langlands correspondence. One advantage of the Tate normalization is that if the isomorphism class of the Weil-Deligne representation σ is in fact defined over a finite extension E of $\mathbb{Q}_p$, then the same is true of the isomorphism class of the $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representation $\pi_\ell(\sigma)$.

For definiteness, we recall the definition of $\pi_\ell(\sigma)$ in the case when $\sigma \cong \chi_1 \bigoplus \chi_2$ for two characters χ_i of $\mathbb{Q}_\ell^\times \cong \mathrm{W}_\ell^{\mathrm{ab}}$ (the isomorphism being provided by the local Artin reciprocity map). If $\chi_1\chi_2^{-1} \neq |\ \ |^{\pm 1}$ (where $|\ \ |$ denotes the absolute value character of $\mathbb{Q}_\ell^\times$), then

$$\pi_\ell(\sigma) := \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_\ell)}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \chi_1 \otimes \chi_2 |\ \ |\right)_{\mathrm{sm}},$$

while if $\chi_1\chi_2^{-1} = |\ \ |$ (resp. $|\ \ |^{-1}$), then $\pi_\ell(\sigma) := \chi_1 \circ \det$ (resp. $\chi_2 \circ \det$). (As we recall in more detail below, in these latter two cases, the smooth induction of $\chi_1 \otimes \chi_2$ is not irreducible.) The collection of representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ attached to such σ is known as the principal series. In particular, the collection of representations $\pi_\ell(\sigma)$ attached to those σ for which χ_1 and χ_2 are both unramified is known as the unramified principal series. Its members can be characterized intrinsically as those irreducible admissible smooth $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representations that contain a non-zero $\mathrm{GL}_2(\mathbb{Z}_\ell)$-fixed vector.

For later use, it will be convenient to recall some additional terminology used in the classification of irreducible admissible smooth $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representations. Note that the parabolic induction $\left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_\ell)}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \underline{1} \otimes \underline{1}\right)_{\mathrm{sm}}$ evidently contains the trivial character of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ as a subrepresentation. The quotient of $\left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_\ell)}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \underline{1} \otimes \underline{1}\right)_{\mathrm{sm}}$ by this one dimensional subrepresentation is an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ called the Steinberg representation [19, §8], which we will denote by St. The twists of St are known collectively as the special representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. They correspond under the local Langlands correspondence to those Weil-Deligne representations that are reducible but not semi-simple.

The irreducible admissible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ that are neither principal series nor special are called cuspidal. They are characterized intrinsically by the property that their matrix coefficients are compactly supported modulo the centre of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. They correspond under the local Langlands correspondence to the irreducible Weil-Deligne representations.

2.1.1. *A modified local Langlands correspondence.* In what follows we will need a slightly modified version of the local Langlands correspondence. Recall that an irreducible admissible smooth representation π of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ over $\overline{\mathbb{Q}}_p$ is called generic if π admits a Whittaker model, or equivalently, if it is infinite-dimensional (see Thm. 2.14 of [40], together with the discussion on p. 62 of this reference). Otherwise we say that π is non-generic. The only non-generic irreducible admissible smooth representations are those of the form $\chi \circ \det$, where χ is a $\overline{\mathbb{Q}}_p^\times$-valued character of $\mathbb{Q}_\ell^\times$ [40, Prop. 2.7].

If σ is a two dimensional Frobenius semi-simple representation of WD_ℓ for which $\pi_\ell(\sigma)$ is generic, then we define $\pi_\ell^{\mathrm{m}}(\sigma) := \pi_\ell(\sigma)$. The only σ for which $\pi_\ell(\sigma)$ is non-generic are those of the form

$$\sigma \cong \begin{pmatrix} \chi|\ \ |^{-1} & 0 \\ 0 & \chi \end{pmatrix},$$

where χ is some character of $\mathbb{Q}_\ell^\times$ (uniquely determined by σ), and $|\ \ |$ denotes the absolute value character of $\mathbb{Q}_\ell^\times$; as we recalled above, the usual Tate normalized local Langlands correspondence is defined on such σ by $\pi_\ell(\sigma) := \chi \circ \det$. For such σ, we define $\pi_\ell^{\mathrm{m}}(\sigma) := \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_\ell)}^{\mathrm{GL}_2(\mathbb{Q}_\ell)} \chi|\ \ |^{-1} \otimes \chi|\ \ |\right)_{\mathrm{sm}}$, which is a reducible but indecomposable infinite dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. It is a non-split extension of the character $\chi \circ \det$ by the twist $\mathrm{St} \otimes \chi$ [19, §8].

2.1.2. *From local Galois representations to* $\mathrm{GL}_2(\mathbb{Q}_\ell)$*-representations.* We finish this subsection by recalling the manner in which the local Langlands correspodence may be used to attach admissible representations of the groups $\mathrm{GL}_2(\mathbb{Q}_\ell)$ to two dimensional Galois representations. Let V be a continuous two dimensional representation of $G_{\mathbb{Q}_\ell}$ defined over $\overline{\mathbb{Q}}_p$. If ℓ is a prime distinct from p, then we may attach a Weil-Deligne representation

$$\sigma(V) : \mathrm{WD}_\ell \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

to V via a recipe of Deligne [29, §8] (see also [58, Thm. 4.2.1]). We may furthermore Frobenius semi-simplify $\sigma(V)$ (see [29, 8.6]) to obtain a Frobenius semi-simple Weil-Deligne representation $\sigma^{\mathrm{ss}}(V)$. We then write $\pi_\ell(V) := \pi_\ell(\sigma^{\mathrm{ss}}(V))$, and also $\pi_\ell^{\mathrm{m}}(V) := \pi_\ell^{\mathrm{m}}(\sigma^{\mathrm{ss}}(V))$. If $\ell = p$ and V is furthermore potentially semi-stable, then we may attach a Weil-Deligne representation

$$\sigma(V) : \mathrm{WD}_p \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

to V via a recipe of Fontaine [36, §2.3.7]. (See also [59], which gives a succinct description of the construction for representations having coefficients in $\overline{\mathbb{Q}}_p$.) Again, we may Frobenius semi-simplify to obtain a Frobenius semi-simple Weil-Deligne representation $\sigma^{\mathrm{ss}}(V)$, and we write $\pi_p(V) := \pi_p(\sigma^{\mathrm{ss}}(V))$, and also $\pi_p^{\mathrm{m}}(V) := \pi_p^{\mathrm{m}}(\sigma^{\mathrm{ss}}(V))$.

If V is a two-dimensional continuous $G_{\mathbb{Q}_\ell}$-representation over $\overline{\mathbb{Q}}_p$ for which $\pi_\ell(V)$ is defined (i.e. V is arbitrary if $\ell \neq p$, and V is potentially semi-stable if $\ell = p$), then we say that V is generic if $\pi_\ell(V)$ is generic, or equivalently, if $\pi_\ell(V) = \pi_\ell^{\mathrm{m}}(V)$. Otherwise, we will say that V is non-generic.

2.1.3. **Remark.** If V is defined over the finite extension E of $\mathbb{Q}_p$, then the isomorphism class of the Weil-Deligne representation $\sigma^{\mathrm{ss}}(V)$ is defined over E, and thus so is the isomorphism class of each of $\pi_p(V)$ and $\pi_p^{\mathrm{m}}(V)$ (since we have used the Tate normalization for the local Langlands correspondence). It follows that each of these representations may be defined over E. (See [20, Prop. 3.2] for the analogous statement in the case of $\mathrm{GL}_n(F)$, for any finite extension F of $\mathbb{Q}_\ell$, and any $n \geq 1$.)

2.2. **Representations of $\mathrm{GL}_2(\mathbb{A}_f)$ attached to two dimensional global Galois representations.** One may use the local Langlands correspondence to attach representations of adèlic groups to global Galois representations. Suppose that V is a two dimensional $\overline{\mathbb{Q}}_p$-vector space equipped with a continuous action of $G_{\mathbb{Q}}$ that is unramified away from a finite number of primes. For each prime ℓ distinct from p, the local Langlands correspondence recalled in the preceding subsection gives rise to an irreducible admissible smooth representation $\pi_\ell(V) := \pi_\ell(V_{|D_\ell})$ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$, defined over $\overline{\mathbb{Q}}_p$. Since V is unramified at almost all primes ℓ, the representation $\pi_\ell(V)$ lies in the unramified principal series for almost all ℓ, and so we may form the restricted tensor product

$$\pi^p(V) := \bigotimes_{\ell \neq p}{}' \, \pi_\ell(V),$$

which is an irreducible admissible smooth representation of $\mathrm{GL}_2(\mathbb{A}_f^p)$ defined over $\overline{\mathbb{Q}}_p$.

We also introduce notation for the modified version of this construction, in which we use the modified local Langlands correspondence described in 2.1.1. Namely, we define

$$\pi^{\mathrm{m},p}(V) := \bigotimes_{\ell \neq p}{}' \, \pi_\ell^{\mathrm{m}}(V).$$

This is an admissible smooth $\mathrm{GL}_2(\mathbb{A}_f^p)$-representation.

If $V_{|D_p}$ is furthermore potentially semi-stable, then we may form the irreducible admissible smooth representation $\pi_p(V) := \pi_p(V_{|D_p})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. We may then define

$$\pi(V) := \pi_p(V) \otimes \pi^p(V) = \bigotimes_{\ell}{}' \, \pi_\ell(V)$$

(the restricted tensor product now running over all primes). This is an irreducible admissible smooth representation of $\mathrm{GL}_2(\mathbb{A}_f)$. (We could also define a modified representation $\pi^{\mathrm{m}}(V)$, but we will not have need of this.)

2.2.1. **Remark.** Remark 2.1.3 shows that if V is defined over the finite extension E of $\mathbb{Q}_p$, then so are $\pi^p(V)$, $\pi^{\mathrm{m},p}(V)$, and (if $V_{|D_p}$ is potentially semi-stable) $\pi(V)$.

2.3. **Galois representations attached to newforms.** If f is a cuspidal newform of weight $k \geq 1$ and conductor N defined over $\overline{\mathbb{Q}}_p$, then associated to f is a two dimensional irreducible continuous representation V_f of the absolute Galois group $G_{\mathbb{Q}}$ (constructed by Shimura [54] when $k = 2$, by Deligne [28] when $k > 2$, and by Deligne-Serre [30] when $k = 1$). This representation is characterized by the following condition: it is unramified outside of the primes dividing Np, and for each $\ell \nmid Np$, the geometric Frobenius at ℓ has the characteristic polynomial

$$X^2 - a_\ell X + \chi(\ell)\ell^{k-1}; \tag{4}$$

here a_ℓ denotes the ℓth Hecke eigenvalue of f, and $\chi : (\mathbb{Z}/N)^\times \to \mathbb{Q}_p^\times$ denotes the nebentypus character of f. As we now recall, one can formulate a more precise relationship between the Galois representation V_f and the newform f, which takes into account the local properties of each of them at all primes (including those dividing Np), in terms of their associated $\mathrm{GL}_2(\mathbb{A}_f)$-representations.

On the one hand, there is a natural action of the group $\mathrm{GL}_2(\mathbb{A}_f)$ on the space of cuspforms over $\overline{\mathbb{Q}}_p$ of weight k and arbitrary level which makes the space of modular forms an admissible smooth representation of $\mathrm{GL}_2(\mathbb{A}_f)$. Under this action, the newform f generates an irreducible representation $\pi(f)$.[4] On the other hand, V_f is unramified away from a finite number of primes, and is known to be potentially semi-stable at p, and so we may associate to V_f an irreducible

[4]To be precise about our choice of normalization, if $\pi_u(f)$ denotes the unitary $\mathrm{GL}_2(\mathbb{A}_f)$-representation attached to f, then $\pi(f) := \pi_u(f) \otimes |\ \ |^{(2-k)/2}$.

admissible smooth representation $\pi(V_f)$ of $\mathrm{GL}_2(\mathbb{A}_f)$ via the procedure of the preceding Subsection.

2.3.1. **Theorem.** *There is a* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant isomorphism* $\pi(f) \xrightarrow{\sim} \pi(V_f)$.

This theorem expresses the compatibility of the local and global Langlands correspondence. As already intimated, it provides a significant strengthening of the formula (4) for the characteristic polynomials of Frobenius acting on V_f. (One recovers that formula from the isomorphism of the theorem by considering the action just of those groups $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for $\ell \nmid Np$.)

Both the construction of the representations V_f and the proof of Theorem 2.3.1 rely on an analysis of the étale cohomology of modular curves. We briefly recall some of the details in the following subsections.

2.4. **Cohomology of modular curves.** If K_f is an open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ (which we think of as a "level"), then we let $Y(K_f)$ denote the open modular curve over $\mathbb{Q}$ that classifies elliptic curves with a K_f-level structure (i.e. elliptic curves E equipped with a K_f-orbit of isomorphisms $(\mathbb{Q}/\mathbb{Z})^2 \xrightarrow{\sim} E_{\mathrm{tor}} := \varinjlim_n E[n]$). This curve is a fine moduli space provided that the intersection $\mathrm{SL}_2(\mathbb{Z}) \bigcap K_f$ is torsion free. Under this assumption on K_f, let pr denote the projection to $Y(K_f)$ from the universal elliptic curve over $Y(K_f)$, and define $\mathcal{V}_3 = R^1\mathrm{pr}_*\mathbb{Z}_p$. For any integer $k \geq 2$, set $\mathcal{V}_k := \mathrm{Sym}^{k-2}\mathcal{V}_3$; this is a free of rank $k-1$ étale local system of $\mathbb{Z}_p$-modules on $Y(K_f)$.

If $K'_f \subset K_f$ is an inclusion of levels, then there is an induced surjection $Y(K'_f) \to Y(K_f)$ ("pass to the underlying K_f-level structure"), and the pull-back under this map of the local system $\mathcal{V}_k$ on the target is naturally isomorphic to the corresponding local system $\mathcal{V}_k$ on the source (which justifies our omission of the level K_f from the notation for $\mathcal{V}_k$).

There is a natural right action of $\mathrm{GL}_2(\mathbb{A}_f)$ on the directed system of curves $Y(K_f)$, and the local systems $(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$ are equivariant with respect to this action. This action, and the stated equivariance, are most easily seen on the level of complex points. The Riemann surface of $\mathbb{C}$-valued points of $Y(K_f)$ is equal to

$$Y(K_f)(\mathbb{C}) := \mathrm{GL}_2(\mathbb{Q})\backslash\big(\mathbb{C} \smallsetminus \mathbb{R} \times \mathrm{GL}_2(\mathbb{A}_f)\big)/K_f$$

(where $\mathrm{GL}_2(\mathbb{Q})$ acts on $\mathbb{C} \smallsetminus \mathbb{R}$ via linear fractional transformations, and on $\mathrm{GL}_2(\mathbb{A}_f)$ via left multiplication, while K_f acts trivially on $\mathbb{C} \smallsetminus \mathbb{R}$, and on $\mathrm{GL}_2(\mathbb{A}_f)$ via right multiplication). For each integer $k \geq 2$ the pullback of $\mathcal{V}_k$ to $Y(K_f)(\mathbb{C})$ is a local system in the complex topology admitting the description

$$\mathcal{V}_{k/Y(K_f)(\mathbb{C})} := \mathrm{GL}_2(\mathbb{Q})\backslash\big(\mathbb{C} \smallsetminus \mathbb{R} \times \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{Sym}^{k-2}\mathbb{Z}_p^2\big)/K_f$$

(where the double quotient is taken with respect to the action map

$$\mathrm{GL}_2(\mathbb{Q}) \times \big(\mathbb{C} \setminus \mathbb{R} \times \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{Sym}^{k-2}\mathbb{Z}_p^2\big) \times K_f \to \mathbb{C} \setminus \mathbb{R} \times \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{Sym}^{k-2}\mathbb{Z}_p^2$$

defined via $\gamma \times (\tau, g, v) \times k \mapsto (\gamma\tau, \gamma g k, k_p^{-1} v)$; here k_p denotes the pth component of the element $k \in K_f$). The right action of $\mathrm{GL}_2(\mathbb{A}_f)$ on the complex curves $Y(K_f)(\mathbb{C})$ and on the sheaves $(\mathcal{V}_{k/Y(K_f)(\mathbb{C})})_{\overline{\mathbb{Q}}_p}$ is then defined via

$$(\tau, g) \times g' \mapsto (\tau, gg')$$

and

$$(\tau, g, v) \times g' \mapsto (\tau, gg', (g'_p)^{-1} v)$$

for $g' \in \mathrm{GL}_2(\mathbb{A}_f)$ (where g'_p denotes the pth component of g'). We refer to [42, §3] for a detailed discussion of the $\mathrm{GL}_2(\mathbb{A}_f)$-action on the curves $Y(K_f)$ and the local systems $(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$ over $\mathbb{Q}$ that underlies the action just described on the $\mathbb{C}$-points.

For any integer n, and any choice of $* \in \{\emptyset, c, \mathrm{par}\}$, we let H^n_* denote étale cohomology computed with respect to the support condition $*$: i.e. either no support condition, compact supports, or parabolic cohomology (that is, the image of compactly supported cohomology in cohomology). Now fix an integer $k \geq 2$, and define

$$H^1_*(\mathcal{V}_k) := \varinjlim_{K_f} H^1_*(Y(K_f)_{/\overline{\mathbb{Q}}}, \mathcal{V}_k).$$

This space is equipped with a natural continuous action of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and hence so is the base-change $H^1_*(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$. The right action of $\mathrm{GL}_2(\mathbb{A}_f)$ on the direct system of curves $Y(K_f)$, and the equivariance of the local systems $(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$ under this action, induces a left action of $\mathrm{GL}_2(\mathbb{A}_f)$ on $H^1_*(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$ that commutes with the $G_{\mathbb{Q}}$-action on this space.

2.5. **Cohomological realization of the classical Langlands correspondence.** Let us fix a weight $k \geq 2$. Eichler-Shimura theory, which describes the space $H^1_{\mathrm{par}}(\mathcal{V}_k)$ (with its $\mathrm{GL}_2(\mathbb{A}_f)$-action) in terms of the space of cuspforms of weight k (see [42, §2] for an account in representation-theoretic language), shows that $H^1_{\mathrm{par}}(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$ is semi-simple as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation, that its simple direct summands are precisely isomorphic to the representations $\pi(f)$ (as f runs over all cuspidal newforms of the given weight k), and that each representation $\pi(f)$ appears with multiplicity two in $H^1_{\mathrm{par}}(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$. Thus we may write

$$H^1_{\mathrm{par}}(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p} \xrightarrow{\sim} \bigoplus_f M_f \otimes_{\overline{\mathbb{Q}}_p} \pi(f),$$

where $M_f = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi(f), H^1_{\mathrm{par}}(\mathcal{V}_k))$ is a uniquely determined continuous two dimensional representation of $G_{\mathbb{Q}}$ over $\overline{\mathbb{Q}}_p$.

Since each multiplicity space M_f is a Galois subrepresentation of the étale cohomology of a curve with coefficients in the local system $(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p}$ of geometric origin, it is unramified away from a finite number of primes, and is potentially semi-stable at p. Thus we may define the $\mathrm{GL}_2(\mathbb{A}_f)$-representation $\pi(M_f)$ using the procedure described in Subsection 2.1. The key result concerning the multiplicity spaces M_f then states that for each f there is a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant isomorphism

$$\pi(M_f) \xrightarrow{\sim} \pi(f). \tag{5}$$

By considering the local factors of these two representations just at primes $\ell \nmid Np$, one finds that M_f is a model for V_f, the Galois representation associated to f. (Indeed, this is the approach taken by Deligne to constructing the representation V_f.) The isomorphism (5) then implies Theorem 2.3.1 (for f of weight $k \geq 2$). The following theorem summarizes the situation.

2.5.1. **Theorem.** *For any given $k \geq 2$, there is a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant isomorphism*

$$H^1_{\mathrm{par}}(\mathcal{V}_k)_{\overline{\mathbb{Q}}_p} \xrightarrow{\sim} \bigoplus_f V_f \otimes_{\overline{\mathbb{Q}}_p} \pi(V_f),$$

where f runs over all cuspidal newforms defined over $\overline{\mathbb{Q}}_p$ of weight k.

The isomorphism (5) was established in general by Carayol [18] (for the local factors at the primes $\ell \neq p$) and Saito [47] (for the local factors at p), building on the work of many people, including Eichler, Shimura, Igusa, Deligne, and Langlands.

Note that since V_f is determined up to isomorphism by $\pi(V_f) \xrightarrow{\sim} \pi(f)$, we see that the V_f are non-isomorphic for distinct f. Combining the conjecture of Fontaine and Mazur on geometric Galois representations [37, Conj. 1] and Langlands' conjecture relating automorphic forms and motives [20], one expects that (for fixed k) the representations V_f range over all irreducible continuous two dimensional p-adic representations of $G_{\mathbb{Q}}$ that are unramified outside of a finite number of primes, and that are potentially semi-stable at p with Hodge-Tate weights $(1-k, 0)$ (cf. [37, Conj. 3c]). Thus Theorem 2.5.1 and the Fontaine-Mazur conjecture together imply the following conjecture, on which Conjecture 1.1.1 is modelled.[5]

2.5.2. **Conjecture.** *If V is an irreducible continuous $G_{\mathbb{Q}}$-representation over $\overline{\mathbb{Q}}_p$ that is semi-stable at p with Hodge-Tate weights $(1-k, 0)$ for some $k \geq 2$, and is unramified outside of a finite set of primes, then the space $\mathrm{Hom}_{G_{\mathbb{Q}}}(V, H^1_{\mathrm{par}}(\mathcal{V}_k))$ (which is naturally a $\mathrm{GL}_2(\mathbb{A}_f)$-representation) is isomorphic to the irreducible smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation $\pi(V)$ defined in Subsection 2.2 above.*

[5] Note that in this conjecture we don't explicitly require that V should be odd, since this should follow automatically from the other assumptions on V.

3. The local p-adic Langlands conjecture for $\mathrm{GL}_{2/\mathbb{Q}_p}$

In this section we recall some ideas of Breuil and others related to a possible local p-adic Langlands conjecture.

3.1. **Motivation.** Let ℓ be a prime, and let V be a two dimensional continuous representation of the decomposition group $G_{\mathbb{Q}_\ell}$ over $\overline{\mathbb{Q}}_p$. As was recalled in Subsection 2.1, applying either the recipe of Deligne if $\ell \neq p$, or the recipe of Fontaine if $\ell = p$ and V is potentially semi-stable, we may attach a Frobenius semi-simple Weil-Deligne representation $\sigma^{\mathrm{ss}}(V) : \mathrm{WD}_\ell \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ to V, which in turn corresponds via local Langlands to an admissible smooth representation $\pi_\ell(V)$. The representation $\sigma^{\mathrm{ss}}(V)$ is in turn determined up to isomorphism by $\pi_\ell(V) := \pi_\ell(\sigma^{\mathrm{ss}}(V))$.

In the case when $\ell \neq p$, Deligne's procedure for constructing $\sigma(V)$ from V is reversible, and so if the original representation V was itself Frobenius semi-simple (as is conjectured to be the case when V arises as the restriction to $G_{\mathbb{Q}_\ell}$ of a global Galois representation attached to a cuspidal newform – this is a particular case of a conjecture of Tate [57]) then since W_ℓ is dense in $G_{\mathbb{Q}_\ell}$, we see that V is determined by $V_{|W_\ell}$. Altogether, we see that if V is Frobenius semi-simple, then it is determined up to isomorphism by the associated $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representation $\pi_\ell(V)$.

On the other hand, if $\ell = p$ and V is potentially semi-stable, then the construction of $\sigma(V)$ from V involves passing to the potentially semi-stable Dieudonné module of V (which is an admissible filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module), and then forgetting the Hodge filtration. In general, one can equip a given $(\varphi, N, G_{\mathbb{Q}_p})$-module with a filtration so as to make it an admissible filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module in many different ways. Thus in this case V is typically not determined by $\pi_\ell(V)$.

Breuil has conjectured that there should be a "local p-adic Langlands conjecture", which attaches to V a certain representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a p-adic Banach space. This representation $B(V)$ should determine V up to isomorphism. Breuil originally limited his conjecture to the case when V is potentially semi-stable; however, in light of developments in the field (in particular, the results of Colmez [27]), it seems reasonable to conjecture the existence of such a correspondence for arbitrary continuous two dimensional representations of $G_{\mathbb{Q}_p}$. Although there is so far no precise formulation of the conjecture (even in the case when V is potentially semi-stable), one can formulate a list of at least some of the properties that it should be required to satisfy. We will do this below, after first recalling some basic definitions regarding Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

3.2. **Admissible unitary Banach representations of** $\mathrm{GL}_2(\mathbb{Q}_p)$**.** Let B be an E-Banach space equipped with a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

3.2.1. **Definition.** We call B admissible unitary if there exists a norm that determines the topology of B, such that the unit ball $L \subset B$ of with respect to this norm is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant, and such that the induced $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $L/\varpi L$ is an admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathcal{O}_E/\varpi$. (Recall that ϖ denotes a uniformizer in $\mathcal{O}_E$.)

The notion of admissibility introduced here is due to Schneider and Teitelbaum [50]. We refer the reader to that reference for a number of equivalent characterizations of admissibility.

3.2.2. **Definition.** Let B be an admissible unitary Banach space of $\mathrm{GL}_2(\mathbb{Q}_p)$ and let L be a choice of $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant unit ball in B. If the admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $L/\varpi L$ is of finite length, then we let $\overline{B}$ denote the semi-simplification of $L/\varpi L$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. (Note that if the hypothesis holds for one choice of L then it holds for any such choice, and the semi-simple admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\overline{B}$ is then well-defined up to isomorphism independently of the choice of L.)

3.3. **Statement of the conjecture.** As we recalled above, in the classical local Langlands correspondence, the matching between isomorphism classes of Frobenius semi-simple two dimensional Weil-Deligne representations at p and isomorphism classes of irreducible admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$ representations (with respect to the unitary normalization) is uniquely determined by the requirement that the local L- and ε-factors attached to a pair of matched objects should coincide. No such precise formulation of a local p-adic Langlands conjecture has yet been given. However, following the ideas of Breuil, a consensus seems to have emerged that a statement of the following type should be true.

3.3.1. **Conjecture.** *There is a local p-adic Langlands correspondence that associates to each continuous two dimensional representation V of $G_{\mathbb{Q}_p}$ over E a corresponding non-zero admissible unitary Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E, well-defined up to $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant topological isomorphism. This correspondence satisfies the following properties:*

1. *If V and V' are continuous two dimensional representations of $G_{\mathbb{Q}_p}$ over E, then V and V' are isomorphic if and only $B(V)$ and $B(V')$ are $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariantly topologically isomorphic.*
2. *If V has determinant χ, then $B(V)$ has central character $\chi\varepsilon$.*
3. *For any continuous character $\chi : G_{\mathbb{Q}_p} \to E^\times$, there is an isomorphism*
$$B(V \otimes \chi) \xrightarrow{\sim} B(V) \otimes (\chi \circ \det)$$
(where on the right hand side we regard χ as a character of $\mathbb{Q}_p^\times$ via local class field theory).
4. *The association of $B(V)$ to V is compatible with extending scalars to any finite extension of E.*

(5) *The representation* $B(V)$ *satisfies the hypothesis of Definition 3.2.2, and* $\overline{B}(V)$ *is associated to the semi-simplification* $\overline{V}$ *of the reduction mod* ϖ *of* V *via the local mod* ϖ *Langlands correspondence of* [10, Déf. 1.1].[6]

(6) *If* V *is irreducible, then* $B(V)$ *is topologically irreducible. In general, if* V^{ss} *denotes the semi-simplification of* V*, then the representations* $B(V)$ *and* $B(V^{\mathrm{ss}})$ *have isomorphic topological semi-simplifications.*

(7) *The two dimensional representation* V *is potentially semi-stable, with distinct Hodge-Tate weights, if and only if the subspace* $B(V)_{\mathrm{lalg}}$ *of locally algebraic vectors in* $B(V)$ *is non-zero. Furthermore, if these conditions hold, then (letting* $w_1 < w_2$ *denote the Hodge-Tate weights of* V*) the subrepresentation* $B(V)_{\mathrm{lalg}}$ *is isomorphic to the locally algebraic representation*

$$\widetilde{\pi}_p(V) := \pi_p^{\mathrm{m}}(V) \otimes \left(\mathrm{Sym}^{w_2-w_1-1} E^2\right) \otimes \det{}^{w_1+1}$$

(where $\pi_p^{\mathrm{m}}(V)$ *is the admissible smooth representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *attached to* V *via the modified classical local Langlands correspondence, as explained in 2.1.2; note that by Remark 2.1.3 this representation is defined over* E*).*

(8) *The set of refinements of* V *is related to the structure of* $J_{\mathrm{P}(\mathbb{Q}_p)}(B(V)_{\mathrm{an}})$ *in the following manner: for any character* $\eta \otimes \psi \in \widehat{\mathrm{T}}(E)$ *there is an equality of dimensions*

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V) = \dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}).$$

Here $J_{\mathrm{P}(\mathbb{Q}_p)}$ *denotes the locally analytic Jacquet module functor of* [32] *(see also Subsection 5.2 below), while* $\mathrm{Ref}^{\eta\otimes\psi}(V)$ *and* $\mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}})$ *are both projective spaces over* E*, whose definitions can be found in Definitions 4.1.7 and 5.2.3 below respectively.*

The preceding conjecture merits several remarks.

3.3.2. **Remark.** Conditions (1), (2), (3), and (4) are standard requirements of any Langlands-type conjecture. (The appearance of the factor ε in condition (2) reflects our particular choice of normalization.) By imposing condition (5), expressing the compatibility of the conjectured correspondence with reduction modulo ϖ, we have incorporated one of Breuil's original motivations for introducing a local p-adic Langlands conjecture.

3.3.3. **Remark.** Condition (6) reflects one way in which the conjectured local p-adic Langlands correspondence is quite different from the classical correspondence, in that the representations $B(V)$ will in some instances be topologically reducible. In fact the analysis of the reducible case that we make in Section 6

[6] We normalize this mod ϖ correspondence so that it satisfies the analogue of condition (2) above.

below shows that for any correspondence satisfying Conjecture 3.3.1, the representation $B(V)$ is irreducible (resp. indecomposable) if and only if V is irreducible (resp. indecomposable).

3.3.4. **Remark.** Condition (7) relates the conjectural p-adic correspondence to the classical Langlands correspondence. Note that for a potentially semi-stable representation V with distinct Hodge-Tate weights, the locally algebraic representation $\widetilde{\pi}_p(V)$ encodes the Hodge-Tate weights of V, together with the Weil-Deligne representation $\sigma(V)$ attached to V by Fontaine's recipe – or equivalently, the $(\varphi, N, G_{\mathbb{Q}_p})$-module underlying $D_{\mathrm{pst}}(V)$. [7] Breuil's idea is that the Banach space $B(V)$ (at least when V is irreducible) should be regarded as a completion of $\widetilde{\pi}_p(V)$ with respect to a certain $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm (depending on V), and that the extra data of this norm should actually determine the Hodge filtration on $D_{\mathrm{pst}}(V)$.

In some instances the representation $\widetilde{\pi}_p(V)$ admits only one non-zero unitary completion up to isomorphism. If this is the case, and if Conjecture 3.3.1 holds, then (taking into account the preceding discussion together with condition (1) of Conjecture 3.3.1) we see that V would have to be uniquely determined by the $(\varphi, N, G_{\mathbb{Q}_p})$-module that underlies $D_{\mathrm{pst}}(V)$ (i.e. the Hodge filtration would have to be uniquely determined, up to isomorphism, by the weak admissibility condition). This latter condition holds precisely when V is irreducible and potentially crystalline and $\sigma(V)$ is abelian (equivalently, $\pi_p(V)$ is principal series), and indeed for such V, Breuil has conjectured that $\widetilde{\pi}_p(V)$ does admit a unique non-zero unitary completion (up to isomorphism); see Conjecture 5.1.5 below. (This conjecture has been proved by Berger and Breuil [8] for Frobenius semi-simple V, as we recall in Theorem 5.1.6 below.)

If $\pi_p(V)$ is special or cuspidal then V is not determined by the $(\varphi, N, G_{\mathbb{Q}_p})$-module underlying $D_{\mathrm{pst}}(V)$, and so Conjecture 3.3.1 implies that $\widetilde{\pi}_p(V)$ should admit a whole family of admissible unitary completions, corresponding to the different possible ways of putting a Hodge filtration on the $(\varphi, N, G_{\mathbb{Q}_p})$-module underlying $D_{\mathrm{pst}}(V)$ so as to make it a weakly admissible filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module. Such a family has been constructed in the case when $\pi_p(V)$ is special, i.e. when V is potentially semi-stable but not potentially crystalline. (See Theorem 5.1.13 below.)

3.3.5. **Remark.** As the discussions of Sections 5 and 6 below will make clear, Condition (8) is an analogue in the p-adic setting of the requirement in the classical Langlands correspondence that reducible Weil-Deligne representations

[7] Note that, in contrast to the ℓ-adic setting, no information is lost about the φ-action on $D_{\mathrm{pst}}(V)$ in the process of Frobenius semi-simplifying when one passes from V to $\pi_p(V)$. Indeed, in those cases when $\sigma(V)$ contains two copies of the same character, the weak admissibility of $D_{\mathrm{pst}}(V)$ implies that the φ-action on $D_{\mathrm{pst}}(V)$ is necessarily *not* semi-simple; cf. the proof of [23, Thm. 3.1].

be matched with subrepresentations of parabolically induced representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

3.3.6. **Remark.** The conditions of Conjecture 3.3.1 are not enough to uniquely specify the local p-adic correspondence. As we have explained in Remark 3.3.4, these conditions imply that if V is potentially semi-stable and $\pi_p(V)$ is special or cuspidal, then $\widetilde{\pi}_p(V)$ should admit a family of admissible unitary completions corresponding to the various potentially semi-stable representations V' for which $\widetilde{\pi}_p(V') = \widetilde{\pi}_p(V)$. However, there is no specification in our conjecture of the particular way that the representations V' and the completions of $\widetilde{\pi}_p(V)$ (assuming that they do exist) should be matched.

Nevertheless, we expect that there will be a "natural" way to match each of the representations V' under consideration with a certain specific completion of $\widetilde{\pi}_p(V)$. (For example, in the case when $\pi_p(V)$ is special, the various representations V' are classified by their $\mathcal{L}$-invariants, and Breuil has defined a family of completions of $\widetilde{\pi}_p(V)$ that also depend on a parameter $\mathcal{L}$ – see [12] and also 5.1.7 below.) One would like to find a condition (or conditions) to add to Conjecture 3.3.1 that would uniquely determine the correspondence (analogous to the condition on L-factors and ε-factors in the classical correspondence). Recent work of Colmez suggests one possible such condition (see the following remark).

3.3.7. **Remark.** When V is irreducible and trianguline the correspondence $V \mapsto B(V)$ has been specified – "by hand", as it were – by Breuil and Colmez [12, 27]. It has also been specified for most reducible potentially crystalline V by Breuil and the author [6, 14]. We will recall the definitions of the representations $B(V)$ in the cases where they have been defined in Section 6 below, and also consider the extent to which this explicit correspondence satisfies the conditions of Conjecture 3.3.1.

When V is irreducible, trianguline, and not potentially crystalline, Colmez has shown that there is in fact an intrinsic relationship between V and the representation $B(V)$ that is associated to V; namely, one can recover $B(V)$ as a $\mathrm{P}(\mathbb{Q}_p)$-representation directly from the overconvergent (φ, Γ)-module $D^\dagger(V)$ attached to V. Building on Colmez's methods, Berger and Breuil have done the same for those V that are crystalline and Frobenius semi-simple. It seems likely that some suitable formulation of this relationship between V and $B(V)$ will be a key condition in the ultimate statement of the local p-adic correspondence. However, we have not wanted to try to predict the general form that this relationship should take, and for this reason have not incorporated Colmez's results into the statement of Conjecture 3.3.1.

3.3.8. **Remark.** One point emphasized by Breuil in his initial postulation of a local p-adic correspondence is that the structure of the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation on the space of locally analytic vectors $B(V)_{\mathrm{an}}$ should reflect the p-adic Hodge theory of the Galois representation V (see [12, §1.3]). This requirement does not appear to have been formalized as of yet, and so we have not incorporated it as one of the conditions in Conjecture 3.3.1. However, in Subsection 6.7 we will give a conjectural description of the space $B(V)_{\mathrm{an}}$ in the case when V is trianguline. Comparing this conjecture with the work of Colmez, Berger, and Breuil discussed in the preceding remark, which relates $B(V)$ to $D^{\dagger}(V)$, suggests that there may be an analogous relationship between $B(V)_{\mathrm{an}}$ and $D_{\mathrm{rig}}(V)$ (see [3, 27] for a definition of the latter object). Establishing such a relationship could be one approach to formalizing Breuil's proposed link between $B(V)_{\mathrm{an}}$ and the p-adic Hodge theory of V.

I should note, though, that there is one condition relating $B(V)_{\mathrm{an}}$ to the p-adic Hodge theory of V that can be formulated precisely. Namely, it should be the case that $B(V)_{\mathrm{an}}$ admits a generalized infinitesimal character, and this character (suitably normalized) should match with the Hodge-Sen-Tate weights of V. (I would like to thank Michael Harris for pointing this out to me.) Our conjecture on the structure of $B(V)_{\mathrm{an}}$ for trianguline V is compatible with this requirement.

3.3.9. **Remark.** As in the classical case, one test of the correctness of the proposed local correspondence will be its compatibility with the global situation. Thus there is an interplay between the local Conjecture 3.3.1 and the local-global compatibility Conjecture 1.1.1. The various results proved in Subsections 7.8, 7.9, and 7.10 below in the direction of Conjecture 1.1.1 may all be regarded as evidence for the correctness of the formulation of the local conjecture.

3.3.10. **Remark.** In this remark we discuss the possible "surjectivity" of the local p-adic correspondence. We begin with the following definition (which extends to the characteristic zero situation terminology introduced in [2, §7] by Barthel and Livné for mod ϖ representations of $\mathrm{GL}_2(\mathbb{Q}_p)$).

3.3.11. **Definition.** Let B be a topologically irreducible admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. We say that B is ordinary if it is a subquotient of $\left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\right)_{\mathrm{cont}}$ for some pair of unitary E-valued characters χ_1 and χ_2 of $\mathbb{Q}_p^{\times}$. Otherwise, we say that B is supersingular.

The ordinary representations are precisely the ones appearing in the list of Corollary 5.3.6 below, together with those of the form $\eta \circ \det$ for unitary characters η of $\mathbb{Q}_p^{\times}$ (as follows from Proposition 5.3.4 below).

Condition (8) of Conjecture 3.3.1, when combined with the results of Subsection 4.4 and with Proposition 5.2.1 and Lemma 5.3.3 (2) below, implies that if V is irreducible then $B(V)$ is supersingular. This observation naturally suggests the following question.

3.3.12. **Question.** If B is a supersingular topologically irreducible admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, then is B isomorphic to a representation of the form $B(V)$ for some irreducible continuous two dimensional $G_{\mathbb{Q}_p}$-representation V?

Given the current state of our knowledge, it seems reasonable to hope that this question has an affirmative answer. Certainly, a large number of the conjectures that we make in Sections 5 and 6 below are premised on the expectation that various known phenomena in the context of $G_{\mathbb{Q}_p}$-representations will be mirrored by corresponding phenomena in the context of admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations.

Finally, we note that the preceding discussion shows that the ordinary representations are never of the form $B(V)$ for an irreducible $G_{\mathbb{Q}_p}$-representation V. In fact we will see in the discussion of Section 6 that they appear as constituents of the representations $B(V)$ attached to reducible V. This parallels the case of the local mod ϖ Langlands correspondence defined in [10, Déf. 1.1].

4. Refinements and trianguline local Galois representations

In this section we first introduce a notion of refinement of a continuous two dimensional representation V of $G_{\mathbb{Q}_p}$ that generalizes the one introduced by Mazur for crystalline representations in [45]. This notion is closely related to that of trianguline representations: indeed, one can define the trianguline continuous two dimensional representation of $G_{\mathbb{Q}_p}$ to be those that admit a refinement. We next classify the possible refinements of a trianguline two dimensional representation, before finally discussing the classification (due to Colmez [27]) of trianguline two dimensional representations themselves.

4.1. **Refinements.** Let V be a continuous two dimensional representation of $G_{\mathbb{Q}_p}$ over E.

4.1.1. **Definition.** A refinement of V is a triple $R = (\eta, \alpha, r)$, where:

(1) η is a continuous character $G_{\mathbb{Q}_p} \to E^\times$ such that $V \otimes \eta^{-1}$ has at least one Hodge-Sen-Tate weight equal to 0;
(2) $\alpha \in E^\times$;
(3) r is a non-zero $G_{\mathbb{Q}_p}$-equivariant E-linear map $V^\vee \otimes \eta \to (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\varphi=\alpha}$.

Note that if (η, α, r) is a refinement of V, then in particular $(E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\varphi=\alpha} \neq 0$, and so $\alpha \in \mathcal{O}_E \setminus \{0\}$. Note also that we may regard the morphism r as a non-zero element of $D^+_{\mathrm{crys}}(V \otimes \eta^{-1})^{\varphi=\alpha}$.

4.1.2. **Definition.** We say that a pair of refinements $R = (\eta, \alpha, r)$ and $R' = (\eta', \alpha', r')$ of V are equivalent if there exists $\beta \in \mathcal{O}_E^\times$ and $0 \neq x \in (E \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_p))^{\varphi=\beta}$ such that $r' = x \cdot r$ (and hence such that $\eta' = \eta\,\mathrm{ur}(\beta^{-1})$ and $\alpha' = \alpha\beta$).

Recall that $T(E) = (E^\times)^2$ (regarded as the diagonal torus of $\mathrm{GL}_2(E)$).

4.1.3. **Definition.** If $R = (\eta, \alpha, r)$ is a refinement of V, we define the associated abelian Weil group representation to be the map $\sigma(R) : \mathrm{W}_p^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{Q}_p^\times \to T(E)$ defined via the pair of characters $(\eta\mathrm{ur}(\alpha), (\det V)\eta^{-1}\mathrm{ur}(\alpha)^{-1})$.

By construction the determinant of $\sigma(R)$ coincides with the restriction to $\mathrm{W}_p^{\mathrm{ab}}$ of the determinant of V. Also, $\sigma(R)$ depends only on R up to equivalence. In fact we have the following more precise result.

4.1.4. **Lemma.** *Suppose that V is not of the form $\eta \bigoplus \eta$ for some continuous E-valued character η of $G_{\mathbb{Q}_p}$. If R and R' are two refinements of V, then $\sigma(R) = \sigma(R')$ if and only if R and R' are equivalent.*

Proof. We have already observed that $\sigma(R) = \sigma(R')$ if R and R' are equivalent. We now prove the converse. Write $R = (\eta, \alpha, r)$ and $R' = (\eta', \alpha', r')$. By definition, $\sigma(R) = \sigma(R')$ if and only if $\eta\mathrm{ur}(\alpha) = \eta'\mathrm{ur}(\alpha')$. If the latter holds, then (since η and η' are Galois characters, and hence unitary) we see that $\beta := \alpha'/\alpha \in \mathcal{O}_E^\times$. Fix $0 \neq x \in (E \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_p))^{\varphi=\beta}$. Replacing R by the equivalent refinement $(\eta\mathrm{ur}(\beta^{-1}), \alpha\beta, x \cdot r)$, we see that we may assume that $\eta = \eta'$ and $\alpha = \alpha'$. If r and r' are not linearly dependent over E, then we find that $D^+_{\mathrm{crys}}(V \otimes \eta^{-1})^{\varphi=\alpha}$ is two dimensional, and thus that $V \otimes \eta^{-1}$ is a crystalline representation such that $D_{\mathrm{crys}}(V \otimes \eta^{-1}) = D^+_{\mathrm{crys}}(V \otimes \eta^{-1})^{\varphi=\alpha}$. It follows directly that $V \otimes \eta^{-1}$ is isomorphic to the direct sum of two copies of some crystalline E-valued character of $G_{\mathbb{Q}_p}$, and thus that V is isomorphic to the direct sum of two copies of some continuous character of $G_{\mathbb{Q}_p}$. □

The exceptional case of the preceding lemma is easily analyzed by hand:

4.1.5. **Lemma.** *If $V = \eta \bigoplus \eta$ for some continuous representation $\eta : G_{\mathbb{Q}_p} \to E^\times$, then any equivalence class of refinements of V contains a unique refinement of the form $(\eta, 1, r)$ where r is a non-zero homomorphism $V \otimes \eta^{-1} = E \bigoplus E \to E = (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\varphi=1}$. Conversely, any such r gives rise to a refinement $(\eta, 1, r)$ of V. Thus the equivalence classes of refinements of V are indexed by the points of $\mathbb{P}^1(E)$.*

Proof. If (η', α', r') is a refinement of V, then $\eta'\eta^{-1}$ has Hodge-Sen-Tate weight zero, and also admits an embedding into $E \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$. Thus it must be unramified. Thus (η', α', r') is equivalent to a refinement of the form $(\eta, 1, r)$, which is then clearly unique in its equivalence class. The remainder of the lemma is clear. □

4.1.6. **Definition.** We say that a continuous two dimensional $G_{\mathbb{Q}_p}$-representation V is trianguline if it admits at least one refinement.

The notion of a trianguline $G_{\mathbb{Q}_p}$-representation (of arbitrary finite dimension) was introduced by Colmez in [27, Déf. 5.1] (cf. [27, Prop. 5.3] for a proof of the equivalence between the definition given here in the two dimensional case and the definition given by Colmez).

4.1.7. **Definition.** *Let V be a continuous two dimensional representation of $G_{\mathbb{Q}_p}$ over E.*

(1) *Let* $\mathrm{Ref}(V)$ *denote the set of equivalence classes of refinements.*
(2) *For any* $\sigma \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))$, *set* $\mathrm{Ref}^{\sigma}(V) := \{[R] \in \mathrm{Ref}(V) \mid \sigma(R) = \sigma\}$. *(Here $[R]$ denotes an equivalence class of refinements, with representative R.)*

If we fix σ, then $\mathrm{Ref}^{\sigma}(V)$ is either empty or a point, except in the exceptional case $V = \eta \bigoplus \eta$ considered in Lemma 4.1.5, in which case $\mathrm{Ref}^{\eta\otimes\eta}(V) \cong \mathbb{P}^1(E)$. Thus we regard $\mathrm{Ref}^{\sigma}(V)$ as a projective space over E of dimension -1, 0, or 1.

4.1.8. *Twisting refinements.* Let $\psi : G_{\mathbb{Q}_p} \to E^{\times}$ be a continuous character. If $R = (\eta, \alpha, r)$ is a refinement of V, then $R \otimes \psi := (\eta \otimes \psi, \alpha, r)$ is a refinement of $V \otimes \psi$. The passage from R to $R \otimes \psi$ evidently induces a bijection between the set of (equivalence classes of) refinements of V and of $V \otimes \psi$. In particular, V is trianguline if and only if $V \otimes \psi$ is. The formation of Weil group representations is clearly compatible with twisting: i.e. $\sigma(R \otimes \psi) = \sigma(R) \otimes \psi$.

4.2. **Classifying refinements: the non-potentially semi-stable case.** We begin by recalling the following proposition, due to Colmez.

4.2.1. **Proposition.** *Let V be a two dimensional continuous representation of $G_{\mathbb{Q}_p}$ over E, and suppose that η_1 and η_2 are two continuous characters of $G_{\mathbb{Q}_p}$ over E such that $D_{\mathrm{crys}}(V \otimes \eta_1^{-1})$ and $D_{\mathrm{crys}}(V \otimes \eta_2^{-1})$ are both non-zero. If the weights of η_1 and η_2 do not differ by an integer, then V is the direct sum of two continuous characters.*

Proof. This a restatement of [27, Prop. 5.10]. □

The following corollary of the preceding proposition will provide the key to classifying the possible refinements of two dimensional $G_{\mathbb{Q}_p}$-representations.

4.2.2. **Corollary.** *If V is a trianguline two dimensional continuous representation of $G_{\mathbb{Q}_p}$ over E that is not a twist of a representation of Hodge-Tate type, and does not split as the direct sum of two continuous characters of $G_{\mathbb{Q}_p}$, then V admits a unique equivalence class of refinements.*

Proof. Since V is trianguline, it admits a refinement $R = (\eta, \alpha, r)$. Replacing V by $V \otimes \eta^{-1}$, we may assume that V has a Hodge-Sen-Tate weight of zero, but is not Hodge-Tate, and that $\eta = \underline{1}$, so that r is a non-zero $G_{\mathbb{Q}_p}$-equivariant morphism $r : V^\vee \to (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\varphi=\alpha}$. If $(0, w)$ are the Hodge-Sen-Tate weights of V, then since V is not Hodge-Tate, either $w = 0$ or $w \notin \mathbb{Z}$.

Now suppose that $R' = (\eta', \alpha', r')$ is another refinement of V. Since V has Hodge-Sen-Tate weights $(0, w)$, we see that the weight of η' is either 0 or w. As $(\underline{1}, \alpha, r)$ and (η', α', r') are each refinements of V, we also see that $D_{\mathrm{crys}}(V)$ and $D_{\mathrm{crys}}(V \otimes (\eta')^{-1})$ are both non-zero. Recalling that w is either zero or non-integral, and that by assumption V does not split as the direct sum of two characters, we infer from Proposition 4.2.1 that η' must in fact be of weight zero.

If we choose $0 \neq x \in (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{G_{\mathbb{Q}_p} = \eta'^{-1}}$, then $x \cdot r'$ induces a $G_{\mathbb{Q}_p}$-equivariant map $V^\vee \to E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}$. Thus both r and $x \cdot r'$ induce elements of $D_{\mathrm{dR}}(V)$. Since V is assumed to not be Hodge-Tate, it is in particular not de Rham, and thus $x \cdot r'$ must be a scalar multiple of r. Consequently η' must be a crystalline character of weight zero, which is to say, an unramified character. Thus we see that R' is equivalent to R. □

4.2.3. **Definition.** We say that a refinement $R = (\eta, \alpha, r)$ of V is ultracritical if the composite $r : V^\vee \otimes \eta \to (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\varphi=\alpha} \subset E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}} \xrightarrow{\theta} \mathbb{C}_p$ vanishes. (The terminology is motivated by a comparison of these refinements with the "critical" refinements introduced in Definition 4.4.7 below; see in particular Definition 4.5.2 and Lemma 4.5.3.)

Clearly the property of being ultracritical depends only on the equivalence class of the refinement R. If R is ultracritical, then $V^\vee \otimes \eta$ is necessarily Hodge-Tate, with weights $(0, w)$ for some $w > 0$. One may then define another refinement $R(-w) := (\eta \cdot \varepsilon^{-w}, \alpha p^{-w}, t^{-w} \cdot r)$ of V (here t has its usual meaning as an element of $E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}}$), whose associated Weil group representation is defined via the characters $(\eta \mathrm{ur}(\alpha) z^{-w}, \det(V)\eta^{-1}\mathrm{ur}(\alpha)^{-1} z^w)$.

4.2.4. **Proposition.** *If V is Hodge-Tate and trianguline, but not potentially semi-stable, then V admits a unique ultracritical refinement R, and any refinement of V is equivalent either to R or to $R(-w)$.*

Proof. Since V is assumed to be Hodge-Tate but not potentially semi-stable, its Hodge-Sen-Tate weights are distinct integers, say $w_1 < w_2$. If $R = (\eta, \alpha, r)$ is any refinement of V, then since $V \otimes \eta^{-1}$ has zero as a Hodge-Sen-Tate weight, we see that η is Hodge-Tate of weight either w_1 or w_2, hence potentially semi-stable, and thus $V \otimes \eta^{-1}$ is also Hodge-Tate but not potentially semi-stable – and so also not de Rham [3]. Thus $D_{\mathrm{dR}}(V \otimes \eta^{-1})$ is at most one dimensional over E. The argument of [41, Thm. 6.9] shows that in fact $D_{\mathrm{dR}}(V \otimes \eta^{-1})$ is precisely one dimensional over E, and is supported in filtration degree either 0 (if η is of weight w_1) or $w_2 - w_1$ (if η is of weight w_2). In particular, if $D_{\mathrm{crys}}(V \otimes \eta^{-1})$ is non-zero, then it is one dimensional, and supported in filtration degree either 0 or $w_2 - w_1$. Furthermore, if η and η' are two characters of weights $w, w' \in \{w_1, w_2\}$, and if $D_{\mathrm{crys}}(V \otimes \eta^{-1}) \neq 0$, then $D_{\mathrm{crys}}(V \otimes \eta'^{-1}) \neq 0$ if and only if $\eta'\eta^{-1}$ is crystalline, and hence the product of an unramified character and $\varepsilon^{w'-w}$.

Altogether, we see that if V is trianguline, then for each w_i ($i = 1, 2$) we may find a character η_i of weight w_i, uniquely determined up to multiplication by an unramified character, such that $D^+_{\mathrm{crys}}(V \otimes \eta_i^{-1})$ is non-zero (and hence one dimensional over E), and is supported in filtration degree 0 (respectively $w_2 - w_1$) if $i = 1$ (resp. $i = 2$), and such that $\eta_2 = \varepsilon^{w_2 - w_1}\eta_1$. Thus if we let α_i denote the eigenvalue of φ on $D^+_{\mathrm{crys}}(V \otimes \eta_i^{-1})$, and let r_i denote a non-zero element of this space, then we obtain refinements $R_i := (\eta_i, \alpha_i, r_i)$ of V such that any refinement of V is equivalent to one of the R_i. Note that R_2 is ultracritical, and that $R_1 = R_2(-(w_2 - w_1))$. This proves the lemma. □

4.3. **Classifying refinements: the potentially semi-stable case.** We now consider the case of potentially semi-stable representations.

4.3.1. **Proposition.** *If V is a potentially semi-stable continuous two dimensional representation of $G_{\mathbb{Q}_p}$ over E, then V is trianguline if and only if the Weil-Deligne representation $\sigma(V)$ contains a one dimensional subrepresentation that can be defined over E. If V is furthermore indecomposable, then the refinements of V are in bijection with such one dimensional Weil-Deligne subrepresentations of $\sigma(V)$.*

Proof. Replacing V by a twist if necessary, we may assume that V has Hodge-Tate weights 0 and $w \leq 0$. Thus if $R = (\eta, \alpha, r)$ is any refinement of V, the Hodge-Tate weight of η is either 0 or w. Suppose that $w < 0$, and that the Hodge-Tate weight of η equals w. Then the Hodge-Tate weights of $V^\vee \otimes \eta$ are w and 0, and the existence of the non-zero homomorphism $r : V^\vee \otimes \eta \to (E \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^+$ shows that $V^\vee \otimes \eta$ is an extension of a character of Hodge-Tate weight 0 by a character of Hodge-Tate weight w. Since $V^\vee \otimes \eta$ is potentially semi-stable by assumption, this extension must split, and thus $V^\vee \otimes \eta$, and so also V itself, must be a direct sum of two characters. In this case $\sigma(V)$ is also a direct sum of two characters, each definable over E, and V is certainly trianguline. (See Proposition 4.4.5 for a detailed analysis of this case.)

Thus we may suppose for the rest of the proof that V is indecomposable, in which case if $R = (\eta, \alpha, r)$ is any refinement of V, the character η is of Hodge-Tate weight zero. The homomorphism r then corresponds to a non-zero element of $D_{\mathrm{crys}}(V \otimes \eta^{-1})^{\varphi = \alpha}$, or equivalently, a non-zero element of $\sigma(V)^{\mathrm{W}_p = \eta\mathrm{ur}(\alpha), N = 0}$. Conversely, any such non-zero element gives rise to a refinement (since our assumption that V is indecomposable with Hodge-Tate weights $w \leq 0$ implies that

$$D^+_{\mathrm{crys}}(V \otimes \eta^{-1})^{\varphi = \alpha} = D_{\mathrm{crys}}(V \otimes \eta^{-1})^{\varphi = \alpha}$$

for any η of Hodge-Tate weight zero and any eigenvalue α). The proposition follows. □

4.3.2. **Remark.** If V is potentially semi-stable, but not potentially crystalline, and indecomposable, then from the proposition we see that V is trianguline, and that V admits a unique refinement R, up to equivalence. If $w_1 < w_2$ are the Hodge-Tate weights of V, then the reader can easily compute that $\sigma(R) = (\eta, \eta\,|\;\;|^{-1}\, z^{w_1 - w_2})$ for some character η of $\mathbb{Q}_p^\times$.

4.3.3. **Remark.** If V is potentially crystalline, trianguline, and indecomposable, then V admits two equivalence classes of refinements if V is Frobenius semi-simple, and one equivalence class otherwise. Let $w_1 \leq w_2$ be the Hodge-Tate weights of V. In the Frobenius semi-simple case, if R_1 and R_2 are two inequivalent refinements of V, and if $\sigma(R_1) = (\eta, \psi)$, then the reader can easily check that $\sigma(R_2) = (\psi z^{w_2 - w_1}, \eta z^{w_1 - w_2})$. If V is not Frobenius semi-simple, and if R is a refinement of V, then $\sigma(R) = (\eta, \eta z^{w_1 - w_2})$ for some character η of $\mathbb{Q}_p^\times$.

4.4. **Classifying refinements: the reducible case.**

4.4.1. **Lemma.** *Let $R = (\eta, \alpha, r)$ be a refinement of V. Consider the following conditions:*

1. *One has $\alpha \in \mathcal{O}_E^\times$;*
2. *For some (equivalently, any) refinement $R' = (\eta', \alpha', r')$ equivalent to R, one has $\alpha' \in \mathcal{O}_E^\times$;*
3. *The equivalence class of R contains a refinement of the form $R' = (\eta', 1, r')$, which is then necessarily unique up to scaling r' by an element of $E^\times$;*
4. *The image of r is one dimensional, and R is not ultracritical;*
5. *For some (equivalently, any) refinement $R' = (\eta', \alpha', r')$ equivalent to R, the image of r' is one dimensional, and R' is not ultracritical;*
6. *V is reducible.*

Then (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) $\Longrightarrow$ (6).

Proof. It is clear from the definitions that items (1), (2), and (3) are equivalent, and also that items (4) and (5) are equivalent. Since $(E \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}^+)^{\varphi=1} = E$ is one dimensional, we see that (3) implies (5). Also, if the image of r is one dimensional, then V must be reducible, and so (5) implies (6). This image is then a crystalline character of $G_{\mathbb{Q}}$ which embeds into $(E \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}^+)^{\varphi=\alpha}$, and so is of non-negative Hodge-Tate weight. If R is not ultracritical, then it must in fact be of Hodge-Tate weight zero, and hence unramified. Thus $\alpha \in \mathcal{O}_E^\times$, and we see that (4) implies (1). □

4.4.2. **Definition.** We say that a refinement $R = (\eta, \alpha, r)$ of V is ordinary if it satisfies the equivalent conditions (1), (2), (3), (4), and (5) of the preceding lemma.

The preceding lemma shows that any refinement equivalent to an ordinary refinement of V is also ordinary, and that V admits an ordinary refinement only if V is reducible. More precisely:

4.4.3. **Lemma.** *If R is an ordinary refinement of V, and if $\sigma(R) = (\eta, \psi)$ is the associated Weil group representation, then each of η and ψ is a unitary character of $\mathbb{Q}_p^\times$ (and thus extends uniquely to a continuous character of $G_{\mathbb{Q}_p}$), and V is an extension of ψ by η.*

Proof. Consider the unique refinement of the form $(\eta, 1, r)$ in the equivalence class of R. The map r then realizes E (with trivial Galois action) as a quotient of $V^\vee \otimes \eta$, and thus η embeds as a one dimensional subrepresentation of V. Since $\psi = \det(V)\eta^{-1}$ by definition, the lemma follows. □

We now prove a series of results that completely classify the refinements of reducible two dimensional continuous representations V of $G_{\mathbb{Q}_p}$. We begin with the case when V is reducible but indecomposable.

4.4.4. **Proposition.** *If V may be written as a non-split extension*

$$0 \to \eta \to V \to \psi \to 0,$$

where η and ψ are two continuous characters of $G_{\mathbb{Q}_p}$, then V admits a unique equivalence class of ordinary refinements (and so in particular is trianguline). Furthermore, if we we let w denote the weight of $\psi\eta^{-1}$, then:

(1) *If $w \notin \mathbb{Z}$ (so that V is not of Hodge-Tate type up to twist), then V admits just one equivalence class of refinements, namely the class of ordinary refinements;*
(2) *If $w \in \mathbb{Z}_{>0}$ (so that V is of Hodge-Tate type, but not potentially semi-stable, up to a twist), then V admits two equivalence classes of refinements, namely the class of ordinary refinements, and the class of ultra-critical refinements;*
(3) *If $w \in \mathbb{Z}_{<0}$ (so that V is potentially semi-stable up to a twist), then:*
 (a) *If V is furthermore potentially crystalline up to a twist, then V admits two equivalence classes of refinements, namely the class of ordinary refinements, and one class of non-ordinary refinements;*
 (b) *If V is not potentially crystalline up to a twist, then V admits a unique equivalence class of refinements, namely the ordinary equivalence class;*
(4) *If $w = 0$ (so that either V is unramified and Frobenius non-semi-simple, up to a twist, or else V is not of Hodge-Tate type up to a twist), then V admits a unique equivalence class of refinements, namely the ordinary equivalence class.*

Proof. The fact that V admits a unique equivalence class of ordinary refinements follows immediately from Lemma 4.4.3 and our assumption on V, and claims (1) and (2) are direct consequences of Propositions 4.2.2 and 4.2.4 respectively.

To prove claim (3), we may replace V by a twist if necessary, and so assume that V is potentially semi-stable. The claim then follows from Proposition 4.3.1.

The non-Hodge-Tate case of claim (4) again follows from Proposition 4.2.2. Finally, if V is a twist of an unramified but non-semi-simple representation, then, replacing V by a twist, if necessary, we may assume that in fact V is an unramified non-split extension $0 \to \eta \to V \to \eta \to 0$. (The two characters appearing in V are necessarily equal, since V is unramified but non-split.) Now suppose that $R = (\eta', \alpha, r)$ is a refinement of V. Since V has both Hodge-Tate weights equal to zero, the same must be true of η'. Similarly, since η is a crystalline character, the same must be true of η'. Thus η' is in fact unramified, and so we may replace R by an equivalent refinement R' of the form $R' = (\underline{1}, \alpha', r')$. The map r' is a $G_{\mathbb{Q}_p}$-equivariant non-zero homomorphism $V^\vee \to (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\varphi=\alpha'}$. Since V (and so $V^\vee$) is not semi-simple, the map r cannot be an embedding. Thus R' is necessarily an ordinary refinement of V, as claimed. □

We now consider the case when V is the direct sum of distinct characters.

4.4.5. **Proposition.** *Suppose that* $V = \eta_1 \bigoplus \eta_2$ *for a pair of distinct characters* $\eta_1 \neq \eta_2$ *of* $G_{\mathbb{Q}_p}$. *Then* V *admits exactly two equivalence classes of ordinary refinements. Furthermore:*

(1) *If the Hodge-Sen-Tate weight of* $\eta_1\eta_2^{-1}$ *is either zero or non-integral, then* V *admits exactly two equivalence classes of refinements, namely the two ordinary equivalence classes;*
(2) *If the Hodge-Sen-Tate weight of* $\eta_1\eta_2^{-1}$ *is a non-zero integer, then* V *admits three equivalence classes of refinements, namely the two ordinary equivalence classes and a unique equivalence class of ultracritical refinements.*

Proof. Lemma 4.4.3 shows that V admits exactly two equivalence classes of ordinary refinements. Suppose now that $R = (\eta, \alpha, r)$ is a non-ordinary refinement of V. Then r is a non-zero $G_{\mathbb{Q}_p}$-equivariant morphism $r : V^\vee \otimes \eta = \eta\eta_1^{-1} \bigoplus \eta\eta_2^{-1} \to (E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}})^{\phi=\alpha}$. Since $\eta_1 \neq \eta_2$, we see that r must vanish on one of the direct summands in its source (and on exactly one, since it is non-zero). Since R is non-ordinary, Lemma 4.4.1 (4) implies that it must be ultracritical. In particular $\eta_1\eta_2^{-1}$ must be of non-zero integral Hodge-Sen-Tate weight. Interchanging η_1 and η_2 if necessary, we may in fact assume that $\eta_2\eta_1^{-1}$ has Hodge-Tate weight $w > 0$. We now show (under this assumption) that $\eta_1 \bigoplus \eta_2$ does in fact admit an ultracritical refinement, and that this refinement is unique up to equivalence.

If (η, α, r) is an ultracritical refinement, then we see that $\eta\eta_1^{-1}$ (resp. $\eta\eta_2^{-1}$) must be of weight w (resp. weight 0), and that r must induce an embedding $\eta\eta_1^{-1} \to E \otimes_{\mathbb{Q}_p} B^+_{\mathrm{crys}}$ (and vanish on $\eta\eta_2^{-1}$). Thus η is a character with the property that $\eta\eta_1^{-1}$ is crystalline of weight w. Clearly such an η exists, and any two differ by multiplication by an unramified character. Thus we can construct an ultracritical refinement of $\eta_1 \bigoplus \eta_2$, and any two such are equivalent. □

4.4.6. **Remark.** If V is the direct sum of two copies of the same character of $G_{\mathbb{Q}_p}$, then the refinements of V are classified by Lemma 4.1.5.

We conclude this section by introducing some terminology.

4.4.7. **Definition.** If V is a reducible two dimensional continuous representation of $G_{\mathbb{Q}_p}$ over E, then we say that a refinement R of V is critical if it is not ordinary. (The terminology "critical" is used in analogy with the terminology "critical slope" as it is employed in the theory of modular forms.[8])

Any refinement equivalent to a critical refinement is again critical. The preceding results show that a reducible representation V admits at most one equivalence class of critical refinements. In order for V to admit such a class, V should be Hodge-Tate with distinct Hodge-Tate weights up to a twist, and either not potentially semi-stable up to a twist, or else potentially crystalline up to a twist.

4.5. **The classification of trianguline representations.** Colmez has classified the trianguline representations [27, Thm. 0.5]. We will recall his result here, but will rephrase it in the language of refinements. Note that if R is any refinement of a trianguline two dimensional representation V, and if we write $\sigma(R) = (\eta, \psi)$, then $\eta\psi$ is unitary, while $\mid \eta(p) \mid \leq 1$ (and hence in fact $\mid \eta(p) \mid < 1$ unless both η and ψ are unitary). The cited result of Colmez implies in particular that any continuous representation $\sigma \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))$ satisfying these conditions is of the form $\sigma(R)$ for some refinement R of some trianguline two dimensional representation V. Furthermore, it allows us to classify the representations V that give rise to a particular σ.

We begin with some definitions.

4.5.1. **Definition.** Let $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$ denote the set of maps $\sigma = (\eta, \psi) \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))$ such that $\eta\psi$ is unitary and $\mid \eta(p) \mid \leq 1$.

4.5.2. **Definition.** (1) We say that $\sigma = (\eta, \psi) \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$ is unitary if η and ψ are unitary.

[8] A classical finite slope p-stablized Hecke eigenform of weight $k \geq 2$ over $\overline{\mathbb{Q}}_p$ is said to be of critical slope if its slope is equal to $k-1$ (its largest possible value). Any such form is the "evil twin" of an ordinary p-stabilized eigenform; see for example the discussion of [14, §4.1], as well as the proof of Theorem 7.6.1 (3) below.

(2) We say that $\sigma = (\eta, \psi) \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$ is critical if $\eta\psi^{-1}$ is of integral Hodge-Tate weight $w > 0$, and if $\sigma(-w) := (\eta z^{-w}, \psi z^w)$ again lies in $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$, and is furthermore unitary.

(3) We say that $\sigma = (\eta, \psi) \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$ is ultracritical if $\eta\psi^{-1}$ is of integral Hodge-Tate weight $w > 0$, and if $\sigma(-w) := (\eta z^{-w}, \psi z^w)$ again lies in $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$, and is not unitary.

This terminology is almost compatible with the corresponding terminology for refinements.

4.5.3. **Lemma.** *Let V be a trianguline continuous two dimensional representation of $G_{\mathbb{Q}_p}$ over E, and R a refinement of V.*

(1) *$\sigma(R)$ is ultracritical if and only if V is irreducible and R is ultracritical;*

(2) *$\sigma(R)$ is critical if and only if V is reducible and R is a critical refinement of V.*

Proof. The reader can check this by going through the classification of all possible refinements given in the preceding sections. □

4.5.4. **Theorem.** *Let $\sigma = (\eta, \psi) \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$, and suppose that σ is neither unitary nor critical. Then there is a trianguline two dimensional continuous representation V of $G_{\mathbb{Q}_p}$ over E admitting a refinement R such that $\sigma(R) = \sigma$, and any such V is irreducible. Furthermore:*

(1) *If $\eta\psi^{-1}$ is not of the form εz^n for some $n > 0$, then V is unique up to isomorphism.*

(2) *If $\eta\psi^{-1} = \varepsilon z^n$ for some $n > 0$, then the set of isomorphism classes of such V can be indexed by the elements $\mathcal{L} \in \mathbb{P}^1(E)$ in such a way that, if $\mathcal{L} \in E$, then the members of the corresponding isomorphism class are twists of a semi-stable representation of Hodge-Tate weights $(0, -1-n)$ with $\mathcal{L}$-invariant equal to $\mathcal{L}$, while if $\mathcal{L} = \infty$, then the members of the corresponding isomorphism class are twists of a non-generic crystalline representation of Hodge-Tate weights $(0, -1-n)$.*

Proof. The results of Subsection 4.4 show that the Weil group representation attached to any refinement of a reducible representation is either unitary or critical. Thus if V exists it must be irreducible. Suppose first that σ is not ultracritical. Fix $\mathcal{L} = \infty$ if σ does not satisfy the condition of (2); otherwise let $\mathcal{L}$ be any element of $\mathbb{P}^1(E)$. The pair $s = (\sigma, \mathcal{L})$ is then an element of the space $\mathcal{S}_{\mathrm{irr}}$ considered in [27], and the corresponding $G_{\mathbb{Q}_p}$-representation $V(s)$ (see [27, p. 5]) admits a refinement R with $\sigma(R) = \sigma$ (see the proof of [27, Prop. 5.3]). Conversely, it follows from [27, Prop. 5.3] that if V admits a refinement R with $\sigma(R) = \sigma$, then $V \cong V(s)$ for some $s \in \mathcal{S}_{\mathrm{irr}}$ lying over σ. This proves the theorem for non-ultracritical σ.

If σ is ultracritical, then applying the theorem to $\sigma(-w)$ we obtain a representation V admitting a refinement R such that $\sigma(R) = \sigma(-w)$. It follows that V is Hodge-Tate up to twist, but not potentially semi-stable up to twist, and thus by Proposition 4.2.4 we see that V admits an ultracritical refinement R' such that $\sigma(R') = \sigma$. This completes the proof of the theorem in general. □

The case when $\sigma = (\eta, \psi) \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$ is either unitary or critical is straightforward. When σ is unitary, regarding η and ψ as Galois characters, we have seen that the direct sum $V = \eta \bigoplus \psi$ admits a refinement R such that $\sigma(R) = \sigma$. If σ is critical, and w is the weight of $\eta\psi^{-1}$, then we have seen that $\eta z^{-w} \bigoplus \psi z^{w}$ admits a critical (in fact ultracritical) refinement R such that $\sigma(R) = \sigma$.

The following results describe the possible indecomposable V admitting such a refinement R for which $\sigma(R) = \sigma$. (Note that Lemmas 4.4.1 and 4.5.3 imply that any such V is reducible.)

4.5.5. **Proposition.** *If $\sigma = (\eta, \psi)$ is a unitary element of $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$, then there exists an indecomposable trianguline two dimensional continuous representation V of $G_{\mathbb{Q}_p}$ over E admitting a refinement R such that $\sigma(R) = \sigma$. Furthermore:*

(1) *If $\eta\psi^{-1} \neq \underline{1}, \varepsilon$, then V is unique up to isomorphism.*
(2) *If $\eta\psi^{-1} = \varepsilon$, then the set of isomorphism classes of such V can be indexed by the elements $\mathcal{L} \in \mathbb{P}^1(E)$ in such a way that, if $\mathcal{L} \in E$, then the members of the corresponding isomorphism class are twists of a semi-stable representation of Hodge-Tate weights $(0, -1)$ with $\mathcal{L}$-invariant equal to $\mathcal{L}$, while if $\mathcal{L} = \infty$, then the members of the corresponding isomorphism class are twists of a non-generic crystalline representation of Hodge-Tate weights $(0, -1)$.*
(3) *If $\eta = \psi$, then the set of isomorphism classes of such V can be indexed by the elements of $\mathbb{P}^1(E)$. There is a unique (up to isomorphism) such V that is Hodge-Tate up to twist, which is then in fact crystalline (indeed, even unramified) up to twist.*

Proof. There is an isomorphism $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\psi, \eta) \xrightarrow{\sim} H^1(G_{\mathbb{Q}_p}, \eta\psi^{-1})$ (where Ext^1 is computed in the category of continuous $G_{\mathbb{Q}_p}$-representations, and H^1 indicates continuous Galois cohomology). Using Tate local duality and the local Euler characteristic formula (for example) one finds that $H^1(G_{\mathbb{Q}_p}, \eta\psi^{-1})$ is one-dimensional unless $\eta\psi^{-1} = \underline{1}$ or ε; in these latter two cases it is two-dimensional. It follows that in the situation of (1), any two non-trivial extensions of ψ by η are isomorphic as $G_{\mathbb{Q}_p}$-representations. This proves (1).

The elements of $H^1(G_{\mathbb{Q}_p}, \varepsilon)$, and hence the corresponding extensions of $\underline{1}$ by ε, are easily computed via Kummer theory. We recall the results of this well-known computation: one finds that any extension of $\underline{1}$ by ε is semi-stable, and that the non-trivial extensions are classified (up to isomorphism as $G_{\mathbb{Q}_p}$-representations) by their $\mathcal{L}$-invariants (with $\mathcal{L} = \infty$ corresponding to the unique isomorphism class of crystalline extensions). This proves (2).

The element of $H^1(G_{\mathbb{Q}_p}, \underline{1})$, and hence the corresponding extensions of $\underline{1}$ by itself, are easily computed via local class field theory. Since this computation is also well-known, we again content ourselves with recalling the outcome: one finds that (up to isomorphism as $G_{\mathbb{Q}_p}$-representations) there is a unique non-trivial extension of $\underline{1}$ by itself which is Hodge-Tate, namely the non-trivial unramified extension. This proves (3). □

4.5.6. **Proposition.** *If* $\sigma = (\eta, \psi)$ *is a critical element of* $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$*, then there exists exactly two isomorphism classes of indecomposable trianguline two dimensional continuous representations* V *of* $G_{\mathbb{Q}_p}$ *over* E *admitting a refinement* R *such that* $\sigma(R) = \sigma$*. There is a unique such isomorphism class of* V *whose members are furthermore potentially semi-stable up to a twist (and the members of this class are then in fact potentially crystalline, up to a twist).*

Proof. Applying Proposition 4.5.5 to $\sigma' := (\psi z^w, \eta z^{-w})$, we may find V that is potentially crystalline up to a twist and admits a critical refinement R with $\sigma(R) = \sigma$. Furthermore we see that V is unique up to isomorphism. (Note that when σ' falls into case (2) of that result, only the representation corresponding to $\mathcal{L} = \infty$ admits a critical refinement.)

Applying Proposition 4.5.5 to $\sigma(-w) := (\eta z^{-w}, \psi z^w)$ yields a representation V that admits an ultracritical refinement R with $\sigma(R) = \sigma$. Again we see that V is unique up to isomorphism among the indecomposable representations admitting such a refinement that are not potentially semi-stable up to a twist. □

4.5.7. **Terminology.** We say that a trianguline two dimensional continuous $G_{\mathbb{Q}_p}$-representation "admits an $\mathcal{L}$-invariant" if and only if it is a twist of either a semi-stable, but non-crystalline, representation, or of an indecomposable non-generic crystalline representation.

The trianguline representations that admit an $\mathcal{L}$-invariant are precisely the representations that are classified by Theorem 4.5.4 (2) and Proposition 4.5.5 (2).

5. Some invariants of admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

The problem of classifying topologically irreducible admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations seems currently unapproachable in general, but there are two situations in which we can make some progress: namely if the admissible unitary representation B being classified satisfies one of the following conditions:

(6) The subspace B_{lalg} of locally algebraic vectors in B is non-zero;

(7) The Jacquet module $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$ is non-zero.

In this section we discuss some results and conjectures towards the classification of topologically irreducible admissible unitary representations satisfying one of these conditions.

We first discuss the problem of classifying such representations B satisfying (6). The non-trivial results in this case are due to Colmez [26] and Berger and Breuil [8], and all progress has been motivated by Breuil's initial postulation of the local p-adic Langlands correspondence in the potentially semi-stable case [12]. After a brief review of some properties of the Jacquet module functor $J_{\mathrm{P}(\mathbb{Q}_p)}$, we then turn to the problem of classifying those B satisfying (7). In addition to the theory of the Jacquet module, our discussion relies heavily on the results of Colmez describing the universal unitary completions of certain locally analytic induced representations [27].

While the two problems that we discuss are in principle independent of one another, since B can satisfy either one of conditions (6) or (7) without satisfying the other, all the results obtained so far related to the classification of representations satisfying (6) apply only to representations that also satisfy (7). Nevertheless, even in the cases where no progress has been made, the local p-adic Langlands correspondence suggests the form that the classification should take. (See Conjecture 5.1.19 below.) On the other hand, the representations that satisfy (7) but not (6) can be completely classified, a result that can also be viewed as a manifestation of the local p-adic Langlands correspondence. (See Remark 5.3.13.)

5.1. **Admissible unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ containing locally algebraic vectors.** Suppose that B is a topologically irreducible admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation over E for which $B_{\mathrm{lalg}} \neq 0$. Then there exists an irreducible admissible smooth representation U of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E, an irreducible algebraic representation W of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E, and a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding

$$\pi := U \otimes_E W \to B.$$

In particular, π (when regarded as a topological $\mathrm{GL}_2(\mathbb{Q}_p)$ representation by being equipped with is finest locally convex topology) admits a non-zero universal unitary completion $\widehat{\pi}$ (in the sense of [34]), and $\widehat{\pi}$ maps continuously and $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariantly into B with dense image (since B is topologically irreducible).

5.1.1. **Question.** In the above situation, is the map $\widehat{\pi} \to B$ necessarily surjective?

It seems reasonable to hope that the answer to this question is "yes". (For example, this is the case if $\widehat{\pi}$ is admissible. [9] Proposition 5.1.10 below gives an example where this is again the case, although $\widehat{\pi}$ is not admissible.) Assuming that this is so, we see that B is a topologically irreducible admissible quotient of $\widehat{\pi}$, and that the problem of classifying those B for which $B_{\mathrm{lalg}} \neq 0$ reduces to the following two problems: (a) classify those irreducible admissible locally algebraic representations π of $\mathrm{GL}_2(\mathbb{Q}_p)$ for which $\widehat{\pi} \neq 0$; (b) for such π, classify the topologically irreducible admissible quotients of $\widehat{\pi}$. Problem (a) is close to being solved in its entirety, and substantial progress has been made on problem (b). In order to discuss the situation in more detail, as above we write $\pi = U \otimes_E W$ where U is an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and W is an irreducible algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We may then write $W = \left(\mathrm{Sym}^{w_2-w_1-1} E^2\right) \otimes \det^{w_1+1}$ for some uniquely determined integers $w_1 < w_2$.

We first note the following lemma.

5.1.2. **Lemma.** *If B is a finite dimensional irreducible unitary representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ then B is the trivial representation of $\mathrm{SL}_2(\mathbb{Q}_p)$.*

Proof. Any finite dimensional continuous representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ is necessarily locally analytic [52, LG §5.9, Thm. 2], and hence is locally algebraic, since it is finite dimensional. It follows from [31, Prop. 4.2.8] that we may factor B as a tensor product $B \cong U \otimes W$ where U is a finite dimensional irreducible smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ and W is an irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. It is well-known that U is then necessarily trivial (see for example [40, Prop. 2.7]), and thus we see that the irreducible algebraic representation W admits an $\mathrm{SL}_2(\mathbb{Q}_p)$-invariant norm. But this implies that W is also trivial, whence the lemma. □

This lemma has the following immediate corollary.

5.1.3. **Corollary.** *The only finite dimensional irreducible unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are the characters $\chi \circ \det$ (where χ is a unitary character of $\mathbb{Q}_p^\times$).*

From this corollary we see that in the above discussion we may restrict our attention to the case when π (or equivalently U) is infinite dimensional. Thus we

[9] This is a consequence of the following result: **Theorem.** *If $\phi : U \to V$ is a G-equivariant continuous morphism between two Banach space representations over E of the compact p-adic locally analytic group G, and if U is admissible, then ϕ has closed image.* This is well-known if V is assumed to be admissible (cf. [31, Prop. 6.2.9]) but in fact holds true without that assumption, as Breuil observed in the course of proving [12, Prop. 4.4.4 (v)]. Here is a sketch of the proof: Replacing V with the closure of the image of ϕ, we may assume that ϕ has dense image, so that the topological dual map $\phi' : V' \to U'$ is injective. Since U' is finitely generated over $E[[G]] := E \otimes_{\mathcal{O}_E} \mathcal{O}_E[[G]]$ (by our assumption that U is admissible), the same is true of V', since $\mathcal{O}_E[[G]]$, and hence $E[[G]]$, is Noetherian [44, V.2.2.4]. Thus V is also admissible (cf. [31, Def. 6.2.1]), and so ϕ has closed image, as claimed.

assume this from now on. We also assume that π has a unitary central character, since this is one obvious necessary condition for $\widehat{\pi}$ to be non-zero.

5.1.4. *The principal series case.* Suppose that U is principal series (and infinite dimensional), so that we may write $U = (\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{sm}}$ with $\chi_1\chi_2^{-1} \neq 1, |\ |^{-2}$. The assumption that U has unitary central character corresponds to the condition

$$| \chi_1(p)\chi_2(p)p^{w_1+w_2+1} | = 1. \tag{8}$$

In this case, the theory of Jacquet modules provides an additional necessary condition for $\widehat{\pi}$ to be non-zero, namely

$$| p^{w_2}\chi_1(p) | , | p^{w_2+1}\chi_2(p) | \leq 1 \tag{9}$$

(see Lemma 5.2.4 below).

The following conjecture is due to Breuil. (See Remark 3.3.4 for motivation.)

5.1.5. **Conjecture.** *If conditions (8) and (9) hold then $\widehat{\pi}$ is non-zero, and is a topologically irreducible admissible unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation.*

In particular, in this case we expect that the only topologically irreducible admissible unitary quotient of $\widehat{\pi}$ is $\widehat{\pi}$ itself.

5.1.6. **Theorem.** *Conjecture 5.1.5 is true provided that $\chi_1 \neq \chi_2 |\ |^{-1}$.*

Proof. We may describe π as the locally algebraic induction

$$\pi = (\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 z^{w_2} \otimes \chi_2 z^{w_1+1})_{\mathrm{lalg}} \cong (\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \varepsilon^{-1} z^{w_2+1} \otimes \chi_1 \varepsilon z^{w_1})_{\mathrm{lalg}},$$

where the isomorphism is provided by the theory of intertwining operators. If either $| p^{w_2}\chi_1(p) | = 1$ or $| p^{w_2+1}\chi_2(p) | = 1$, then π is equal to the locally algebraic induction of a unitary character, in which case it is straightforward to check that $\widehat{\pi}$ is equal to the corresponding continuous induction, and that this continuous induction is topologically irreducible and admissible unitary. (See [14, Prop. 2.2.1], and also the first paragraph of the proof of Proposition 5.3.4 below.) On the other hand, if neither of these conditions hold, and if $\chi_1 \neq \chi_2 |\ |^{-1}$, then the claim of the theorem follows from the main result of [8]. $\square$

5.1.7. *The special case.* Let us suppose now that U is special, say $U = \chi \circ \det \otimes \mathrm{St}$ for some character χ. The condition that π have a unitary central character then becomes

$$| \chi(p)p^{(w_1+w_2+1)/2} | = 1. \tag{10}$$

5.1.8. **Theorem.** *Suppose that condition (10) holds.*

(1) *If furthermore* W *is one dimensional (i.e.* $w_2 = w_1 + 1$*), then* $\widehat{\pi}$ *is non-zero, topologically irreducible, and admissible unitary.*
(2) *If* W *is not one dimensional (i.e.* $w_2 > w_1 + 1$*) then* $\widehat{\pi}$ *is non-zero, is not topologically irreducible, and is not admissible unitary.*

Proof. In case (1) we see that π is the twist of St by a unitary character, and the claim of the theorem follows from the particular case $\pi = \mathrm{St}$, which is proved in [12, Lem. 4.5.1].

In case (2), the fact that $\widehat{\pi}$ is non-zero is due to Teitelbaum [60] (resp. Grosse-Klönne [38]) when W is odd (resp. even) dimensional. The fact that $\widehat{\pi}$ is not admissible is proved in [12, Prop. 4.6.5]. The fact that $\widehat{\pi}$ is not topologically irreducible follows from the explicit construction of the series of non-zero topologically irreducible admissible unitary quotients described below. □

In the situation of case (2) of the preceding Theorem, Breuil has described a specific series of quotients of $\widehat{\pi}$, depending on a parameter $\mathcal{L} \in \mathbb{P}^1(E)$, which he conjectured, and Colmez has proved, to be non-zero, topologically irreducible, and admissible unitary. We briefly recall their construction.

First note that it is no loss of generality to replace π by a unitary twist (since twisting by a unitary character commutes with passage to the universal unitary completion). Thus we may assume that $w_2 = 0$, and so write $w = w_1 < -1$. In the cases when w is even we furthermore extend E if necessary so that $\sqrt{p} \in E$. Then, further twisting π if necessary, we may reduce to the case

$$\pi = \mathrm{St} \otimes |\det|^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee}.$$

Given $\mathcal{L} \in \mathbb{P}^1(E)$, choose $(u, v) \in E^2 \setminus \{(0,0)\}$ so that $u/v = \mathcal{L}$, and define the two dimensional representation $\sigma(\mathcal{L})$ of $\overline{\mathrm{P}}(\mathbb{Q}_p)$ via

$$\sigma(\mathcal{L})\Big(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\Big) := \begin{pmatrix} 1 & u \,\mathrm{ord}_p(\frac{a}{d}) + v \log_p(\frac{a}{d}) \\ 0 & 1 \end{pmatrix},$$

where ord_p denotes the p-adic valuation (normalized so that $\mathrm{ord}_p(p) = 1$) and $\log_p$ denotes the Iwasawa p-adic logarithm (so $\log_p(p) = 0$). Note that up to isomorphism, $\sigma(\mathcal{L})$ depends only on the value of $u/v = \mathcal{L}$.

Let χ_w denote the character

$$\chi_w := |\ |^{(1+w)/2} \otimes |\ |^{(-1+w)/2} z^w \varepsilon = |\ |^{(1+w)/2} \otimes |\ |^{(1+w)/2} z^{1+w}$$

of $\overline{\mathrm{P}}(\mathbb{Q}_p)$, and consider the locally analytic induction

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma(\mathcal{L}) \otimes \chi_w\big)_{\mathrm{an}}. \tag{11}$$

Since $\sigma(\mathcal{L})$ is an extension of the trivial character of $\overline{\mathrm{P}}(\mathbb{Q}_p)$ by itself, we see that (11) sits in a short exact sequence

(12)
$$0 \longrightarrow \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} \otimes \mid\ \mid^{(1+w)/2} z^{1+w}\big)_{\mathrm{an}} \longrightarrow \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma(\mathcal{L}) \otimes \chi_w\big)_{\mathrm{an}} \xrightarrow{\mathrm{pr}} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} \otimes \mid\ \mid^{(1+w)/2} z^{1+w}\big)_{\mathrm{an}} \longrightarrow 0.$$

The representation $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} \otimes \mid\ \mid^{(1+w)/2} z^{1+w}\big)_{\mathrm{an}}$ contains

$$\mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee} \tag{13}$$

as a subrepresentation. Following [12], we define

$$\Sigma(1-w, \mathcal{L}) = \mathrm{pr}^{-1}(\mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee}) / \mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee}$$

(i.e the quotient by the copy of (13) sitting in the first term of (12) of the preimage under the projection pr of the copy of (13) sitting in the third term); then $\Sigma(1-w, \mathcal{L})$ is a locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that sits in an exact sequence

(14)
$$0 \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} \otimes \mid\ \mid^{(1+w)/2} z^{1+w}\big)_{\mathrm{an}} / \mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{w-1} E^2\big)^{\vee} \to \Sigma(1-w, \mathcal{L}) \to \mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee} \to 0.$$

(Our notation follows that of [12], possibly up to a unitary twist.) Since $\sigma(\mathcal{L})$ is a non-split extension of the trivial character by itself, one easily sees that the exact sequence containing $\Sigma(1-w, \mathcal{L})$ is also non-split.

The representation $\Sigma(1-w, \mathcal{L})$ has the topological Jordan-Hölder series

(15)
$$0 \subset \pi = \mathrm{St} \otimes \mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee} = \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} \otimes \mid\ \mid^{(1+w)/2} z^{1+w}\big)_{\mathrm{lalg}} / \mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee} \subset \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} \otimes \mid\ \mid^{(1+w)/2} z^{1+w}\big)_{\mathrm{an}} / \mid \det \mid^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee} \subset \Sigma(1-w, \mathcal{L}),$$

related to which we have the following easy lemma.

5.1.9. **Lemma.** *Each of the inclusions after the first in* (15) *is non-split, and has a cokernel whose universal unitary completion vanishes.*

Proof. That the second inclusion is non-split is standard. Its cokernel is isomorphic to

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mid\ \mid^{(1+w)/2} z^{w} \otimes \mid\ \mid^{(1+w)/2} z\big)_{\mathrm{an}},$$

whose universal unitary completion vanishes by Proposition 5.2.1 and Lemma 5.2.4 below.

We observed above that the short exact sequence (14) is non-split, which implies that the third inclusion is non-split. Its cokernel is isomorphic to $|\ |^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee}$, which is finite dimensional but not a character (since $w < -1$), and so has vanishing universal unitary completion by Corollary 5.1.3. □

It follows from the preceding lemma that the maps of universal unitary completions induced by the second and third inclusions have dense image. In fact we have the following less elementary proposition.

5.1.10. **Proposition.** *The second inclusion of* (15) *induces an isomorphism of universal unitary completions, while the third inclusion induces a surjection of universal unitary completions.*

Proof. The first claim follows from [34, Prop. 2.5]. (See also Proposition 5.3.8 below.) The third claim is proved in [13, Cor. 3.3.4] in the case $\mathcal{L} \neq \infty$. In the case when $\mathcal{L} = \infty$, it is easily checked (see the remarks below). □

5.1.11. **Definition.** *For each* $\mathcal{L} \in \mathbb{P}^1(E)$*, let* $B(1-w, \mathcal{L})$ *denote the universal unitary completion of* $\Sigma(1-w, \mathcal{L})$*.*

The preceding proposition shows that $B(1-w, \mathcal{L})$ is a quotient of $\widehat{\pi}$.

5.1.12. **Remark.** The completion $B(1-w, \mathcal{L})$ may be described explicitly as a suitable quotient of a certain space of functions on $\mathbb{Q}_p$ of class $\mathcal{C}^{(-1-w)/2}$; see [13, Cor. 3.3.4] for the case $\mathcal{L} \neq \infty$.

5.1.13. **Theorem.** *Each of the* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations* $B(1-w, \mathcal{L})$ *is non-zero, topologically irreducible, and admissible unitary. Furthermore, the representations corresponding to distinct values of* $\mathcal{L}$ *are non-isomorphic.*

Proof. In the case when $\mathcal{L} \neq \infty$, the statement of the theorem is due to Colmez [26]. In the case when $\mathcal{L} = \infty$, it follows from the results of Berger and Breuil [8], taking into account the isomorphism (16) below. □

Thus the spaces $B(1-w, \mathcal{L})$ yield a series of topologically irreducible admissible unitary quotients of $\widehat{\pi}$ indexed by the elements $\mathcal{L} \in \mathbb{P}^1(E)$. This suggests the following question.

5.1.14. **Question.** Do the spaces $B(1-w, \mathcal{L})$ (for $\mathcal{L} \in \mathbb{P}^1(E)$) exhaust the topologically irreducible admissible unitary quotients of $\widehat{\pi}$?

One can at least say that no other such quotients have been described in the literature to date.

5.1.15. **Remark.** The case when $\mathcal{L} = \infty$ is a little different from the case when $\mathcal{L}$ is finite, since the representation $\sigma(\infty)$ is smooth. In fact, one can check that there is an isomorphism

$$(16) \qquad \Sigma(1-w,\infty) \xrightarrow{\sim} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ \ |^{(-1+w)/2} \otimes |\ \ |^{(3+w)/2}\, z^{1+w}\big)_{\mathrm{an}}.$$

Thus in this case the space of locally algebraic vectors in $\Sigma(1-w,\infty)$ is larger than π; namely one has an isomorphism

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ \ |^{-1} \otimes |\ \ |\big)_{\mathrm{sm}} \otimes |\det|^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^{\vee} \xrightarrow{\sim} \Sigma(1-w,\infty)_{\mathrm{lalg}}.$$

5.1.16. **Remark.** In the case when w is odd and $\geq -p$, Breuil and Mézard have constructed a certain torsion free sheaf of $\mathcal{O}_E$-modules on the p-adic upper half plane whose space of global sections is the unit ball in the topological dual of $B(1-w,\mathcal{L})$ [15]. They have used this description of $B(1-w,\mathcal{L})$ to compute the mod ϖ representation $\overline{B}(1-w,\mathcal{L})$.

5.1.17. *The cuspidal case.* We now assume that $\pi = U \otimes W$ with U irreducible and cuspidal and W irreducible algebraic. We continue to suppose that the central character of π is unitary. In this context we have the following easy proposition.

5.1.18. **Proposition.** *The universal unitary completion $\widehat{\pi}$ is non-zero, and is not admissible.*

Proof. The non-vanishing of $\widehat{\pi}$ follows directly from the fact that the matrix coefficients of U are compactly supported modulo the centre (together with the assumption that the central character of π is unitary); see [34, Prop. 1.18]. It is then easily checked (by using the construction of U as a compact induction, for example) that in fact $\widehat{\pi}$ is not admissible. □

In this case nothing general seems to be known about the possible topologically irreducible admissible unitary quotients of $\widehat{\pi}$. One can use global considerations to show that π does admit some non-zero admissible unitary completions, which one presumes (but cannot prove as of yet) are topologically irreducible quotients of $\widehat{\pi}$. (See Remark 7.7.16 below, and recall that we expect Question 5.1.1 to have a positive answer in general.)

Motivated by the local p-adic Langlands conjecture, Breuil has made the following conjecture (cf. Remark 3.3.4 above and Remark 6.1.9 below).

5.1.19. **Conjecture.** *The universal unitary completion $\widehat{\pi}$ admits a family of non-zero, mutually non-isomorphic, topologically irreducible, admissible unitary quotients, indexed by the elements of $\mathbb{P}^1(E)$.*

An optimist might guess that in fact this conjectural series of quotients exhausts all topologically irreducible, admissible unitary quotients of $\widehat{\pi}$.

5.2. **Jacquet modules.** Recall that if U is any topological E-vector space of compact type equipped with a locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, then the Jacquet module $J_{\mathrm{P}(\mathbb{Q}_p)}(U)$ is a certain locally analytic representation of $\mathrm{T}(\mathbb{Q}_p)$ over E functorially associated to U [32]. We do not recall the definition here (see the discussion following [32, Prop. 0.3]), but do recall that $J_{\mathrm{P}(\mathbb{Q}_p)}$ is additive and left exact. In the remainder of this subsection we describe the additional properties of the functor $J_{\mathrm{P}(\mathbb{Q}_p)}$ that we will require.

The following result describes the Jacquet modules of induced representations.

5.2.1. **Proposition.** *Fix* $\chi_1 \otimes \chi_2 \in \widehat{T}(\mathbb{Q}_p)$.

(1) *If* $\chi_1\chi_2^{-1}$ *is of integral Hodge-Tate weight* $w \geq 0$, *and if* $\chi_1\chi_2^{-1} \neq z^w \mid \ \mid^{-1}$, *then*

$$J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{lalg}}\big) \cong (\chi_1 \mid \ \mid \otimes \chi_2 \mid \ \mid^{-1}) \bigoplus (\chi_2 z^w \otimes \chi_1 z^{-w}).$$

(2) *If* $\chi_1\chi_2^{-1}$ *is of integral Hodge-Tate weight* $w \geq 0$, *and if* $\chi_1\chi_2^{-1} = z^w \mid \ \mid^{-1}$, *then* $J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{lalg}}\big)$ *is a non-split extension of the character* $\chi_1 \mid \ \mid \otimes \chi_2 \mid \ \mid^{-1} = \chi_2 z^w \otimes \chi_1 z^{-w}$ *by itself.*

(3) *If* $\chi_1\chi_2^{-1}$ *is of integral Hodge-Tate weight* $w \geq 0$ *then the inclusion*

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}} \subset \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$$

induces an isomorphism

$$J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{lalg}}\big) \xrightarrow{\sim} J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{an}}\big).$$

(4) *If* $\chi_1\chi_2^{-1}$ *is not of non-negative integral Hodge-Tate weight then*

$$J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{an}}\big) \cong \chi_1 \mid \ \mid \otimes \chi_2 \mid \ \mid^{-1}.$$

Proof. Claims (1) and (2) follow from [32, Prop. 4.3.6] and the well-known structure of the Jacquet modules of smooth induced representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (cf. [19, Lem. 7.1.1 (a)]). The proofs of claims (3) and (4) (in a much more general context) will appear in [35] (and are in any case straightforward computations working from the definition of $J_{\mathrm{P}(\mathbb{Q}_p)}$). ☐

The following corollary is an immediate consequence of the preceding proposition.

5.2.2. **Corollary.** *For every* $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$ *there is an embedding of* $\mathrm{T}(E)$-*representations*

$$\chi_1 \mid \ \mid \otimes \chi_2 \mid \ \mid^{-1} \rightarrow J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)_{\mathrm{an}}\big),$$

uniquely determined up to multiplication by a non-zero scalar.

5.2.3. **Definition.** Fix a compact type locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation U defined over E.

(1) Define $\mathrm{Exp}(U)$ to be the set of one dimensional $\mathrm{T}(\mathbb{Q}_p)$-invariant subspaces of $J_{\mathrm{P}(\mathbb{Q}_p)}(U)$.
(2) For any line $L \in \mathrm{Exp}(U)$, write $\chi(L) \in \widehat{\mathrm{T}}(E)$ to denote the character via which $\mathrm{T}(\mathbb{Q}_p)$ acts on L.
(3) For any character $\chi \in \widehat{\mathrm{T}}(E)$, write $\mathrm{Exp}^{\chi}(U) := \{L \in \mathrm{Exp}(U) \mid \chi(L) = \chi\}$.

If we fix a character $\chi \in \widehat{\mathrm{T}}(E)$, then $\mathrm{Exp}^{\chi}(U)$ has the structure of a projective space (possibly of infinite dimension, or possibly of dimension -1, i.e. empty), namely it is the projectivization of the χ-eigenspace $J^{\chi}_{\mathrm{P}(\mathbb{Q}_p)}(U)$ in $J_{\mathrm{P}(\mathbb{Q}_p)}(U)$. Thus $\mathrm{Exp}(U)$ is a disjoint union of projective spaces, indexed by the characters $\chi \in \widehat{\mathrm{T}}(E)$.

Recall the identification $\widehat{\mathrm{T}}(E) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))$ discussed in Subsection 1.3. We let $\widehat{\mathrm{T}}(E)_+$ denote the subset of $\widehat{\mathrm{T}}(E)$ that corresponds to $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{W}_p, \mathrm{T}(E))_+$ (as defined in Definition 4.5.1) under this identification, and we will use this identification to apply the terminology of Definition 4.5.2 to elements of $\widehat{\mathrm{T}}(E)_+$.

The following result imposes a limitation on the exponents that can appear in $J_{\mathrm{P}(\mathbb{Q}_p)}(U)$ in the cases of interest to us.

5.2.4. **Lemma.** *If B is an E-Banach space equipped with a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for which B_{an} is of compact type, and if $J^{\chi}_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}) \neq 0$ for some character $\chi = \chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$, then $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \in \widehat{\mathrm{T}}(E)_+$.*

Proof. This is a consequence of [32, Lem. 4.4.2]. ☐

The following theorem provides a kind of converse to Corollary 5.2.2.

5.2.5. **Theorem.** *Let B be an E-Banach space equipped with a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for which B_{an} is of compact type, let $L \in \mathrm{Exp}(B_{\mathrm{an}})$, and write $\chi(L) = \chi_1 \otimes \chi_2$. Suppose furthermore that the character $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ is neither critical nor ultracritical in the sense of Definition 4.5.2. Then there is a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map*

$$(17) \qquad \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)_{\mathrm{an}} \to B_{\mathrm{an}},$$

uniquely determined up to multiplication by a non-zero scalar, so that the inclusion $L \subset J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$ is obtained by applying the functor $J_{\mathrm{P}(\mathbb{Q}_p)}$ to (17) and *composing the resulting map with the inclusion*

$$\chi_1 \otimes \chi_2 \subset J_{\mathrm{P}(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |)_{\mathrm{an}}\big)$$

provided by Corollary 5.2.2.

Proof. As in [34, §2], we identify $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)_{\mathrm{an}}$ with a space of locally analytic E-valued functions on $\mathbb{Q}_p$ satisfying a certain regularity condition at infinity. We let $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{sm}}$ (resp. $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{an}}$) denote the closed $\mathrm{P}(\mathbb{Q}_p)$-invariant (resp. $(\mathfrak{gl}_2, \mathrm{P}(\mathbb{Q}_p))$-invariant) subspace of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)_{\mathrm{an}}$ consisting of locally constant (resp. locally analytic) functions that are compactly supported on $\mathbb{Q}_p$. (Here $\mathfrak{gl}_2$ denotes the Lie algebra of $\mathrm{GL}_2(\mathbb{Q}_p)$.) If $\chi_1\chi_2^{-1}$ is furthermore of integral Hodge-Tate weight $w \geq 0$, then we let $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{lalg}}$ denote the $(\mathfrak{gl}_2, \mathrm{P}(\mathbb{Q}_p))$-invariant subspace of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{an}}$ consisting of functions that are locally polynomial of degree $\leq w$.

The action of $\mathfrak{gl}_2$ on $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{an}}$ induces a natural $(\mathfrak{gl}_2, \mathrm{P}(\mathbb{Q}_p))$-equivariant map

$$\text{(18)}\quad \mathrm{U}\mathfrak{gl}_2 \otimes_{\mathrm{U}\mathfrak{p}} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{sm}} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{an}}$$

(where $\mathfrak{p}$ denotes the Lie algebra of $\mathrm{P}(\mathbb{Q}_p)$ and $\mathrm{U}\mathfrak{r}$ indicates the universal enveloping algebra of the Lie algebra $\mathfrak{r}$), which is an isomorphism if $\chi_1\chi_2^{-1}$ is not of non-negative integral Hodge-Tate weight, and which induces a surjection

$$\text{(19)}\quad \mathrm{U}\mathfrak{gl}_2 \otimes_{\mathrm{U}\mathfrak{p}} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{sm}} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{lalg}}$$

if $\chi_1\chi_2^{-1}$ is of non-negative integral Hodge-Tate weight.

The defining adjointness property of $J_{\mathrm{P}(\mathbb{Q}_p)}$ [32, Thm. 3.5.6] yields a natural isomorphism

$$\text{(20)}\quad \mathrm{Hom}_{(\mathfrak{gl}_2,\mathrm{P}(\mathbb{Q}_p))}(\mathrm{U}\mathfrak{gl}_2 \otimes_{\mathrm{U}\mathfrak{p}} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ \ |^{-1} \otimes \chi_2 |\ \ |\big)(\mathbb{Q}_p)_{\mathrm{sm}}, B_{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{T}(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2, J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})).$$

If $\chi_1\chi_2^{-1}$ is not of non-negative integral Hodge-Tate weight, then upon combining the isomorphism (18) with [14, Prop. 2.1.4] (which is a special case of the main result of [35]) we find that passing to Jacquet modules yields an isomorphism

$$\text{(21)}\quad \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p))}\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)_{\mathrm{an}}, B_{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{T}(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2, J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})),$$

proving the theorem in this case.

If $\chi_1\chi_2^{-1}$ is of non-negative integral Hodge-Tate weight, then by assumption $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ is neither critical nor ultacritical. Taking this together with the fact that B is unitary, it follows from [32, Thm. 4.4.5] that the canonical lift to B_{an} of any element in $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$ (see [32, 0.9]) is locally SL_2-algebraic. It follows directly from this, and from the construction of the map (20), that any map in the source of (20) necessarily factors through the surjection (19), yielding an isomorphism

$$\text{(22)}\quad \mathrm{Hom}_{(\mathfrak{gl}_2,\mathrm{P}(\mathbb{Q}_p))}\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)(\mathbb{Q}_p)_{\mathrm{lalg}}, B_{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{T}(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2, J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})).$$

The same argument that proves [34, Prop. 2.5] (which is a variation of the classical argument of Amice-Vélu [1] and Vishik [62], and relies again on the fact that $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ is neither critical nor ultacritical) shows that the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p))}\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)_{\mathrm{an}}, B_{\mathrm{an}}) \to \mathrm{Hom}_{(\mathfrak{gl}_2,\mathrm{P}(\mathbb{Q}_p))}\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)(\mathbb{Q}_p)_{\mathrm{lalg}}, B_{\mathrm{an}})$$

is an isomorphism. Composing this isomorphism with the isomorphism (22), we find that in this case passing to Jacquet modules again yields an isomorphism (21). This completes the proof of the theorem. □

The following supplement to Theorem 5.2.5 treats the case when $\mathrm{Exp}(B(V)_{\mathrm{an}})$ contains a line L such that $\chi(L)$ is critical or ultracritical.

5.2.6. **Proposition.** *Let B be an E-Banach space equipped with a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for which B_{an} is of compact type, let $L \in \mathrm{Exp}(B_{\mathrm{an}})$, and write $\chi(L) = \chi_1 \otimes \chi_2$. Suppose that $\chi_1\chi_2^{-1}$ is of integral Hodge-Tate weight $w \geq 0$.*

1. *If $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ is critical, then either there exists $L' \in \mathrm{Exp}(B_{\mathrm{an}})$ such that $\chi(L')$ equals one of $\chi_1 z^{-1-w} \otimes \chi_2 z^{1+w}$ or $\chi_2 |\ | z^w \otimes \chi_1 |\ |^{-1} z^{-w}$, or else $\chi_1 = \chi_2$ and B contains a copy of the one dimensional $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant representation $\chi_1 \circ \det$.*
2. *If $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ is ultracritical, then*

$$\dim \mathrm{Exp}^{\chi_1 \otimes \chi_2}(B(V)_{\mathrm{an}}) \leq \dim \mathrm{Exp}^{\chi_1 z^{-1-w} \otimes \chi_2 z^{1+w}}(B(V)_{\mathrm{an}}).$$

In particular, there exists $L' \in \mathrm{Exp}(B_{\mathrm{an}})$ *such that* $\chi(L')$ *equals* $\chi_1 z^{-1-w} \otimes \chi_2 z^{1+w}$.

Proof. We suppose that we are in either case (1) or case (2), that is, that $\chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\, \varepsilon^{-1}$ is either critical or ultracritical. We will use the terminology and results recalled in the proof of the preceding theorem. Let $v \in B_{\mathrm{an}}$ be the canonical lift of a non-zero element in L. Suppose first that v is locally SL_2-algebraic. Then under the adjunction isomorphism (20), the inclusion

$$L \to J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}) \tag{23}$$

arises from a $(\mathfrak{gl}_2, \mathrm{P}(\mathbb{Q}_p))$-equivariant map

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\big)(\mathbb{Q}_p)_{\mathrm{lalg}} \to B_{\mathrm{an}}.$$

One can show without difficulty that the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p))}\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\big)_{\mathrm{lalg}}, B_{\mathrm{an}}\big) \to \mathrm{Hom}_{(\mathfrak{gl}_2, \mathrm{P}(\mathbb{Q}_p))}\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\big)(\mathbb{Q}_p)_{\mathrm{lalg}}, B_{\mathrm{an}}\big)$$

is an isomorphism (for any character $\chi_1 \otimes \chi_2$ such that $\chi_1 \chi_2^{-1}$ is of integral Hodge-Tate weight, and any continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space representation B; this is a variant of [14, Lem. A.2.4]). Thus the map (23) is obtained by applying $J_{\mathrm{P}(\mathbb{Q}_p)}$ to a non-zero continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\big)_{\mathrm{lalg}} \to B_{\mathrm{an}}. \tag{24}$$

If this map is injective, then it induces an injection on Jacquet modules, and Proposition 5.2.1 now shows that we obtain a copy of the character $\chi_2 \,|\ \,|\, z^w \otimes \chi_1 \,|\ \,|^{-1} z^{-w}$ in $J_{\mathrm{P}(\mathbb{Q}_p)}(B)$, and so Lemma 5.2.4 implies that $\chi_2 z^w \otimes \chi_1 z^{-w}$, and hence also $\chi_2 z^w \otimes \chi_1 z^{-w} \varepsilon^{-1}$, lies in $\widehat{\mathrm{T}}(E)_+$. Since $\chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\, \varepsilon^{-1}$ is assumed to be either critical or ultracritical, so that $\chi_1 z^{-w} \varepsilon^{-1} \otimes \chi_2 z^w (= \chi_1 \,|\ \,|^{-1} z^{-1-w} \otimes \chi_2 \,|\ \,|\, \varepsilon^{-1} z^{1+w})$ also lies in in $\widehat{\mathrm{T}}(E)_+$, we conclude that in fact $\chi_1 z^{-w} \varepsilon^{-1} \otimes \chi_2 z^w$ is unitary, and thus that $\chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\, \varepsilon^{-1}$ is critical. Hence we are necessarily in case (1).

The source of (24) is irreducible (and consequently this map is necessarily injective) unless $\chi_1 \chi_2^{-1} \,|\ \,|^{-1} = |\ \,|^{\pm 1} z^w$. Again using the fact that $\chi_1 \,|\ \,|^{-1} \otimes \chi_2 \,|\ \,|\, \varepsilon^{-1} = \chi_1 \,|\ \,|^{-1} \otimes \chi_2 z^{-1}$ is either critical or ultracritical, we see that the source of (24) is irreducible unless $w = 0$ and $\chi_1 = \chi_2$. Thus if (24) is not injective, then $\chi_1 = \chi_2$ and (24) must factor through the quotient $\chi_1 \circ \det$ of its source, yielding a copy of this one dimensional $\mathrm{GL}_2(\mathbb{Q}_p)$-representation inside B. Clearly χ_1 must then be unitary, and so we must again be in case (1). This completes our discussion in the case when v is locally SL_2-algebraic.

Suppose now that the canonical lift v of a non-zero element of L is not locally SL_2-algebraic. If $X_- := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{gl}_2$, then [32, Prop. 4.4.4] shows that $(X_-)^{1+w}v$ is the canonical lift of a non-zero element of $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi_1 z^{-1-w}\otimes\chi_2 z^{1+w}}(B_{\mathrm{an}})$. In particular, this completes the proof of case (1). Since in case (2) no element of $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi_1\otimes\chi_2}$ has a locally SL_2-algebraic canonical lift (as we have seen above), we see that the action of $(X_-)^{1+w}$ on canonical lifts in fact induces an injection $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi_1\otimes\chi_2}(B_{\mathrm{an}}) \to J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi_1 z^{-1-w}\otimes\chi_2 z^{1+w}}(B_{\mathrm{an}})$. This yields the claimed inequality in case (2), and completes the proof of the proposition. □

5.3. **Admissible unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with non-zero Jacquet modules.** In this subsection we consider the classification of those topologically irreducible admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations B for which $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}) \neq 0$. We begin with the following proposition.

5.3.1. **Proposition.** *If B is an admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation such that $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}) \neq 0$, then we may find a finite extension E' of E such that $\mathrm{Exp}(E' \otimes_E B_{\mathrm{an}}) \neq \emptyset$ (or equivalently, there is a character $\chi \in \widehat{\mathrm{T}}(E')$ such that $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi}(E' \otimes_E B_{\mathrm{an}}) \neq 0$).*

Proof. Since B is admissible unitary, the locally analytic representation B_{an} is admissible, and so $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$ is an essentially admissible $\mathrm{T}(\mathbb{Q}_p)$-representation [32, Thm. 0.5], corresponding to a rigid analytic coherent sheaf $\mathcal{F}$ on $\widehat{\mathrm{T}}$ (as explained in [31, §6.4] – the global sections of $\mathcal{F}$ are naturally dual to $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$). Since $J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}) \neq 0$, the coherent sheaf $\mathcal{F}$ is non-zero, and so has a non-empty support. If $\chi \in \widehat{\mathrm{T}}(E')$ (for some finite extension E' of E) is a point in the support of $\mathcal{F}$, then the fibre $\mathcal{F}_\chi \neq 0$. This fibre is naturally dual to $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi}(E' \otimes_E B_{\mathrm{an}})$, and so we find that this latter space is also non-zero. □

In light of the preceding result, we restrict our attention from now on to topologically irreducible admissible unitary B such that $\mathrm{Exp}(B_{\mathrm{an}}) \neq \emptyset$. From Theorem 5.2.5 we see that the first step in classifying such B will be to describe the universal unitary completions of locally analytic induced representations.

We consider first the induction of a unitary character, and begin by introducing some additional notation.

5.3.2. **Definition.** *Let $\widehat{\mathrm{St}}$ denote the universal unitary completion of the Steinberg representation* St.

5.3.3. **Lemma.** (1) *The Banach space $\widehat{\mathrm{St}}$, with its natural $\mathrm{GL}_2(\mathbb{Q}_p)$-action, is topologically irreducible and admissible unitary.*

(2) *The inclusion* $\mathrm{St} \subset \widehat{\mathrm{St}}$ *induces an isomorphism*

$$|\ | \otimes |\ |^{-1} = J_{\mathrm{P}(\mathbb{Q}_p)}(\mathrm{St}) \xrightarrow{\sim} J_{\mathrm{P}(\mathbb{Q}_p)}(\widehat{\mathrm{St}}_{\mathrm{an}}).$$

Proof. Part (1) was already observed in Theorem 5.1.8 (1). To prove claim (2), begin with the short exact sequence

$$0 \to \underline{1} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \underline{1} \otimes \underline{1}\big)_{\mathrm{sm}} \to \mathrm{St} \to 0.$$

Passing to universal unitary completions yields the short exact sequence of admissible unitary representations

$$0 \to \underline{1} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \underline{1} \otimes \underline{1}\big)_{\mathrm{cont}} \to \widehat{\mathrm{St}} \to 0 \tag{25}$$

(cf. the proof of the following proposition). Passing to locally analytic vectors (which is an exact functor on admissible unitary representations [51, Thm. 7.1]) yields the short exact sequence

$$0 \to \underline{1} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \underline{1} \otimes \underline{1}\big)_{\mathrm{an}} \to \widehat{\mathrm{St}}_{\mathrm{an}} \to 0.$$

Passing to Jacquet modules (and taking into account Proposition 5.2.1) yields an exact sequence of $\mathrm{T}(\mathbb{Q}_p)$-representations

$$0 \to \underline{1} \to \underline{1} \bigoplus |\ | \otimes |\ |^{-1} \to J_{\mathrm{P}(\mathbb{Q}_p)}(\widehat{\mathrm{St}}_{\mathrm{an}}).$$

Although $J_{\mathrm{P}(\mathbb{Q}_p)}$ is not a right exact functor in general, it is not difficult to check that in this case the preceding exact sequence is exact on the right as well, proving claim (2). □

The next proposition gives a precise description of the universal unitary completion of the locally analytic induction of a unitary character.

5.3.4. **Proposition.** *Let* $\chi = \chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$ *be a unitary character.*

(1) *The continuous injection*

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$$

identifies the latter representation with the universal unitary completion of the former.

(2) *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ *is admissible unitary.*

(3) *If* $\chi_1 \neq \chi_2$ *then* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ *is topologically irreducible.*

(4) *If* $\chi_1 = \chi_2$ *then* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ *sits in a non-split exact sequence*

$$0 \to \chi_1 \circ \det \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}} \to \chi_1 \circ \det \otimes \widehat{\mathrm{St}} \to 0.$$

Proof. Claim (1) follows directly from the fact that the character χ is unitary; it is a straightforward generalization of the standard fact in p-adic analysis that the space of locally analytic E-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ is dense in the space of continuous E-valued functions with respect to the sup norm (which is the special case of claim (1) when χ is the trivial character). We note that when $\chi_1 \otimes \chi_2^{-1}$ is of non-negative integral Hodge-Tate weight, the same reasoning (locally polynomial functions are dense in continuous functions) shows that $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ may be regarded as the universal unitary completion of the locally algebraic induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}}$.

It was already observed in Subsection 1.3 that the continuous induction is admissible. This proves claim (2).

Suppose now that $\chi_1 \neq \chi_2$. If $\chi_1 \otimes \chi_2^{-1}$ is of non-negative integral Hodge-Tate weight then claim (3) is proved in [14, Prop. 2.2.1]. If not, then the same proof works, but is even simpler, since $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$ is topologically irreducible. If $\chi_1 = \chi_2$, then claim (4) results from twisting the short exact sequence (25) by $\chi_1 = \chi_2$. □

5.3.5. **Remark.** Parts (2), (3), and (4) of this proposition have also been observed by Schneider and Teitelbaum.

5.3.6. **Corollary.** *If B is a topologically irreducible admissible unitary Banach space representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *such that* $\mathrm{Exp}^{\chi_1|\ |\otimes\chi_2|\ |^{-1}}(B_{\mathrm{an}}) \neq \emptyset$ *for some unitary character* $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$ *then either:*

(1) $B \xrightarrow{\sim} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ *(if $\chi_1 \neq \chi_2$); or*

(2) $B \xrightarrow{\sim} \chi_1 \otimes \widehat{\mathrm{St}}$ *(if $\chi_1 = \chi_2$).*

Proof. It follows from Theorem 5.2.5 and Proposition 5.3.4 (1) that there is a non-zero continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}} \to B.$$

Since the source and target of this map are admissible unitary (the source by Proposition 5.3.4 (2) and the target by assumption), we see that it has closed image [31, Prop. 6.2.9], and hence is surjective. The Corollary then follows from Proposition 5.3.4 (3) and (4). □

The next result, which treats the universal unitary completions of the locally analytic inductions of non-unitary characters, lies significantly deeper than Proposition 5.3.4.

5.3.7. **Theorem.** *Let $\chi = \chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$ be such that the following conditions hold:*

(1) $\chi_1 \otimes \chi_2$ *is not unitary;*
(2) *The ratio* $\chi_1\chi_2^{-1}$ *does not have non-negative integral Hodge-Tate weight.*

Then the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$ *is a non-zero, topologically irreducible, admissible unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation.*

Proof. Arguing as in the proof of [8, Thm. 4.3.1] (cf. the proof of [14, Thm. 2.2.2]), one sees that the universal unitary completion in question may be identified with the space $B(s)$ of [27], taking s to be the character $\chi_1 \otimes \chi_2\varepsilon$. The theorem is thus a restatement of [27, Thm. 0.12]. □

5.3.8. **Proposition.** *Let* $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$ *be such that* $\chi_1\chi_2^{-1}$ *is of integral Hodge-Tate weight* $w \geq 0$*. If* $\chi_1 \otimes \chi_2\varepsilon^{-1}$ *is neither critical nor ultracritical, then the inclusion* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}} \subset \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$ *induces an isomorphism on universal unitary completions.*

Proof. This is proved in [34, Prop. 2.5], and independently in [8]. (Although the case $\chi_1\chi_2^{-1}z^{-w} = |\ |^{-2}$ is not explicitly considered in [34], the proof as written applies to it perfectly well.) □

5.3.9. **Remark.** If $\chi_1\chi_2^{-1} = z^w$ for some $w > 0$ then the surjection

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}}/\chi_1 \otimes \big(\mathrm{Sym}^w E^2\big)^\vee$$

induces an isomorphism on universal unitary completions, since Corollary 5.1.3 implies that the universal unitary completion of $|\ |^{-w/2} \otimes \big(\mathrm{Sym}^w E^2\big)^\vee$ vanishes.

5.3.10. **Definition.** If $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$ is such that $\chi_1 \otimes \chi_2\varepsilon^{-1}$ is neither unitary, critical, nor ultracritical, then we write $B(\chi_1 \otimes \chi_2)$ to denote the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$.

5.3.11. **Remark.** In the context of the preceding definition, if we write $r := \mathrm{ord}_p(\chi_1(p))$, then the completion $B(\chi_1 \otimes \chi_2)$ may be described explicitly as a suitable quotient of a certain space of functions on $\mathbb{Q}_p$ of class $\mathcal{C}^r$. (See [13, §3.3] and [8, Thm. 4.3.1]; as we remarked in the proof of Theorem 5.3.7, the second cited result extends to apply to any of the completions $B(\chi_1 \otimes \chi_2)$ under consideration.)

5.3.12. **Remark.** If, in the context of the preceding definition, the character $\chi_1 \otimes \chi_2$ satisfies condition (2) of Theorem 5.3.7, then that theorem shows that $B(\chi_1 \otimes \chi_2)$ is topologically irreducible and admissible unitary. Suppose on the other hand that $\chi_1\chi_2^{-1}$ is of integral Hodge-Tate weight $w \geq 0$. If $\chi_1\chi_2^{-1}z^{-w}$ is furthermore $\neq \underline{1}, |\ |^{-1}, |\ |^{-2}$, then Proposition 5.3.8 shows that $B(\chi_1 \otimes \chi_2)$ may be identified with the unitary universal completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{lalg}}$, which is a unitary twist of a representation to which Theorem 5.1.6 applies. Thus that Theorem shows that in this case $B(\chi_1 \otimes \chi_2)$ is again topologically irreducible and admissible unitary. (If Conjecture 5.1.5 holds in full generality, then we see that the same is true without the requirement that $\chi_1\chi_2^{-1}z^{-w} \neq |\ |^{-1}$.) If $\chi_1\chi_2^{-1}z^{-w} = |\ |^{-2}$, then Remark 5.1.15 shows that $B(\chi_1 \otimes \chi_2)$ is a unitary twist of $B(2-w, \infty)$, and so by Theorem 5.1.13 is also topologically irreducible and admissible unitary. Finally, if $\chi_1\chi_2^{-1}z^{-w} = \underline{1}$ (in which case necessarily $w > 0$, since $\chi_1 \otimes \chi_2$ is non-unitary by assumption) then Theorem 5.1.8, Proposition 5.3.8 and Remark 5.3.9 together show that $B(\chi_1 \otimes \chi_2)$ is neither topologically irreducible nor admissible unitary.

5.3.13. **Remark.** For characters $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$ such that $\chi_1 \otimes \chi_2\varepsilon^{-1}$ is neither unitary, critical, nor ultracritical, we have seen that the representation $B(\chi_1 \otimes \chi_2)$ is (or should be) admissible unitary and topologically irreducible precisely when $\chi_1 \otimes \chi_2\varepsilon^{-1}$ corresponds to a unique trianguline two dimensional $G_{\mathbb{Q}_p}$-representation with respect to the classification scheme of Theorem 4.5.4. This can be viewed as evidence in favour of a "p-adic local Langlands philosophy".

5.3.14. **Corollary.** *Let B be a topologically irreducible admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\mathrm{Exp}^{\chi_1|\ |\otimes\chi_2|\ |^{-1}}(B_{\mathrm{an}}) \neq \emptyset$ for some character $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$ satisfying the following conditions:*

(1) *$\chi_1 \otimes \chi_2\varepsilon^{-1}$ is neither unitary, critical, nor ultracritical;*
(2) *$\chi_1\chi_2^{-1}z^{-w} \neq \underline{1}, |\ |^{-1}$ for any integer $w \geq 0$.*

Then B is isomorphic to $B(\chi_1 \otimes \chi_2)$.

Proof. Note that by Lemma 5.2.4 we have $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$. The result now follows by an argument similar to that used to prove Corollary 5.3.6, applying Remark 5.3.12 in place of Proposition 5.3.4. □

If Conjecture 5.1.5 were known to be true in full generality then the preceding corollary would hold with condition (2) replaced by the weaker condition that $\chi_1\chi_2^{-1} \neq z^w$ for any $w > 0$.

5.3.15. **Remark.** Suppose that B is a topologically irreducible admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that is not one dimensional (and hence is infinite dimensional by Corollary 5.1.3) such that $\mathrm{Exp}^{\chi_1|\ |\otimes\chi_2|\ |^{-1}}(B_{\mathrm{an}}) \neq \emptyset$ for a character $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$ for which $\chi_1 \otimes \chi_2\varepsilon^{-1}$ is critical (resp. ultracritical). Part (1) (resp. part (2)) of Proposition 5.2.6 then shows that there is also a unitary character (resp. a character satisfying the conditions of Theorem 5.3.7) $\psi_1 \otimes \psi_2 \in \widehat{\mathrm{T}}(E)$ for which $\mathrm{Exp}^{\psi_1\otimes\psi_2}(B_{\mathrm{an}}) \neq \emptyset$; the representation B is then classified by Corollary 5.3.6 (resp. Corollary 5.3.14).

5.3.16. **Lemma.** *If* $\chi_1 \otimes \chi_2 \varepsilon^{-1} \in \widehat{\mathrm{T}}(E)_+$ *is neither unitary, critical, nor ultracritical, if* $\chi_1\chi_2^{-1}$ *is of integral Hodge-Tate weight* $w \geq 0$, *and if* $\chi_1\chi_2^{-1}z^{-w} \neq \underline{1}, |\ |^{-2}$, *then there is an* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant topological isomorphism* $B(\chi_1 \otimes \chi_2) \cong B(\chi_2 |\ |^{-1} z^w, \chi_1 |\ | z^{-w})$.

Proof. Proposition 5.3.8 shows that $B(\chi_1\otimes\chi_2)$ (resp. $B(\chi_2 |\ |^{-1} z^w, \chi_1 |\ | z^{-w})$) may be identified with the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1\otimes\chi_2\big)_{\mathrm{lalg}}$ (resp. $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 |\ |^{-1} z^w \otimes \chi_1 |\ | z^{-w}\big)_{\mathrm{lalg}}$). The theory of intertwining operators for smooth parabolic induction yields an isomorphism between these two locally algebraic inductions. □

We close this section by considering one more universal unitary completion, this time in a critical case.

5.3.17. **Definition.** *Let* $B(2,\infty)$ *denote the universal unitary completion of the smooth induction* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ |^{-1} \otimes |\ |\big)_{\mathrm{sm}}$.

(The notation will be explained in Subsection 6.5 below.)

5.3.18. **Lemma.** *If* B *is a Banach space equipped with a unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-action, then any continuous* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant injection* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ |^{-1} \otimes |\ |\big)_{\mathrm{sm}} \to B$ *extends uniquely to a continuous* $\mathrm{GL}_2(\mathbb{Q}_p)$*-invariant injection* $B(2,\infty) \to B$.

Proof. By the very definition of the universal unitary completion, any continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map $\phi : \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ |^{-1} \otimes |\ |\big)_{\mathrm{sm}} \to B$ extends uniquely to a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant map $\widetilde{\phi} : B(2,\infty) \to B$. Taking $\mathcal{L} = \infty$ in the discussion of Subsection 6.5 below shows that the representation $B(2,\infty)$ sits in a non-split short exact sequence

$$0 \to \widehat{\mathrm{St}} \to B(2,\infty) \to \underline{1} \to 0.$$

Since $\widehat{\mathrm{St}}$ is topologically irreducible, by Lemma 5.3.3 (1), we see that if $\widetilde{\phi}$ is not injective, then its restriction to $\widehat{\mathrm{St}}$ vanishes. But this implies that the restriction of ϕ to the subrepresentation St of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ |^{-1} \otimes |\ |\big)_{\mathrm{sm}}$ vanishes, and so ϕ is not injective either. □

6. The local correspondence for trianguline representations

In this section we will describe the explicit correspondence $V \mapsto B(V)$ in the trianguline case that was discussed in the introduction and in Remark 3.3.7. We will see that in fact the conditions of Conjecture 3.3.1 serve to almost entirely determine this correspondence. For example, among the irreducible trianguline

V, the representations that are potentially semi-stable, but not potentially crystalline, up to a twist, are the only ones for which $B(V)$ is not completly determined by Conjecture 3.3.1. (This reflects the fact that these are the only irreducible trianguline representations whose classification requires Hodge-theoretic, in addition to Weil-Deligne, invariants; cf. Remark 3.3.4 and Theorem 4.5.4.)

For reducible V the situation is more complicated. If V is the direct sum of two characters then we can specify a corresponding $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $B(V)$, and this representation is completely determined by Conjeture 3.3.1. However if V is indecomposable then we are not able to specify a candidate for $B(V)$ in general (although we can predict what structure it should have), due mainly to our lack of knowledge about the structure of extensions in the category of admissible unitary representations.

Although the case of reducible V may seem somewhat marginal in relation to the problem of constructing the local p-adic correspondence in general, we have found it useful to study this case carefully for two reasons: firstly, the analysis in this case illustrates how the conditions of Conjecture 3.3.1 can be used to narrow down the possibilities for the associated $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $B(V)$. Secondly, the extensions that one expects to find on the level of Banach spaces in the case of reducible indecomposable V provide a model for the extensions that are conjectured to appear after passing to the subspace of locally analytic vectors in the case of irreducible V (as we explain in Subsection 6.7 below).

I would like to close this introductory passage by mentioning that the discussion of Subsection 6.5 below has been strongly influenced by remarks and suggestions of Breuil.

6.1. V **is irreducible.** If V is an irreducible continuous two dimensional trianguline representation of $G_{\mathbb{Q}_p}$, and if R is a refinement of V, then R is neither ordinary nor critical. Furthermore, we may and do choose R so that it is not ultracritical. Write $\sigma(R) = (\eta, \psi)$. According to condition (8) of Conjecture 3.3.1, the admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated to $B(V)$ should satisfy $\mathrm{Exp}^{\eta|\ |\otimes\psi\varepsilon|\ |^{-1}}(B(V)_{\mathrm{an}}) \neq \emptyset$. Suppose first that $\eta\psi^{-1} \neq |\ | z^w$ for any $w > 1$. Assuming that Conjecture 5.1.5 holds if necessary, we see that it follows from Corollary 5.3.14 that $B(V)$ must be isomorphic to $B(\eta \otimes \psi\varepsilon)$. This motivates us to make the following definitions.

6.1.1. **Definition.** If R is a non-ultracritical refinement of an irreducible trianguline representation, then we define $B(R) := B(\eta \otimes \psi\varepsilon)$.

Note that the definition of $B(R)$ depends only on R up to equivalence (since this is true of $\sigma(R)$).

6.1.2. **Definition.** Let V be an irreducible continuous two dimensional trianguline $G_{\mathbb{Q}_p}$-representation admitting a non-ultracritical refinement R such that, writing $\sigma(R) = (\eta, \psi)$, we have $\eta\psi^{-1} \neq |\ \ | z^w$ for any $w > 1$. Then we define $B(V) := B(R)$.

6.1.3. **Remark.** If V admits two inequivalent refinements R_1 and R_2 both satisfying the conditions of the preceding definition, then V is necessarily a twist of a Frobenius semi-simple potentially crystalline representation, and if we write $\sigma(R_1) = (\eta, \psi)$, and let $w > 0$ denote the Hodge-Tate weight of $\eta\psi^{-1}$, then $\sigma(R_2) = (\psi z^w, \eta z^{-w})$ (see Remark 4.3.3). Lemma 5.3.16 yields an isomorphism $B(R_1) \cong B(R_2)$, and so we see that $B(V)$ is well-defined, independent of the choice of R.

6.1.4. **Remark.** If in the context of Definition 6.1.2 the representation V is potentially semi-stable, then $\eta\psi^{-1}$ has Hodge-Tate weight $w > 0$ (since V is irreducible). Thus Proposition 5.3.8 shows that $B(V)$ can also be defined as the universal unitary completion of the locally algebraic representation $\widetilde{\pi}_p(V)$. (See Conjecture 3.3.1 (7) for the definition of $\widetilde{\pi}_p(V)$.) Thus the definition of $B(V)$ given here coincides with that given by Breuil in [12, §1.3] (perhaps up to a unitary twist, reflecting a difference between our choice of normalization of the correspondence and his).

6.1.5. **Remark.** If in the context of Definition 6.1.2 the representation V is not potentially semi-stable up to a twist, so that $\eta\psi^{-1}$ is not of positive integral Hodge-Tate weight, then the definition of $B(V)$ given here coincides with that given by Colmez [27] (again up to a possible unitary twist, reflecting a difference in normalizations). As was noted in the proof of Theorem 5.3.7, the proof of this follows the same lines as the proof of [8, Thm. 4.3.1].

The only irreducible trianguline V that do not admit a refinement satisfying the conditions of Definition 6.1.2 are those that are twists of semi-stable non-crystalline representations. If V is of this form, then adjoining $\sqrt{p}$ to E if necessary in the case when $w_2 - w_1$ is even, and replacing V by a suitable twist (and taking into account Conjecture 3.3.1 (3) and (4)) we may assume that V is semi-stable with Hodge-Tate weights $w < -1$ and 0, and that the eigenvalues of φ on $D_{\mathrm{st}}(V)$ are equal to $p^{(-1-w)/2}$ and $p^{(1-w)/2}$. The representation V then admits a refinement R that is unique up to equivalence, and $\sigma(R) = (|\ \ |^{(1+w)/2}, |\ \ |^{(-1+w)/2} z^w)$. Proposition 5.3.8 and Remark 5.3.9 show that $B(R)$ may be identified with the universal unitary completion of $\widetilde{\pi}_p(V) = \mathrm{St} \otimes |\ \ |^{(1+w)/2} \otimes \big(\mathrm{Sym}^{-1-w} E^2\big)^\vee$. The representation $\widetilde{\pi}_p$ determines $D_{\mathrm{st}}(V)$ as a (φ, N)-module, but it does not determine the Hodge filtration on $D_{\mathrm{st}}(V)$. This filtration depends on an additional parameter $\mathcal{L} \in E$ (the $\mathcal{L}$-invariant of E – in defining which we follow the convention of [26, 0.2]) that measures the relative position of the Hodge filtration on $D_{\mathrm{st}}(V)$ and the filtration induced by the kernel of N. The semi-stable representation V is thus classified by the locally algebraic representation $\widetilde{\pi}_p(V)$ together with the invariant $\mathcal{L}$.

6.1.6. **Definition.** If V is as in the preceding discussion, with $\mathcal{L}$-invariant equal to $\mathcal{L}$, then we define $B(V)$ to be the corresponding quotient $B(1-w,\mathcal{L})$ of $B(R)$, as defined in Definition 5.1.11.

6.1.7. **Remark.** If V is a twist of a non-generic crystalline representation, then adjoining $\sqrt{p}$ to E if necessary in the case when $w_2 - w_1$ is even, and replacing V by a suitable twist, we may assume that V is crystalline with Hodge-Tate weights $w < -1$ and 0, and that the eigenvalues of φ on $D_{\mathrm{crys}}(V)$ are equal to $p^{(-1-w)/2}$ and $p^{(1-w)/2}$. The representation V admits two equivalence classes of refinements. We may choose representatives R and R' so that $\sigma(R) = (|\ \ |^{(1+w)/2}, |\ \ |^{(-1+w)/2}\, z^w)$ and $\sigma(R') = (|\ \ |^{(-1+w)/2}, |\ \ |^{(1+w)/2}\, z^w)$. Definition 6.1.2 applies to V, and stipulates that $B(V) := B(R')$. On the other hand, Remark 5.1.15 shows that $B(V)$ is also isomorphic to the quotient $B(1-w,\infty)$ of $B(R)$. Thus we may treat all the cases in which V admits an $\mathcal{L}$-invariant in a uniform fashion.

6.1.8. **Remark.** If V is any irreducible trianguline representation admitting a refinement R with $\sigma(R) = (\eta,\psi)$, and if we write $r := \mathrm{ord}_p(\eta(p))$, then $B(V)$ may be described explicitly as a suitable quotient of a certain space of functions on $\mathbb{Q}_p$ of class $\mathcal{C}^r$; cf. Remarks 5.1.12 and 5.3.11. The proofs of the results of Colmez and of Berger and Breuil recalled in Remark 3.3.7, and in Theorems 5.1.6, 5.1.13, and 5.3.7, rely heavily on this explicit description of $B(V)$.

6.1.9. **Remark.** Let V be a potentially semi-stable continuous two dimensional $G_{\mathbb{Q}_p}$-representation for which $\pi_p(V)$ is cuspidal. (Equivalently, V is potentially crystalline, but becomes crystalline only over a non-abelian extension of V.) Although V is not trianguline, this case is quite similar to the case in which $\pi_p(V)$ is special. As in that case, V is not uniquely determined by the associated locally algebraic representation $\widetilde{\pi}_p(V)$. Indeed, while this representation determines the $(\varphi, G_{\mathbb{Q}_p})$-module underlying $D_{\mathrm{pcrys}}(V)$, as well as the Hodge numbers of the filtration on $D_{\mathrm{pcrys}}(V)$, there is a $\mathbb{P}^1(E)$ worth of possible choices for the Hodge filtration itself on $D_{\mathrm{dR}}(V)$. This provides the motivation for Conjecture 5.1.19: one hopes to find a series of topologically irreducible admissible unitary quotients of the universal unitary completion of $\widetilde{\pi}_p(V)$ parameterized by the same set $\mathbb{P}^1(E)$, and then to define $B(V)$ to be that quotient of $\widehat{\pi}$ indexed by the point of $\mathbb{P}^1(E)$ corresponding to the Hodge filtration on $D_{\mathrm{dR}}(V)$.

6.2. V **is the direct sum of two characters.** Suppose that $V = \eta \bigoplus \psi$ for two continuous E-valued characters η and ψ of $G_{\mathbb{Q}_p}$. If $\eta \neq \psi$, then V admits two inequivalent ordinary refinements R_1 and R_2 such that $\sigma(R_1) = (\eta,\psi)$ and $\sigma(R_2) = (\psi,\eta)$, and so Conjecture 3.3.1 (8) stipulates that we have an embedding

$$\eta\,|\ \ | \otimes \psi\varepsilon\,|\ \ |^{-1} \bigoplus \psi\,|\ \ | \otimes \eta\varepsilon\,|\ \ |^{-1} \to J_{\mathrm{P}(\mathbb{Q}_p)}(B(V)_{\mathrm{an}}). \tag{26}$$

If $\eta = \psi$, then Lemma 4.1.5 shows that we have $\mathrm{Ref}^{\eta\otimes\eta}(V) \cong \mathbb{P}^1(E)$. Thus Conjecture 3.3.1 (8) stipulates that

$$\mathrm{Exp}^{\eta|\ |\otimes\eta\varepsilon|\ |^{-1}}(B(V)_{\mathrm{an}}) \cong \mathbb{P}^1(E).$$

Thus we must again have an embedding of the form (26). Theorem 5.2.5 and Proposition 5.3.4 (1) show that the embedding (26) must be induced by a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism

$$(27) \qquad \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\otimes\psi\varepsilon\big)_{\mathrm{cont}} \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi\otimes\eta\varepsilon\big)_{\mathrm{cont}} \to B(V).$$

6.2.1. **Lemma.** *Suppose that* $\eta\psi^{-1} \neq \varepsilon^{\pm 1}$. *If* $B(V)$ *satisfies conditions (5) and (8) of Conjecture 3.3.1, then (27) is necessarily an isomorphism.*

Proof. The reduction modulo ϖ of V is equal to $\overline{\eta} \bigoplus \overline{\psi}$ (the sum of the reductions modulo ϖ of the characters η and ψ). Under the modulo ϖ local Langlands correspondence, this representation is matched with the semi-simplification of the direct sum

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \overline{\eta}\otimes\overline{\psi\varepsilon}\big)_{\mathrm{sm}} \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \overline{\psi}\otimes\overline{\eta\varepsilon}\big)_{\mathrm{sm}}.$$

This direct sum is also the reduction modulo ϖ of the source of (27). Thus Conjecture 3.3.1 (5) implies that the reductions modulo ϖ of the source and target of (27) have isomorphic semi-simplifications.

Since $\eta\psi^{-1} \neq \varepsilon^{\pm 1}$ the two direct summands in the source of (27) are irreducible, and so (27) must be an embedding. The observation of the preceding paragraph implies that it is in fact an isomorphism. □

We now consider the case when $\eta\psi^{-1} = \varepsilon^{\pm 1}$. Interchanging η and ψ if necessary, we may assume that in fact $\eta\psi^{-1} = \varepsilon$.

6.2.2. **Lemma.** *If* $\eta\psi^{-1} = \varepsilon$, *and if* $B(V)$ *satisfies conditions (3), (5), (7), and (8) of Conjecture 3.3.1, then there is an isomorphism*

$$\eta\circ\det\otimes B(2,\infty) \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\varepsilon^{-1}\otimes\eta\varepsilon\big)_{\mathrm{cont}} \xrightarrow{\sim} B(V).$$

Proof. Twisting V by η^{-1} (and taking into account Conjecture 3.3.1 (3)), we see that we may in fact assume that $\eta = \underline{1}$, so that $V = \underline{1} \bigoplus \varepsilon^{-1}$. Conjecture 3.3.1 (7) then stipulates that we should have an isomorphism

$$(28) \qquad \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ \ |^{-1}\otimes|\ \ |\big)_{\mathrm{sm}} \xrightarrow{\sim} B(V)_{\mathrm{lalg}}.$$

As a consequence we find that St is a subrepresentation of $B(V)$, and so (27) cannot be an isomorphism. Also, by Lemma 5.3.18, the isomorphism (28) induces a closed embedding

$$(29) \qquad B(2,\infty) \to B(V).$$

Since (27) is not an isomorphism, Proposition 5.3.4 (3) and (4) show that (27) factors through an embedding

$$\widehat{\mathrm{St}} \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}} \to B(V). \tag{30}$$

Considering reductions modulo ϖ as in the proof of the preceding lemma, we see that the cokernel of this embedding must be one dimensional. Thus the copy of $\widehat{\mathrm{St}}$ appearing in $B(V)$ by virtue of the embedding (29) must coincide with the copy of $\widehat{\mathrm{St}}$ appearing in $B(V)$ by virtue of the embedding (30), and we see that $B(V)$ is isomorphic to the direct sum of the sources of these two embeddings, amalgamated along their common subrepresentation $\widehat{\mathrm{St}}$. The lemma follows. □

In light of the preceding results, we make the following definition.

6.2.3. **Definition.** (1) If $V = \eta \bigoplus \psi$ and $\eta\psi^{-1} \neq \varepsilon^{\pm 1}$ then we define

$$B(V) := \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{cont}} \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{cont}}.$$

(2) If $V = \eta \bigoplus \psi$ and $\eta\psi^{-1} = \varepsilon$ then we define

$$B(V) := \eta \circ \det \otimes B(2,\infty) \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\varepsilon^{-1} \otimes \eta\varepsilon\big)_{\mathrm{cont}}.$$

Note that this definition is an obvious extension of the one given in [14] in the case when η and ψ are Hodge-Tate.

6.3. *V* **is reducible and indecomposable – first case.** Write V as a non-split extension

$$0 \to \eta \to V \to \psi \to 0, \tag{31}$$

and suppose that $\eta\psi^{-1} \neq \varepsilon^{\pm 1}$. Combining Definition 6.2.3 (1) with condition (6) of Conjecture 3.3.1 we see the topological Jordan-Hölder factors $B(V)$ should be $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{cont}}$ and $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{cont}}$. (Note that both of these representations are topologically irreducible, by Proposition 5.3.4 (3).) Since V admits an ordinary refinement R such that $\sigma(R) = (\eta,\psi)$, condition (8) of Conjecture 3.3.1 requires that $\mathrm{Exp}^{\eta|\ |\otimes\psi\varepsilon|\ |^{-1}}(B(V)_{\mathrm{an}}) \neq \emptyset$, and thus by Theorem 5.2.5 and Proposition 5.3.4 (1) there should be an embedding

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{cont}} \to B(V).$$

Altogether we see that if $B(V)$ is to satisfy the conditions of Conjecture 3.3.1 then it must sit in a non-split short exact sequence

$$0 \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{cont}} \to B(V) \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{cont}} \to 0. \tag{32}$$

(This sequence must be non-split, by condition (1) of Conjecture 3.3.1, since by assumption V is not the direct sum of η and ψ.)

The preceding considerations, together with Proposition 4.5.5, motivate the following conjecture.

6.3.1. **Conjecture.** *Let η and ψ be a pair of unitary characters of $\mathbb{Q}_p^\times$.*

(1) *If $\eta\psi^{-1} \neq \underline{1}, \varepsilon^{\pm 1}$ then the space of extensions*

$$\mathrm{Ext}^1\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{cont}}, \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{cont}}\big),$$

computed in the category of admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations over E, is one dimensional.*

(2) *For any unitary character η of $\mathbb{Q}_p^\times$, the spaces of extensions*

$$\mathrm{Ext}^1\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \eta\varepsilon\big)_{\mathrm{cont}}, \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \eta\varepsilon\big)_{\mathrm{cont}}\big),$$

computed in the category of admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations over E that have central character $\eta^2\varepsilon$, is two dimensional.*

6.3.2. **Remark.** In case (1) of the preceding conjecture, any element of the Ext^1 under consideration will automatically have central character $\eta\psi\varepsilon$, since the representations $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\otimes\psi\varepsilon\big)_{\mathrm{cont}}$ and $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi\otimes\eta\varepsilon\big)_{\mathrm{cont}}$ are topologically irreducible and non-isomorphic. Thus there is not need to explicitly impose the condition that the extensions under consideration admit a central character.

If the preceding conjecture is correct, then for a non-split extension (31) with η and ψ satisfying the condition of Conjecture 6.3.1 (1) we would define $B(V)$ to be any non-zero element of the corresponding Ext^1 (any two such elements being topologically isomorphic as $\mathrm{GL}_2(\mathbb{Q}_p)$-representations, by the conjecture). If on the other hand $\eta = \psi$, then set of isomorphism classes of $G_{\mathbb{Q}_p}$-representations that sit in a non-split extension (31) is parameterized by the points of $\mathbb{P}^1(E)$, and we expect to be able to define the correspondence $V \mapsto B(V)$ so as to match these various isomorphism classes with the corresponding $\mathbb{P}^1(E)$-worth of topological isomorphism classes of non-zero elements of the Ext^1 considered in Conjecture 6.3.1 (2).

If V is potentially crystalline and generic, up to a twist, and if $\eta \neq \psi$ (which puts us in case (1) of Conjecture 6.3.1), then Breuil has constructed an extension of the desired type by taking the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^w \otimes \eta\varepsilon z^{-w}\big)_{\mathrm{an}}$ (where $w > 0$ is the Hodge-Tate weight of $\eta\psi^{-1}$) [6, 14]. (Note that the character $\psi z^w \otimes \eta z^{-w}$ is critical.) Anticipating that Conjecture 6.3.1 (1) is correct, we follow these references in making the following definition.

6.3.3. **Definition.** If $0 \to \eta \to V \to \psi \to 0$ is a non-split extension that is potentially crystalline and generic, up to a twist, and if furthermore $\eta \neq \psi$, then we define $B(V)$ to be the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^w \otimes \eta\varepsilon z^{-w}\big)_{\mathrm{an}}$.

6.4. V **is reducible and indecomposable – second case.** Suppose that V is a non-split extension of the form $0 \to \psi\varepsilon^{-1} \to V \to \psi \to 0$. Twisting V by ψ^{-1} (and taking into account Conjecture 3.3.1 (3)), we may assume that V is in fact of the form

$$0 \to \varepsilon^{-1} \to V \to \underline{1} \to 0. \tag{33}$$

The representation V is then unique up to isomorphism, and is Hodge-Tate but not potentially semi-stable. Definition 6.2.3 (2) and Conjecture 3.3.1 (6) show that the topological Jordan-Hölder factors of $B(V)$ should be $\widehat{\mathrm{St}}$, $\underline{1}$, and $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}}$. By Conjecture 3.3.1 (7) we should have $B(V)_{\mathrm{lalg}} = 0$, and thus neither $\underline{1}$ nor $\widehat{\mathrm{St}}$ can be subobjects of $B(V)$. Consequently there must be an inclusion

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}} \to B(V). \tag{34}$$

By Conjecture 3.3.1 (8), together with Proposition 4.4.4 (2), we expect that

$$J_{\mathrm{P}(\mathbb{Q}_p)}\big(B(V)_{\mathrm{an}}\big) = |\ \ |\,\varepsilon^{-1} \otimes \varepsilon\,|\ \ |^{-1} \bigoplus \underline{1} \otimes \underline{1}.$$

If we let B denote the cokernel of (34), then the topological Jordan-Hölder factors of B must be $\widehat{\mathrm{St}}$ and $\underline{1}$, and (taking into account Proposition 5.2.1 (4) and the fact that $J_{\mathrm{P}(\mathbb{Q}_p)}$ is left exact) there is an injection

$$\underline{1} \otimes \underline{1} \to J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}).$$

These facts and Proposition 5.2.6 (or better, its proof) together imply that B contains a copy either of $\underline{1}$ or else of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ \ |^{-1} \otimes |\ \ |\big)_{\mathrm{sm}}$. In the latter case we deduce from Lemma 5.3.18 and our knowledge of the topological Jordan-Hölder factors of B that B is isomorphic to $B(2,\infty)$. One can show that there are no non-trivial extensions of $\underline{1}$ by $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}}$, and so since $\underline{1}$ is not a subrepresentation of $B(V)$, we conclude that $\underline{1}$ cannot be a subrepresentation of B either. Thus indeed we must have $B \cong B(2,\infty)$, and so $B(V)$ must sit in a non-split short exact sequence

$$0 \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}} \to B(V) \to B(2,\infty) \to 0. \tag{35}$$

This suggests the following conjecture.

6.4.1. **Conjecture.** *The space of extensions*

$$\mathrm{Ext}^1\big(B(2,\infty), \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}}\big),$$

computed in the category of admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations over* E*, is one dimensional.*

6.4.2. **Remark.** As there are no non-zero continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant homomorphisms from $B(2,\infty)$ to $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\varepsilon^{-1}\otimes\varepsilon\big)_{\mathrm{cont}}$, any element of the Ext^1 under consideration necessarily has trivial central character.

If this conjecture were true, we would then define $B(V)$ to be the extension whose existence it predicts.

6.5. V **is reducible and indecomposable – third case.** Suppose now that V is a non-split extension of the form $0\to\eta\to V\to\eta\varepsilon^{-1}\to 0$. Twisting V by η^{-1} (and recalling condition (3) of Conjecture 3.3.1), we see that it suffices to consider non-split extensions of the form

$$0\to\underline{1}\to V\to\varepsilon^{-1}\to 0. \tag{36}$$

Such an extension is either semi-stable (but not crystalline), and classified by the value of its $\mathcal{L}$-invariant, or else is non-generic crystalline (and uniquely determined up to isomorphism) – the case $\mathcal{L}=\infty$. As in the preceding case, Definition 6.2.3 (2) together with condition (6) of Conjecture 3.3.1 shows that the topological Jordan-Hölder factors of $B(V)$ should be $\widehat{\mathrm{St}}$, $\underline{1}$, and $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\varepsilon^{-1}\otimes\varepsilon\big)_{\mathrm{cont}}$.

In the case when $\mathcal{L}=\infty$, so that V is non-generic crystalline, Conjecture 3.3.1 (7) together with Lemma 5.3.18 shows that there must be an injection $B(2,\infty)\to B(V)$, and thus that $B(V)$ must be a non-split extension

$$0\to B(2,\infty)\to B(V)\to\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\varepsilon^{-1}\otimes\varepsilon\big)_{\mathrm{cont}}. \tag{37}$$

(Non-split because of Conjecture 3.3.1 (1) and the fact that V is not the direct sum of $\underline{1}$ and ε^{-1}.)

6.5.1. **Lemma.** *The universal unitary completion of* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,|\ |^{-1}\otimes|\ |\big)_{\mathrm{an}}$ *is a non-split extension of the form (37).*

Proof. The natural injection

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,|\ |^{-1}\otimes|\ |\big)_{\mathrm{sm}}\to\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,|\ |^{-1}\otimes|\ |\big)_{\mathrm{an}}$$

induces an injection of $B(2,\infty)$ into this completion We leave it to the reader to identify the cokernel as $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\varepsilon^{-1}\otimes\varepsilon\big)_{\mathrm{cont}}$. □

6.5.2. **Conjecture.** *The space of extensions*

$$\mathrm{Ext}^1\big(\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\varepsilon^{-1}\otimes\varepsilon\big)_{\mathrm{cont}},B(2,\infty)\big),$$

computed in the category of admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations over* E*, is one dimensional.*

6.5.3. **Remark.** Since there no non-zero continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant homomorphisms from $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}}$ to $B(2,\infty)$, any element of the Ext^1 under consideration necessarily has trivial central character.

Presuming that the conjecture is correct, we follow [14, Rem. 2.3.1] in making the following definition.

6.5.4. **Definition.** If V is the non-split crystalline extension of the form (36) then define $B(V)$ to be the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ |^{-1} \otimes |\ |\big)_{\mathrm{an}}$.

Now suppose that V is a semi-stable, non-crystalline, extension of the form (36). In this case Conjecture 3.3.1 (7) requires us to have an isomorphism $B(V)_{\mathrm{lalg}} \xrightarrow{\sim}$ St. Thus we have must have an injection $\widehat{\mathrm{St}} \to B(V)$, whose cokernel has $\underline{1}$ and $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\big)_{\mathrm{cont}}$ as topological Jordan-Hölder factors. For such V a candidate for $B(V)$ has not yet been specified in the literature. However, we will state a conjecture regarding its structure.

We begin by following [12], and constructing a family of non-split extensions of the trivial character by $\widehat{\mathrm{St}}$ depending on a parameter $\mathcal{L} \in \mathbb{P}^1(E)$. Recall the representation $\sigma(\mathcal{L})$ of $\overline{\mathrm{P}}(\mathbb{Q}_p)$ constructed in 5.1.7 above. This is evidently a unitary representation of $\overline{\mathrm{P}}(\mathbb{Q}_p)$ which extends the trivial character of $\overline{\mathrm{P}}(\mathbb{Q}_p)$ by itself, and so the continuous induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma(\mathcal{L})\big)_{\mathrm{cont}}$ is an admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, which sits in a short exact sequence

$$(38)\quad 0 \longrightarrow \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \underline{1} \otimes \underline{1}\big)_{\mathrm{cont}} \longrightarrow \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma(\mathcal{L})\big)_{\mathrm{cont}} \xrightarrow{\mathrm{pr}} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \underline{1} \otimes \underline{1}\big)_{\mathrm{cont}} \longrightarrow 0.$$

Recall that the trivial character $\underline{1}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ is a subrepresentation of the continuous induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \underline{1} \otimes \underline{1}\big)_{\mathrm{cont}}$. Following [12], we define $B(2,\mathcal{L}) = \mathrm{pr}^{-1}(\underline{1})/\underline{1}$ (i.e the quotient by the copy of $\underline{1}$ sitting in the first term of (38) of the preimage under the projection pr of the copy of $\underline{1}$ sitting in the third term); then $B(2,\mathcal{L})$ is an admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that sits in an exact sequence

$$0 \to \widehat{\mathrm{St}} \to B(2,\mathcal{L}) \to \underline{1} \to 0.$$

(The notation follows that used in [12].) Since $\sigma(\mathcal{L})$ is a non-split extension of the trivial character by itself, one easily sees that the exact sequence containing $B(2,\mathcal{L})$ is also non-split. In the case $\mathcal{L} = \infty$, the reader can check that $B(2,\infty)$ is topologically isomorphic to the representation defined by Definition 5.3.17 above (which explains our choice of notation in that definition).

We also remark that if $\mathcal{L} \neq \infty$, then the evident inclusion $\mathrm{St} \to B(2,\mathcal{L})_{\mathrm{lalg}}$ is an isomorphism (since the function $\log_p$, which intervenes in the definition of $\sigma(\mathcal{L})$, is not locally algebraic).

For any extension of the form (36) we then expect the representation $B(V)$ to sit in a non-split short exact sequence of admissible unitary representations

$$0 \to B(2,\mathcal{L}) \to B(V) \to \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon\right)_{\mathrm{cont}} \longrightarrow 0, \tag{39}$$

where $\mathcal{L}$ is the $\mathcal{L}$-invariant of V. Nothing is known about the existence or classification of such extensions except in the case of $\mathcal{L} = \infty$, where such an extension is constructed in Lemma 6.5.1 above. Nevertheless, we might hope that the obvious generalization of Conjecture 6.5.2, with ∞ replaced by an arbitrary element $\mathcal{L} \in \mathbb{P}^1(E)$, holds.

We close this section with the following Lemma, which bears on the question of whether the representation $B(V)$, if it were to be defined as an extension of the form (39), would satisfy conditions (7) and (8) of Conjecture 3.3.1.

6.5.5. **Lemma.** *If* $\mathcal{L} \in E$ *then the evident inclusion* $\mathrm{St} \to B(2,\mathcal{L})$ *induces isomorphisms* $\mathrm{St} \xrightarrow{\sim} B(2,\mathcal{L})_{\mathrm{lalg}}$ *and* $|\ \ | \otimes |\ \ |^{-1} = J_{\mathrm{P}(\mathbb{Q}_p)}(\mathrm{St}) \xrightarrow{\sim} J_{\mathrm{P}(\mathbb{Q}_p)}(B(2,\mathcal{L})_{\mathrm{an}})$.

Proof. Both assertions are easily checked by the reader, taking into account the truth of the analogous assertions for $\widehat{\mathrm{St}}$ in place of $B(2,\mathcal{L})$ as well as the fact that the function $\log_p$ (which intervenes in the definition of $\sigma(\mathcal{L})$, and thus of $B(2,\mathcal{L})$) is not locally algebraic. ☐

6.6. **Compatibility with Conjecture 3.3.1.** If V is a continuous two dimensional trianguline representation of $G_{\mathbb{Q}_p}$ over E, then the discussion of the preceding subsections yields a precise definition of $B(V)$ provided V satisfies one of the following (mutually exclusive) conditions:

(1) V is irreducible (Definitions 6.1.2 and 6.1.6);
(2) V is a direct sum of two characters (Definition 6.2.3);
(3) V is reducible, indecomposable, and is a twist of a Frobenius semi-simple potentially crystalline representation (Definitions 6.3.3 and 6.5.4).

In the remainder of this subsection we discuss the extent to which this explicitly defined correspondence in the trianguline case satisfies the conditions of Conjecture 3.3.1. We first note that conditions (2) through (8) of that conjecture refer only to properties of $B(V)$ for some particular V, while condition (1) involves comparing the representations $B(V)$ for all V. Thus in any verification of condition (1), we will have limit the scope of that statement to those V for which

$B(V)$ has actually been defined. We also note that conditions (2), (3), and (4) are built into the construction of the correspondence in the trianguline case, and so there is no need to discuss them further.

We consider first the reducible case. To avoid irritating circumlocutions, we will assume that the correspondence $V \mapsto B(V)$ has actually been constructed for all reducible V (i.e. that the various extensions whose existence has been conjectured in Subsections 6.2 through 6.5 have been actually shown to exist). In those cases when $B(V)$ has actually been constructed (i.e. if V is a direct sum of two characters, or is Frobenius semi-simple potentially crystalline up to a twist), our remarks will apply unconditionally; in the remaining case, they will apply as soon as $B(V)$ has been constructed, provided that it is an extension of the form conjectured above.

It is clear from the constructions that if V and W are two reducible continuous two dimensional $G_{\mathbb{Q}_p}$-representations then $B(V)$ is topologically isomorphic to $B(W)$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation if and only if V and W are isomorphic $G_{\mathbb{Q}_p}$-representations. (In the case when V and W both admit an $\mathcal{L}$-invariant, use the fact that the representations $B(2, \mathcal{L})$ are non-isomorphic for distinct values of $\mathcal{L}$, since this is true of the representations $\sigma(\mathcal{L})$ that are used in their construction.) Condition (5) is satisfied in this case (essentially by the definition of $B(V)$ and of the mod ϖ correspondence), as are the remaining conditions (6) (by construction), (7) (by a direct computation of locally algebraic vectors), and (8) (by a direct computation of Jacquet modules).

We now turn to the case of irreducible V.

6.6.1. **Theorem.** (1) *If V is an irreducible trianguline two dimensional continuous $G_{\mathbb{Q}_p}$-representation, and if V is not a twist of a potentially crystalline Frobenius non-semi-simple representation, then $B(V)$ is infinite dimensional and topologically irreducible, and satisfies Schur's lemma: any $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant continuous endomorphism of $B(V)$ is scalar.*

(2) *If V and W are trianguline two dimensional continuous representations of $G_{\mathbb{Q}_p}$, each satisfying the condition of part (1), then $B(V)$ is topologically isomorphic to $B(W)$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation if and only if V and W are isomorphic as $G_{\mathbb{Q}_p}$-representations.*

Proof. Given the definition of $B(V)$ in the various cases, it follows from Theorems 5.1.6, 5.1.13, and 5.3.7 that $B(V)$ is non-zero and topologically irreducible. Note that since $B(V)$ is a non-zero quotient of the universal unitary completion of an irreducible infinite dimensional locally analytic representation, it must in fact be infinite dimensional. The claim regarding Schur's lemma is [8, Prop. 3.4.5]. (Actually, this reference only treats the case of Frobenius semi-simple potentially crystalline V, but the argument applies equally well to the situations considered in [26, 27].) This proves part (1), while part (2) is a restatement of [27, Thm. 0.11]. □

Part (1) of the preceding result verifies condition (6) of Conjecture 3.3.1 for those representations to which it applies. Part (2) shows that the correspondence $V \mapsto B(V)$ distinguishes non-isomorphic irreducible trianguline representations that satisfy the condition of part (1).

6.6.2. **Remark.** It is expected that an appropriate variation of the methods used in the proof of the preceding result should apply to establish the remaining unproved cases of Conjecture 5.1.5. If this expectation were to be realized, then part (1) of the preceding theorem would apply without exception to all irreducible trianguline two dimensional representations. These same methods are similarly expected to allow one to extend part (2) to cover the outstanding cases. (This is implicit in the statement of [27, Thm. 0.11]; cf. [27, Rem. 0.13].)

We also note the following lemma. Note that it does not require that V satisfy the conditions of part (1) of the preceding theorem.

6.6.3. **Lemma.** *Let V be an irreducible trianguline representation, let R be a non-ultracritical refinement of V, and write $\sigma(R) = (\eta, \psi)$. Let B be an E-Banach space equipped with an admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-action. If $B(V) \to B$ is a non-zero continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism, then $J^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}}) \neq 0$.*

Proof. Let w denote the Hodge-Sen-Tate weight of $\eta\psi^{-1}\varepsilon^{-1}$. We begin by defining a locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation U attached to the pair (η, ψ), namely

$$U = \begin{cases} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \text{ if } w \notin \mathbb{Z}_{\geq 0}; \\ \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{lalg}} \text{ if } w \in \mathbb{Z}_{\geq 0} \text{ and } \eta\psi^{-1}\varepsilon^{-1} \neq z^w; \\ \eta \circ \det \otimes \mathrm{St} \otimes \big(\mathrm{Sym}^w E^2\big)^\vee \text{if } w \in \mathbb{Z}_{>0} \text{ and } \eta\psi^{-1}\varepsilon^{-1} = z^w. \end{cases}$$

(Since V is irreducible, the character $\eta \otimes \psi\varepsilon^{-1}$ is not unitary, and so falls into exactly one of the three cases.) In each case we see from the definition of $B(V)$ that there is a continuous injection $U \to B(V)$ with dense image, while Proposition 5.2.1 shows that there is an inclusion

$$\eta|\ \ | \otimes \psi\varepsilon|\ \ |^{-1} \subset J_{\mathrm{P}(\mathbb{Q}_p)}(U). \tag{40}$$

Except in the case when $w \in \mathbb{Z}_{\geq 0}$ and $\eta\psi^{-1}\varepsilon^{-1} = |\ \ |^{-2} z^w$, the representation U is furthermore topologically irreducible. Thus the non-zero map $B(V) \to B$ induces an injection $U \to B_{\mathrm{an}}$, and hence passing to Jacquet modules yields an injection $J_{\mathrm{P}(\mathbb{Q}_p)}(U) \to J_{\mathrm{P}(\mathbb{Q}_p)}(B_{\mathrm{an}})$. Composing this injection with the inclusion (40) gives the lemma.

Suppose finally that we are in the exceptional case where $\eta\psi^{-1}\varepsilon^{-1} = |\ |^{-2} z^w$, and that the map $U \to B_{\mathrm{an}}$ is not an injection. It then induces an injection of the finite dimensional quotient $(\eta |\ |) \circ \det \otimes \big(\mathrm{Sym}^w E^2\big)^\vee$ of U into B_{an}. However the Jacquet module of this quotient is precisely $\eta |\ | \otimes \psi\varepsilon |\ |^{-1}$. Thus the lemma follows in this case also. □

In light of Remark 6.6.2, for the remainder of this subsection we will write as if Theorem 6.6.1 (or equivalently Conjecture 5.1.5) held true in full generality. The cautious reader may add the necessary caveats.

6.6.4. **Lemma.** *Let V be an irreducible trianguline representation and let $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)_+$ satisfy one of the following conditions:*

(1) $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ *is unitary;*
(2) $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ *is critical.*

Then $\mathrm{Exp}^{\chi_1 \otimes \chi_2}(B(V)_{\mathrm{an}}) = \emptyset$.

Proof. Let R be a non-ultracritical refinement of V, and write $\sigma(R) = (\eta, \psi)$. Since V is irreducible, $\eta \otimes \psi$ is neither unitary, critical, nor ultracritical (by Lemma 4.5.3). By construction, together with Theorem 6.6.1, the representation $B(V)$ is a non-zero quotient of the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}$. Taking into account the isomorphism (21) established in the course of the proof of Theorem 5.2.5, we see that $J_{\mathrm{P}(\mathbb{Q}_p)}^{\eta |\ | \otimes \psi\varepsilon |\ |^{-1}}(B(V)_{\mathrm{an}}) \neq 0$.

Theorem 6.6.1 (1) shows that $B(V)$ is topologically irreducible and infinite dimensional, and thus it cannot contain any one dimensional subrepresentations. Proposition 5.2.6 thus implies that if $\mathrm{Exp}^{\chi_1 \otimes \chi_2}(B(V)_{\mathrm{an}}) \neq \emptyset$ for a character $\chi_1 \otimes \chi_2$ satisfying condition (2), then there is also a character $\chi_1' \otimes \chi_2'$ satisfying condition (1) for which $\mathrm{Exp}^{\chi_1' \otimes \chi_2'}(B(V)_{\mathrm{an}}) \neq \emptyset$. Thus we may assume that $\chi_1 \otimes \chi_2$ satisfies condition (1). Corollary 5.3.6 then implies that $B(V)$ may be identified with one of the representations appearing in the statement of that corollary. But one checks (again using Proposition 5.2.1) that none of these contains a copy of $\eta |\ | \otimes \psi\varepsilon |\ |^{-1}$ in its Jacquet module (since $\eta \otimes \psi$ is neither unitary nor critical). This contradicts the conclusion of the first paragraph, and the lemma follows. □

The preceding lemma, when combined with the computation of the Jacquet modules of the representations $B(V)$ attached to reducible V, implies that if V is reducible and W is irreducible trianguline, then $B(V)$ and $B(W)$ are not isomorphic. Combined with Theorem 6.6.1 (2), it shows that the correspondence

$V \mapsto B(V)$ for trianguline representations satisfies Conjecture 3.3.1 (1), if one restricts the range of that condition to trianguline representations.

It remains to discuss conditions (5), (7), and (8) in the case of irreducible trianguline V. Condition (5) has been verified by Berger [5] (improving on earlier results of various authors [7, 9, 10, 15]). We now turn to condition (7). Write $B(V)_{\mathrm{exp.lalg}}$ to denote the expected subspace of locally algebraic vectors in $B(V)$, according to Conjecture 3.3.1; that is $B(V)_{\mathrm{exp.lalg}} = 0$ if V is not potentially semi-stable with distinct Hodge-Tate weights, and $B(V)_{\mathrm{exp.lalg}} = \widetilde{\pi}_p(V)$ if V is potentially semi-stable with distinct Hodge-Tate weights. It is clear from the construction of $B(V)$, and the fact that it is non-zero (by Theorem 6.6.1), that there is an injection $B(V)_{\mathrm{exp.lalg}} \subset B(V)_{\mathrm{lalg}}$. However it doesn't seem to be known whether or not this inclusion is an equality. For example, if U (resp. W) is a special or cuspidal irreducible admissible smooth representation (resp. an irreducible algebraic representation) of $\mathrm{GL}_2(\mathbb{Q}_p)$, and if the central characters of $\pi := U \otimes W$ and of $B(V)$ coincide, then it doesn't seem possible to rule out the existence of an injection $\pi \to B(V)_{\mathrm{lalg}}$ whose image is not contained in $B(V)_{\mathrm{exp.lalg}}$, since the possible admissible unitary completions of π have not been clasified. (If V is the restriction to $G_{\mathbb{Q}_p}$ of the global Galois representation attached to a finite slope overconvergent eigenform, then we can rule out this possibility, and thus verify condition (7) for such V – see Remark 7.10.6 below.)

Our knowledge regarding condition (8) is similarly partial. Namley, we have the following result.

6.6.5. **Proposition.** *Let V be a trianguline irreducible two dimensional continuous $G_{\mathbb{Q}_p}$-representation, and $\eta \otimes \psi \in \widehat{\mathrm{T}}(E)_+$.*

(1) *If $\eta \otimes \psi$ is not ultracritical, and if $\eta\psi^{-1}$ is not of the form εz^n for some $n > 0$, then*

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V) = \dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}).$$

(2) *If $\eta\psi^{-1} = \varepsilon z^n$ for some $n > 0$, then*

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V) \leq \dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}),$$

with equality if $B(W)_{\mathrm{exp.lalg}} = B(W)_{\mathrm{lalg}}$ for all twists W of V.

(3) *If $\eta \otimes \psi$ is ultracritical, then*

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V) \geq \dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}).$$

Proof. If $\eta \otimes \psi$ is unitary or critical then $\mathrm{Ref}^{\eta\otimes\psi}(V) = \emptyset$ by Lemmas 4.4.1 and 4.5.3, while $\mathrm{Exp}^{\eta|\ \ |\otimes\psi|\ \ |\varepsilon^{-1}}(V) = \emptyset$ by Lemma 6.6.4. Thus we may suppose that $\eta \otimes \psi$ is neither unitary nor critical.

We begin by establishing the inequality

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V) \leq \dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}) \tag{41}$$

for all non-ultracritical characters $\eta \otimes \psi$ (assumed to be neither unitary nor critical). First note that if $\mathrm{Ref}^{\eta\otimes\psi}(V)$ is empty, then this inequality is trivially satisfied. Thus we may as well suppose that there exists a a non-ultracritical refinement R of V such that $\sigma(R) = \eta \otimes \psi$. Lemma 6.6.3, applied to the identity map from $B(V)$ to itself, then shows that the right-hand side of (41) is non-negative. Thus (41) holds in all cases.

Now suppose that $\eta \otimes \psi$ satisfies the conditions of (1). It then follows from Remark 5.3.12 that $B(\eta \otimes \psi\varepsilon)$ is topologically irreducible and admissible unitary. By Theorem 4.5.4 there exits a unique (up to isomorphism) irreducible trianguline two dimensional representation W of $G_{\mathbb{Q}_p}$ such that $\mathrm{Ref}^{\eta\otimes\psi}(W) \neq \emptyset$; then by definition $B(W) = B(\eta \otimes \psi\varepsilon)$. If V and W are not isomorphic then by construction of W we have $\mathrm{Ref}^{\eta\otimes\psi}(V) = \emptyset$. Also Theorem 6.6.1 (2) shows that $B(V)$ and $B(W)$ are not isomorphic, and so by Corollary 5.3.14 we see that $\mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}) = \emptyset$. Thus the equality of (1) holds in this case. Suppose on the other hand that V and W are isomorphic. Then we must show that $\dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}) = 0$. This follows from Theorem 5.2.5 and the fact that $B(V)$ satisfies Schur's lemma. This completes the proof of (1).

Suppose next that $\eta\psi^{-1} = \varepsilon z^n$ for some $n > 0$. The inequality of (2) is then a special case of (41). Write $\pi := \eta \otimes \mathrm{St} \otimes (\mathrm{Sym}^n E^2)^\vee$. Twisting both V and (η, ψ) appropriately, we may assume that η is of integral Hodge-Tate weight, and thus that π is locally algebraic. We then have

$$\begin{aligned}\dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}) &= \dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(B(\eta\otimes\psi\varepsilon), B(V)) - 1 \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi, B(V)) - 1 = \dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi, B(V)_{\mathrm{lalg}}) - 1,\end{aligned}$$

the first and second equality following from Theorem 5.2.5 and the fact that $B(\eta\otimes\psi\varepsilon)$ is the universal unitary completion of π, and the third from the fact that π is a locally algebraic representation. If we assume that $B(V)_{\mathrm{lalg}}$ has the structure predicted by Conjecture 3.3.1 (7), then one easily checks that $\dim \mathrm{Ref}^{\eta\otimes\psi}(V) = \dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi, B(V)_{\mathrm{lalg}}) - 1$. (Both sides of the equation equal 0.) This gives equality in (2).

Finally, let us assume that $\eta \otimes \psi$ is ultracritical, and let $w > 0$ denote the Hodge-Tate weight of $\eta\psi^{-1}$. Proposition 5.2.6 yields an inequality

$$\dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(B(V)_{\mathrm{an}}) \leq \dim \mathrm{Exp}^{\eta|\ \ |z^{-w}\otimes\psi\varepsilon|\ \ |^{-1}z^w}(B(V)_{\mathrm{an}}). \tag{42}$$

On the other hand, we have an equality

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V) = \dim \mathrm{Ref}^{\eta z^{-w}\otimes\psi z^w}(V)$$

(as follows from Proposition 4.2.4 and the fact that both dimensions are necessarily -1 if V is not Hodge-Tate up to twist, or is potentially semi-stable up to twist). Thus (3) follows from (1), applied to $\eta z^{-w} \otimes \psi z^{w}$. □

If V is the restriction to $G_{\mathbb{Q}_p}$ of the $G_{\mathbb{Q}}$-representation attached to a finite slope overconvergent eigenform, then we will show in Remark 7.10.6 below that in fact $B(V)$ does satisfy condition (8).

Note that our inability to prove an equality rather than an inequality in general in part (2) of this result is again due to our lack of knowledge of the possible admissible unitary completions of representations of the form Special ⊗ algebraic. One way to bypass this difficulty, and to establish conditions (7) and (8) completely for the representations $B(V)$ under consideration, would be to have an explicit description of $B(V)_{\mathrm{an}}$ (which would then allow us to calculate both $J_{\mathrm{P}(\mathbb{Q}_p)}(B(V)_{\mathrm{an}})$ and $B(V)_{\mathrm{lalg}}$). In the following subsection we will present a conjectural description of $B(V)_{\mathrm{an}}$ of the desired type.

6.7. **Conjectures on locally analytic vectors.** In this subsection we describe the expected structure of the locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $B(V)_{\mathrm{an}}$ associated to any trianguline continuous two dimensional representation V of $G_{\mathbb{Q}_p}$ over E. We consider various cases in turn.

6.7.1. *V is irreducible, and does not admit an $\mathcal{L}$-invariant.* According to Definition 6.1.2, in this case $B(V)$ is defined to be $B(R)$ for an appropriately chosen refinement R of V. (Any refinement that is not ultracritical will do.) Thus, if we write $\sigma(R) = (\eta, \psi)$, then $B(V)$ is equal to the universal unitary completion of the locally analytic induction

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}, \tag{43}$$

and so there is a natural morphism

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \to B(V)_{\mathrm{an}}, \tag{44}$$

whose image is dense in $B(V)$ by construction.

6.7.2. **Lemma.** *The map (44) is injective (provided that $B(V)$ is non-zero).*

Proof. If $\eta\psi^{-1}$ is not of positive integral Hodge-Tate weight, then the locally analytic induction (43) is topologically irreducible, and the lemma immediately follows. If instead $\eta\psi^{-1}$ is of Hodge-Tate weight w, for some integer $w > 0$, then (43) contains the locally algebraic induction

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{lalg}} \tag{45}$$

as its unique non-zero proper submodule. Since V is irreducible, R is not an ordinary refinement. Also, R is not ultracritical by assumption. It follows from Lemma 4.5.3 that $1 >| \eta(p) |>| p |^w$, and thus by Proposition 5.3.8 we see that the locally algebraic induction (45) is itself dense in $B(V)$. Thus the lemma follows in this case also. $\square$

As we have noted in the preceding subsection, it is known in almost all cases that $B(V)$ is indeed non-zero.

6.7.3. **Conjecture.** *The cokernel of* (44) *is isomorphic to* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}}$, *and thus the space of locally analytic vectors* $B(V)_{\mathrm{an}}$ *sits in a short exact sequence of locally analytic* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations*

$$0 \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \to B(V)_{\mathrm{an}} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}} \to 0. \tag{46}$$

6.7.4. **Lemma.** *Any extension of the form (46) is necessarily non-split.*

Proof. If the extension (46) were to split, then the locally analytic induction

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}} \tag{47}$$

would admit an embedding into the unitary representation $B(V)$. On the other hand, since the pair of characters (η, ψ) arises from a non-ordinary refinement (non-ordinary because V is assumed irreducible), we see that $| \eta(p) |< 1$, and thus that $| \psi(p) |> 1$. A consideration of Jacquet modules (more precisely, Lemma 5.2.4) shows that (47) does not admits a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm. Thus the extension under consideration must be non-split, as claimed. $\square$

Since we are in the case when V does not admit an $\mathcal{L}$-invariant, the trianguline representation V is completely determined by R, and so we expect that the extension appearing in Conjecture 6.7.3 should not depend on any additional invariant. This suggests that, up to isomorphism, there is a unique non-trivial extension of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}}$ by $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}$ in the category of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. (Reasoning analogous to that presented in Remark 6.3.2 shows that any such extension would necessarily admit a central character.)

6.7.5. *Justification for and elaboration on Conjecture 6.7.3.* Some justification for this conjecture arises by comparison with the reducible case; see [6, Cor 7.2.6] and 6.7.10 below. In Proposition 7.6.5 below, we will show in the case when V arises from a twist of a finite slope overconvergent eigenform that is non-classical, but of integral weight (so that V is Hodge-Tate, but not potentially semi-stable, up to twist, by [41, Thm. 6.6]), that the space $B(V)_{\mathrm{an}}$ *does* contain as a subrepresentation an extension of the locally algebraic representation

$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{lalg}}$ by $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}$. This provides some evidence for the conjecture.

Consider now the case when V is potentially crystalline and generic (so that it satisfies our assumption of not admitting an $\mathcal{L}$-invariant) and also Frobenius semi-simple. In this case it is known that $B(V)_{\mathrm{an}}$ contains an extension of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}}$ by $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}$, as we now recall.

It will be harmless to replace V by a twist (since this just replaces $B(V)$ by the same twist), and so we may and do assume that V is in fact potentially crystalline with Hodge-Tate weights 0 and w, for some integer $w < 0$. Since V is Frobenius semi-simple, it admits a unique (up to equivalence) pair of inequivalent refinements R_1 and R_2. If we write $\sigma(R_1) = (\eta, \psi)$, then $\sigma(R_2) = (\psi z^{-w}, \eta z^{w})$. Considering (44) with R taken to be R_1 and R_2 in turn, we obtain maps

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \to B(V)_{\mathrm{an}} \tag{48}$$

and

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^{-w} \otimes \eta z^{w}\varepsilon\big)_{\mathrm{an}} \to B(V)_{\mathrm{an}}, \tag{49}$$

which are injective by Lemma 6.7.2 (since $B(V)$ is non-zero by Theorem 6.6.1 (1)), and which coincide (up to multiplication by a non-zero scalar) on the common locally algebraic subrepresentation of their sources

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{lalg}} \xrightarrow{\sim} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^{-w} \otimes \eta z^{w}\varepsilon\big)_{\mathrm{lalg}}$$

(the isomorphism being provided by the theory of intertwining operators for smooth principal series). Thus we obtain an injection

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \widetilde{\bigoplus} \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^{-w} \otimes \eta z^{w}\varepsilon\big)_{\mathrm{an}} \to B(V)_{\mathrm{an}}, \tag{50}$$

where $\widetilde{\bigoplus}$ indicates that we form the amalgamated sum of the two summands over their common locally algebraic subrepresentation. Since the quotient

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^{-w} \otimes \eta z^{w}\varepsilon\big)_{\mathrm{an}} / \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^{-w} \otimes \eta z^{w}\varepsilon\big)_{\mathrm{lalg}}$$

is isomorphic to $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}}$, we see that the source of (50) is an extension of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}}$ by $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}$. Thus in this case Conjecture 6.7.3 is equivalent to conjecturing that (50) is an isomorphism. The author learnt this case of Conjecture 6.7.3 from Breuil. (See [8, Conj. 5.4.4].)

6.7.6. *V is irreducible, and admits an $\mathcal{L}$-invariant.* As in 5.1.7, we replace V by a twist, extending E if necessary, so that V has Hodge-Tate weights 0 and w, and so that $B(V)$ is equal to the universal unitary completion of $\Sigma(1-w,\mathcal{L})$.

It follows directly from Lemma 5.1.9 that the tautological map

$$\Sigma(1-w,\mathcal{L}) \to B(V)_{\mathrm{an}} \tag{51}$$

is an injection. We then make the following conjecture, which is the natural analogue of Conjecture 6.7.3 in the present context.

6.7.7. **Conjecture.** *The cokernel of* (51) *is isomorphic to* $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ \ |^{(-1+w)/2} z^w \otimes |\ \ |^{(3+w)/2} z\big)_{\mathrm{an}}$*, and thus the space of locally analytic vectors* $B(V)_{\mathrm{an}}$ *sits in a short exact sequence of locally analytic* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations*

$$0 \to \Sigma(1-w,\mathcal{L}) \to B(V)_{\mathrm{an}} \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\ \ |^{(-1+w)/2} z^w \otimes |\ \ |^{(3+w)/2} z\big)_{\mathrm{an}} \to 0. \tag{52}$$

6.7.8. **Lemma.** *Any extension of the form (46) is necessarily non-split.*

Proof. This follows by the same argument as used in the proof of Lemma 6.7.4. □

6.7.9. *V is reducible.* The description of $B(V)_{\mathrm{an}}$ in this case is straightforward, if we assume that $B(V)$ has the structure conjectured in Subsections 6.2 through 6.5, using the following two facts: the space of locally analytic vectors in a continuous induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{cont}}$ coincides with the corresponding locally analytic induction $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)_{\mathrm{an}}$; passing to locally analytic vectors is an exact functor [51, Thm. 7.1].

If $V = \eta \bigoplus \psi$ and $\eta\psi^{-1} \neq \varepsilon^{\pm 1}$, so that $B(V)$ is as defined in Definition 6.2.3 (1), then we will have

$$B(V)_{\mathrm{an}} = \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}}.$$

If $V = \eta \bigoplus \eta\varepsilon^{-1}$, so that $B(V)$ is as defined in Definition 6.2.3 (2), then we will have

$$B(V)_{\mathrm{an}} = \eta \circ \det \otimes B(2,\infty)_{\mathrm{an}} \bigoplus \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\varepsilon^{-1} \otimes \eta\varepsilon\big)_{\mathrm{an}}.$$

As is discussed below, for any value of $\mathcal{L} \in \mathbb{P}^1(E)$, the space $\eta \circ \det \otimes B(2,\mathcal{L})_{\mathrm{an}}$ is an extension of $\eta \circ \det$ by $\eta \circ \det \otimes \widehat{\mathrm{St}}_{\mathrm{an}}$.

If V is a non-split extension $0 \to \eta \to V \to \psi \to 0$, and $\eta\psi^{-1} \neq \varepsilon$, so that $B(V)$ is a non-split extension of the form (32), then $B(V)_{\mathrm{an}}$ is an extension of the form (46) which is itself necessarily non-split (since $B(V)$ may be recovered from $B(V)_{\mathrm{an}}$ by passing to universal unitary completions).

Finally, if V is an extension of the form $0 \to \eta \to V \to \eta\varepsilon^{-1} \to 0$, so that $B(V)$ is a non-split extension

$$0 \to \eta \circ \det \otimes B(2,\mathcal{L}) \to B(V) \to \eta \circ \det \otimes (\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varepsilon^{-1} \otimes \varepsilon)_{\mathrm{cont}} \to 0,$$

where $\eta \circ \det \otimes B(2,\mathcal{L})$ sits in the non-split short exact sequence

$$0 \to \eta \circ \det \otimes \widehat{\mathrm{St}} = \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \eta\big)_{\mathrm{an}} / \eta \circ \det \to \eta \circ \det \otimes B(2,\mathcal{L}) \to \eta \circ \det \to 0$$

whose isomorphism class is classified by the $\mathcal{L}$-invariant of V (in $\mathbb{P}^1(E)$), then $B(V)_{\mathrm{an}}$ and $\eta \circ \det \otimes B(2,\mathcal{L})_{\mathrm{an}}$ sit in extensions

$$0 \to \eta \circ \det B(2,\mathcal{L})_{\mathrm{an}} \to B(V)_{\mathrm{an}} \to (\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\varepsilon^{-1} \otimes \eta\varepsilon)_{\mathrm{an}} \to 0,$$

and

$$0 \to \eta \circ \det \otimes \widehat{\mathrm{St}}_{\mathrm{an}} = \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \eta\big)_{\mathrm{an}} / \eta \circ \det \\ \to \eta \circ \det \otimes B(2,\mathcal{L})_{\mathrm{an}} \to \eta \circ \det \to 0,$$

both of which are non-split (for the same reason as in the previous case, namely that $B(V)$ (resp. $\eta \circ \det \otimes B(2,\mathcal{L})$) can be recovered from $B(V)_{\mathrm{an}}$ (resp. $\eta \circ \det \otimes B(2,\mathcal{L})_{\mathrm{an}}$) by passing to universal unitary completions).

6.7.10. *Additional remarks.* In the case when V is indecomposable (but possibly reducible) and does not admit an $\mathcal{L}$-invariant, we have seen that $B(V)_{\mathrm{an}}$ should always be a non-split extension of the form (46). When V is not of Hodge-Tate type, or is of Hodge-Tate type but the two Hodge-Tate weights of V coincide, so that neither $\eta \otimes \psi\varepsilon$ nor $\psi \otimes \eta\varepsilon$ is of non-negative integral Hodge-Tate weight, both the locally analytic inductions appearing in (46) are topologically irreducible [49, Thm. 6.1], and so there is (conjecturally) no further "internal structure" in $B(V)_{\mathrm{an}}$. However, if V is Hodge-Tate up to twist, with distinct weights, then either

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{lalg}} \tag{53}$$

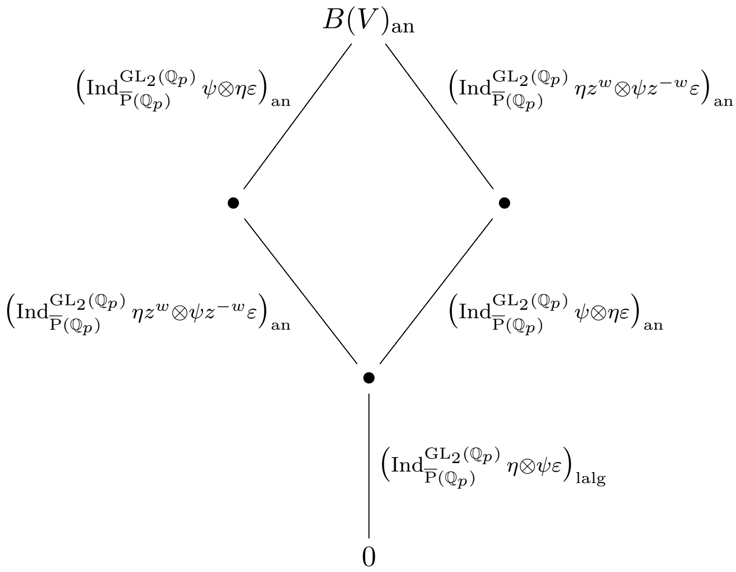
or

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{an}} \tag{54}$$

contains a locally algebraic subrepresentation, and so $B(V)_{\mathrm{an}}$ has (conjecturally) topological length three. Set $w \neq 0$ to be the Hodge-Tate weight of $\eta\psi^{-1}$.

In the case when $w < 0$, so that V is furthermore potentially crystalline up to twist, it is (53) that contains a locally algebraic representation, and in 6.7.5 we saw that (when V is irreducible) $B(V)_{\mathrm{an}}$ should be the amalgamated sum of two locally analytic inductions over their common locally algebraic subrepresentation. In the case when V is reducible, so that $B(V)$ is defined by Definition 6.3.3, one can compute directly that $B(V)_{\mathrm{an}}$ has this same structure [6, Cor. 7.2.6].

The lattice of closed subrepresentations of $B(V)_{\mathrm{an}}$ thus (conjecturally) has the following structure:



(where the labels indicate the corresponding topological Jordan-Hölder factors).

In the case when $w > 0$, so that V is not potentially crystalline up to twist, it is (54) that contains a locally algebraic subrepresentation. Thus the extension (46) contains a submodule W which is an extension

$$0 \to \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\eta\otimes\psi\varepsilon\right)_{\mathrm{an}} \to W \to \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\,\psi\otimes\eta\varepsilon\right)_{\mathrm{lalg}} \to 0,$$

which must also be non-split. (When V is irreducible, this follows by the same argument that proves lemma 6.7.4. When V is reducible, so that $\psi\otimes\eta\varepsilon$ is unitary, we see that if this extension were split, then the same would be true of (46) itself, by Proposition 5.3.8.) The lattice of closed subrepresentations of $B(V)_{\mathrm{an}}$ thus (conjecturally) has the following structure:

$$\begin{array}{l} B(V)_{\mathrm{an}} \\ \Big| \; \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi z^{-w} \otimes \eta z^{w} \varepsilon\right)_{\mathrm{an}} \\ \bullet \\ \Big| \; \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta \varepsilon\right)_{\mathrm{lalg}} \\ \bullet \\ \Big| \; \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi \varepsilon\right)_{\mathrm{an}} \\ 0 \end{array}$$

(where again we have labelled the various topological Jordan-Hölder factors). We will see in Subsection 7.6 below (see Proposition 7.6.5 and its proof) that in the case when V arises from an overconvergent eigenform of finite slope, the appearance of the locally algebraic representation in the middle of the composition series, rather than at the bottom, corresponds to the existence of the operator θ^w on overconvergent forms of weight $1 - w$.

7. Local-global compatibility

The goal of this section is to discuss in some detail Conjecture 1.1.1 of the introduction.

7.1. p-adic $\mathrm{GL}_2(\mathbb{A}_f)$-representations associated to p-adic representations of $G_{\mathbb{Q}}$. Suppose that V is a two dimensional continuous representation of $G_{\mathbb{Q}}$ defined over E, which is unramified away from a finite number of primes. As in Subsection 2.2, we may attach the admissible smooth $\mathrm{GL}_2(\mathbb{A}_f^p)$-representation $\pi^{\mathrm{m},p}(V)$ to V, which by Remark 2.2.1 is defined over E.

Assuming that the local p-adic Langlands correspondence predicted by Conjecture 3.3.1 exists, we may also associate to $V_{|D_p}$ the admissible unitary Banach space representation $B(V_{|D_p})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E.

7.1.1. **Definition.** Write $\Pi(V) := B(V_{|D_p}) \otimes_E \pi^{\mathrm{m},p}(V)$, equipped with the inductive tensor product topology (in the sense of [48, §17]).

The locally convex E-space $\Pi(V)$ is equipped with an action of $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}_f^p)$ which makes it an admissible continuous representation of this group (in the sense of [31, Def. 7.2.1]).

7.1.2. **Remark.** We remark for future reference that if $V = V_f$ for some classical cuspidal newform f, then $\pi^p(V) = \pi^{\mathrm{m},p}(V)$; equivalently, the local representations $\pi_\ell(V) \cong \pi_\ell(f)$ are all generic [40, p. 354]. Thus for such V we may also write $\Pi(V) = B(V_{|D_p}) \otimes_E \pi^p(V)$. Since the formation of $\Pi(V)$ is compatible with twisting in an evident sense, the same remark hold for those V that are twists of V_f.

Among all the admissible continuous representations of $\mathrm{GL}_2(\mathbb{A}_f)$ one would like to single out those representations of the form $\Pi(V)$ as being "automorphic". Ideally, one would have an *a priori* definition of what it means for an admissible continuous $\mathrm{GL}_2(\mathbb{A}_f)$-representation to be automorphic (perhaps in terms of a space of "p-adic automorphic forms" equipped with an admissible continuous $\mathrm{GL}_2(\mathbb{A}_f)$-action), and then one could phrase a global p-adic Langlands conjecture to the effect that these are precisely the representations of the form $\Pi(V)$. Unfortunately, no such *a priori* definition is known as of yet; more precisely, one does not actually have a candidate for the space of p-adic automorphic forms for GL_2 over $\mathbb{Q}$. However, in the discussion that follows we will present a workable substitute for this space, which one hopes should detect those $\Pi(V)$ for which V is an odd irreducible representation.

7.2. **Completed cohomology of modular curves.** We recall some constructions from [33]. To begin with, fix a compact open subgroup K^p of $\mathrm{GL}_2(\widehat{\mathbb{Z}}^p)$; we refer to K^p as the "tame level". Fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$. For $A = \mathcal{O}_E$, $\mathcal{O}_E/\varpi^s$ (for some $s > 0$), or E, write $H^1_*(K^p)_A := \varinjlim_{K_p} H^1_*(Y(K^pK_p)_{/\overline{\mathbb{Q}}}, A)$, where the inductive limit is taken over all open subgroups of $\mathrm{GL}_2(\mathbb{Z}_p)$, the cohomology is étale cohomology, and $* = \emptyset$ or c (i.e. we are considering either cohomology with unrestricted supports or cohomology with compact supports.)

7.2.1. **Lemma.** *The $\mathcal{O}_E$-module $H^1_*(K^p)_{\mathcal{O}_E}$ is torsion free and p-adically separated.*

Proof. That $H^1_*(K^p)_{\mathcal{O}_E}$ is torsion free is clear, since this is true of the étale cohomology with coefficients in $\mathcal{O}_E$ of any curve over an algebraically closed field. The claim of separatedness follows from the fact that the map [33, (4.3.4)] is an isomorphism. □

There is a natural inclusion $H^1_*(K^p)_{\mathcal{O}_E} \subset H^1_*(K^p)_E$. Since $H^1_*(K^p)_{\mathcal{O}_E}$ spans $H^1_*(K^p)_E$ as an E-vector space, Lemma 7.2.1 shows that we may put a norm on $H^1_*(K^p)_E$ – the so-called gauge of $H^1_*(K^p)_{\mathcal{O}_E}$ – whose unit ball is equal to $H^1_*(K^p)_{\mathcal{O}_E}$.

7.2.2. **Definition.** Define $\widehat{H}^1_*(K^p)_E$ to be the p-adic Banach space obtained by completing $H^1_*(K^p)_E$ with respect to the gauge of $H^1_*(K^p)_{\mathcal{O}_E}$. We let $\widehat{H}^1_*(K^p)_{\mathcal{O}_E}$ denote the unit ball of $\widehat{H}^1_*(K^p)_E$; it is naturally identified with the p-adic completion of $H^1_*(K^p)_{\mathcal{O}_E}$.

7.2.3. **Lemma.** *For any $s > 0$, there is a natural isomorphism*

$$\widehat{H}^1_*(K^p)_{\mathcal{O}_E}/\varpi^s \xrightarrow{\sim} \varinjlim_{K_p} H^1_*(Y(K_pK^p), \mathcal{O}_E/\varpi^s)$$

(where K_p runs over the directed set of all compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$*).*

Proof. This follows from the isomorphism $\widehat{H}^1_*(K^p)_{\mathcal{O}_E} \xrightarrow{\sim} \widetilde{H}^1_*(K^p)_{\mathcal{O}_E}$ discussed in [33, §4.1]. □

The following corollary results immediately from the preceding lemma, and the definition of $\widehat{H}^1_*(K^p)_{\mathcal{O}_E}$.

7.2.4. **Corollary.** *There is a natural isomorphism*

$$\widehat{H}^1_*(K^p)_{\mathcal{O}_E} \xrightarrow{\sim} \varprojlim_{s} \varinjlim_{K_p} \widehat{H}^1_*(Y(K_pK^p), \mathcal{O}_E/\varpi^s).$$

There is a natural action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $H^1_*(K^p)_E$, induced by the right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on the tower of modular curves $Y(K_pK^p)$ (for varying K_p and fixed K^p). This action preserves $H^1_*(K^p)_{\mathcal{O}_E}$, and thus extends to an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{H}^1_*(K^p)_E$ which preserves $\widehat{H}^1_*(K^p)_{\mathcal{O}_E}$.

7.2.5. **Lemma.** *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-action on $\widehat{H}^1_*(K^p)_E$ equips this Banach space with an admissible unitary representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$*.*

Proof. This is a special case of [33, Thm. 2.2.11]. □

7.2.6. **Definition.** If Σ is a fixed finite set of primes, we write $\mathbb{T}$ (or $\mathbb{T}(\Sigma)$ if we wish to emphasize Σ) to denote the polynomial algebra $\mathcal{O}_E[\{T_\ell, S_\ell\}_{\ell \notin \Sigma}]$.

Let $\Sigma(K^p)$ denote the set of primes at which K^p is ramified, together with p. For any finite set of primes Σ containing $\Sigma(K^p)$, the usual action of $\mathbb{T} := \mathbb{T}(\Sigma)$ on $H^1_*(K^p)_E$ via Hecke operators commutes with the $\mathrm{GL}_2(\mathbb{Q}_p)$-action and preserves the lattice $H^1_*(K^p)_{\mathcal{O}_E}$, and so extends to a continuous action of $\mathbb{T}$ on $\widehat{H}^1_*(K^p)_E$ that commutes with the $\mathrm{GL}_2(\mathbb{Q}_p)$-action and preserves $\widehat{H}^1_*(K^p)_{\mathcal{O}_E}$.

If $K_1^p \subset K_2^p$ is an inclusion of tame levels, then the injection $H^1_*(K_2^p)_E \to H^1_*(K_1^p)_E$ induced by pulling back cohomology classes is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, and induces a closed $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding

$$\widehat{H}^1_*(K_2^p)_E \to \widehat{H}^1_*(K_1^p)_E. \tag{55}$$

(See [33, Prop. 2.2.13].) If Σ is any finite set of primes containing $\Sigma(K_2^p)$, then this embedding is also $\mathbb{T}(\Sigma)$-equivariant.

7.2.7. **Definition.** Define

$$\widehat{H}^1_{*,E} := \varinjlim_{K^p} \widehat{H}^1_*(K^p)_E$$

(the inductive limit being taken over all tame levels K^p), endowed with the inductive limit $\mathrm{GL}_2(\mathbb{Q}_p)$-action, and with the locally convex inductive limit topology.

There is a smooth action of $\mathrm{GL}_2(\mathbb{A}_f^p)$ on $\widehat{H}^1_{*,E}$ commuting with the $\mathrm{GL}_2(\mathbb{Q}_p)$-action, induced by the action of $\mathrm{GL}_2(\mathbb{A}_f^p)$ on the tower of modular curves $Y(K_f)$. (See [33, Thm. 2.2.16] for more details.) This action is compatible with the action of the Hecke algebras $\mathbb{T}\big(\Sigma(K^p)\big)$ on the various spaces $\widehat{H}^1_*(K^p)_E$, in a manner that the following lemma makes precise.

7.2.8. **Lemma.** *If K^p is any tame level, then the natural injection of $\widehat{H}^1_*(K^p)_E$ into $\widehat{H}^1_{*,E}$ induces an isomorphism of continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-representations*

$$\widehat{H}^1_*(K^p)_E \xrightarrow{\sim} (\widehat{H}^1_{*,E})^{K^p}.$$

Furthermore, this isomorphism intertwines the action of T_ℓ (resp. S_ℓ) on the source with the action of the double coset $K^p \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} K^p$ (resp. $K^p \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} K^p$) on the target, for each prime $\ell \neq p$ that is unramified in K^p.

Proof. The first claim follows from [33, Thm. 2.2.16 (ii)]. For the second claim, see the formulas of [33, Prop. 4.4.2]. □

7.2.9. *The $G_{\mathbb{Q}}$-action on $\widehat{H}^1_{*,E}$.* Each of the spaces $H^1_*(K^p)_E$ is also equipped with a continuous action of $G_{\mathbb{Q}}$ that preserves $H^1_*(K^p)_{\mathcal{O}_E}$, and so extends to a continuous action of $G_{\mathbb{Q}}$ on $\widehat{H}^1_*(K^p)_E$ that preserves $\widehat{H}^1_*(K^p)_{\mathcal{O}_E}$. This action commutes with the actions of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathbb{T}$, and is compatible with the embeddings (55). It is unramified outside of $\Sigma(K^p)$. Passing to the inductive limit, we obtain a continuous action of $G_{\mathbb{Q}}$ on $\widehat{H}^1_{*,E}$, commuting with the $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_2(\mathbb{A}_f^p)$-actions.

7.2.10. *Forgetting supports.* Forgetting supports induces $G_{\mathbb{Q}}\times\mathrm{GL}_2(\mathbb{A}_f)$-equivariant morphisms

$$\widehat{H}^1_{c,\mathcal{O}_E} \to \widehat{H}^1_{\mathcal{O}_E} \tag{56}$$

and

$$\widehat{H}^1_{c,E} \to \widehat{H}^1_E \tag{57}$$

that are furthermore surjective. Indeed, for each tame level K^p, the maps

$$\widehat{H}^1_c(K^p)_{\mathcal{O}_E} \to \widehat{H}^1(K^p)_{\mathcal{O}_E} \tag{58}$$

and

$$\widehat{H}^1_c(K^p)_E \to \widehat{H}^1(K^p)_E \tag{59}$$

are surjective. (See [33, Prop. 4.3.9] and its proof).

7.2.11. *Completions of* H^0. We close this subsection with a brief discussion of the spaces $\widehat{H}^0(K^p)_E$ and $\widehat{H}^0_E$, which are defined in a completely analogous fashion to $\widehat{H}^1(K^p)_E$ and $\widehat{H}^1_E$ (replacing the superscript 1 by 0 at each point). The space $\widehat{H}^0(K^p)_E$ is equipped with commuting actions of $G_{\mathbb{Q}}\times\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathbb{T}(\Sigma(K^p))$, while $\widehat{H}^0_E$ is equipped with an action of $G_{\mathbb{Q}}\times\mathrm{GL}_2(\mathbb{A}_f)$. The cup product defines a pairing $\widehat{H}^0_E\otimes\widehat{H}^1_{*,E}\to\widehat{H}^1_{*,E}$ for $*=\emptyset$ or c [33, Prop. 2.2.21].

Unlike the completions of H^1, these completions of H^0 can be described very explicitly. Namely, there is an isomorphism

$$\widehat{H}^0_E \xrightarrow{\sim} \mathcal{C}(\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times, E), \tag{60}$$

where the target of the isomorphism is the space of functions that are continuous in the p-adic variable, and smooth in the prime-to-p-adic variables. We may describe the $G_{\mathbb{Q}}\times\mathrm{GL}_2(\mathbb{A}_f)$-action on $\widehat{H}^0_E$ concretely in terms of this isomorphism: global class field theory yields an isomorphism $G_{\mathbb{Q}}^{\mathrm{ab}}\xrightarrow{\sim}\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times$, and $G_{\mathbb{Q}}$ acts on $\mathcal{C}(\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times, E)$ via the composite of this isomorphism with the action of $\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times$ by translation; the group $\mathrm{GL}_2(\mathbb{A}_f)$ act on $\mathcal{C}(\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times, E)$ via the composite of the determinant map $\det:\mathrm{GL}_2(\mathbb{A}_f)\to\mathbb{A}_f^\times$ with the action of $\mathbb{A}_f^\times$ by translation.

7.3. **Systems of Hecke eigenvalues.** We will recall some basic terminology and facts related to systems of Hecke eigenvalues. Fix a finite set of primes Σ containing p, and write $\mathbb{T}:=\mathbb{T}(\Sigma)$ as above.

7.3.1. **Definition.** If A is an $\mathcal{O}_E$-algebra, then a system of Hecke eigenvalues defined over A (and outside of Σ) is a homomorphism of $\mathcal{O}_E$-algebras $\lambda:\mathbb{T}\to A$.

If λ is a system of Hecke eigenvalues defined over A, and if M is a $\mathbb{T}$-module, then we write $\big(A \otimes_{\mathcal{O}_E} M\big)^{\lambda}$ to denote the subspace of $A \otimes_{\mathcal{O}_E} M$ on which $\mathbb{T}$ acts through the character λ.

7.3.2. **Definition.** Let A be an $\mathcal{O}_E$-algebra, and let V be a free A-module of rank two equipped with a $G_{\mathbb{Q}}$-representation that is unramified outside Σ. We then define a system of Hecke eigenvalues over A via the formula: $\lambda(T_\ell) = \mathrm{trace}(\mathrm{Frob}_\ell^{-1} \mid V)$, $\lambda(\ell S_\ell) = \det(\mathrm{Frob}_\ell^{-1} \mid V)$, for all $\ell \not\in \Sigma$. We refer to λ as the system of Hecke eigenvalues associated to V.

Recall that $\mathbb{F}$ denotes the residue field $\mathcal{O}_E/\varpi$.

7.3.3. **Remark.** As is well-known, if A is a finite extension of either E or $\mathbb{F}$, and if the representation V is continuous, then V is determined up to semi-simplification by λ (by the Čebotarev density and Brauer-Nesbitt theorems).

7.3.4. **Definition.** If λ is a system of Hecke eigenvalues (outside of Σ) defined over an $\mathcal{O}_E$-algebra A, and if $\psi : \widehat{\mathbb{Z}}^\times \to A^\times$ is a character, unramified outside of Σ, then we define $\lambda \otimes \psi$ (the twist of λ by ψ) to be the system of Hecke eigenvalues $T_\ell \mapsto \psi(\ell)\lambda(T_\ell)$, $S_\ell \mapsto \psi(\ell)^2\lambda(S_\ell)$ (for $\ell \not\in \Sigma$).

Note that if λ is the system of Hecke eigenvalues attached to a two dimensional $G_{\mathbb{Q}}$-representation V, then $\lambda \otimes \psi$ is the system of eigenvalues attached to the twist $V \otimes \psi$ (where we identify ψ with a character of $G_{\mathbb{Q}}$ via global class field theory).

7.3.5. **Definition.** Let A be one of the fields E or $\mathbb{F}$, and let $\overline{A}$ denote its algebraic closure. We say that a system of Hecke eigenvalues λ defined over a finite extension of A is Eisenstein if there is a pair of continuous characters $\psi_1, \psi_2 : \widehat{\mathbb{Z}}^\times \to \overline{A}^\times$, unramified outside of Σ, such that λ is the system of Hecke eigenvalues attached to $\psi_1 \bigoplus \psi_2$, regarded as a $G_{\mathbb{Q}}$-representation via global class field theory. (Concretely, this amounts to requiring that $\lambda(T_\ell) = \psi_1(\ell) + \psi_2(\ell)$ and $\lambda(\ell S_\ell) = \psi_1(\ell)\psi_2(\ell)$ for all primes $\ell \not\in \Sigma$.)

7.3.6. **Remark.** Remark 7.3.3 shows that the system of Hecke eigenvalues associated to a continuous two dimensional $G_{\mathbb{Q}_p}$-representation V defined over a finite extension of E (resp. $\mathbb{F}$) is Eisenstein if and only if V becomes reducible over a finite extension of E (resp. $\mathbb{F}$).

7.3.7. **Definition.** We say that a p-torsion free $\mathbb{T}$-module M is Eisenstein if any system of Hecke eigenvalues λ defined over a finite extension E' of E in $\overline{\mathbb{Q}}_p$ for which $(E' \otimes_{\mathcal{O}_E} M)^{\lambda} \neq 0$ is Eisenstein.

There is a simple criterion for a $\mathbb{T}$-module to be Eisenstein.

7.3.8. **Lemma.** *Suppose that M is p-adically separated p-torsion free $\mathbb{T}$-module with the following property: there is a pair of commuting representations $\Psi_1, \Psi_2 :$*

$\widehat{\mathbb{Z}}^\times \to \mathrm{Aut}(M)$*, each unramified outside* Σ *and continuous when* $\mathrm{Aut}(M)$ *is endowed with the weak topology (i.e. the topology of pointwise convergence), such that for all* $\ell \notin \Sigma$ *we have* T_ℓ *acts on* M *via* $\Psi_1(\ell) + \Psi_2(\ell)$ *and* ℓS_ℓ *acts on* M *via* $\Psi_1(\ell)\Psi_2(\ell)$*. Then* M *is Eisenstein.*

Proof. This is essentially what is shown in the proof of [14, Cor. 3.1.3]. We recall the argument. Suppose that λ is a system of Hecke eigenvalues defined over a finite extension E' of E for which $(E' \otimes_{\mathcal{O}_E} M)^\lambda \neq 0$. We again write $\Psi_i(\ell)$ to denote the extension of scalars of $\Psi_i(\ell)$ to each of $\mathcal{O}_{E'} \otimes_{\mathcal{O}_E} M$ and $E' \otimes_{\mathcal{O}_E} M$ $(i = 1, 2)$. The formulas for the action of T_ℓ and S_ℓ in terms of the automorphisms $\Psi_i(\ell)$ show that these automorphisms commute with the action of $\mathbb{T}$, and thus that $(E' \otimes_{\mathcal{O}_E} M)^\lambda$ is stable under the action of the $\Psi_i(\ell)$. Furthermore, on this space we have $\lambda(T_\ell) = \Psi_1(\ell) + \Psi_2(\ell)$ and $\lambda(\ell S_\ell) = \Psi_1(\ell)\Psi_2(\ell)$ for all $\ell \notin \Sigma$. In particular, multiplication by $\lambda(T_\ell)$ and $\lambda(S_\ell)$ preserves the non-zero p-adically separated module $(\mathcal{O}_{E'} \otimes_{\mathcal{O}_E} M)^\lambda$, and so λ must take values in $\mathcal{O}_{E'}$.

Replacing E by E' and M by $(\mathcal{O}_{E'} \otimes_{\mathcal{O}_E} M)^\lambda$, we may assume that $E' = E$, that M is a non-zero $\mathbb{T}$-module, and that λ is an $\mathcal{O}_E$-valued system of Hecke eigenvalues such that $\mathbb{T}$ acts on M through λ. We then find that for each $i = 1, 2$, we have $\Psi_i(\ell)^2 - \lambda(T_\ell)\Psi_i(\ell) + \lambda(\ell S_\ell) = 0$. Replacing E by the compositum of its (finitely many) quadratic extensions, we may assume that each of these quadratics splits over E. Choose $\ell_1, \ldots, \ell_r \in \widehat{\mathbb{Z}}$ that topologically generate $\widehat{\mathbb{Z}}^\times/(\widehat{\mathbb{Z}}^\Sigma)^\times$. Since M is non-zero, and since the automorphisms $\Psi_i(\ell_j)$ commute among themselves, we may find a non-zero element m of M such that each $\Psi_i(\ell_j)$ acts via a scalar on m, and so deduce (since Ψ_1 and Ψ_2 are weakly continuous) that there exist continuous characters $\psi_1, \psi_2 : \widehat{\mathbb{Z}}^\times/(\widehat{\mathbb{Z}}^\Sigma)^\times \to \mathcal{O}_E^\times$ such that $\Psi_i(\ell)m = \psi_i(\ell)m$ $(i = 1, 2)$. We conclude that $\lambda(T_\ell) = \psi_1(\ell) + \psi_2(\ell)$ and $\lambda(\ell S_\ell) = \psi_1(\ell)\psi_2(\ell)$, and thus that λ is Eisenstein. □

For example, one can use this criterion to show that for any tame level K^p unramified outside Σ, the kernel of the map (57) is an Eisenstein $\mathbb{T}$-module. (See Lem. 3.1.2 and Cor. 3.1.3 of [14].)

7.3.9. **Definition.** If K^p is a tame level unramified outside of Σ, then we write $\widetilde{\mathbb{T}}_*(K^p)$ to denote the weak closure of the image of $\mathbb{T}$ in $\mathrm{End}\big(\widehat{H}^1_*(K^p)_E\big)$. (Of course here End denotes continuous endomorphisms.)

We can give a more concrete description of $\widetilde{\mathbb{T}}_*(K^p)$. For any compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$, and any integer $s > 0$, write $\widetilde{\mathbb{T}}_*(K_pK^p)_s$ to denote the image of $\mathbb{T}$ in $\mathrm{End}\big(H^1_*(Y(K_pK^p), \mathcal{O}_E/\varpi^s)\big)$. Since $H^1_*(Y(K_pK^p), \mathcal{O}_E/\varpi^s)$ is a finite $\mathcal{O}_E$-module, we see that $\widetilde{\mathbb{T}}_*(K_pK^p)_s$ is a finite $\mathcal{O}_E$-algebra. Corollary 7.2.4

gives a topological isomorphism

$$\widetilde{\mathbb{T}}_*(K_p) \xrightarrow{\sim} \varprojlim_{s,K_p} \widetilde{\mathbb{T}}_*(K_pK^p)_s. \tag{61}$$

The surjection (59) induces a continuous homomorphism

$$\widetilde{\mathbb{T}}_c(K^p) \to \widetilde{\mathbb{T}}(K^p), \tag{62}$$

which has dense image by construction.

7.3.10. **Lemma.** *The map (62) is an isomorphism of reduced, compact topological $\mathcal{O}_E$-algebras.*

Proof. The isomorphism (61) shows that both the source and target of (62) are profinite, and so compact. Since this map is continuous with dense image, it must be surjective. If we fix the level K_pK^p, and write $\widetilde{\mathbb{T}}_*(K_pK^p) := \varprojlim_s \widetilde{\mathbb{T}}_*(K^pK_p)_s$, then $\widetilde{\mathbb{T}}_*(K^pK_p)$ is naturally identified with the image of $\mathbb{T}$ in $\mathrm{End}\big(H^1_*(Y(K_pK^p), \mathcal{O}_E)\big)$. This image is well-known to be reduced. Thus, rewriting (61) in the form

$$\widetilde{\mathbb{T}}_*(K_p) \xrightarrow{\sim} \varprojlim_{K_p} \widetilde{\mathbb{T}}_*(K_pK^p),$$

we obtain a description of the algebra $\widetilde{\mathbb{T}}_*(K^p)$ as the projective limit of a projective system of reduced rings. This shows that $\widetilde{\mathbb{T}}_*(K^p)$ is reduced.

Let $\widehat{M}_E \subset \widehat{H}^1_c(K^p)_E$ denote the kernel of (59), and write

$$I = \widetilde{\mathbb{T}}_c(K^p) \bigcap \mathrm{Hom}(\widehat{H}^1_c(K^p)_E, \widehat{M}_E), \quad J = \widetilde{\mathbb{T}}_c(K^p) \bigcap \mathrm{Hom}(\widehat{H}^1(K^p)_E, \widehat{M}_E)$$

(where Hom denotes continuous homomorphisms.) Clearly I is the kernel of (62), while $J^2 = 0$. We claim that $I = J$, i.e. that any element of $\widetilde{\mathbb{T}}_c(K^p)$ that annihilates $\widehat{H}^1(K^p)$ also annihilates $\widehat{M}_E$. It will follow that $I^2 = 0$, and thus that $I = 0$, since (as we have just shown) $\widetilde{\mathbb{T}}_c(K^p)$ is reduced.

Let $\mathcal{E}is$ denote the set of pairs of characters $\psi_1, \psi_2 : \widehat{\mathbb{Z}}^\times \to \mathbb{Q}_p^\times$, unramified outside of Σ, associated to the various classical Eisenstein series of weight 2 and tame level K^p. For each $(\psi_1, \psi_2) \in \mathcal{E}is$, there is a continuous homomorphism $\lambda_{\psi_1,\psi_2} : \widetilde{\mathbb{T}}_c(K^p) \to \mathbb{Q}_p$ uniquely determined by the fact that its composite with the natural map $\mathbb{T} \to \widetilde{\mathbb{T}}_c(K^p)$ is equal to the system of Hecke eigenvalues associated to $\psi_1 \bigoplus \psi_2$. (This is just the homomorphism describing the Hecke action on the Eisenstein series attached to (ψ_1, ψ_2); cf. Remark 7.3.14 below.) If I_{ψ_1,ψ_2} denotes the kernel of λ_{ψ_1,ψ_2}, then the annihilator of $\widehat{M}_E$ in $\widetilde{\mathbb{T}}_c(K^p)$ is equal to $\bigcap_{(\psi_1,\psi_2)\in\mathcal{E}is} I_{\psi_1,\psi_2}$. (This follows for example from the explicit description of $\widehat{M}_E$ given in [14].) Thus we must show that if $t \in \widetilde{\mathbb{T}}_c(K^p)$ annihilates $\widehat{H}^1(K^p)_E$, then

$t \in I_{\psi_1,\psi_2}$ for all (ψ_1, ψ_2). This is clear, however, since the classical cohomology space $H^1(K^p)_E$ contains a class with annihilator equal to I_{ψ_1,ψ_2} (since the system of eigenvalues λ_{ψ_1,ψ_2} appears in the cokernel of the map $H^1_c(K^p)_E \to H^1(K^p)_E$). □

7.3.11. **Definition.** Let A be a finite field extension of E (resp. of $\mathbb{F}$). We say that a system of Hecke eigenvalues λ defined over A is promodular (resp. modular) if it can be written as the composite of the natural map $\mathbb{T} \to \widetilde{\mathbb{T}}(K^p)$ with a continuous homomorphism $\widetilde{\lambda} : \widetilde{\mathbb{T}}(K^p) \to A$, for some tame level K^p that is unramified outside Σ.

7.3.12. **Remark.** Lemma 7.3.10 shows that replacing $\widetilde{\mathbb{T}}(K^p)$ with $\widetilde{\mathbb{T}}_c(K^p)$ in the preceding definition would yield an equivalent notion.

7.3.13. **Lemma.** *Let λ be a system of eigenvalues defined over a finite extension E' of E. If $(\widehat{H}^1(K^p)_{E'})^{\lambda} \neq 0$, then λ is promodular.*

Proof. Since $\mathbb{T}$ has dense image in $\widetilde{\mathbb{T}}(K^p)$, we see that any element of $\widetilde{\mathbb{T}}(K^p)$ acts on the vectors of $(\widehat{H}^1(K_p)_{E'})^{\lambda}$ through a scalar. Since $\widetilde{\mathbb{T}}(K^p)$ acts on $\widehat{H}^1(K_p)_{E'}$ via continuous operators, the resulting homomorphism $\widetilde{\lambda} : \widetilde{\mathbb{T}}(K^p) \to E'$ is continuous. Since λ is obtained as the composite of the natural map $\mathbb{T} \to \widetilde{\mathbb{T}}(K^p)$ with $\widetilde{\lambda}$, we see that λ is promodular. □

7.3.14. **Remark.** If f is a Hecke eigenform of weight $k \geq 2$ and tame level K^p, either cuspidal or an Eisenstein series, then the system of Hecke eigenvalues λ attached to f is promodular. If f is of weight 2, then this follows directly from the preceding lemma and Eichler-Shimura theory, which shows that $(H^1(K^p)_{E'})^{\lambda} \neq 0$. If $k > 2$, then $(H^1(K^p)_{E'})^{\lambda} = 0$. Nevertheless, Theorem 7.4.2 below shows that $(\widehat{H}^1_{E'})^{\lambda} \neq 0$, and so the preceding lemma still applies. Any weight 1 eigenform also gives rise to a promodular system of Hecke eigenvalues: taking its product with the members of a sequence of positive weight Eisenstein series that converges p-adically to the constant q-expansion 1, we obtain a sequence of modular forms of weights ≥ 2 whose members are eigenforms modulo increasingly large powers of p that converges (on the level of q-expansions) to f.

7.3.15. **Definition.** Let A be a finite extension of E (resp. of $\mathbb{F}$), and let V be a continuous two dimensional $G_{\mathbb{Q}_p}$-representation defined over A and unramified outside of Σ. We say that V is promodular (resp. modular) if its associated system of Hecke eigenvalues is promodular (resp. modular).

If V is given, but the set of primes Σ is not specified, we will say that V is promodular (or modular) if it is so with respect to some set of primes Σ containing p together with all the ramified primes of V.

7.3.16. **Remark.** Let $\overline{V}$ be an absolutely irreducible continuous two dimensional $G_{\mathbb{Q}}$-representation over $\mathbb{F}$. Then $\overline{V}$ is modular in the above sense if and only if it is modular in the usual sense of being obtained as the reduction mod ϖ of the Galois representation attached to some newform. Furthermore, let X be the deformation space associated to $\overline{V}$ (parameterizing deformations of $\overline{V}$ to $G_{\mathbb{Q}}$-representations over Artinian local $\mathcal{O}_E$-algebras with residue field equal to $\mathbb{F}$), which is a formal scheme over $\mathcal{O}_E$, and let $\mathcal{X}$ be its rigid analytic generic fibre. Let $\mathcal{X}^{\mathrm{mod}}$ denote the rigid analytic Zariski closure in $\mathcal{X}$ of the set of points arising from deformations of $\overline{V}$ attached to classical modular forms of weight ≥ 2 and level unramified outside of Σ. One can check that a lift of $\overline{V}$ to a representation V over a finite extension A of E is promodular in the above sense if and only if it corresponds to an A-valued point of $\mathcal{X}^{\mathrm{mod}}$.

7.3.17. **Proposition.** *Let A be a finite field extension of E (resp. of $\mathbb{F}$). If $\lambda : \mathbb{T} \to A$ is a promodular (resp. modular) system of Hecke eigenvalues, then there is a continuous two dimensional Galois representation V defined over A, unramified outside of Σ, and such that λ is the system of Hecke eigenvalues associated to V. Any such V is necessarily odd.*

Proof. This result is well-known, but we recall the proof. Denote by $\widetilde{\lambda}$ the homomorphism $\widetilde{\mathbb{T}}(K^p) \to A$ associated to λ (for an appropriate choice of tame level K^p). In the case when A is an extension of $\mathbb{F}$, the isomorphism (61) shows that $\widetilde{\lambda}$ factors through $\mathbb{T}(K_pK^p)_1 \to A$ for some sufficiently small K_p. Thus (by the Deligne-Serre lemma [30, Lem. 6.11]) λ can be regarded as the reduction mod ϖ of the system of Hecke eigenvalues attached to some weight 2 newform f, and we take V to be the reduction mod ϖ of V_f. (Note that V is odd, since V_f is.)

Now suppose that A is an extension of E. For each newform (either cuspidal or Eisenstein) f of weight 2 and tame level K^p, Remark 7.3.14 shows that there is a homomorphism $\widetilde{\mathbb{T}}(K^p) \to \overline{\mathbb{Q}}_p$ describing the action of the Hecke operators on f. Let I_f denote its kernel. Since $H^1(K^p)_E$ is dense in $\widehat{H}^1(K^p)_E$, Eichler-Shimura theory show that

$$\bigcap_f I_f = 0 \tag{63}$$

(the intersection running over all f). Attached to each f is an odd continuous two dimensional representation V_f of $G_{\mathbb{Q}}$, unramified outside of Σ, and defined over $(\mathbb{T}/I_f)[1/p]$. (When f is the Eisenstein series associated to a pair of characters (ψ_1, ψ_2), set $V_f = \psi_1 \bigoplus \psi_2$.) Taking the product of all these, we obtain an odd continuous representation V of $G_{\mathbb{Q}}$ into $\mathrm{GL}_2(\prod_f (\widetilde{\mathbb{T}}(K^p)/I_f)[1/p])$ that is unramified outside Σ. From (63) we see that the natural (diagonal) map $\widetilde{\mathbb{T}}(K^p) \to \prod_f (\widetilde{\mathbb{T}}(K^p)/I_f)[1/p]$ realizes $\widetilde{\mathbb{T}}(K^p)$ as a subalgebra of the product. Since the traces on V of elements of $G_{\mathbb{Q}}$ lie in $\widetilde{\mathbb{T}}(K^p)$, one sees that the continuous pseudo-representation attached to V actually takes values in $\widetilde{\mathbb{T}}(K^p)$ (or in $\widetilde{\mathbb{T}}(K^p)\otimes_{\mathcal{O}_E} E$, if $p=2$); cf. the formulas of [24, p. 71]. Specializing this pseudo-representation with respect to the homomorphism $\widetilde{\lambda}$ yields a continuous pseudo-representation of $G_{\mathbb{Q}}$ over A that is unramified outside Σ. This continuous pseudo-representation then underlies a continuous two dimensional odd $G_{\mathbb{Q}}$-representation, unramified outside Σ, which is our desired representation V.

Note that (in either case) if W is any other representation associated to the same system of Hecke eigenvalues λ then V and W have isomorphic semi-simplifications (by Remark 7.3.3), and thus W is also odd. □

7.3.18. **Remark.** In the context of the preceding proposition, it follows from Remarks 7.3.3 and 7.3.6 that if λ is not Eisenstein then V is uniquely determined up to isomorphism, and is absolutely irreducible.

7.3.19. **Lemma.** *If V is a promodular continuous two dimensional representation of $G_{\mathbb{Q}}$ over E, and if $\psi: G_{\mathbb{Q}} \to E^{\times}$ is a continuous character, then $V\otimes\psi$ is again promodular.*

Proof. Choose the tame level K^p so that the system of Hecke eigenvalues λ attached to V arises from a continuous homomorphism $\widetilde{\lambda}: \widetilde{\mathbb{T}}(K^p) \to E$; further shrink K^p if necessary so that ψ, regarded as a character of $\mathbb{Q}_{>0}\backslash\mathbb{A}_f^{\times}$ by global class field theory, is trivial on $\det(K^p)$. Thinking of ψ as an element of $\mathcal{C}(\mathbb{Q}_{>0}\backslash\mathbb{A}_f^{\times}, E)$, let c_ψ denote the cohomology class in $\widehat{H}^0_E$ that corresponds to ψ under the isomorphism (60). Cupping with c_ψ induces an automorphism of $\widehat{H}^1_*(K^p)_E$, which intertwines the Hecke operator T_ℓ (resp. S_ℓ) with the operator $\psi(\ell)T_\ell$ (resp. $\psi(\ell)^2S_\ell$) for any $\ell\in\Sigma(K^p)$. Thus the automorphsm $T_\ell\mapsto\psi(\ell)T_\ell$, $S_\ell\mapsto\psi(\ell)^2S_\ell$ of $\mathbb{T}(\Sigma(K^p))$ induces a corresponding continuous automorphism of $\widetilde{\mathbb{T}}(K^p)$. Composing $\widetilde{\lambda}$ with this automorphism yields a continuous homomorphism $\widetilde{\lambda}'$, whose restriction to $\mathbb{T}(\Sigma(K^p))$ is immediately seen to coincide with $\lambda\otimes\psi$. Thus $\lambda\otimes\psi$, and so also $V\otimes\psi$, is promodular. □

7.4. **Locally algebraic vectors in $\widehat{H}^1_{*,E}$.** We will give an explicit description of the space $\big(\widehat{H}^1_{*,E}\big)_{\mathrm{lalg}}$.

7.4.1. **Definition.** Let $E(1)$ denote a one dimensional vector space over E on which $G_{\mathbb{Q}}\times\mathrm{GL}_2(\mathbb{A}_f)$ acts through the character $\varepsilon\otimes(\varepsilon\circ\det)$. For any $n\in\mathbb{Z}$ let $E(n)=E(1)^{\otimes n}$.

Recall the definition of $H^1_*(\mathcal{V}_k)$ (for $k\geq 2$) from Subsection 2.5.

7.4.2. **Theorem.** *For* $* = \emptyset$ *or* c *there is a natural* $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$*-equivariant isomorphism*

$$\bigoplus_{k \geq 2, n \in \mathbb{Z}} H^1_*(\mathcal{V}_k)_E \otimes_E \left(\mathrm{Sym}^{k-2} E^2\right)^{\vee} \otimes_E E(n) \xrightarrow{\sim} \left(\widehat{H}^1_{*,E}\right)_{\mathrm{lalg}}.$$

(Here $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$ *acts on* $\left(\mathrm{Sym}^{k-2} E^2\right)^{\vee}$ *through its quotient* $\mathrm{GL}_2(\mathbb{Q}_p)$*, and the subscript "*lalg*" indicates the subspace of locally* $\mathrm{GL}_2(\mathbb{Q}_p)$*-algebraic vectors.)*

Proof. This follows from the isomorphism of [33, (4.3.4)], once one notes that

$$\{\left(\mathrm{Sym}^{k-2} E^2\right)^{\vee} \otimes \det{}^n\}_{k \geq 2, n \in \mathbb{Z}}$$

is a complete set of isomorphism class representatives of the irreducible algebraic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. □

If f is a cuspidal newform of some weight k defined over E then the two dimensional $G_{\mathbb{Q}}$-representation V_f may also be defined over E, as may the associated $\mathrm{GL}_2(\mathbb{A}_f)$-representation $\pi(V_f)$ (see Remark 2.2.1). From Theorem 2.5.1 we see that there is a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding

$$V_f \otimes_E \pi(V_f) \to H^1_*(\mathcal{V}_k)_E,$$

for $* = \emptyset$ or c. (We are justified in taking either value of $*$, since the Manin-Drinfeld theorem provides a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant splitting of the surjection $H^1_c(\mathcal{V}_k)_E \to H^1_{\mathrm{par}}(\mathcal{V}_k)_E$.)

7.4.3. **Corollary.** *If* f *is a cuspidal newform of weight* k *defined over* E*, then for either value of* $*$ *(i.e.* $\emptyset$ *or* c*) there is a* $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$*-equivariant embedding*

$$V_f \otimes_E \pi(V_f) \otimes_E \left(\mathrm{Sym}^{k-2} E^2\right)^{\vee} \to \widehat{H}^1_{*,E}.$$

(Here $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$ *acts on* $\left(\mathrm{Sym}^{k-2} E^2\right)^{\vee}$ *through the projection onto* $\mathrm{GL}_2(\mathbb{Q}_p)$*.)*

Proof. This follows from the preceding remarks together with Theorem 7.4.2. □

7.5. **The Jacquet module of $\widehat{H}^1(K^p)_E$.** Fix a tame level K^p. As we recalled in the proof of Proposition 5.3.1, since $\widehat{H}^1(K^p)_E$ is admissible unitary, its Jacquet module $J_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)$ is an essentially admissible $\mathrm{T}(\mathbb{Q}_p)$-representation, and so corresponds to a coherent sheaf $\mathcal{E}(K^p)$ on the rigid analytic space $\widehat{\mathrm{T}}$. (Recall that the global sections of $\mathcal{E}(K^p)$ are naturally identified with the topological dual to $J_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)$.) Write $\Sigma := \Sigma(K^p)$. The action of $\mathbb{T}(\Sigma)$ on $\widehat{H}^1(K^p)_E$ induces an action of $\mathbb{T}(\Sigma)$ on $J_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)$, and hence on $\mathcal{E}(K^p)$. We let $\mathcal{A}(K^p)$ denote the coherent sheaf of commutative algebras over $\widehat{\mathrm{T}}$ generated by the image of $\mathbb{T}(\Sigma)$ in the sheaf of endomorphisms of $\mathcal{E}(K^p)$, and let $\mathrm{Spec}\,\mathcal{A}(K^p)$

denote its relative spectrum over $\mathbb{T}$. Thus $\operatorname{Spec}\mathcal{A}(K^p)$ is a rigid analytic space over E, equipped with a closed immersion

$$\operatorname{Spec}\mathcal{A}(K^p) \to \widehat{\mathrm{T}} \times \operatorname{Spec}\big(E \otimes_{\mathcal{O}_E} \mathbb{T}(\Sigma)\big),$$

whose composite with the projection onto the factor $\widehat{\mathrm{T}}$ is a finite morphism. We denote a typical point of $\big(\operatorname{Spec}\mathcal{A}(K^p)\big)(\overline{\mathbb{Q}}_p)$ by $(\chi_1 \otimes \chi_2, \lambda)$, where $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(\overline{\mathbb{Q}}_p)$, and $\lambda : \mathbb{T}(\Sigma) \to \overline{\mathbb{Q}}_p$.

7.5.1. **Definition.** Let $\mathcal{W}$ denote the rigid analytic space parameterizing the continuous characters of $\mathbb{Z}_p^\times$ (i.e. "weight space", in the usual terminology).

There is a natural action of $\mathcal{W}$ on $\widehat{\mathrm{T}} \times \operatorname{Spec}\big(E \otimes_{\mathcal{O}_E} \mathbb{T}(\Sigma)\big)$ via twisting, given on the level of $\overline{\mathbb{Q}}_p$-valued points by the formula

$$\psi \times (\chi_1 \otimes \chi_2, \lambda) \mapsto (\chi_1\psi \otimes \chi_2\psi, \lambda \otimes \psi)$$

(where we extend ψ to a character of $\mathbb{Q}_p^\times$ by setting $\psi(p) = 1$). The closed subspaces $\operatorname{Spec}\mathcal{A}(K^p)$ of $\widehat{\mathrm{T}} \times \operatorname{Spec}\big(E \otimes_{\mathcal{O}_E} \mathbb{T}(\Sigma)\big)$ are invariant under twisting [33, Prop. 4.4.6].

The following results show that there are $G_{\mathbb{Q}}$-representations attached to the points of $\operatorname{Spec}\mathcal{A}(K^p)$, and that these representations appear in $\widehat{H}^1(K^p)_E$.

7.5.2. **Proposition.** *If* $x = (\chi_1 \otimes \chi_2, \lambda)$ *is a* $\overline{\mathbb{Q}}_p$*-valued point of* $\operatorname{Spec}\mathcal{A}(K^p)$ *(recall that* x *is then in fact an* E'*-valued point for some finite extension* E' *of* E*), then the system of Hecke eigenvalues* λ *is promodular.*

Proof. The existence of x shows that $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi_1\otimes\chi_2}\big((\widehat{H}^1(K^p)_{E'})_{\mathrm{an}}^{\lambda}\big) \neq 0$. In particular, $(\widehat{H}^1(K^p)_{E'})_{\mathrm{an}}^{\lambda} \neq 0$, and so Lemma 7.3.13 shows that λ is promodular. ☐

7.5.3. **Proposition.** *Let* $(\chi_1 \otimes \chi_2, \lambda)$ *be a point of* $\big(\operatorname{Spec}\mathcal{A}(K^p)\big)(E)$ *for which* $\chi_1 |\ |^{-1} \otimes \chi_2 |\ | \varepsilon^{-1}$ *is neither critical nor ultracritical and* λ *is non-Eisenstein. Let* V *be the* $G_{\mathbb{Q}}$*-representation over* E *associated to* λ *(by the previous Proposition together with Proposition 7.3.17).*

(1) *If* $\chi_1 |\ |^{-1} \otimes \chi_2 |\ |$ *is unitary then there is a non-zero* $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant continuous map*

$$V \otimes \big(\operatorname{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)_{\mathrm{cont}} \to \widehat{H}^1(K^p)_E^{\lambda}.$$

(2) *If* $\chi_1 |\ |^{-1} \otimes \chi_2 |\ |$ *is not unitary then there is a non-zero* $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant continuous map*

$$V \otimes B(\chi_1 |\ |^{-1} \otimes \chi_2 |\ |) \to \widehat{H}^1(K^p)_E^{\lambda}.$$

(Recall from Definition 5.3.10 that $B(\chi_1 |\ |^{-1} \otimes \chi_2 |\ |)$ *denotes the universal unitary completion of* $\big(\operatorname{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)_{\mathrm{an}}$*.)*

Proof. The assumption that $(\chi_1 \otimes \chi_2, \lambda)$ lies in $\big(\operatorname{Spec}\mathcal{A}(K^p)\big)(E)$ implies that the (*a priori* finite dimensional) $G_{\mathbb{Q}}$-representation $J^{\chi_1\otimes\chi_2}_{\mathrm{P}(\mathbb{Q}_p)}\big((\widehat{H}^1(K^p)_E)_{\mathrm{an}}\big)^{\lambda}$ is non-zero, and so by the Eichler-Shimura relations (and the irreducibility of V) contains a copy of V. Theorem 5.2.5 thus shows that there is a non-zero $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map

$$V \otimes \big(\operatorname{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 |\ |^{-1} \otimes \chi_2 |\ |\big)_{\mathrm{an}} \to \widehat{H}^1(K^p)^{\lambda}_E.$$

The proposition follows (taking into account Proposition 5.3.4 (1) in case (1)). □

The injection $\widehat{H}^1(K^p)_{\mathrm{lalg}} \to \widehat{H}^1(K^p)_{\mathrm{an}}$ induces a corresponding continuous injection on Jacquet modules. If we take into account the isomorphism of Theorem 7.4.2 and the compatibility of the formation of Jacquet modules with countable direct sums [32, Lem. 3.4.7 (iv)], we may write this injection in the form

$$\bigoplus_{k\geq 2, n\in\mathbb{Z}} J_{\mathrm{P}(\mathbb{Q}_p)}\big(H^1(\mathcal{V}_k)^{K^p} \otimes_E (\mathrm{Sym}^{k-2}E^2)^{\vee} \otimes_E E(n)\big) \to J_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big). \tag{64}$$

If we fix a value of k and n, then the injection $H^1(\mathcal{V}_k)^{K^p} \otimes_E (\mathrm{Sym}^{k-2}E^2)^{\vee} \otimes_E E(n) \to \big(\widehat{H}^1(K^p)_E\big)_{\mathrm{an}}$ is a closed embedding [31, Prop. 6.3.6] (the source being equipped with its finest convex topology), and so induces a closed embedding on Jacquet modules. Passing to the associated coherent sheaves on $\widehat{\mathrm{T}}$, we obtain a surjection of coherent sheaves of $\mathbb{T}(\Sigma)$-modules $\mathcal{E}(K^p) \to \mathcal{E}_{k,n}$ (where we let $\mathcal{E}_{k,n}$ denote the coherent sheaf associated to $J_{\mathrm{P}(\mathbb{Q}_p)}\big(H^1(\mathcal{V}_k)^{K^p} \otimes_E (\mathrm{Sym}^{k-2}E^2)^{\vee} \otimes_E E(n)\big)\big)$. Thus if we let $\mathcal{A}_{k,n}$ denote the coherent algebra of endomorphisms of $\mathcal{E}_{k,n}$ induced by $\mathbb{T}(\Sigma)$, we obtain a closed embedding $\operatorname{Spec}\mathcal{A}_{k,n} \to \operatorname{Spec}\mathcal{A}(K^p)$. Taking the union over all $k \geq 2, n \in \mathbb{Z}$, we find that (64) induces an injective morphism of rigid analytic spaces over $\widehat{\mathrm{T}}$

$$\coprod_{k\geq 2, n\in\mathbb{Z}} \operatorname{Spec}\mathcal{A}_{k,n} \to \operatorname{Spec}\mathcal{A}(K^p), \tag{65}$$

which induces a closed embedding when restricted to each term of the disjoint union in the source.

7.5.4. **Lemma.** *For any fixed value of k and n, the sheaf $\mathcal{A}_{k,n}$ is reduced, and has discrete support on $\widehat{\mathrm{T}}$.*

Proof. Recall that the locally analytic Jacquet module functor is compatible with the classical Jacquet module functor [32, Prop. 4.3.6], and that the classical Jacquet module of an admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is an admissible smooth representation of $\mathrm{T}(\mathbb{Q}_p)$, which corresponds to a sheaf with discrete support on $\widehat{\mathrm{T}}$. Since $H^1(\mathcal{V}_k)^{K^p}_E$ is an admissible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, we conclude that $\mathcal{E}_{k,n}$, and so also $\mathcal{A}_{k,n}$, has discrete support. Since the action of $\mathbb{T}(\Sigma)$ on $H^1(\mathcal{V}_k)_E^{K^p}$ is semi-simple (as one sees from Eichler-Shimura theory), the sheaf $\mathcal{A}_{k,n}$ is furthermore reduced. $\square$

7.5.5. **Definition.** We say that a point of $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)(\overline{\mathbb{Q}}_p)$ is classical if it is in the image of the map on $\overline{\mathbb{Q}}_p$-points induced by (65).

7.5.6. **Definition.** We denote by $\widetilde{D}(K^p)$ the rigid analytic Zariski closure of the set of classical points in $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}}$ (or equivalently, in $\widehat{\mathrm{T}} \times \mathrm{Spec}\big(E \otimes_{\mathcal{O}_E} \mathbb{T}(\Sigma)\big)\big)$. We refer to $\widetilde{D}(K^p)$ as the "eigensurface" of tame level K^p.

7.5.7. **Remark.** Since the source of (65) is reduced (Lemma 7.5.4), we may also define $\widetilde{D}(K^p)$ as the "scheme-theoretic image" (so to speak) of that map.

For a fixed pair (k, n) the space $\mathrm{Spec}\,\mathcal{A}_{k,n}$ is invariant under twisting by locally constant characters in $\mathcal{W}$, while twisting by ε^m induces an isomorphism $\mathrm{Spec}\,\mathcal{A}_{k,n} \xrightarrow{\sim} \mathrm{Spec}\,\mathcal{A}_{k,m+n}$. Thus we see that $\widetilde{D}(K^p)$ is invariant under twisting by characters of integral Hodge-Tate weight in $\mathcal{W}$. Since these characters are Zariski dense in $\mathcal{W}$, we see that $\widetilde{D}(K^p)$ is invariant under the twisting action of $\mathcal{W}$.

The discussion of [33, §4.4] shows that in fact $\widetilde{D}(K^p)$ factors as a product

$$\widetilde{D}(K^p) \cong D(K^p) \times \mathcal{W} \tag{66}$$

where $D(K^p)$ denotes the reduced eigencurve parameterizing finite slope overconvergent p-adic eigenforms of tame level K^p. (This justifies our designation of $\widetilde{D}(K^p)$ as an eigensurface.) This factorization is $\mathcal{W}$-equivariant, with respect to the twisting action by $\mathcal{W}$ on the source, and the action of $\mathcal{W}$ by multiplication on itself on the target.

Let us describe the isomorphism (66) more precisely. Suppose that f is a finite slope overconvergent $\mathbb{T}(\Sigma)[U_p]$-eigenform of tame level K^p and integral weight k. Let $\alpha \in \overline{\mathbb{Q}}_p$ denote U_p-eigenvalue of f, let χ denote the nebentypus of f (so that $\chi : \widehat{\mathbb{Z}}^\times \to \overline{\mathbb{Q}}_p^\times$ is a locally constant character unramified outside of $\Sigma(K^p)$), and let $\lambda : \mathbb{T}(\Sigma) \to \overline{\mathbb{Q}}_p$ denote the system of Hecke eigenvalues attached to f. Let $c_f \in D(K^p)(\overline{\mathbb{Q}}_p)$ be the point attached to f (so that c_f depends only on α and λ – but note that λ encodes χ), and fix some $\psi \in \mathcal{W}(\overline{\mathbb{Q}}_p)$. Under the isomorphism (66), the point $(c_f, \psi) \in \big(D(K^p) \times \mathcal{W}\big)(\overline{\mathbb{Q}}_p)$ corresponds to the point

$$(\mathrm{ur}(\alpha) \mid\, \mid \psi \otimes \mathrm{ur}(\alpha)^{-1}\chi_p^{-1} \mid\, \mid^{-1} \varepsilon^{2-k}\psi, \lambda \otimes \psi) \in \widetilde{D}(K^p)(\overline{\mathbb{Q}}_p) \subset \big(\mathrm{Spec}\,\mathcal{A}_c(K^p)\big)(\overline{\mathbb{Q}}_p).$$

Here χ_p denotes the local component of χ at p, thinking of χ as a character of $\mathbb{A}_f^\times$ via the isomorphism (1) of Subsection 1.3, and as above we extend ψ to a character of $\overline{\mathbb{Q}}_p^\times$ by setting $\psi(p) = 1$; thus the first coordinate

$$\mathrm{ur}(\alpha) \mid \ \mid \psi \otimes \mathrm{ur}(\alpha)^{-1}\chi_p^{-1} \mid \ \mid^{-1} \varepsilon^{2-k}\psi$$

of this point makes sense as an element of $\widehat{\mathrm{T}}$, as it should. (See the discussion of [33, §4.4], and in particular the formulas in the statement of [33, Prop. 4.4.2].)

We also note that under (66) the classical points in $\widetilde{D}(K^p)(\overline{\mathbb{Q}}_p)$ lying in the image of $(\mathrm{Spec}\,\mathcal{A}_{k,n})(\overline{\mathbb{Q}}_p)$ under the map (65) correspond to the points $(c_f, \psi) \in (D(K^p) \times \mathcal{W})(\overline{\mathbb{Q}}_p)$ for which f is a finite slope p-stabilized classical Hecke eigenform of weight k, and ψ is a character of integral Hodge-Tate weight n. (Thus the discreteness assertion of Lemma 7.5.4 is simply a rephrasing of the fact that the collection of classical eigenforms of finite slope and fixed weight k gives rise to a discrete set of points on the eigencurve.)

The following result clarifies the relationship between the spaces $\widetilde{D}(K^p)$ and $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}}$. In its statement we employ the following notational convention: if ψ_1 and ψ_2 are a pair of continuous characters $G_{\mathbb{Q}} \to E$, we write $\psi_1 \oplus \psi_2$ to denote the system of Hecke eigenvalues attached to the $G_{\mathbb{Q}}$-representation $\psi_1 \bigoplus \psi_2$.

7.5.8. **Theorem.** *The image of the closed embedding*

$$\widetilde{D}(K^p) \to \big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}} \tag{67}$$

is a union of connected components of the target, and its complement decomposes as the disjoint union of a collection of one dimensional irreducible connected components. Furthermore, the system of Hecke eigenvalues attached to any point in this complement is Eisenstein.

Proof. We begin by recalling some of the known result about the structure of the spaces $\widetilde{D}(K^p)$ and $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}}$. It is known that the eigencurve $D(K^p)$ is equidimensional of dimension one [16], and thus $\widetilde{D}(K^p)$ is equidimensional of dimension two. On the other hand, it follows from [33, Prop. 2.3] that the spaces $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}}$ are at most two dimensional. Since they are invariant under twisting, each irreducible component is of thus of dimension either one or two. Furthermore, the one dimensional irreducible components (if any exist) are also connected components, since they consist of single $\mathcal{W}$-orbits under twisting. The non-critical slope criterion for classicality [33, Prop. 2.3.6] shows that any two dimensional component of $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}}$ contains a Zariski dense set of classical points, and thus is in fact a component of $\widetilde{D}(K^p)$. If we write C to denote the complement of the image of the closed embedding (67), then the preceding discussion shows that C consists of a union of connected components of $\big(\mathrm{Spec}\,\mathcal{A}(K^p)\big)_{\mathrm{red}}$, each consisting of a single $\mathcal{W}$-orbit, and containing no classical points.

We now recall some notation and constructions from the proof of [33, Thm. 2.1.5]. That argument, specialized to our particular case, yields the existence of a length two complex $\widetilde{S}^0 \to \widetilde{S}^1$ of admissible Banach K_p-representations (for some sufficiently small compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Z}_p)$) whose cohomology groups are naturally isomorphic to $\widehat{H}^0(K^p)_E$ and $\widehat{H}^1(K^p)_E$ respectively, and such that $\widetilde{S}^i \xrightarrow{\sim} \mathcal{C}(K_p, E)^{r_i}$ as a K^p-representation for some $r_i \geq 0$ $(i = 0, 1)$. (Actually, that proposition discuss certain spaces $\widetilde{H}^i$ rather than the completions $\widehat{H}^i$ under discussion here. However, in our situation the spaces $\widehat{H}^i$ and $\widetilde{H}^i$ coincide – see [33, §4.1].) Passing to locally analytic vectors (which we remind the reader is an exact functor [51, Thm. 7.1]) yields a complex $\widetilde{S}^\bullet_{\mathrm{an}}$ of locally analytic K^p-representations, with cohomology groups isomorphic to $\big(\widehat{H}^\bullet(K^p)_E\big)_{\mathrm{an}}$, and such that $\widetilde{S}^i_{\mathrm{an}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{an}}(K_p, E)^{r_i}$. Taking topological duals we obtain a complex of finite rank free $\mathcal{D}^{\mathrm{an}}(K^p, E)$-modules $(\widetilde{S}^\bullet_{\mathrm{an}})'$, with cohomology groups isomorphic to $\big(\widehat{H}^\bullet(K^p)_{E,\mathrm{an}}\big)'$. (Here $\mathcal{D}^{\mathrm{an}}(K^p, E)$ denotes the nuclear Fréchet algebra of locally analytic E-valued distributions on K^p.)

Imagine for a moment that $\widehat{H}^0(K^p)_{E,\mathrm{an}}$ vanished. We would then have a short exact sequence

$$0 \to \big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)' \to (\widetilde{S}^\bullet_{\mathrm{an}})' \to (\widetilde{S}^\bullet_{\mathrm{an}})' \to 0,$$

of $\mathcal{D}^{\mathrm{an}}(K^p, E)$-modules in which the second and third terms are free. It would follow that $\big(\widehat{H}^1(K^p)_{E,\mathrm{an}}\big)'$ is a direct summand of a free $\mathcal{D}^{\mathrm{an}}(K^p, E)$-module, and a variant of [32, Prop. 4.2.36] would show that the support of $\mathcal{E}(K^p)$ in $\widehat{\mathrm{T}}$, and hence also $\operatorname{Spec} \mathcal{A}(K^p)$, is equidimensional of dimension two. The discussion of the first paragraph would then show that (67) is an isomorphism.

Now of course $\widehat{H}^0(K^p)_{E,\mathrm{an}} \neq 0$. However, it is Eisenstein. So, heuristically, if we localize away from the Eisenstein systems of Hecke eigenvalues, we may proceed as if $\widehat{H}^0(K^p)_{E,\mathrm{an}}$ were zero, and so conclude that any component containing a point $\big(\operatorname{Spec} \mathcal{A}(K^p)\big)_{\mathrm{red}}$ associated to a non-Eisenstein system of Hecke eigenvalues is equidimensional of dimension two, yielding the theorem. The details of the argument will appear in a future publication of the author. ☐

7.5.9. **Remark.** It was proved in [14, Prop. 5.3.2] that the embedding (67) induces a bijection on ordinary points.

7.6. **Refinements and the eigensurface.** The following result is essentially a reformulation of [41, Thm., p. 375].

7.6.1. **Theorem.** *Let V be an odd irreducible continuous two dimensional representation of $G_{\mathbb{Q}}$ over E, unramified outside of a finite set of primes. Let K^p be a tame level such that $\Sigma(K^p)$ contains all the primes of ramification of V, and let $\lambda : \mathbb{T}(\Sigma(K^p)) \to E$ denote the system of Hecke eigenvalues attached to V.*

(1) *The following are equivalent:*

(a) *There is a Galois character* $\psi : G_{\mathbb{Q}} \to E^{\times}$*, unramified outside of* p*, such that the twist* $V \otimes \psi$ *is the Galois representation attached to an overconvergent modular form of finite slope and of tame level* K^p *defined over* E*.*

(b) *There is a character* $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$ *such that* $(\chi_1 \otimes \chi_2, \lambda)$ *lies in* $\widetilde{D}(K^p)(E)$*.*

(2) *Furthermore, for any character* $\chi_1 \otimes \chi_2$ *satisfying condition (1)(b), there is a refinement* R *of* $V_{|D_p}$ *such that* $\sigma(R) = (\chi_1 \mid \ \mid^{-1}, \chi_2 \mid \ \mid \varepsilon^{-1})$*. (In particular,* $V_{|D_p}$ *is trianguline.)*

(3) *If* $V_{|D_p}$ *is indecomposable, then the converse to (2) holds: If* V *satisfies the equivalent conditions of (1), then for any refinement* R *of* $V_{|D_p}$*, writing* $\sigma(R) = (\eta, \psi)$*, the point* $(\eta \mid \ \mid \otimes \psi\varepsilon \mid \ \mid^{-1}, \lambda)$ *lies in* $\widetilde{D}(K^p)(\overline{\mathbb{Q}}_p)$*.*

Proof. The equivalence of the two conditions of (1) follows directly from the description (66) of $\widetilde{D}(K^p)$. Clearly the assertion of (2) is invariant under twisting. Thus we may assume that $\chi_1 \otimes \chi_2 = \mathrm{ur}(\alpha) \mid \ \mid \otimes \mathrm{ur}(\alpha)^{-1}\varepsilon \mid \ \mid^{-1} \det V_{|D_p}$, and that V is the Galois representation attached to an overconvergent eigenform f with U_p-eigenvalue α. It then follows from [41, Thm., p. 375] that V admits a refinement R such that $\sigma(R) = (\mathrm{ur}(\alpha), \mathrm{ur}(\alpha)^{-1} \det V_{|D_p})$. This proves (2).

We turn to proving (3). Thus $V_{|D_p}$ is assumed to be indecomposable, and so Propositions 4.2.2 and 4.4.4 (4) show that V admits a unique equivalence class of refinements unless $V_{|D_p}$ is a twist of a Hodge-Tate representation with distinct Hodge-Tate weights. Thus, taking into account (1)(b), and applying a twist if necessary, we see that we may assume that V is attached to an overconvergent eigenform f of integral weight $k \neq 1$. Since $V_{|D_p}$ is assumed to be indecomposable, it follows from [41, Thm. 6.6] that f is classical if and only if $V_{|D_p}$ is potentially semi-stable. Let us consider this case first. Since f is classical, we may and do assume that it is the p-stabilization of a newform.

If $V_{|D_p}$ is not potentially crystalline, or is potentially crystalline but not Frobenius semi-simple, then it admits a unique equivalence class of refinements, and so there is nothing to prove. If $V_{|D_p}$ is potentially crystalline and Frobenius semi-simple, then it has two equivalence classes of refinements. The point on $\widetilde{D}(K^p)$ given by f corresponds to one of these equivalence classes, and we must find a second point lying over V that corresponds to the other. If $V_{|D_p}$ is in fact crystalline, then f is the p-stabilization of a newform of prime-to-p level. If $\widetilde{f}$ denotes the p-stabilized twin of f, then $\widetilde{f}$ gives the sought-after second point. If $V_{|D_p}$ is not crystalline, then f is itself a newform, whose nebentypus ω is non-trivial at p. Let ω_p be the p-part of ω. If we take $\widetilde{f}$ to be the newform associated to the twist $f \otimes \omega_p^{-1}$, then $\widetilde{f}$ is p-stabilized and has finite slope, and gives rise to a point on $\widetilde{D}(K^p)$ with system of Hecke eigenvalues $\lambda \otimes \omega$. Twisting that point by $\omega^{-1} \in \mathcal{W}(E)$ gives the second point lying over V. (See [14, §4.1] for a representation theoretic point of view on this discussion.)

Now suppose that $V_{|D_p}$ is Hodge-Tate but not potentially semi-stable. If $k > 1$, then let g be an overconvergent eigenform of weight $2 - k$ such that $\theta^{k-1} g = f$. (Such a g exists by [21, 22].) If $k < 1$, then let $g = \theta^{1-k} f$. Let x denote the point in $\widetilde{D}(K^p)(E)$ corresponding to f, and let y denote the point in $\widetilde{D}(K^p)(E)$ obtained by twisting the point corresponding to g by the character ε^{1-k}. Then V is the Galois representation attached to both x and y, and we have found two distinct points of $\widetilde{D}(K^p)(E)$ giving rise to V, which by (2) must account for the two equivalence classes of refinements of V. This completes the proof of (3). □

7.6.2. **Conjecture.** *Part (3) of the preceding theorem continues to hold in the case when* $V_{|D_p}$ *is the direct sum of two characters.*

7.6.3. **Remark.** The preceding conjecture can be proved in many cases. Suppose that V satisfies the equivalent conditions of part (1) of the preceding theorem, and also that $V_{|D_p} = \eta \bigoplus \psi$ for two characters η and ψ. If $\eta = \psi$ then $\sigma(R) = (\eta, \eta)$ for every refinement of V and there is nothing to prove. Thus we may also assume that η and ψ are distinct. There are then two cases to consider, namely the two cases of Proposition 4.4.5.

In the first case, $V_{|D_p}$ admits just its two equivalence classes of ordinary refinements. Part (2) of the preceding theorem then shows that the overconvergent eigenform f to whose twist V is attached must be ordinary. Twisting and relabelling if necessary, we may assume that V is attached to f, that η is unramified, and that $\eta(p)$ is the U_p-eigenvalue of f. Thus the point of $\widetilde{D}(K^p)$ given by f corresponds to the ordinary refinement R_1 of $V_{|D_p}$ for which $\sigma(R_1) = (\eta, \psi)$. Let ψ_0 denote the restriction $\psi_{|\mathbb{Z}_p^\times}$, regarded as a character of $G_{\mathbb{Q}}$ unramified outside of p. To prove (3), we must show that there is a (necessarily ordinary) overconvergent form g with U_p-eigenvalue equal to $\psi(p)$, and with associated Galois representation equal to $V \otimes \psi_0^{-1}$. (The point of $\widetilde{D}(K^p)$ obtained by twisting the point corresponding to g by ψ_0 will then have V as associated Galois representation, and will correspond to the refinement R_2 for which $\sigma(R_2) = (\psi, \eta)$.) If the mod ϖ representation $\overline{V}$ attached to V satisfies the necessary hypothesis, then results identifying ordinary universal deformation rings with ordinary Hecke algebras (such as the one stated in [17, §1]) will provide the required form g, and thus verify part (3) of the preceding theorem in this case.

Now suppose that we are in case (2) of Proposition 4.4.5, so that V is potentially crystalline, with distinct Hodge-Tate weights, up to a twist. Since V has three equivalence classes of refinements, we must construct three points on $\widetilde{D}(K^p)$ that give rise to V. Twisting V if necessary, we may thus assume that V is potentially crystalline, and is attached to an overconvergent eigenform f of integral weight $k \neq 1$. If $k < 1$, then $V \otimes \varepsilon^{k-1}$ is attached to $\theta^{1-k} f$, which is of weight $2-k$. Thus we may assume that in fact f is of weight $k > 1$. The Fontaine-Mazur conjecture [37] implies that in fact f should be classical. If this is so, then the argument that proves [14, Thm. 1.1.2] in the split case provides the necessary three points. Essentially, we have two classical points, namely an ordinary form f and an associated slope $k-1$ form $\widetilde{f}$ (just as in the proof of classical case of the preceding theorem) as well as a weight $2-k$ ordinary form g such that $\theta^{k-1} g = \widetilde{f}$. (The existence of g is a consequence of the fact that $V_{|D_p}$ is split [14, Thm. 1.1.3].) The results of Skinner and Wiles [55, 56] show that in fact the Fontaine-Mazur conjecture for V is true provided that η and ψ have distinct reductions modulo ϖ. (The reader should bear in mind that V is assumed to arise from an overconvergent eigenform, so that the residual modularity hypothesis of [56] holds for V.) Thus under this hypothesis (3) is also true. (Note that (3) is actually equivalent to the Fontaine-Mazur conjecture for V: if (3) holds, then we see that some twist of V must arise from an ordinary form of weight $k > 1$, which is necessarily classical.)

7.6.4. **Remark.** Consider the projection

$$\mathrm{pr} : \widetilde{D}(K^p)(\overline{\mathbb{Q}}_p) \to \big(\mathrm{Spec}\, E \otimes_{\mathcal{O}_E} \mathbb{T}(\Sigma)\big)(\overline{\mathbb{Q}}_p)$$

which maps a point to its associated system of Hecke eigenvalues. If λ is a non-Eisenstein system of Hecke eigenvalues lying in the image, corresponding to the irreducible $G_{\mathbb{Q}}$-representation V, then we see from the preceding discussion that the fibre $\mathrm{pr}^{-1}(\lambda)$ consists of one, two, or (admitting Conjecture 7.6.2) three points, depending on whether $V_{|D_p}$ satisfies condition (1), (2), or (3) of the following list:

(1) Either
   (a) $V_{|D_p}$ is indecomposable, and is not Hodge-Tate up to a twist.
   (b) $V_{|D_p}$ is indecomposable, and is potentially semi-stable, but not potentially crystalline and Frobenius semi-simple, up to twist.
   (c) $V_{|D_p}$ is the direct sum of two copies of the same character.
(2) Either
   (a) $V_{|D_p}$ is indecomposable, and is Hodge-Tate, but not potentially semi-stable, up to a twist.
   (b) $V_{|D_p}$ is indecomposable, and is potentially crystalline and Frobenius semi-simple up to a twist.
   (c) $V_{|D_p}$ is a direct sum of two characters, but is not Hodge-Tate up to a twist.
(3) $V_{|D_p}$ is a direct sum of two distinct characters, and is Hodge-Tate up to a twist.

Recall that if one restricts the projection pr to the eigencurve $D(K^p)$ then the behaviour is simpler: the preimage of a non-Eisenstein system of eigenvalues will have either one or two points. Furthermore the second case occurs precisely when the corresponding Galois representation V is Frobenius semi-simple and crystalline, and thus attached to a newform of prime-to-p conductor (with the proviso that we admit Conjecture 7.6.2, or equivalently, the Fontaine-Mazur conjecture, in the case when $V_{|D_p}$ is a direct sum of distinct characters whose reductions mod ϖ coincide; this is the exceptional case of [41, Thm. 6.6]).

We now state a result that gives some evidence for Conjecture 6.7.3 (cf. the discussions of 6.7.5 and 6.7.10). Suppose that V is an irreducible continuous two dimensional $G_{\mathbb{Q}_p}$-representation that is trianguline and Hodge-Tate, but not potentially semi-stable, up to a twist. Suppose furthermore that V is (the restriction to $G_{\mathbb{Q}_p}$ of) the Galois representation attached to a point of $\widetilde{D}(K^p)(E)$. Let R be the non-ultracritical refinement of V, and write $\sigma(R) = (\eta, \psi)$.

7.6.5. **Proposition.** *With the preceding hypotheses and notation, the locally analytic* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $B(V)_{\mathrm{an}}$ *contains a subrepresentation* W *that sits in a short exact sequence*

$$0 \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \to W \to \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{lalg}} \to 0.$$

Proof. Let R' denote the ultracritical refinement of V, so that $\sigma(R') = (\eta z^{-w}, \psi z^{w})$, where $w < 0$ is the Hodge-Tate weight of $\eta\psi^{-1}$. If λ denotes the system of Hecke eigenvalues attached to the overconvergent eigenform giving rise to V, then Theorem 7.6.1 (3) (applied to R') shows that the point $(\eta z^{-w}\,|\;| \otimes \psi z^{w}\varepsilon\,|\;|^{-1}, \lambda)$ lies in $\widetilde{D}(K^p)(E) \subset \big(\mathrm{Spec}\,\mathcal{A}_c(K^p)\big)(E)$. In particular $J_{\mathrm{P}(\mathbb{Q}_p)}^{\eta z^{-w}|\;|\otimes\psi z^{w}\varepsilon|\;|^{-1}}\big(\widehat{H}^1_c(K^p)_{E,\mathrm{an}}\big) \neq 0$. Let $u \in \widehat{H}^1_c(K^p)_{E,\mathrm{an}}$ denote the canonical lift (in the sense of [32, 0.9]) of a non-zero element of this space, and let U denote the closed $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation of $\widehat{H}^1_c(K^p)_E$ generated by u. As was observed in the proof of Proposition 5.2.6, the element $(X_-)^{-w}u \in \widehat{H}^1_c(K^p)_{E,\mathrm{an}}$ is the canonical lift of a non-zero element of $J_{\mathrm{P}(\mathbb{Q}_p)}^{\eta|\;|\otimes\psi\varepsilon|\;|^{-1}}\big(\widehat{H}^1_c(K^p)_E\big)$. Since $(X_-)^{-w}u \in U$, it is in fact the canonical lift of a non-zero element of $J_{\mathrm{P}(\mathbb{Q}_p)}^{\eta|\;|\otimes\psi\varepsilon|\;|^{-1}}\big(U\big)$. By Theorem 5.2.5 there is thus a non-zero map $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}} \to U$. Since U is unitary (being a closed subspace of $\widehat{H}^1_c(K^p)_E$), this extends to a non-zero map

$$B(\eta \otimes \psi\varepsilon) \to U. \tag{68}$$

(Recall that $B(\eta\otimes\psi\varepsilon)$ denotes the universal unitary completion of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta\otimes \psi\varepsilon\big)_{\mathrm{an}}$.)

Since $B(\eta\otimes\psi\varepsilon)$ is topologically irreducible and admissible (by Theorem 5.3.7) the map (68) must be a closed embedding. Let $\overline{u}$ denote the image of u in the cokernel of this embedding. Since $(X_-)^{-w}\overline{u} = 0$, we see that if $\overline{u}$ is non-zero, then it generates a copy of

$$\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2} \eta z^{-w} \otimes \psi z^w \varepsilon\big)_{\mathrm{lalg}} \cong \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2} \psi \otimes \eta\varepsilon\big)_{\mathrm{lalg}}$$

(the isomorphism being supplied by an intertwiner on the smooth factors of these irreducible locally algebraic representations). However Lemma 5.2.4 shows that this locally algebraic representation cannot be contained in a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. Thus necessarily $\overline{u} = 0$, and so (68) is an isomorphism. If W denotes the closed $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation generated by u in $U_{\mathrm{an}} \subset \big(\widehat{H}^1_c(K^p)_E\big)_{\mathrm{an}}$, then W has the extension structure stipulated in the statement of the proposition (since the closed subrepresentation of U_{an} generated by $(X_-)^{-w}u$ is precisely a copy of $\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{an}}$). The proposition follows, once we recall that $B(V) = B(\eta \otimes \psi\varepsilon)$ by definition. □

7.7. **Mapping Galois representations into $\widehat{H}^1_{*,E}$.** Throughout this subsection V will denote an irreducible continuous two dimensional representation of $G_{\mathbb{Q}}$ over E. We let $\overline{V}$ denote the semi-simplification of the reduction modulo ϖ of (some $G_{\mathbb{Q}}$-invariant lattice in) V. (The resulting semi-simple $G_{\mathbb{Q}}$-representation over $\mathbb{F}$ is independent of the choice of lattice.)

7.7.1. **Definition.** If $* \in c, \emptyset$, then we write $\mathcal{M}_*(V) := \mathrm{Hom}_{G_{\mathbb{Q}}}(V, \widehat{H}^1_{*,E})$.

Note that $\mathcal{M}_*(V)$ is a $\mathrm{GL}_2(\mathbb{A}_f)$-invariant closed subspace of the admissible continuous $\mathrm{GL}_2(\mathbb{A}_f)$-representation $V^\vee \otimes_E \widehat{H}^1_{*,E}$, and so is again an admissible continuous $\mathrm{GL}_2(\mathbb{A}_f)$-representation [31, Prop. 7.2.2]. The following lemma states some basic properties of $\mathcal{M}_*(V)$.

7.7.2. **Lemma.** (1) *The natural evaluation map $V \otimes_E \mathcal{M}_*(V) \to \widehat{H}^1_{*,E}$ is a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding.*
(2) *If ψ is any continuous character $\psi : G_{\mathbb{Q}} \to E^\times$, then there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant topological isomorphism $\mathcal{M}(V \otimes \psi)_* \xrightarrow{\sim} \mathcal{M}_*(V) \otimes (\psi \circ \det)$ (where on the right hand side we regard ψ as a character of $\mathbb{A}_f^\times$ via global class field theory).*
(3) *The formation of $\mathcal{M}_*(V)$ is compatible with extension of scalars to finite extensions of E.*

Proof. The evaluation map is tautologically $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-equivariant, and is an injection since V is irreducible (and thus absolutely irreducible, being odd). The proof of (2) is similar to the proof of Lemma 7.3.19. As in that proof, let c_ψ denote the cohomology class in $\widehat{H}^0_E$ that corresponds to ψ under the isomorphism (60). Then $G_{\mathbb{Q}}$ acts on c_ψ through the character ψ, while $\mathrm{GL}_2(\mathbb{A}_f)$ acts on c_ψ through $\psi \circ \det$. Cupping with c_ψ thus induces the claimed isomorphism of (2) (with the inverse isomorphism being provided by cupping with $c_{\psi^{-1}}$). Claim (3) is immediate. □

7.7.3. **Lemma.** *If* $\mathcal{M}_*(V)(K^p) \neq 0$ *(for* $* = \emptyset$ *or* c*) and* $\overline{V}$ *is absolutely irreducible*[10] *then* $\overline{V}$ *is modular.*

Proof. If V appears in $\widehat{H}^1_*(K^p)_E$ for some tame level K^p, then Lemma 7.2.3 shows that $\overline{V}$ appears in $H^1_*(Y(K_pK^p), \mathbb{F})$ for some sufficiently small compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$. The space $M := \mathrm{Hom}_{G_\mathbb{Q}}\big(\overline{V}, H^1_*(Y(K_pK^p), \mathbb{F})\big)$ is naturally a $\mathbb{T} := \mathbb{T}(\Sigma(K^p))$-module. Furthermore, the $\mathbb{T}$-action on this space factors through its quotient $\widetilde{\mathbb{T}}(K_pK^p)_1$ (as defined in Subsection 7.3). Since M is finite dimensional over $\mathbb{F}$, we must have that $(\mathbb{F}' \otimes_\mathbb{F} M)^\lambda \neq 0$ for some finite extension $\mathbb{F}'$ of $\mathbb{F}$ and some system of Hecke eigenvalues λ defined over $\mathbb{F}'$. The system λ is necessarily modular, since it is in fact a system of eigenvalues of $\widetilde{\mathbb{T}}(K_pK^p)_1$. Since $\overline{V}$ is absolutely irreducible, the natural evaluation map $\overline{V} \otimes_\mathbb{F} (\mathbb{F}' \otimes_\mathbb{F} M)^\lambda \to H^1_*(Y(K_pK^p), \mathbb{F}')$ is injective, and the Eichler-Shimura relations then imply that λ coincides with the system of eigenvalues associated to $\overline{V}$. Thus $\overline{V}$ is modular, as claimed. □

7.7.4. **Corollary.** *If* $\mathcal{M}_*(V) \neq 0$*, if* $\overline{V}$ *is absolutely irreducible, and if* $p \neq 2$*, then* V *is odd.*

Proof. The preceding proposition shows that $\overline{V}$ is modular, and thus odd. Since $p \neq 2$, we may conclude that V is odd. □

The following result gives some control over the action of $\mathrm{GL}_2(\mathbb{A}_f^p)$ on $\mathcal{M}_*(V)$.

7.7.5. **Lemma.** *If* K^p *is a tame level for which* $\mathcal{M}_*(V)^{K^p} \neq 0$*, then* V *is unramified outside of* $\Sigma(K^p)$*.*

Proof. To say that $\mathcal{M}_*(V)^{K^p} \neq 0$ is to say that there is a $G_\mathbb{Q}$-equivariant embedding $V \to \widehat{H}^1_*(K^p)_E$. The lemma thus follows from the fact that the $G_\mathbb{Q}$-action on $\widehat{H}^1_*(K^p)_E$ is unramified away from $\Sigma(K^p)$. □

7.7.6. **Corollary.** *If* $\mathcal{M}_*(V) \neq 0$ *then* V *is unramified at all but finitely many primes.*

Proof. If $\mathcal{M}_*(V) \neq 0$, then $\mathcal{M}_*(V)^{K^p} \neq 0$ for some tame level K^p. The claim thus follows from the preceding lemma. □

If K^p is a tame level, and $\Sigma := \Sigma(K^p)$, then the Hecke algebra $\mathbb{T}(\Sigma)$ acts naturally on $\mathcal{M}_*(V)^{K^p}$.

[10] We will see in Proposition 7.7.13 below that under this assumption the spaces $\mathcal{M}_*(V)$ for either choice of $*$ coincide; in particular, if one is non-zero, so is the other.

7.7.7. **Proposition.** *Suppose that we are in the situation of Lemma 7.7.5, and that furthermore V is not a twist of a representation with finite image. Let $\lambda : \mathbb{T}(\Sigma(K^p)) \to E$ denote the system of Hecke eigenvalues attached to V (which is defined, since V is unramified outside of $\Sigma(K^p)$). Then $\mathbb{T}(\Sigma(K^p))$ acts on $\mathcal{M}_*(V)^{K^p}$ through the character λ.*

Proof. The following lemma shows that our hypothesis on V is equivalent to requiring that the projective representation associated to V has infinite image. The proposition thus follows from the (proof of) [14, Prop. 3.2.3]. $\square$

7.7.8. **Lemma.** *The following conditions on a continuous two dimensional Galois representation V defined over $\overline{\mathbb{Q}}_p$ are equivalent:*

(1) *The projective representation attached to V has finite image;*
(2) *V is a twist of a representation with finite image.*

Proof. The claimed equivalence follows from the discussion of [53, §6]. Indeed, if the projective image of V is finite, then [53, Cor., p. 227] shows that we may find a continuous two dimensional representation W of $G_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_p$ which has finite image, and whose associated projective representation coincides with that of V. The discussion of [53, p. 226] then shows that V is a twist of W. $\square$

Proposition 7.7.7 has the following corollary.

7.7.9. **Corollary.** *If $\mathcal{M}_*(V) \neq 0$ for $* = \emptyset$ or c and if V is not a twist of a representation with finite image,*[11] *then V is promodular (and in particular, is odd).*

Proof. If $\mathcal{M}_*(V) \neq 0$ then $\mathcal{M}_*(V)^{K^p} \neq 0$ for some tame level K^p. If λ denotes the system of Hecke eigenvalues attached to V, then the preceding proposition shows that $(\widehat{H}^1_*(K^p)_E)^{\lambda} \neq 0$. It follows from Lemma 7.3.13 that λ is promodular. $\square$

7.7.10. **Remark.** Presumably any irreducible two dimensional Galois representation V over E for which $\mathcal{M}_*(V) \neq 0$ (for $* = \emptyset$ or c) is promodular, and so also odd. In fact these two conditions on V are conjecturally equivalent (assuming always that $\mathcal{M}_*(V) \neq 0$). Indeed, suppose that V is odd and that $\mathcal{M}_*(V) \neq 0$. We know that V is promodular unless it is a twist of a representation with finite image. By Lemmas 7.7.2 (2) and 7.3.19, we may assume that V does in fact have finite image. The Artin conjecture, together with standard converse theorems, then implies that $V \cong V_f$ for some modular form f of weight 1. (This is the so-called Strong Artin conjecture, which is actually a conjecture of Langlands.) Thus Remark 7.3.14 shows that V is promodular.

[11] We will see in Proposition 7.7.13 below that under this assumption the spaces $\mathcal{M}_*(V)$ for either choice of $*$ coincide; in particular, if one is non-zero, so is the other.

The surjection (57) induces a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant continuous map

$$\mathcal{M}_c(V) \to \mathcal{M}(V). \tag{69}$$

7.7.11. **Conjecture.** *The map (69) is a* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant topological isomorphism.*

This conjecture was originally formulated by Breuil in the case when $V = V_f$ for a modular form f of weight ≥ 2. It has been proved in that case by Breuil and the author (see Proposition 7.7.13 (2) below).

7.7.12. **Remark.** Since the Galois action on the kernel of (57) is abelian, it follows directly that (69) is injective; the difficulty is to show that it is surjective. Note that (69) is an isomorphism if and only if the induced map $\mathcal{M}_c(V)^{K^p} \to \mathcal{M}(V)^{K^p}$ is an isomorphism for each tame level K^p.

The following proposition establishes Conjecture 7.7.11 in many cases. Part (1) is due to Breuil; part (2) to Breuil and the author [14, Prop. 3.2.4].

7.7.13. **Proposition.** *Conjecture 7.7.11 is true provided V satisfies either one of the following two conditions:*

(1) $\overline{V}$ *is absolutely irreducible.*
(2) V *is not a twist of a representation with finite image.*

Proof. In case (1), it follows from Lemma 7.7.3 that $\overline{V}$ is modular. Let $\overline{\lambda} : \mathbb{T}(\Sigma) \to \mathbb{F}$ denote the corresponding system of eigenvalues, let $\mathfrak{m}$ denote the kernel of $\overline{\lambda}$, and let subscript $\mathfrak{m}$ denote localization at $\mathfrak{m}$. It is clear that the injection

$$\mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, \widehat{H}^1_*(K^p)_{E,\mathfrak{m}}\big) \to \mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, \widehat{H}^1_*(K^p)_E\big) =: \mathcal{M}_*(V)$$

is in fact an isomorphism (cf. the proof of Lemma 7.7.3). Also, since $\overline{V}$ is absolutely irreducible, and since the kernel $\widehat{M}$ of (59) is Eisenstein [14, Cor. 3.1.3], we see that the natural map $\widehat{H}^1_c(K^p)_{E,\mathfrak{m}} \to \widehat{H}^1(K^p)_{E,\mathfrak{m}}$ is an isomorphism. This proves (1).

Case (2) is proved in [14]. We present a variant on that proof here. Let $\lambda : \widetilde{\mathbb{T}}(K^p) \to \mathcal{O}_E$ denote the (promodular) system of Hecke eigenvalues attached to V. It follows from Proposition 7.7.7 that

$$\mathcal{M}_*(V) = \mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, (\widehat{H}^1_*(K^p)_E)^{\lambda}\big).$$

We claim that the natural map

$$\big(\widehat{H}^1_c(K^p)_E\big)^{\lambda} \to \big(\widehat{H}^1(K^p)_E\big)^{\lambda} \tag{70}$$

is an isomorphism.

If $\widehat{M}_{\mathcal{O}_E}$ denotes the kernel of (58), then (70) sits in the short exact sequence of $\widetilde{\mathbb{T}}(K^p)$-modules

$$0 \to \widehat{M}^{\lambda} \to \big(\widehat{H}^1_c(K^p)_E\big)^{\lambda} \to \big(\widehat{H}^1(K^p)_E\big)^{\lambda} \\ \to \varprojlim_s \operatorname{Ext}^1\big(\widetilde{\mathbb{T}}(K^p)/I, \widehat{M}_{\mathcal{O}_E}/\pi^s\big)\otimes_{\mathcal{O}_E} E,$$

where I denotes the kernel of λ, and the Ext^1s are computed in the category of $\widetilde{\mathbb{T}}(K^p)$-modules. On the one hand, the final term in this sequence is annihilated by I. On the other hand, since $\widehat{M}_{\mathcal{O}_E}$ satisfies the conditions of Lemma 7.3.8, by [14, Lem. 3.1.2], the same is also true of this term, and so that lemma implies that this term is Eisenstein. Since λ is not Eisenstein, this term must vanish. The term $\widehat{M}^{\lambda}$ must also vanish, since $\widehat{M}$ is Eisenstein. This shows that (70) is an isomorphism, as required. □

The following result describes the structure of the space $\mathcal{M}_*(V)_{\text{lalg}}$ of locally $\mathrm{GL}_2(\mathbb{Q}_p)$-algebraic vectors in $\mathcal{M}_*(V)$.

7.7.14. **Proposition.** *The space $\mathcal{M}_*(V)_{\text{lalg}}$ is non-zero (for either choice of $* \in \{\emptyset, c\}$) if and only if V is the twist by a power of ε of the representation associated to a classical newform f over E of weight $k \geq 2$.*

If these equivalent conditions hold (so that in particular $V_{|D_p}$ is potentially semi-stable with distinct Hodge-Tate weights, and the locally algebraic representation $\widetilde{\pi}_p(V_{|D_p})$ is defined – see Conjecture 3.3.1 (7)) then the following additional results also hold:

(1) *There is a canonical $\mathrm{GL}_2(\mathbb{A}^p_f)$-equivariant isomorphism*

$$\pi^p(V) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{\pi}_p(V_{|D_p}), \mathcal{M}(V)\big), \tag{71}$$

which induces a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant isomorphism

$$\widetilde{\pi}_p(V_{|D_p}) \times \pi^p(V) \xrightarrow{\sim} \mathcal{M}(V)_{\text{lalg}}. \tag{72}$$

(2) *For any $0 \neq \xi \in \pi^p(V)$, regarded as an element of*

$$\operatorname{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{\pi}_p(V_{|D_p}), \mathcal{M}(V)\big)$$

via the isomorphism of (1), the closure of $\xi\big(\widetilde{\pi}_p(V_{|D_p})\big)$ in $\mathcal{M}(V)$ is an admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation which is independent of ξ, up to a canonical isomorphism. Denote this completion of $\widetilde{\pi}_p(V_{|D_p})$ by $\widehat{\pi}_p(V)$.

(3) *There is a canonical closed $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding*

$$\widehat{\pi}_p(V) \otimes_E \pi^p(V) \to \mathcal{M}(V), \tag{73}$$

whose image coincides with the closure in $\mathcal{M}(V)$ of $\mathcal{M}(V)_{\text{lalg}}$.

(4) *The isomorphism (71) induces a canonical* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant isomorphism* $\pi^p(V) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}_p(V), \mathcal{M}(V)\big)$ *(where* Hom *denotes continuous homomorphisms).*

Proof. There is a natural isomorphism $\mathcal{M}_*(V)_{\mathrm{lalg}} \xrightarrow{\sim} \mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, \widehat{H}^1_{*,E,\mathrm{lalg}}\big)$. The first claim of the theorem thus follows from Theorems 7.4.2 and 2.5.1, once we recall that the kernel of the natural map $H^1_c(\mathcal{V}_k)_E \to H^1_{\mathrm{par}}(\mathcal{V}_k)_E$ (resp. the cokernel of the natural map $H^1_{\mathrm{par}}(\mathcal{V}_k)_E \to H^1(\mathcal{V}_k)_E$) contains no absolutely irreducible two dimensional Galois representations, for any $k \geq 2$.

Suppose for the remainder of the proof that in fact $V \cong V_f \otimes \varepsilon^n$ for some newform f of weight $k \geq 2$ and some integer n. The isomorphism (72) then follows directly from the same theorems as were cited above. Since $\widetilde{\pi}_p(V_{|D_p})$ is a locally algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, the inclusion $\mathcal{M}(V)_{\mathrm{lalg}} \subset \mathcal{M}(V)$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{\pi}_p(V_{|D_p}), \mathcal{M}(V)_{\mathrm{lalg}}\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{\pi}_p(V_{|D_p}), \mathcal{M}(V)\big).$$

Combined with (72), this yields the isomorphism (71).

The isomorphism (71) in turn induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding

$$\widetilde{\pi}_p(V_{|D_p}) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f^p)}(\pi^p(V), \mathcal{M}(V)).$$

Since $\pi^p(V)$ is irreducible, and so in particular finitely generated, we see that the target of this embedding is an admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. Thus if $\widehat{\pi}_p(V)$ denotes the closure of the image of this embedding, we see that $\widehat{\pi}_p(V)$ is an admissible unitary completion of $\widetilde{\pi}_p(V_{|D_p})$. The natural evaluation map

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f^p)}(\pi^p(V), \mathcal{M}(V)) \otimes_E \pi^p(V) \to \mathcal{M}(V)$$

is tautologically $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant, and is injective, since $\pi^p(V)$ is absolutely irreducible. Thus it restricts to a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant injection (73), which must in fact be a closed embedding, by [31, Prop. 7.2.2 (iii)]. Its image is the closure of $\mathcal{M}(V)_{\mathrm{lalg}}$, as follows from the explicit description of $\mathcal{M}(V)_{\mathrm{lalg}}$ provided by (72). This proves part (3).

If $0 \neq \xi \in \pi^p(V)$, then the closure of $\xi\big(\widetilde{\pi}_p(V_{|D_p})\big)$ coincides with the image of $\widehat{\pi}_p(V) \otimes \xi$ under (73). Since that map is an injection, we see that this closure is canonically identified with $\widehat{\pi}_p(V)$. This gives (2).

Finally, we have the series of maps

$$\pi^p(V) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}_p(V), \mathcal{M}(V)\big) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{\pi}_p(V_{|D_p}), \mathcal{M}(V)\big) \xrightarrow{\sim} \pi^p(V),$$

where the first map is the injection induced by the embedding (73), the second map is induced by restriction, which is also an injection (since the inclusion

$\widetilde{\pi}_p(V_{|D_p}) \to \widehat{\pi}_p(V)$ has dense image), and the third map is the isomorphism (71). By construction, the composite of these maps is the identity, and so each must be an isomorphism. This gives (4). □

7.7.15. **Remark.** If f is any classical cuspidal newform f over E of weight $k \geq 2$, then $\widehat{\pi}_p(V_f)$ is conjectured to be topologically irreducible. Indeed, when $V_{f\,|D_p}$ is irreducible, it is conjectured to coincide with $B(V_{f\,|D_p})$, the admissible unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space representation attached to $V_{f\,|D_p}$ via Conjecture 3.3.1 (see Conjecture 7.8.1 below). This is known when $V_{f\,|D_p}$ is trianguline and Frobenius semi-simple (using the definition of $B(V_{f\,|D_p})$ given in Subsection 6.1); see Theorem 7.10.1 below. On the other hand, if $V_{|D_p}$ is reducible, then $\widehat{\pi}_p(V)$ coincides with the universal unitary completion of $\widetilde{\pi}_p(V_{D_p})$ (since this universal unitary completion is admissible unitary and topologically irreducible, by Lemma 5.3.3 and Proposition 5.3.4). Thus in this case we find that $\widehat{\pi}_p(V)$ is a topologically irreducible closed subrepresentation of the reducible representation $B(V_{f\,|D_p})$ (at least in those cases when a definition of $B(V_{f\,|D_p})$ has been given in Section 6.)

7.7.16. **Remark.** Let π be a locally algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ of the form $\pi := U \otimes (\mathrm{Sym}^{k-2} E^2)^\vee$, where U is an absolutely irreducible admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. The preceding proposition shows that if π can be realized as $\widetilde{\pi}_p(V_{f\,|D_p})$ for some cuspidal newform f of weight k, then π has a non-zero universal unitary completion, and in fact admits a non-zero admissible unitary completion, namely $\widehat{\pi}_p(V_f)$. If we suppose that U is cuspidal and that π has unitary central character, then we can always realize π in the form $\widetilde{\pi}_p(V_{f\,|D_p})$ (at least after making a twist and replacing E with a finite extension); thus we have verified the remarks made following the proof of Proposition 5.1.18.

Our final result of this subsection provides a characterization of the representations V attached to points of the eigensurface $\widetilde{D}(K^p)$ in terms of the structure of $\mathcal{M}(V)^{K^p}$.

7.7.17. **Proposition.** *For any tame level K^p, the set of exponents*

$$\mathrm{Exp}\big(\mathcal{M}(V)_{\mathrm{an}}^{K^p}\big)$$

is non-empty if and only if V is a twist of a representation associated to a finite slope overconvergent eigenform of tame level K^p defined over E.

Proof. If $\mathcal{M}(V)^{K^p}$ vanishes, there is nothing to prove. Thus we may as well assume that this space is non-zero, and thus (by Lemma 7.7.5) that V is unramified outside $\Sigma(K^p)$. Write $\mathbb{T} := \mathbb{T}(\Sigma(K^p))$, and λ denote the system of Hecke eigenvalues associated to V.

Suppose first that $J_{\mathrm{P}(\mathbb{Q}_p)}^{\chi_1 \otimes \chi_2}\big(\mathcal{M}(V)_{\mathrm{an}}^{K^p}\big) \neq 0$ for some $\chi_1 \otimes \chi_2 \in \widehat{\mathrm{T}}(E)$. This is then a non-trivial finite dimensional E-vector space equipped with an action of

$\mathbb{T}$. Lemma 7.7.2 (1) yields a $G_{\mathbb{Q}}$-equivariant $\mathbb{T}$-linear injection

$$V \otimes J^{\chi_1\otimes\chi_2}_{\mathrm{P}(\mathbb{Q}_p)}\big(\mathcal{M}(V)^{K^p}_{\mathrm{an}}\big) \to J^{\chi_1\otimes\chi_2}_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)_E\big). \tag{74}$$

Since the source of (74) is non-trivial and finite dimensional, the Eichler-Shimura relations show that it contains a $\mathbb{T}$-eigenvector with system of eigenvalues λ. Consequently, we find that $J^{\chi_1\otimes\chi_2}_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)^{\lambda}_E\big) \neq 0$, and thus that the point $(\chi_1 \otimes \chi_2, \lambda)$ lies in $\big(\operatorname{Spec}\mathcal{A}(K^p)\big)(E)$, and hence in $\widetilde{D}(K^p)(E)$ (by Theorem 7.5.8, since λ is not Eisenstein). Thus V is a twist of a $G_{\mathbb{Q}}$-representation attached to a finite slope overconvergent eigenform of tame level K^p defined over E. Conversely, if V is such a twist, then Proposition 7.5.3 (or better, its proof) shows that $\operatorname{Exp}(\mathcal{M}(V)^{K^p}_{\mathrm{an}}) \neq \emptyset$. □

7.8. **The local-global compatibility conjecture.** If V is an irreducible continuous representation of $G_{\mathbb{Q}}$ over E, unramified outside of a finite number of primes, then as in the preceding subsection we will let $\overline{V}$ denote the semi-simplification of the reduction modulo ϖ of (some $G_{\mathbb{Q}}$-invariant lattice in) V.

We now restate Conjecture 1.1.1.

7.8.1. **Conjecture.** *If V is an odd irreducible continuous representation of $G_{\mathbb{Q}}$ over E, unramified outside of a finite number of primes, then there is a* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant topological isomorphism*

$$\mathcal{M}(V) \cong \Pi(V)$$

(where $\Pi(V)$ is the admissible continuous $\mathrm{GL}_2(\mathbb{A}_f)$*-representation associated to V via Definition 7.1.1).*

7.8.2. **Remark.** In light of Remark 7.7.10 and Conjecture 7.7.11, we have restricted our attention to odd Galois representations and to the multiplicity space $\mathcal{M}(V)$.

We also state the following conjecture, which is well-posed independent of any assumption about the existence of a local p-adic correspondence defined on all $G_{\mathbb{Q}_p}$-representations.

7.8.3. **Conjecture.** *If V is an odd irreducible continuous representation of $G_{\mathbb{Q}}$ over E, then there is an admissible unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation $\Pi_p(V)$ such that $\mathcal{M}(V) \xrightarrow{\sim} \Pi_p(V) \otimes_E \pi^{\mathrm{m},p}(V)$. Furthermore, the association of $\Pi_p(V)$ to V satisfies the following properties:*

(1) *If V and V' are two odd irreducible continuous two dimensional representations of $G_{\mathbb{Q}}$ over E, then $V_{|D_p} \cong V'_{|D_p}$ as $G_{\mathbb{Q}_p}$-representations if and only if there is a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant topological isomorphism $\Pi_p(V) \cong \Pi_p(V')$.*

(2) *The representation* $\Pi_p(V)$ *has central character corresponding to the local Galois character* $(\det V)_{|G_{\mathbb{Q}_p}}\varepsilon$ *via local class field theory.*

(3) *The representation* $\Pi_p(V)$ *satisfies the hypothesis of Definition 3.2.2, and* $\overline{\Pi}_p(V)$ *is associated to the semi-simplification of* $\overline{V}_{|D_p}$ *with respect to the local mod* ϖ *Langlands correspondence of* [10].

(4) *If* $V_{|D_p}$ *is irreducible then* $\Pi_p(V)$ *is topologically irreducible.*

(5) *The representation* $V_{|D_p}$ *is potentially semi-stable, with distinct Hodge-Tate weights, if and only if the subspace* $\Pi_p(V)_{\mathrm{lalg}}$ *of locally algebraic vectors of* $\Pi_p(V)$ *is non-zero. Furthermore, if these conditions hold, then the subspace* $\Pi_p(V)_{\mathrm{lalg}}$ *coincides with the locally algebraic representation* $\widetilde{\pi}_p(V_{|D_p})$ *attached to* $V_{|D_p}$ *as defined in condition (7) of Conjecture 3.3.1.*

(6) *For any character* $\eta\otimes\psi\in\widehat{\mathrm{T}}(E)$ *there is an equality of dimensions*

$$\dim \mathrm{Ref}^{\eta\otimes\psi}(V)=\dim \mathrm{Exp}^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}(\Pi_p(V)_{\mathrm{an}}).$$

(7) *If* $V_{|D_p}$ *is trianguline, then there is a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant topological isomorphism* $\Pi_p(V)\cong B(V_{|D_p})$*, where* $B(V_{|D_p})$ *is the admissible unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation attached to* $V_{|D_p}$ *by the discussion of Section 6.*[12]

7.8.4. **Remark.** Lemma 7.7.2 shows that the formation of $\Pi_p(V)$ would necessarily be compatible with twisting, and with change of scalars.

7.8.5. **Remark.** As follows from the discussion of the preceding sections, condition (7) is by no means independent of the preceding conditions. For example, if $V_{|D_p}$ is irreducible and trianguline, and is either not potentially semi-stable up to a twist, or else is potentially crystalline and Frobenius semi-simple up to a twist, then conditions (4) and (6) imply that $\Pi_p(V)\cong B(V_{|D_p})$. Similarly, if $V_{|D_p}$ is the direct sum of two characters, then conditions (3), (5), and (6) imply that $\Pi_p(V)\cong B(V_{|D_p})$.

7.8.6. **Remark.** If $V=V_f$ for some classical newform f of weight $k\geq 2$, then Proposition 7.7.14 shows that $\Pi_p(V)$, if it exists, must contain $\widehat{\pi}_p(V)$. If furthermore $V_{|D_p}$ is irreducible (equivalently, if f is not a twist of a form that is ordinary at p), then condition (4) of the conjecture implies that $\Pi_p(V)$ should equal $\widehat{\pi}_p(V)$, or equivalently, that $\mathcal{M}(V)_{\mathrm{lalg}}$ should be dense in $\mathcal{M}(V)$.

The following result provides some evidence for Conjecture 7.8.3 at the primes away from p.

7.8.7. **Proposition.** *If* V *is not a twist of a representation of finite image, then for any tame level* K^p*, the inclusion* $\mathcal{M}(V)^{K^p}\subset\mathcal{M}(V)$*, together with the action*

[12] For those reducible indecomposable $V_{|D_p}$ for which $B(V_{|D_p})$ has not yet been defined, we understand this statement to mean just that $\Pi_p(V)$ is an extension of the form conjectured in Subsections 6.4 and 6.5 above.

of $\mathrm{GL}_2(\mathbb{A}_f^{\Sigma(K^p)})$ *on* $\mathcal{M}(V)$*, induces a natural* $\mathrm{GL}_2(\mathbb{A}_f^{\Sigma(K^p)})$*-equivariant map*

$$\Big(\bigotimes_{\ell\notin\Sigma(K^p)}{}'\ \pi_\ell^{\mathrm{m}}(V)\Big)\otimes_E \mathcal{M}(V)^{K^p}\to\mathcal{M}(V) \tag{75}$$

(where $\mathrm{GL}_2(\mathbb{A}_f^{\Sigma(K^p)})$ *acts through its action on the first factor). If* $\overline{V}$ *is furthermore absolutely irreducible, then (75) is an embedding.*

Proof. Write $\Sigma := \Sigma(K^p)$. If $\mathcal{M}(V)^{K^p}$ vanishes then there is nothing to prove, so we may assume that $\mathcal{M}(V)^{K^p}\neq 0$. Lemma 7.7.5 then shows that V is unramified outside Σ. Thus, letting λ denote the system of Hecke eigenvalues attached to V, we have that

$$\bigotimes_{\ell\notin\Sigma}{}'\,\pi_\ell^{\mathrm{m}}(V)\cong\big(c-\mathrm{Ind}_{\mathrm{GL}_2(\widehat{\mathbb{Z}}^\Sigma)}^{\mathrm{GL}_2(\mathbb{A}_f^\Sigma)}\underline{1}\big)_{\mathrm{sm},\lambda}. \tag{76}$$

(Here $c-\mathrm{Ind}$ denotes the compactly supported induction, and the subscript λ indicates that we take the maximal quotient on which $\mathbb{T}(\Sigma)$ acts through λ.) The induced representation appearing in (76) is universal for maps to unramified representations of $\mathrm{GL}_2(\mathbb{A}_f^\Sigma)$ on which $\mathbb{T}(\Sigma)$ acts via λ; more precisely, for any smooth $\mathrm{GL}_2(\mathbb{A}_f^\Sigma)$-representation W over E, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f^\Sigma)}\big((c-\mathrm{Ind}_{\mathrm{GL}_2(\widehat{\mathbb{Z}}^\Sigma)}^{\mathrm{GL}_2(\mathbb{A}_f^\Sigma)}\underline{1})_{\mathrm{sm},\lambda},W\big)\xrightarrow{\sim}\big(W^{\mathrm{GL}_2(\widehat{\mathbb{Z}}^\Sigma)}\big)^\lambda.$$

We write K^p as the product of its ramified and unramified parts $K^p_\Sigma\times\mathrm{GL}_2(\widehat{\mathbb{Z}}^\Sigma)$, and then apply this isomorphism with W taken to be $\mathcal{M}(V)^{K^p_\Sigma}$. Since Proposition 7.7.7 shows that the Hecke algebra $\mathbb{T}(\Sigma)$ acts on the non-zero space $\mathcal{M}(V)^{K^p}$ through λ, we obtain the map (75).

Suppose now that $\overline{V}$ is absolutely irreducible. For any fixed value of ℓ, either $\pi_\ell^{\mathrm{m}}(V)=\pi_\ell(V)$ is irreducible, or else $\pi_\ell^{\mathrm{m}}(V)$ is the non-split extension of a character of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ by an irreducible (special) representation. Thus if (75) is not an injection, we find that $\mathcal{M}(V)^{K^p}$ contains a one dimensional $\mathrm{GL}_2(\mathbb{Q}_\ell)$-invariant subspace for some $\ell\notin\Sigma$. If we denote this one dimensional subspace by L, then Lemma 7.7.2 (1) gives a $G_\mathbb{Q}\times\mathrm{GL}_2(\mathbb{Q}_\ell)$-equivariant embedding $V\otimes_E L\to\widehat{H}^1(K^p)_E$. If we let $\overline{L}$ denote the reduction mod ϖ of L, then reducing this embedding mod ϖ (and taking into account Lemma 7.2.3) yields a $G_\mathbb{Q}\times\mathrm{GL}_2(\mathbb{Q}_\ell)$-invariant embedding

$$\overline{V}\otimes_\mathbb{F}\overline{L}\to\varinjlim_{K_p}H^1(Y(K_pK^p),\mathbb{F}). \tag{77}$$

Thus for some sufficiently small compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ we see that $H^1(Y(K_pK^p),\mathbb{F})$ contains a copy of $\overline{V}$ which is invariant under $\mathrm{GL}_2(\mathbb{Q}_\ell)$. This contradicts Ihara's lemma [39] (see also the proof of Theorem 4.1 in [46]).

(We require the hypothesis that $\overline{V}$ is absolutely irreducible because Ihara's lemma is a statement about the cohomology of closed modular curves.) □

7.9. **Consequences of the local-global compatibility conjecture.** The following result makes explicit some of the consequences of the preceding conjecture. (The ? are to indicate that each assertion is dependent on Conjecture 7.8.3.)

7.9.1. **Proposition.** *Conjecture 7.8.3 has the following consequences. (As usual we let V denote an odd irreducible continuous two dimensional representation of $G_{\mathbb{Q}}$ over E.)*

- $(1)_?$ *If N is the tame conductor of V, then V appears as a $G_{\mathbb{Q}}$-subrepresentation of $\widehat{H}^1(K_1^p(N))_E$. (Here $K_p^1(N) := \{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}^p) \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}$.)*
- $(2)_?$ *V is pro-modular.*
- $(3)_?$ *The representation $V_{|D_p}$ of $G_{\mathbb{Q}_p}$ is potentially semi-stable with distinct Hodge-Tate weights if and only if V is a twist by a power of ε of the Galois representation attached to a classical cuspidal Hecke eigenform of weight $k \geq 2$.*
- $(4)_?$ *The representation $V_{|D_p}$ of $G_{\mathbb{Q}_p}$ is trianguline if and only if V is a twist of a Galois representation attached to a p-adic overconvergent cuspidal Hecke eigenform of finite slope.*

Proof. If V has tame conductor N, then $\pi^p(V)^{K_1^p(N)} \neq 0$. Thus the conjecture implies that $\mathcal{M}(V)^{K_1^p(N)} \neq 0$, giving $(1)_?$. If Σ denotes the set of primes dividing Np, and if λ denotes the system of eigenvalues attached to V, then $\mathbb{T}(\Sigma)$ acts on $\pi^{\mathrm{m},p}(V)^{K_1^p(N)}$ through λ. The conjecture thus implies that $\big(\mathcal{M}(V)^{K_1^p(N)}\big)^{\lambda} \neq 0$, and so $(2)_?$ follows from Lemma 7.3.13. Part $(3)_?$ follows immediately from condition (5) of the conjecture together with Proposition 7.7.14, while part $(4)_?$ follows from condition (6) of the conjecture together with Proposition 7.7.17. □

7.9.2. **Remark.** Note that part $(1)_?$ of the preceding proposition provides a strong converse to Lemma 7.7.5.

7.9.3. **Remark.** Part $(2)_?$ is in many cases a theorem of Böckle. Indeed, he has shown in a large number of cases for which $\overline{V}$ is absolutely irreducible that the modular points are Zariski dense in the deformation space of $\overline{V}$, and thus that V is necessarily promodular [11].

7.9.4. **Remark.** As we remarked in the introduction, part $(3)_?$ is a consequence of the Fontaine-Mazur conjecture [37, Conj. 3c], while part $(4)_?$ is related to a conjecture of Kisin [41, Conj. 11.8].

7.9.5. **Remark.** The converse to $(2)_?$ above seems quite plausible: namely, if one assumes that V is promodular, it seems reasonable to expect that $\mathcal{M}(V) \neq$

0. When combined with the results of Böckle mentioned in Remark 7.9.3, this provides a strong source of motivation for Conjecture 7.8.3.

7.9.6. **Remark.** If we combine $(1)_?$ with the proof of $(4)_?$, we obtain the following result: if V is an irreducible $G_{\mathbb{Q}}$-representation attached to a finite slope overconvergent eigenform of some tame level, and if N is the tame conductor of V, then V is in fact attached to a finite slope overconvergent eigenform of tame level N (giving a kind of theory of newforms for finite slope overconvergent eigenforms). Can one prove this unconditionally? Note that in many cases when $\overline{V}$ is absolutely irreducible, the work of Böckle mentioned in Remark 7.9.3 identifies the localization of $\mathbb{T}(K_1^p(N))$ at the maximal ideal corresponding to $\overline{V}$ with the universal deformation ring of $\overline{V}$. Thus one can conclude in these cases that V is attached to a p-adic modular form of tame level N, and the problem is to show that this p-adic modular form can be taken to be overconvergent. If $V_{|D_p}$ is reducible then one can do this, since the ordinary part of $\mathbb{T}(K_1^p(N))$ becomes identified with the ordinary universal deformation ring (under appropriate assumptions on $\overline{V}$ – see [17, §1] for example).

7.10. **Some partial results.** Recall from Theorem 7.6.1 that if V is a twist of the Galois representation attached to an overconvergent cuspidal Hecke eigenform of finite slope, then $V_{|D_p}$ is trianguline, so that we may define $B(V_{|D_p})$ (at least for most such $V_{|D_p}$) according to the discussion of Subsection 6.

The next two results are in the direction of part (7) of Conjecture 7.8.3. Note though that they only apply to Galois representations V that are assumed to be attached to finite slope overconvergent eigenforms (up to a twist), and so do not shed any light on part $(4)_?$ of Proposition 7.9.1.

The first theorem deals with the case when V actually arises (up to a twist) from a classical newform (which by [41, Thm. 6.6] is essentially equivalent to assuming that $V_{|D_p}$ is potentially semi-stable up to a twist). In the case of V satisfying condition (a) of the theorem, it strengthens a part of [13, Thm. 1.1.2] (this strengthening being made possible by virtue of the results of [8, 26]). For V satisfying condition (b), it is essentially a restatement of [14, Thm. 1.1.2 (i)]. (To facilitate the comparison with Conjecture 7.8.3, we recall from Remark 7.1.2 that $\pi^p(V) = \pi^{\mathrm{m},p}(V)$ in the context of the theorem.)

7.10.1. **Theorem.** *Let V be a twist of the Galois representation attached to a classical cuspidal Hecke eigenform of finite slope and weight $k \geq 2$. Assume furthermore that the following conditions hold:*

(a) If $V_{|D_p}$ is irreducible, then one (equivalently every) potentially semi-stable twist of $V_{|D_p}$ is Frobenius semi-simple.
(b) If $V_{|D_p}$ is reducible, then it is indecomposable, and is potentially crystalline up to a twist.

Then there is a $\mathrm{GL}_2(\mathbb{A}_f^p)$*-equivariant isomorphism*

$$\pi^p(V) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(B(V_{|D_p}), \mathcal{M}(V)\big).$$

Furthermore, all the non-zero morphisms in the target of this isomorphism are injective.

Proof. Replacing V by an appropriate twist, and appealing to Lemma 7.7.2 (2), we see that it is no loss of generality to assume that V is attached to a classical newform f of weight $k \geq 2$, and that either $V_{|D_p}$ is irreducible and Frobenius semi-simple, or else is reducible, indecomposable, and potentially crystalline. Let $\widehat{\pi}_p(V)$ denote the completion of $\widetilde{\pi}_p(V_{|D_p})$ attached to V by Proposition 7.7.14.

Suppose first that $V_{|D_p}$ is irreducible. We will show that $B(V_{|D_p}) \cong \widehat{\pi}_p(V)$. The claimed isomorphism will then follow from Proposition 7.7.14 (4).

If $V_{|D_p}$ is potentially crystalline, then $B(V_{|D_p})$ is equal to the universal unitary completion of $\widetilde{\pi}_p(V_{|D_p})$ (cf. Remark 6.1.4), and is topologically irreducible (Theorem 5.1.6), and so we are done. If $V_{|D_p}$ is not potentially crystalline, then the main result of [13] gives a non-zero homomorphism $B(1-w,\mathcal{L})\otimes\eta \to \widehat{\pi}_p(V)$, where $1-w$ is the weight of f, η is a character, and $\mathcal{L}$ is an (*a priori* unknown) element of E. By construction the image of this map contains $\widetilde{\pi}_p(V_{|D_p})$, and so is dense in $\widehat{\pi}_p(V)$. Since $B(1-w,\mathcal{L})$ is admissible unitary and topologically irreducible (Theorem 5.1.13) this map must be an isomorphism. In particular, $\widehat{\pi}_p(V)$ is also topologically irreducible. As Colmez explained to me, it then follows from [26] that $\mathcal{L}$ must in fact equal the $\mathcal{L}$-invariant of $V_{|D_p}$, and thus that $B(V_{|D_p}) \cong \widehat{\pi}_p(V)$.

Colmez's argument is as follows: Let $\mu_f^{\pm}$ denote the Mazur-Tate-Teitelbaum distributions defining the p-adic $\mathcal{L}$-function attached to f. Essentially by definition these distributions may be regarded as elements of the topological dual to the universal unitary completion B of $\widetilde{\pi}_p(V)$. As is explained in [13, §5] (see also [34, §5]) it follows from the construction of $\mu_f^{\pm}$ via modular symbols that they in fact lie in the topological dual of the quotient $\widehat{\pi}_p(V)$ of B. On the other hand, Kato's approach to defining p-adic L-functions, via the method of "Coleman power series" and explicit reciprocity laws, when reinterpreted in the language of (ϕ,Γ)-modules (as in [25]) and combined with the main result of [26], shows that these same distributions $\mu_f^{\pm}$ also lie in the topological dual of the quotient $B(V_{|D_p})$ of B. Since both $B(V_{|D_p})$ and $\widehat{\pi}_p(V)$ are topologically irreducible quotients of B, we see that they must be quotients by the same closed subrepresentation, and thus that $\widehat{\pi}_p(V) \cong B(V_{|D_p})$.

Consider now the case when $V_{|D_p}$ is reducible, but indecomposable. Recall from Remark 7.7.15 that $\widehat{\pi}_p(V)$ is a closed subrepresentation of $B(V_{|D_p})$. It follows from [14, Thm. 1.1.2 (ii)] that any non-zero element of $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(B(V_{|D_p}), \mathcal{M}(V)\big)$

must be injective, and thus that restricting homomorphisms yields an injection

$$(78)\quad \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(B(V_{|D_p}),\mathcal{M}(V)\big)\to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}_p(V),\mathcal{M}(V)\big)\xrightarrow{\sim}\pi^p(V)$$

(the isomorphism being provided by Proposition 7.7.14 (4)). Since $\pi^p(V)$ is an irreducible $\mathrm{GL}_2(\mathbb{A}_f^p)$-representation, and since [14, Thm. 1.1.2 (i)] shows that the source is non-zero, we see that (78) is in fact an isomorphism. Thus in this case we again obtain the desired isomorphism. □

7.10.2. **Remark.** Suppose in the context of the preceding theorem that $V_{|D_p}$ is the direct sum of two characters, say $V_{|D_p}=\eta\bigoplus\psi$. Since V is associated to a classical cuspform, all of the local factors of the representation of $\mathrm{GL}_2(\mathbb{A}_f)$ associated to V are generic [40, p. 354], and so in particular $\eta\psi^{-1}\neq\varepsilon^{\pm1}$. Thus $B(V_{|D_p}) := \big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\eta\otimes\psi\varepsilon\big)_{\mathrm{cont}}\bigoplus\big(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\psi\otimes\eta\varepsilon\big)_{\mathrm{cont}}$. If we label the summands η and ψ in such a way that $\eta\psi^{-1}$ has Hodge-Tate weight $k-1$, then the first of these summands is the universal unitary completion of $\widetilde{\pi}_p(V_{|D_p})$. One thus has isomorphisms of smooth $\mathrm{GL}_2(\mathbb{A}_f^p)$-representations

$$(79)\quad \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\eta\otimes\psi\varepsilon)_{\mathrm{cont}},\mathcal{M}(V)\big)$$
$$\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{\pi}_p(V_{|D_p}),\mathcal{M}(V)\big)\xrightarrow{\sim}\pi_p$$

(the first isomorphism following from the universal property of the universal unitary completion, and the second from Theorem 7.4.2). Part (7) of Conjecture 7.8.3 implies that there should be a similar isomorphism of $\mathrm{GL}_2(\mathbb{A}_f^p)$-representations

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\psi\otimes\eta\varepsilon)_{\mathrm{cont}},\mathcal{M}(V)\big)\overset{?}{\xrightarrow{\sim}}\pi_p.$$

It follows from [14, Thm. 1.1.2 (i)] that the source of this conjectural isomorphism is non-zero (and in fact that it has a non-zero space of K^p-invariants, if K^p denotes the tame level of the newform f giving rise to V). However neither that result nor its proof seems to extend in any immediate way to determine the structure of $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\psi\otimes\eta\varepsilon)_{\mathrm{cont}},\mathcal{M}(V)\big)$ as a $\mathrm{GL}_2(\mathbb{A}_f^p)$-representation.

The next result deals with the case when V is attached to a non-classical finite slope overconvergent eigenform.

7.10.3. **Theorem.** *If V is a twist of a Galois representation attached to a non-classical overconvergent eigenform of finite slope and tame level K^p, and if $V_{|D_p}$ is irreducible, then there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding $B(V_{|D_p})\to\mathcal{M}(V)^{K^p}$.*

Proof. If R is a non-ultracritical refinement of $V_{|D_p}$, then Theorem 7.6.1 and Proposition 7.5.3 together yield a map $B(R)\to\mathcal{M}(V)^{K^p}$. Since $V_{|D_p}$ is irreducible, and since the eigenform to which it is associated (up to a twist) is not classical, we infer from [41, Thm. 6.6] that $V_{|D_p}$ is not potentially semi-stable up to a twist. Thus $B(V_{|D_p}) := B(R)$, and the theorem follows. □

7.10.4. **Remark.** The various restrictions on the structure of $V_{|D_p}$ in the statements of the preceding two theorems are not actually so serious. For example, if Conjecture 5.1.5 were to be proved in full generality, then we could eliminate condition (a) from the statement of Theorem 7.10.1. In any case, it is conjectured that $V_{|D_p}$ is always Frobenius semi-simple when V is the Galois representation attached to a classical newform.

Let us now consider the case when $V_{|D_p}$ is reducible, but does not satisfy condition (b) of Theorem 7.10.1. If $V_{|D_p}$ is indecomposable, and if the conjectured extensions of Subsections 6.4 and 6.5 were constructed, so that $B(V_{|D_p})$ could be defined, then an argument analogous to that of [14, §5.6] would provide an embedding $B(V_{|D_p}) \subset \mathcal{M}(V)$ (at least if $V_{|D_p}$ is an extension of distinct characters). (Note that in the case when V admits an $\mathcal{L}$-invariant, one would build on the main result of [13], which provides an embedding of the appropriate twist of $B(2, \mathcal{L})$ into $\mathcal{M}(V)$.)

Suppose on the other hand that $V_{|D_p}$ is split. If V is attached to a classical form of weight $k \geq 2$, then (as was implicitly recalled in Remark 7.10.2 above) it follows from [14, Thm. 1.1.2 (i)] that the analogue of Theorem 7.10.3 holds for V. If instead V is attached either to a form of weight 1 or to a non-classical form, then the method of proof of [14, Thm. 1.1.2 (i)] will extend to establish an analogue of Theorem 7.10.3 for V provided that one can prove the existence of p-adic companion forms in these contexts. (Essentially, one is given an ordinary eigenform corresponding to one equivalence class of ordinary refinements, and one has to construct an ordinary eigenform that corresponds to the other equivalence class of ordinary refinements.) The isomorphism between the ordinary deformation ring and the ordinary Hecke ring attached to $\overline{V}$ provided by [17, §1] (for example) will establish the existence of such a companion form, provided that $\overline{V}$ satisfies the necessary hypotheses for that result to apply (and so Theorem 7.10.3 does extend to cover these cases).

7.10.5. **Remark.** In contrast to the cases considered in Theorem 7.10.1, in Theorem 7.10.3 we do not have any control over the multiplicity with which $B(V_{|D_p})$ appears in $\mathcal{M}(V)$. (As was noted in Remark 7.9.6, it seems that one doesn't even know for which tame levels K^p one has $\mathcal{M}(V)^{K^p} \neq 0$.)

7.10.6. **Remark.** The preceding results, combined with Proposition 7.7.14, show that if V is the $G_{\mathbb{Q}}$-representation attached to a finite slope overconvergent eigenform for which $V_{|D_p}$ is irreducible, and is not a twist of a potentially semi-stable Frobenius non-semi-simple representation, then the representation $B(V_{|D_p})$ satisfies condition (7) of Conjecture 3.3.1. It follows that $B(V_{|D_p})$ also satisfies condition (8) of that conjecture (cf. Proposition 6.6.5, and note that Proposition 7.6.5 allows us to replace the inequality in part (3) of that result by equality for the representations under consideration).

The final result of our paper gives a kind of converse to the preceding theorems. It includes as a special case a part of [13, Thm. 1.1.2].

7.10.7. **Theorem.** *Assume that V is not a twist of a representation having finite image, and let W be a trianguline continuous two dimensional representation of $G_{\mathbb{Q}_p}$ for which $B(W)$ has been defined (i.e. W satisfies one of the three conditions stated at the beginning of Subsection 6.6).*

(1) *If there exists a non-zero continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map $B(W) \to \mathcal{M}(V)$, then V is a twist of a representation attached to an overconvergent eigenform of finite slope, and there is an isomorphism $(V_{|D_p})^{\mathrm{ss}} \cong W^{\mathrm{ss}}$.*
(2) *If $V_{|D_p}$ is indecomposable, if W is not the direct sum of two copies of the same character, and if the map of (1) is furthermore an embedding, then $V_{|D_p} \cong W$.*

Proof. Suppose that we are given a non-zero map $B(W) \to \mathcal{M}(V)$. Our assumption on V, together with Proposition 7.7.7, shows that $\mathbb{T}(\Sigma(K^p))$ acts on $\mathcal{M}(V)$ through the system of eigenvalues λ associated to V. Thus there is induced a non-zero map

$$B(W) \to \widehat{H}^1(K^p)^{\lambda}_{E}. \tag{80}$$

Suppose first that W is irreducible, and let R be a non-ultracritical refinement of W. If we write $\sigma(R) = (\eta, \psi)$, then Lemma 6.6.3 shows that

$$J^{\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}}_{\mathrm{P}(\mathbb{Q}_p)}\big(\widehat{H}^1(K^p)^{\lambda}_{E,\mathrm{an}}\big) \neq 0. \tag{81}$$

Thus $(\eta|\ \ |\otimes\psi\varepsilon|\ \ |^{-1}, \lambda)$ lies in $\big(\operatorname{Spec}\mathcal{A}(K^p)\big)(E)$, and hence in $\widetilde{D}(K^p)(E)$ (by Theorem 7.5.8, since λ is non-Eisenstein). In particular we see that V is attached to a twist of a finite slope overconvergent eigenform. Theorem 7.6.1 (2) then shows that there is a refinement R' of $V_{|D_p}$ such that $\sigma(R') = (\eta, \psi)$. Thus Theorem 4.5.4 implies that $V_{|D_p}$ and W are isomorphic, unless $\eta\psi^{-1}\varepsilon^{-1} = z^w$ for some $w > 0$, in which case $V_{|D_p}$ and W each admit an $\mathcal{L}$-invariant. In this case we can apply the argument of Colmez given in the proof of Theorem 7.10.1. Namely, after twisting each of V and W, we may assume that $V_{|D_p}$ is semi-stable, and thus (by [41, Thm. 6.6]) that V is attached to a classical newform f. The Mazur-Tate-Teitelbaum distributions $\mu_f^{\pm}$, which are naturally distributions on $B(R)$, are then seen to induce distributions on both $B(V_{|D_p})$ and $B(W)$. Since these are both topologically irreducible quotients of $B(R)$, they must coincide, and so again $V_{|D_p} \cong W$. (If the $\mathcal{L}$-invariants of $V_{|D_p}$ and W are finite, then we could instead appeal to [13, Thm. 1.1.2].)

We summarize the case when W is reducible, leaving the reader to fill in the details. As we observed in Subsection 6.6, the representation $B(W)$ satisfies condition (8) of Conjecture 3.3.1. Thus if the map (80) is injective, then condition (81) holds for every refinement R of W, and hence (again applying Theorem 7.6.1 (2)) we find that $\mathrm{Ref}^{\sigma}(W) \neq \emptyset$ for some $\sigma \in \mathrm{Hom}_{\mathrm{cont}}(W_p, \mathrm{T}(E))$ implies that also $\mathrm{Ref}^{\sigma}(V_{|D_p}) \neq \emptyset$. If we assume furthermore that $V_{|D_p}$ is indecomposable, and that W is not the direct sum of two copies of the same character, then, given our other restrictions on W, and recalling Propositions 4.5.5 and 4.5.6 and Lemmas 4.4.1 and 4.4.3, we conclude that indeed $V_{|D_p} \cong W$.

Suppose now that (80) is not necessarily injective. Writing W as an extension of ψ by η (and interchanging ψ and η if necessary in the case when W is split), and taking into account the definition of $B(W)$, we may then assume that either (a) the map (80) induces an injection $\big(\mathrm{Ind}_{\mathrm{P}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \eta \otimes \psi\varepsilon\big)_{\mathrm{cont}} \to \mathcal{M}(V)$, (b) the map (80) induces an injection $\big(\mathrm{Ind}_{\mathrm{P}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi \otimes \eta\varepsilon\big)_{\mathrm{cont}} \to \mathcal{M}(V)$, (c) $\eta\psi^{-1} = \varepsilon$, and (80) induces an injection $\eta \circ \det \otimes B(2,\infty) \to \mathcal{M}(V)$, or (d) $W = \eta\psi^{-1} = \varepsilon$, and (80) induces an injection $\eta \circ \det = \big(\eta \circ \det \otimes B(2,\infty)\big)/\big(\eta \circ \det \otimes \widehat{\mathrm{St}}\big) \to \mathcal{M}(V)$. In cases (a) and (c) (resp. (b)), a consideration of Jacquet modules shows that $V_{|D_p}$ admits a refinement R such that $\sigma(R) = (\eta, \psi)$ (resp. $\sigma(R) = (\psi, \eta)$). This implies that V is reducible, and that $(V_{|D_p})^{\mathrm{ss}} \cong W^{\mathrm{ss}}$. In case (d), $\mathcal{M}(V)$ would contain the one dimensional representation $\eta \circ \det$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is impossible. (Such a representation is locally algebraic up to a twist, but Theorem 7.4.2, together with the fact that the local factors of the $\mathrm{GL}_2(\mathbb{A}_f)$-representation attached to a cuspidal newform are generic [40, p. 354], shows that $\widehat{H}^1(K^p)_{E,\mathrm{lalg}}$ cannot contain a one dimensional $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.) Thus case (d) cannot occur. □

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Matthew Emerton
Mathematics Department
Northwestern University, 2033 Sheridan Rd.
Evanston, IL 60208
E-mail: emerton@math.northwestern.edu