Extensions of Truncated Discrete Valuation Rings

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Dedicated to Professor Jean-Pierre Serre on the Occasion of His 80th Birthday

Abstract: An equivalence is established between the category of at most $a$-ramified finite separable extensions of a complete discrete valuation field $K$ and the category of at most $a$-ramified finite extensions of the “length-$a$ truncation” $\mathcal{O}_K/m_K^a$ of the integer ring of $K$.

1. Introduction

Let $K$ be a complete discrete valuation field (abbr. cdvf in the following), $\mathcal{O}_K$ its valuation ring, and $m_K$ its maximal ideal. Let $a$ be an integer $\geq 1$. In this paper, we prove that the category $\mathcal{FE}_{\leq a}^K$ of finite étale $K$-algebras with ramification “bounded by $a$” (cf. Def. 3.1) depends only on $\mathcal{O}_K/m_K^a$. More precisely, let $m$ be any rational number such that $0 < m \leq a$ and put $\mathcal{A} = \mathcal{O}_K/m_K^a$. We give an equivalence of $\mathcal{FE}_{\leq m}^K$ with a category $\mathcal{FFP}_{\leq m}^\mathcal{A}$ of finite flat principal $\mathcal{A}$-algebras with ramification “bounded by $m$” (cf. Def. 3.2). The morphisms in $\mathcal{FFP}_{\leq m}^\mathcal{A}$ are defined (cf. Def. 3.3) by using Hattori’s functor ([6]); they are the usual $\mathcal{A}$-algebra homomorphisms modulo a certain equivalence relation.

For each object $L$ in $\mathcal{FE}_{\leq m}^K$, let $\mathcal{O}_L$ be the integral closure of $\mathcal{O}_K$ in $L$. Then the quotient ring $T(L) := \mathcal{O}_L/m_K^a\mathcal{O}_L$ is an object of $\mathcal{FFP}_{\leq m}^\mathcal{A}$ (Cor. 3.5). This correspondence $L \mapsto T(L)$ is functorial, and thus we obtain a functor

\[ T : \mathcal{FE}_{\leq m}^K \to \mathcal{FFP}_{\leq m}^\mathcal{A}. \]

Our main result in this paper is:

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2We mean by a principal $\mathcal{A}$-algebra an $\mathcal{A}$-algebra of which every ideal is generated by one element. All algebras in this paper are commutative.
Theorem 1.1. The functor $T$ is an equivalence of categories.

Remarks. (i) The case of $a = 1$ in the Theorem is well-known (cf. [12], Chap. III, Sect. 5). Indeed, if $m \leq 1$, the objects of $\mathcal{FE} \leq m$ are direct products of finite unramified extensions of $K$, and the Theorem implies that the objects of $\mathcal{FFP} \leq m_A$ are étale over $A$. Thus our main interest is in the case $a > 1$.

(ii) Let $G_K = \text{Gal}(\bar{K}/K)$ denote the absolute Galois group of $K$, and $G^a_K$ its $a$th ramification subgroup defined by Abbes and Saito ([2], [3]). The category $\mathcal{FE}_K^{\leq m}$ is, and hence $\mathcal{FFP}_A^{\leq m}$ is also, a Galois category whose fundamental group is $G_K/G^m_K$ by the very definition of the ramification filtration (cf. Sect. 3). Note that $\mathcal{FE}_K^{\leq m}$ is equivalent also to the category of coverings of $\text{Spec}(O_K)$ with ramification bounded by $m^a_K$ ([7], Def. 2.3); in the terminology of op. cit., we have $\pi_1(\text{Spec}(O_K), m^a_K) = G_K/G^m_K$.

A finite étale $K$-algebra is the direct product of a finite number of finite separable extension fields of $K$. Similarly, a finite flat principal $A$-algebra is the direct product of a finite number of local objects (cf. [9], Th. 1.1, Th. 1.2). Since the boundedness of ramification of direct products of $K$- and $A$-algebras may be considered componentwise, the above Theorem is equivalent with the following Corollary, in which $\mathcal{FE}_K^{\leq m}$ (resp. $\mathcal{FFP}_A^{\leq m}$) denotes the full subcategory of $\mathcal{FE}_K^{\leq m}$ (resp. $\mathcal{FFP}_A^{\leq m}$) consisting of local rings.

Corollary 1.2. The functor $T$ induces an equivalence $\mathcal{FE}_K^{\leq m} \simeq \mathcal{FFP}_A^{\leq m}$.

This extends a theorem of Deligne ([4], Th. 2.8) to the imperfect residue field case, except that our construction of the category $\mathcal{FFP}_A^{\leq m}$ for $A = O_K/m^a_K$ depends on the cdvf $K$ and hence our result is somewhat weaker than the “true” generalization of Deligne’s theorem. We expect, however, that the category $\mathcal{FFP}_A^{\leq m}$ depends only on the isomorphism class of $A$ as a ring (such a ring as $A = O_K/m^a_K$ is called a truncated discrete valuation ring; see Sect. 2). If this is the case, we may define the Galois group $G_A$ of $A$ to be $G_K/G^m_K$ (or equivalently, to be the fundamental group of the Galois category $\mathcal{FFP}_A^{\leq m}$) together with the ramification subgroups $G^m_A := G^m_K/G^m_K$, where $K$ is any cdvf such that $A \simeq O_K/m^a_K$. The filtered group $G_A$ should depend (up to inner automorphisms) only on the isomorphism class of $A$ as a ring. It is natural to ask the converse:

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3Note also that Deligne uses a category, instead of $\mathcal{FFP}_A^{\leq m}$, of certain triples which have a priori less information than the objects of $\mathcal{FFP}_A^{\leq m}$.
Extensions of Truncated Discrete Valuation Rings

**Question.** If $A$ and $A'$ are two truncated discrete valuation rings of length $a$ and if there is an isomorphism $\gamma : G_A \rightarrow G_{A'}$ of groups such that $\gamma(G^m_A) = G^m_{A'}$ for all $m \leq a$, then is it true that $A \simeq A'$ as a ring?

This problem is a version of the Grothendieck conjecture in anabelian geometry. It will certainly be necessary to assume that the residue fields of $A$ and $A'$ are either finite or of some “anabelian” nature. For the case of local fields (or, the case of “$a = \infty$” and finite residue fields), see [10] and [1].

In Section 2, we study basic properties of truncated discrete valuation rings. After recalling some basics of the ramification theory of Abbes-Saito ([2], [3]) and Hattori ([6]), we construct the category $\mathcal{F} \mathcal{F} \mathcal{P} \mathcal{P}_{\leq m}^A$ and prove the Theorem in Section 3.

Throughout this paper, $K$ is a complete discrete valuation field with residual characteristic $p > 0$. We denote by $\mathcal{O}_K$ the valuation ring of $K$, $m_K$ the maximal ideal of $\mathcal{O}_K$, $\pi_K$ a uniformizing element of $K$, and $\tilde{K}$ a fixed separable closure of $K$. For any étale $K$-algebra $L$, we denote by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in $L$. For $A$-algebras $B$ and $B'$, we denote by $\text{Hom}_A(B, B')$ the set of $A$-algebra homomorphisms $B \rightarrow B'$. We use the following abbreviations:

- $\text{cdvf} :=$ complete discrete valuation field,
- $\text{cdvr} :=$ complete discrete valuation ring,
- $\text{tdvr} :=$ truncated discrete valuation ring.

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### 2. Truncated discrete valuation rings

A $\text{tdvr}$ is an Artinian local ring whose maximal ideal is generated by one element. The length of a $\text{tdvr}$ $A$ is the length of $A$ as an $A$-module. It is known that a $\text{tdvr}$ $A$ is principal, and any ideal is of the form $m_A^i$ for some $i \geq 0$ if $m_A$ is the maximal ideal of $A$. Any generator $\pi_A$ of $m_A$ is said to be a uniformizer of $A$. Any non-zero element $x$ of $A$ can be written as $x = u \pi_A^i$ with $u \in A^\times$, $\pi_A$ a uniformizer of $A$, and $0 \leq i < \text{length}(A)$ (with the convention $0^0 = 1$ if $\text{length}(A) = 1$). If $\text{length}(A) > 1$ (resp. $\text{length}(A) = 1$), we mean by an extension $B/A$ of $\text{tdvr}$’s a local ring homomorphism $A \rightarrow B$ of $\text{tdvr}$’s via which $B$ is flat over $A$ (resp. an extension $B/A$ of fields); thus we refrain from calling a
homomorphism such as $A \hookrightarrow A[t]/(t^n)$ an extension if $A$ is a field. An extension $B/A$ is said to be finite if $B$ is finite as an $A$-module. If $a > 1$, an $A$-algebra is a finite extension of $A$ if and only if it is finite, flat, principal and local. In general, the objects of the category $\text{FFP}_{A}^{\leq m}$ are finite extensions of the tdvr $A$. The ramification index $e_{B/A}$ of a homomorphism $f : A \rightarrow B$ of tdvr’s is defined to be the integer $e$ such that $f(m_A)B = m_B^e$ (with the convention $e_{B/A} = 1$ if $\text{length}(A) = 1$). Note that the homomorphism $f$ is an extension of tdvr’s if and only if one has the equality $\text{length}(B) = e_{B/A} \text{length}(A)$ (cf. [4], Sect. 1.4 and [8], Exer. 22.1).

**Lemma 2.1.** Let $B$ and $C$ be extensions of $A$. Then any $A$-algebra homomorphism $f : B \rightarrow C$ is an extension.

**Proof.** We have to show that $\text{length}(C) = e_{C/B} \text{length}(B)$. We may assume that $\text{length}(A) > 1$. Let $m_A$, $m_B$ and $m_C$ be respectively the maximal ideals of $A$, $B$ and $C$. By the definition of ramification index, we have $m_AB = m_B^{e_{B/A}}$, $m_AC = m_C^{e_{C/A}}$, and $f(m_B)C = m_C^{e_{C/B}}$. The equality $m_C^{e_{C/A}} = f(m_B^{e_{B/A}})C$ (the ideal generated by $m_A$) implies that $e_{C/A} = e_{C/B} e_{B/A}$. Since $B$ and $C$ are extensions of $A$, we have $\text{length}(C) = e_{C/A} \text{length}(A) = e_{C/B} e_{B/A} \text{length}(A) = e_{C/B} \text{length}(B)$. □

If $K$ is a cdvf, then $O_K/m_A^a$ is a tdvr for any integer $a \geq 1$. If $L/K$ is a finite extension of cdvf’s, then $B = O_L/m_A^aO_L$ is a finite extension of $A = O_K/m_A^a$. Conversely, it is known that any tdvr is a quotient of a cdvr ([9], Th. 3.3). More precisely, we have:

**Proposition 2.2.** (i) Let $A$ be a tdvr with residue field $k$ of characteristic $p \geq 0$, and let $a$ be the length of $A$. Then there exists a cdvr $O$ such that $A$ is isomorphic to $O/m^a$, where $m$ is the maximal ideal of $O$. If $pA = 0$, then this $O$ can be taken to be the power series ring $k[\pi]$; if $pA \neq 0$, then $O$ as above must be finite over a Cohen $p$-ring ([5], 0IV, 19.8) with residue field $k$. (If $pA = 0$ and $p \neq 0$, then both types of $O$ are possible.)

(ii) Let $K$ be a cdvf and let $A = O_K/m_A^a$ with $a \geq 1$. For any finite extension $B/A$ of tdvr’s, there exist a finite separable extension $L/K$ and an isomorphism.
ψ : \(O_L/\mathfrak{m}_K^a O_L \rightarrow B\) such that the diagram

\[
\begin{array}{ccc}
O_L/\mathfrak{m}_K^a O_L & \xrightarrow{\psi} & B \\
\uparrow & & \uparrow \\
O_K/\mathfrak{m}_K^a & \longrightarrow & A
\end{array}
\]

is commutative, where the left vertical arrow is the one induced by \(O_K \hookrightarrow O_L\).

**Proof.** (i) Let \(W\) be a Cohen \(p\)-ring with residue field \(k\). The reduction map \(W \rightarrow k\) lifts by the formal smoothness of \(W\) to a local ring homomorphism \(W \rightarrow A\) ([5], 0IV, 19.8.6).

If \(pA = 0\), the map \(W \rightarrow A\) factors through the residue field \(k\), which makes \(A\) a \(k\)-algebra. Then there exists a surjective \(A\)-algebra homomorphism \(k[\pi] \rightarrow A\) which maps \(\pi\) to \(\pi_A\), where \(\pi_A\) is a uniformizer of \(A\). Hence \(A\) is isomorphic to \(k[\pi]/(\pi^a)\) (cf. [9], Th. 3.1).

In the general case, we can write \(A\) as a quotient of the polynomial ring \(W[X]\) by sending \(X\) to \(\pi_A\). Then we obtain a surjection onto \(A\) from a cdvr \(O\) which is finite over \(W\) by the same procedure as in the proof of (ii) below.

(ii) Since \(B\) is finite over \(A = O_K/\mathfrak{m}_K^a\), there exists a surjective \(O_K\)-algebra homomorphism \(\phi : R \rightarrow B\) from a polynomial ring \(R = O_K[X_1, \ldots, X_n]\) onto \(B\). Let \(\mathfrak{m} = \phi^{-1}(\mathfrak{m}_B)\) and \(R_{\mathfrak{m}}\) the localization of \(R\) at the maximal ideal \(\mathfrak{m}\). Then \(R_{\mathfrak{m}}\) is a regular local ring of Krull dimension \(n + 1\) ([5], 0IV, 17.3.7), and \(\phi\) extends to a surjective \(O_K\)-algebra homomorphism \(\varphi : R_{\mathfrak{m}} \rightarrow B\). By abuse of notation, we denote also by \(\mathfrak{m}\) the maximal ideal of \(R_{\mathfrak{m}}\). Put \(n = \text{Ker} (\varphi)\). We identify the residue field \(k'\) of \(R_{\mathfrak{m}}\) with that of \(B\) via \(\varphi\). Since \(\varphi(\mathfrak{m}^2) = \mathfrak{m}_B^2\), the map \(\varphi\) induces a surjective \(k'\)-linear map \(\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2\) and its kernel is \((n + \mathfrak{m}^2)/\mathfrak{m}^2 \cong n/(n \cap \mathfrak{m}^2)\). Thus we have an exact sequence

\[
0 \rightarrow n/(n \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2 \rightarrow 0.
\]

Assume \(a \geq 2\), as the case \(a = 1\) can be treated similarly and more easily. Then \(\dim_{k'}(\mathfrak{m}_B/\mathfrak{m}_B^2) = 1\) and \(\dim_{k'}(n/(n \cap \mathfrak{m}^2)) = n\). Choose a regular system of parameters \((w, f_1, \ldots, f_n)\) of \(R_{\mathfrak{m}}\) such that \(\varphi(w)\) gives a basis of \(\mathfrak{m}_B/\mathfrak{m}_B^2\) and \(f_1, \ldots, f_n \in n\) give a basis of \(n/(n \cap \mathfrak{m}^2)\). Let \(\mathfrak{p}\) be the ideal of \(R_{\mathfrak{m}}\) generated by \(f_1, \ldots, f_n\). Then by [5], 0IV, 17.1.7, the quotient ring \(O = R_{\mathfrak{m}}/\mathfrak{p}\) is a regular local ring of dimension 1 and hence a discrete valuation ring. It contains \(O_K\) since \(\varphi\) maps \(\pi_K\) to a non-zero non-unit in \(B\), and is finite over \(O_K\). Hence it is a cdvr.
Since \( n \supset p \), the map \( \varphi \) factors through \( \mathcal{O} \). Thus we see the diagram (1) commutes (with \( \mathcal{O} \) in place of \( \mathcal{O}_L \)). Since \( B \) is flat over \( A \), the induced homomorphism \( \psi \) is bijective.

To make the fraction field \( L \) of \( \mathcal{O} \) separable over \( K \), we “deform” the prime ideal \( p \) if necessary. By multiplying the \( f_i \) with some \( u \in R \setminus \mathfrak{m} \), we may assume that all \( f_i \) are in the polynomial ring \( R \). Note that the composite map \( R \hookrightarrow R_\mathfrak{m} \to R_\mathfrak{m}/p = \mathcal{O} \) is surjective by Nakayama’s lemma, since its image generates \( B = \mathcal{O}/\mathfrak{m}_K^2 \mathcal{O} \). Let \( q \) be its kernel, so that \( \mathcal{O} = R/q \). We have \( qR_\mathfrak{m} = p \), i.e., \( q \) is generated by \( f_1, \ldots, f_n \) locally at \( \mathfrak{m} \). By the Jacobian criterion ([11], V, Sect. 2, Th. 5), the \( K \)-algebra \( L \) is separable (i.e., the \( \mathcal{O}_K \)-algebra \( \mathcal{O} \) is étale at the generic point of \( \text{Spec}(\mathcal{O}) \)) if and only if the Jacobian \( \det \left( \frac{\partial f_i}{\partial X_j} \right)_{1 \leq i,j \leq n} \neq 0 \pmod{q} \).

Let \( g_i := f_i + x_i X_i \) with \( x \in \mathfrak{m}_K^2 \). Then, since \( g_i \in n \) and \( g_i \equiv f_i \pmod{n \cap \mathfrak{m}^2} \), the ideal \( p' = (g_1, \ldots, g_n) \) of \( R_\mathfrak{m} \) has similar properties as \( p \) so that the quotient ring \( \mathcal{O}' := R_\mathfrak{m}/p' \) is a cdvr which contains \( \mathcal{O}_K \) and surjects onto \( B \). Moreover, if \( J := \left( \frac{\partial f_i}{\partial X_j} \right)_{1 \leq i,j \leq n} \) we have

\[
\det \left( \frac{\partial g_i}{\partial X_j} \right)_{1 \leq i,j \leq n} = \det(xI_n + J) = x^n + \text{Tr}(J)x^{n-1} + \cdots + \det(J).
\]

Considering this modulo \( q \) and noticing that \( \mathcal{O}_K \subset \mathcal{O} = R/q \), we find an \( x \in \mathfrak{m}_K^n \) such that \( \det \left( \frac{\partial g_i}{\partial X_j} \right) \neq 0 \pmod{q} \). Then the fraction field of \( \mathcal{O}' \) is separable over \( K \).

3. Ramification

Let \( G_K \) be the absolute Galois group of \( K \). A. Abbes and T. Saito ([2], [3]) defined a decreasing filtration \((G_K^m)_{m \geq 0}\) by closed normal subgroups \( G_K^m \) of \( G_K \) indexed with rational numbers \( m \geq 0 \), in such a way that \( \cap_{m \geq 0} G_K^m = 1 \), \( G_K^0 = G_K \) and \( G_K^1 \) is the inertia subgroup of \( G_K \). The filtration coincides with the classical upper numbering ramification filtration shifted by one if the residue field of \( K \) is perfect (see [12], Chap. IV, Sect. 3, for the classical case). It is defined by using certain functors \( F \) and \( F^m \) from the category \( \mathcal{FE}_K \) of finite étale \( K \)-algebras to the category \( S_K \) of finite \( G_K \)-sets. We recall here the definition of \( F \) and \( F^m \) assuming for simplicity that \( m \) is a positive integer. Let \( L \) be a finite étale \( K \)-algebra, and let \( \mathcal{O}_L \) be the integral closure of \( \mathcal{O}_K \) in \( L \). We define \( F(L) := \text{Hom}_K(L, \overline{K}) = \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K) \). The functor \( F \) gives an anti-equivalence of \( \mathcal{FE}_K \) with \( S_K \), thereby making \( \mathcal{FE}_K \) a Galois category. To define
Extensions of Truncated Discrete Valuation Rings

$F^m$, we proceed as follows: An embedding of $\mathcal{O}_L$ is a pair $(B, B \to \mathcal{O}_L)$ consisting of an $\mathcal{O}_K$-algebra $B$ which is formally of finite type and formally smooth over $\mathcal{O}_K$ and a surjection $B \to \mathcal{O}_L$ of $\mathcal{O}_K$-algebras which induces an isomorphism $B/m_B \to \mathcal{O}_L/m_L$, where $m_B$ and $m_L$ are respectively the radicals of $B$ and $\mathcal{O}_L$ (cf. [3], Def. 1.1). Let $I$ be the kernel of the surjection $B \to \mathcal{O}_L$. Define an affinoid algebra $B^m$ over $K$ by $B^m = B[I/\pi_K^m]^\wedge \otimes_K K$, where $\wedge$ means the $\pi_K$-adic completion. Let $X^m(B \to \mathcal{O}_L)$ be the affinoid variety $\text{Sp}(B^m)$ associated with $B^m$. For any affinoid variety $X$ over $K$, let $\pi_0(X_R)$ denote the set $\lim_{\pi_K} \pi_0(X \otimes_K K')$ of geometric connected components, where $K'$ runs through the finite separable extensions of $K$. Then we define the functor $F^m$ by

$$F^m(L) := \lim_{(B \to \mathcal{O}_L)} \pi_0(X^m(B \to \mathcal{O}_L)_R),$$

where $(B \to \mathcal{O}_L)$ runs through the category of embeddings of $\mathcal{O}_L$. The projective system in the right-hand side is constant. The finite set $F(L)$ can be identified with a subset of $X^m(B \to \mathcal{O}_L)(\bar{R})$, and this causes a natural surjective map $F(L) \to F^m(L)$. Thus the category $\mathcal{F}E_{\leq m}^K$ of finite étale $K$-algebras with ramification bounded by $m$ forms a Galois full-subcategory of $\mathcal{F}E_K$ whose fundamental group is $G_K/G_K^m$ ([2], Prop. 2.1) as noted in the Introduction. Note that the above definition of “ramification bounded by $m$” coincides with Deligne’s one in [4] when $L$ is a field and $\mathcal{O}_L$ is monogenic over $\mathcal{O}_K$ (cf. [2], Prop. 6.7).

Let $a$ be an integer $\geq 1$, and put $A = \mathcal{O}_K/m_K^a$. For each rational number $0 < m \leq a$, Hattori ([6]) defined another functor $F^m$ from the category of finite flat $A$-algebras to the category $\mathcal{S}_K$ of finite $G_K$-sets. We next recall the definition of $F^m$ assuming for simplicity that $m$ is a positive integer. Let $B$ be a finite flat $A$-algebra. An embedding of $B$ is a pair $(B, B \to B)$ consisting of an $\mathcal{O}_K$-algebra $B$ which is formally of finite type and formally smooth over $\mathcal{O}_K$ and a surjection $B \to B$ of $\mathcal{O}_K$-algebras which induces an isomorphism $B/m_B \to B/m_B$, where $m_B$ and $m_B$ are respectively the radicals of $B$ and $B$. Let $I$ be the kernel of the surjection $B \to B$. Define an affinoid algebra $B^m$ over $K$ by $B^m = B[I/\pi_K^m]^\wedge \otimes_K K$. Let $X^m(B \to B)$ be the affinoid variety $\text{Sp}(B^m)$ associated with $B^m$. Then we define the functor $F^m$ by

$$F^m(B) := \lim_{(B \to B)} \pi_0(X^m(B \to B)_R),$$

where $(B \to B)$ runs through the category of embeddings of $B$. In general, we have $\mathcal{g}F^m(B) \leq \text{rank}_A(B)$. Two key definitions in this paper are the following:
Definition 3.2. Let $B$ be a finite flat $A$-algebra. We say that the ramification of $B$ is bounded by $m$ if \( \sharp F^m(B) = \text{rank}_A(B) \).

Definition 3.3. For any rational number $m$ with $0 < m \leq a$, we define $\mathcal{FFP}_{\leq m}^A$ to be the category whose objects are finite flat principal $A$-algebras with ramification bounded by $m$ and whose morphisms are defined as follows: For any $B$ and $B'$ in $\mathcal{FFP}_{\leq m}^A$, set
\[
\text{Hom}_{\mathcal{FFP}_{\leq m}^A}(B, B') \colonequals \text{Hom}_{\mathcal{S}_K}(F^m(B'), F^m(B)).
\]
We also define $\mathcal{FFP}_{\leq m}^A$ to be the full-subcategory of $\mathcal{FFP}_{\leq m}^A$ consisting of local objects.

To prove Theorem 1.1, we recall the following lemma due to Hattori ([6], Lem. 1):

Lemma 3.4. Let $L$ be a finite étale $K$-algebra, and $a$ an integer $\geq 1$. If $B = O_L/m_a^aO_L$, then we have $F^m(B) = F^m(L)$ as an object of $\mathcal{S}_K$ for any rational number $0 < m \leq a$.

This is because one may choose a common $B$ in the embeddings $(B, B \to O_L)$ and $(B, B' \to O_L)$, so that, if $m \leq a$, we have $X^m(B \to O_L) = X^m(B' \to O_L)$.

By Definitions 3.1 and 3.2, we have:

Corollary 3.5. For any rational number $0 < m \leq a$, the ramification of $B$ is bounded by $m$ if and only if the ramification of $L$ is bounded by $m$.

Now we can prove Theorem 1.1. The essential surjectivity of the functor $T : \mathcal{FE}_{\leq m}^K \to \mathcal{FFP}_{\leq m}^A$ follows from (ii) of Proposition 2.2 and Corollary 3.5, since any object of $\mathcal{FFP}_{\leq m}^A$ is a direct product of finite extensions of $A$. To prove the full-faithfulness of $T$, let $L$ and $L'$ be two objects in $\mathcal{FE}_{\leq m}^K$, and let $B = T(L)$ and $B' = T(L')$. Since the functor $F^m$ gives an anti-equivalence of the Galois category $\mathcal{FE}_{\leq m}^K$ with a full-subcategory of $\mathcal{S}_K$, we have
\[
\text{Hom}_{\mathcal{FE}_{\leq m}^K}(L, L') \simeq \text{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)).
\]
By Lemma 3.4, we have
\[
\text{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) = \text{Hom}_{\mathcal{S}_K}(F^m(B'), F^m(B)).
\]
It follows from our definition (2) of Hom in $\mathcal{FFP}_{\leq m}^A$ that
\[
\text{Hom}_{\mathcal{FE}_{\leq m}^K}(L, L') = \text{Hom}_{\mathcal{FFP}_{\leq m}^A}(B, B').
\]
This completes the proof of the Theorem.
Remark. The relation of $\text{Hom}_A(B, B')$ to the Hom sets appearing in the above proof is summarized by the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_K(L, L') & \xrightarrow{\cong}_{F^m} & \text{Hom}_{S_K}(F^m(L'), F^m(L)) \\
\downarrow & & \downarrow \\
\text{Hom}_A(B, B') & \xrightarrow{\cong}_{F^m} & \text{Hom}_{S_K}(F^m(B'), F^m(B)),
\end{array}
\]

where the left vertical arrow is the reduction mod $m^\mu_K$ of $\text{Hom}_{O_K}(O_L, O_{L'})$. This shows that the map $F^m : \text{Hom}_A(B, B') \to \text{Hom}_{F, F, p, \leq m}(B, B')$ is surjective and compatible with the composition of morphisms. It can be shown that this map identifies the set $\text{Hom}_{F, F, p, \leq m}(B, B')$ with the quotient of $\text{Hom}_A(B, B')$ by an equivalence relation $\sim$ defined as follows: Put $\bar{A} = O_R/m^\mu_R O_R$ and let $\mathcal{X}^m$ be the affinoid variety associated with an embedding of $B$. Recall that there exists a natural surjective map $\mathcal{X}^m(R) \to \text{Hom}_A(B, \bar{A})$ with connected fibers ([2], Lem. 3.2), so that its inverse yields a well-defined map $\xi : \text{Hom}_A(B, \bar{A}) \to \pi_0(\mathcal{X}^m_R)$. Then we have a map

\[
\text{Hom}_A(B, B') \times \text{Hom}_A(B', \bar{A}) \to \pi_0(\mathcal{X}^m_R)
\]

which maps $(f, \alpha)$ to $\xi(\alpha \circ f)$. For $f$ and $f'$ in $\text{Hom}_A(B, B')$, define

\[
f \sim f' \iff \xi(\alpha \circ f) = \xi(\alpha \circ f') \quad \text{for all } \alpha \in \text{Hom}_A(B', \bar{A}).
\]

It can also be shown that, if $B'$ is local, then for given $f$ and $f'$, the equality $\xi(\alpha \circ f) = \xi(\alpha \circ f')$ holds for all $\alpha \in \text{Hom}_A(B', \bar{A})$ if it holds for some $\alpha$.

References


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