Some Improved Caffarelli-Kohn-Nirenberg Inequalities with General Weights and Optimal Remainders*

Yaotian Shen and Zhihui Chen

Dedicated to Professor J.J.Kohn on the occasion of his 75th birthday

Abstract: In this paper, we establish some improved Caffarelli-Kohn-Nirenberg inequalities with general weights and optimal remainders. Moreover, we give a positive answer to an open problem raised by Abdellaoui et al. [1].

Keywords: Hardy-Sobolev inequality, general weight, optimal remainders

1 Introduction

Let $p > 1$ be a constant. In 1920, Hardy [7] showed that, for any positive $f(x) \in L^p(0, \infty)$,

$$\int_0^\infty \left[ \frac{F(x)}{x} \right]^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p \, dx,$$

where $F(x) = \int_0^x f(t) \, dt$, and the constant $\left( \frac{p}{p-1} \right)^p$ is optimal.

Received February 6, 2008.

2000 MR Subject Classification, 46E35, 35J40

*Project supported by the National Natural Science Foundation of China (No. 10771074, 10726060) and the Natural Science Foundation of Guangdong Province (No. 04020077)
In 1933, Leray [8] gave the following multidimensional version of Hardy’s inequality

\[
\int_{\mathbb{R}^2 \setminus B_1(0)} \frac{u^2}{|x|^2 \ln^2 |x|} \, dx \leq 4 \int_{\mathbb{R}^2 \setminus B_1(0)} |\nabla u|^2 \, dx, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus B_1(0)) \tag{1.1}
\]

\[
\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx \leq \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad u \in C_0^\infty(\mathbb{R}^N), \quad N \geq 3 \tag{1.2}
\]

We may call the above two inequalities Hardy-Leray inequality, which is called Hardy-Sobolev inequality in the literature (see [1]). For any bounded domain \( \Omega \subset B_R(0) \) including origin, \( B_R(0) \) denotes a ball in \( \mathbb{R}^N \) with radius \( R \) and centered at 0, Shen [9] obtained (1.1) with \( \ln |x| \) being replaced by \( \ln R/|x| \).

Brézis and Vázquez [3] obtained a remainder term for the Hardy-Leray’s inequality. More precisely, if \( 1 \leq q < \frac{2N}{N-2}, \quad N \geq 3 \), there exists a constant \( C(q,|\Omega|) > 0 \) such that

\[
\int_{\Omega} |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geq C(q,|\Omega|) \left( \int_{\Omega} |u|^q \, dx \right)^{2/q}, \quad u \in H_0^1(\Omega) \tag{1.3}
\]

They raised some open problems in [3], and the second one states whether there is a further improvement in the direction of this inequality.

Vázquez and Zuazua [16], among other results, improved the previous inequality by showing that if \( 1 < q < 2 \), there exists a constant \( C(q,|\Omega|) > 0 \) such that, for each \( u \in H_0^1(\Omega) \),

\[
\int_{\Omega} |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geq C(q,|\Omega|) \left( \int_{\Omega} |\nabla u|^q \, dx \right)^{2/q} \tag{1.4}
\]

The Caffarelli-Kohn-Nirenberg inequality [4] shows that, if \( 1 < p < N \) and \( \gamma < \frac{N-p}{p} \), for any \( u \in C_0^\infty(\Omega) \),

\[
c_p \int_{\Omega} |x|^{-p(\gamma+1)} |u|^p \, dx \leq \int_{\Omega} |x|^{-p\gamma} |\nabla u|^p \, dx \tag{1.5}
\]

where \( \Omega \) is allowed to be the whole space \( \mathbb{R}^N \).

Wang and Willem [17] obtained the Caffarelli-Kohn-Nirenberg inequality with
optimal remainder, that is, if \( \emptyset \subset B_R(0) \), then for any \( u \in H^1_0(\emptyset) \),
\[
\int_{\emptyset} |x|^{-2\gamma} |\nabla u|^2 \, dx - \left[ \frac{N - 2(\gamma + 1)}{2} \right]^2 \int_{\emptyset} |x|^{-2(\gamma + 1)} |u|^2 \, dx \\
\geq C \int_{\emptyset} |x|^{-2\gamma} (\ln R/|x|)^{-2} |\nabla u|^2 \, dx \quad (1.6)
\]
It is optimal in the sense that \((\ln R/|x|)^{-1}\) cannot be replaced by \(g(x)(\ln R/|x|)^{-1}\) with \(g\) satisfying \(|g(x)| \to \infty\) as \(|x| \to 0\). If \(\gamma = 0\), (1.6) gives a positive answer to the second open problem of [3] in some sense. The authors proved another result which works for bounded domains as well as exterior domains, that is,
\[
\int_{\emptyset} |x|^{-2\gamma} |\nabla u|^2 \, dx - \left[ \frac{N - 2(\gamma + 1)}{2} \right]^2 \int_{\emptyset} |x|^{-2(\gamma + 1)} |u|^2 \, dx \\
\geq \frac{1}{4} \int_{\emptyset} |x|^{-2(\gamma + 1)} (\ln R/|x|)^{-2} u^2 \, dx,
\]
where \(\gamma \leq \frac{N-p}{p}, \emptyset \subset B_R(0)\) or \(\emptyset \subset B^C_R(0)\). Moreover, the constant \(\frac{1}{4}\) is also sharp.

Abdellaoui et al. [1] proved that if \(1 < q < p < N\), then for any \( u \in C_0^\infty(\emptyset) \),
\[
\int_{\emptyset} |x|^{-\gamma p} |\nabla u|^p \, dx - \left[ \frac{N - p(\gamma + 1)}{p} \right]^p \int_{\emptyset} \frac{|u|^p}{|x|^{p(\gamma + 1)}} \, dx \geq C \int_{\emptyset} |x|^{-\gamma r} |\nabla u|^q \, dx \quad (1.7)
\]
where \(q < r < +\infty\) if \(\gamma \leq 0\), or \(r < p + \rho(N,p,q,\gamma)\) for some positive constant \(\rho\) if \(\gamma > 0\). The authors point out that it seems to be an open problem to obtain the best weight for (1.7) as in (1.6), in the case \(p \neq 2\). In this paper, we give a positive answer to this open problem. In fact, we obtain the Caffarelli-Kohn-Nirenberg inequality with general weights and remainder term. Because the weight is general, we also obtain the corresponding inequality with weight \(|x|^{-\gamma p}\) in the case of \(N = p > 1\). When \(N = p = 2\), this problem has been discussed in [14].

Now we introduce the weighted Sobolev space. Let \(\phi\) be a positive continuous function with \(\phi(|x|) \in L(B_\delta(0))\) for some positive \(\delta\), and define
\[
\tilde{h}(r_1, r_2) = c_0 \int_{r_1}^{r_2} (\phi r^{-N-1})^{-1/(p-1)} \, dr
\]
for \(0 \leq r_1 \leq r_2 \leq \infty\), where \(c_0\) is a given positive constant. In this paper, we consider the following two cases:
(A1) \( \tilde{h}(r, \infty) < \infty \) for all \( r > 0 \) and \( \tilde{h}(0, \infty) = \infty \);

(A2) \( \tilde{h}(r, \infty) = \infty \) and \( \tilde{h}(0, D) = \infty \) for some \( r, D > 0 \).

**Definition 1.** Let \( p > 1 \), we denote by \( W^{1,p}_0(\Omega, \phi) \) the completion of \( C_0^\infty(\Omega) \) with respect to the norm

\[
\|u\|_{1,p,\phi} = \left( \int_\Omega \phi(r)|\nabla u|^p \, dx \right)^{1/p}
\]

where \( r = |x| \).

**Example 1.** Let \( \phi = r^{-p\gamma} \) and \( 0 \in \Omega \subset B_D(0) \). If \( \gamma < \frac{N-p}{p} \), then (A1) happens, and \( W^{1,p}_0(\Omega, |x|^{-p\gamma}) \) is identical with \( D^{1,p}_{0,\gamma}(\Omega) \) in [1]. If \( \gamma = \frac{N-p}{p} \), then (A2) happens, and \( W^{1,p}_0(\Omega, |x|^{-p\gamma}) \) has not been discussed before.

In what follows, for short, we use \( \phi \) for \( \phi(r) \) or \( \phi(|x|) \), etc.

Set

\[
\tilde{h} = \begin{cases} 
\tilde{h}(r, \infty), & \text{if (A1) holds} \\
\tilde{h}(r, D), & \text{if (A2) holds}
\end{cases}
\]

If \( N > p \) and \( \phi \equiv 1 \), then (A1) holds, therefore \( \tilde{h}(|x|) = |x|^{\frac{p-N}{p-1}} \) is a fundamental solution for the \( p \)-Laplace operator. For general weight \( \phi \), function \( h = \tilde{h}^{(p-1)/p} \) satisfies in the sense of distribution

\[
-\Delta_{\phi,p} u = \text{div}(\phi|\nabla u|^{p-2} \nabla u) = \psi|u|^{p-2} u
\]

where \( \psi = \left( \frac{p-1}{p} \right)^p \phi \left( \frac{-h'}{h} \right)^p = \phi \left( \frac{-h'}{h} \right)^p \), that is, \( h \) is a weak solution of the Euler-Lagrange equation (1.8) of the functional

\[
I_{1,\phi}(u) = \int_\Omega (\phi|\nabla u|^p - \psi|u|^p) \, dx
\]

In [10][11][12] it has been proved that if \( \phi, \psi \) are positive functions in \( C^1(0,a) \) and satisfy the Bernoulli equation

\[
(\phi^{1/p} \psi^{1-1/p})' + \frac{N-1}{r} \phi^{1/p} \psi^{1-1/p} = p \psi
\]

then for any \( u \in C_0^\infty(\Omega) \),

\[
\int_\Omega \psi|u|^p \, dx \leq \int_\Omega \phi|\nabla u|^p \, dx,
\]
and the constant 1 is optimal, where \( a = +\infty \) and \( \Omega = \mathbb{R}^N \) if (A1) holds, or \( a = D \) and \( \Omega \subset B_D(0) \) if (A2) holds. Because \( \bar{h} \) is a fundamental solution of operator \( -\Delta_{p,\phi} \), in other words, \( h \) is a distribution solution of equation (1.8), we know \( \psi \) can be expressed by \( \bar{h} \) and \( \phi \) or by \( h \) and \( \phi \) as follows

\[
\psi = \left( \frac{p-1}{p} \right)^p \phi \left( \frac{-\bar{h}'}{\bar{h}} \right)^p = \phi \left( \frac{-h'}{h} \right)^p
\]

**Theorem 1.1** ([5], Theorem 1.1). Let \( \Omega \) be \( \mathbb{R}^N \) if (A1) holds or \( \Omega \) be a bounded domain included in \( B_D(0) \) if (A2) holds. Suppose that \( \phi \) is continuous and set

\[
h = \left( c_0 \int_r^a (\phi r^{N-1})^{-1/(p-1)} \, dr \right)^{(p-1)/p}
\]  

where \( a = +\infty \) if (A1) holds or \( a = D \) if (A2) holds. Then for any \( u \in W_0^{1,p}(\Omega, \phi) \)

\[
\int_{\Omega} \phi \left( \frac{-\bar{h}'}{\bar{h}} \right)^p |u|^p \, dx \leq \int_{\Omega} \phi |\nabla u|^p \, dx
\]

where the constant 1 is optimal.

**Remark 1.1.** (A1) or (A2) implies the integrability of \( \phi \left( \frac{-\bar{h}'}{\bar{h}} \right)^p \) in \( B_D(0) \).

**Theorem 1.2.** Let \( p > 1 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). Suppose \( \phi \) is continuous satisfying (A1) or (A2), \( h \) is defined by (1.11). Set

\[
h_1 = \begin{cases} 
\frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)}, & \text{if (A1) holds} \\
\frac{p}{(p-1)c_0} \ln h(r), & \text{if (A2) holds}
\end{cases}
\]

then

1. There exists a positive constant \( D_0 \leq D \) such that for any \( u \in W_0^{1,p}(\Omega, \phi) \)

\[
\int_{\Omega} \phi |\nabla u|^p \, dx - \int_{\Omega} \psi |u|^p \, dx \geq \frac{p}{2(p-1)c_0^2} \int_{\Omega} \psi h_1^{-2} |u|^p \, dx
\]

where \( \psi = \phi \left( \frac{-\bar{h}'}{\bar{h}} \right)^p \).

2. The constants in (1.13) are optimal, that is,

\[
\frac{p}{2(p-1)c_0^2} = \inf_{W_0^{1,p}(\Omega, \phi)} \frac{I_\phi(u)}{\int_{\Omega} \psi h_1^{-2} |u|^p \, dx}
\]
Remark 1.2. Let $\phi = r^{-p\gamma}$ with $\gamma < \frac{N-p}{p}$ and $c_0 = \frac{N-p(\gamma+1)}{p-1}$. It follows from (1.11) and (1.12) that

$$h = r^{-\frac{N-p(\gamma+1)}{p}}, \quad \psi = \left(\frac{N-p(\gamma+1)}{p}\right)^p r^{-p(\gamma+1)}, \quad h_1 = \ln \frac{D}{r}$$

hence we obtain by (1.13)

$$\int_{\Omega} \left( |x|^{-p}\nabla u|^p - \left(\frac{N-p(\gamma+1)}{p-1}\right)^p |x|^{-p(\gamma+1)}|u|^p \right) dx \geq \frac{p-1}{2p} \left(\frac{N-p(\gamma+1)}{p-1}\right)^{p-2} \int_{\Omega} |x|^{-p(\gamma+1)}(\ln D/|x|)^{-2}|u|^p dx$$

which is identical with Theorem A in [2] when $\gamma = 0$.

Remark 1.3. Theorem 1.2 improves the results of [13][15].

Theorem 1.3. Under the hypothesis of Theorem 1.2, we have

i) \( \int_{\Omega} \phi |\nabla u|^p - \psi |u|^p \, dx \geq C \int_{\Omega} \phi h_1^{-2} |\nabla u|^p \, dx, \quad \forall u \in W^{1,p}_0(\Omega, \phi) \) \hspace{1cm} (1.14)

ii) The inequality (1.14) is optimal in the sense that $h_1^{-2}$ can not be replaced by any weight of the form $g(x)h_1^{-2}$ where $g(x)$ is a positive function such that $g(x) \rightarrow \infty$ as $x \rightarrow 0$.

Remark 1.4. Taking $\phi = r^{-p\gamma}$ in Theorem 1.3, if $\gamma < \frac{N-p}{p}$, then

$$\int_{\Omega} \left( |x|^{-p}\nabla u|^p - \left(\frac{N-p(\gamma+1)}{p-1}\right)^p |x|^{-p(\gamma+1)}|u|^p \right) dx \geq C \int_{\Omega} |x|^{-p(\ln D/|x|)^{-2}}|\nabla u|^p dx$$

for any $u \in W^{1,p}_0(\Omega, \phi)$. This is a positive answer to the open problem in [1]. If $\gamma = \frac{N-p}{p}$, then

$$\int_{\Omega} \left( |x|^{-p\gamma}\nabla u|^p - \left(\frac{p-1}{p}\right)^p |x|^{-p(\gamma+1)}(\ln D/|x|)^{-p}|u|^p \right) dx \geq C \int_{\Omega} |x|^{-p(\gamma+1)}(\ln D/|x|)^{-p}\ln (\ln D/|x|)^{-2}|\nabla u|^p dx$$

for any $u \in W^{1,p}_0(\Omega, \phi)$, where $D' > eD$. This solves the problem for the case of $\gamma = \frac{N-p}{p}$ which has not been discussed before.
Remark 1.5. Wang and Willem [17] proved (1.6) by using a change of variable that appear in [6]. However, to prove Theorem 1.3, we use a change of variables that appear in [14] \((p = 2)\), which involves the function \(\bar{h}\) or the distribution solution \(h\).

Remark 1.6. Theorem 1.3 gives a positive answer to the second open problem of [3] in the case of general weights.

2 Some Lemmas and Corollaries

**Lemma 2.1** ([1]). For all \(\zeta_1, \zeta_2 \in \mathbb{R}^N\), the following inequalities hold

i) if \(p \leq 2\),

\[
|\zeta_2|^p - |\zeta_1|^p - p|\zeta_1|^{p-2}\langle \zeta_1, \zeta_2 - \zeta_1 \rangle \geq c(p) \frac{|\zeta_2 - \zeta_1|^2}{(|\zeta_1| + |\zeta_2|)^{2-p}} \tag{2.1}
\]

ii) if \(p > 2\),

\[
|\zeta_2|^p - |\zeta_1|^p - p|\zeta_1|^{p-2}\langle \zeta_1, \zeta_2 - \zeta_1 \rangle \geq c(p)|\zeta_2 - \zeta_1|^p \tag{2.2}
\]

Direct calculations give the following results:

**Lemma 2.2.** Assume \(h\) satisfies (1.11). If \((A_1)\) or \((A_2)\) happens, then

\[
\text{div} \left( \phi h^{\alpha}(-h')^{p-1} \frac{X}{|X|} \right) = (1 - \alpha) \phi h^{\alpha-1}(-h')^p \tag{2.3}
\]

**Lemma 2.3.** Let \(h = \left( c_0 \int_r^{\infty} (\phi r^{N-1})^{1/(p-1)} \frac{dr}{(p-1)^p} \right)^{(p-1)/p} \). Then

i) the function \(h\) satisfies the Euler-Lagrange equation

\[- \text{div}(\phi |\nabla h|^{p-2} \nabla h) = \psi h^{p-1}, \quad x \in \mathbb{R}^N \setminus \{0\}\]

and in weak sense,

\[
\int_{\mathbb{R}^N} \phi |\nabla h|^{p-2} \nabla h \nabla \zeta \, dx = \int_{\mathbb{R}^N} \psi h^{p-1} \zeta \, dx, \quad \zeta \in C_0^\infty(\mathbb{R}^N)
\]

where \(\psi = \phi \left( -\frac{h'}{p} \right)^p\);
ii) the function
\[ \tilde{h} = h^{p/(p-1)} = c_0 \int_r^\infty (\phi r^{N-1})^{-1/(p-1)} \, dr \]
satisfies in the sense of distribution
\[ -\text{div}(\phi |\nabla \tilde{h}|^{p-2} \nabla \tilde{h}) = \left( \frac{p}{p-1} \right)^{p-1} \omega_N \delta(x) \]
where \( \delta(x) \) is the Dirac measure and \( \omega_N \) denotes the volume of the unit ball in \( \mathbb{R}^N \). In other words, \( \tilde{h} \) is a fundamental solution for operator \(-\Delta_{\phi,p}\) defined as before.

**Corollary 2.4.** Under the hypothesis of Theorem 1.2, if \( \alpha > 0 \), then for any \( u \in W^{1,p}_0(\Omega, \phi) \),
\[ \int_\Omega \psi h_1^{-\alpha} |u|^p \, dx \leq \int_\Omega \phi h_1^{-\alpha} |\nabla u|^p \, dx \]

**Proof.** Assume \((A_1)\) holds. Set \( \tilde{\phi} = \phi h_1^{-\alpha} \), then
\[ \frac{\tilde{h}'}{h} = \frac{p-1}{p} \frac{\phi h_1^{-\alpha} r^{N-1}}{\int_r^D (\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)} \, dr} \]

By Theorem 1.1, we have
\[ \int_\Omega \tilde{\psi} |u|^p \, dx \leq \int_\Omega \tilde{\phi} |\nabla u|^p \, dx \]
where \( \tilde{\psi} = \phi (-\frac{\tilde{h}'}{h})^p \). We claim that
\[ \psi h_1^{-\alpha} \leq \tilde{\psi} \]
that is
\[ \phi h_1^{-\alpha} (-\frac{\tilde{h}'}{h})^p \leq \phi h_1^{-\alpha} (-\frac{\tilde{h}'}{h})^p \]
and this complete the proof. In the following we prove this claim. Since \( h_1 \) is decreasing, we have
\[ \int_r^D (\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)} \, dr \leq h_1^{\alpha/(p-1)} \int_r^D (\phi r^{N-1})^{-1/(p-1)} \, dr \]
Multiplying by \((\phi r^{N-1})^{-1/(p-1)}\), we obtain
\[ \frac{(\phi r^{N-1})^{-1/(p-1)}}{\int_r^D (\phi r^{N-1})^{-1/(p-1)} \, dr} \leq \frac{(\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)}}{\int_r^D (\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)} \, dr} \]
Hence

$$- \frac{h'}{h} \leq - \frac{\bar{h}'}{\bar{h}}$$

that is, the claim is true.

## 3 Proof of Theorem

**Proof of Theorem 1.2 (1).** We proceed to make use of a suitable vector field as in [2]. Define a vector field as follows

$$T = \phi \left( - \frac{h'}{h} \right)^{p-1} (1 + c_0^{-1} \eta + a \eta^2) \nabla r$$

where $a$ is a free parameter to be chosen later and $\eta = h_1^{-1}$. By Lemma 2.1, we have

$$\text{div} T \geq \phi \left( - \frac{h'}{h} \right)^p \left[ (p + pc_0^{-1} \eta + ap \eta^2) + \frac{p \eta^2}{(p-1)c_0} + \frac{2ap \eta^3}{(p-1)c_0} \right] \tag{3.1}$$

Next we compute $(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)}$. We set for convenience

$$g(\eta) = (1 + c_0^{-1} \eta + a \eta^2)^{p/(p-1)}$$

When $\eta > 0$ is small, the Taylor expansion of $g(\eta)$ about $\eta = 0$ gives

$$g(\eta) = 1 + \frac{p}{(p-1)c_0} \eta + \frac{1}{2} \left( \frac{p}{(p-1)^2c_0^2} + \frac{2pa}{p-1} \right) \eta^2$$

$$+ \frac{1}{6} \left( \frac{p(2-p)}{(p-1)^3c_0^3} + \frac{6pa}{(p-1)^2c_0} \right) \eta^3 + O(\eta^4)$$

and so

$$(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} = \phi \left( - \frac{h'}{h} \right)^p \left[ (p-1) + \frac{p}{c_0} \eta \right.$$

$$+ \left( \frac{p}{2(p-1)c_0^2} + pa \right) \eta^2 + \left( \frac{p(2-p)}{(p-1)^2c_0^2} + \frac{pa}{(p-1)c_0} \right) \eta^3 + O(\eta^4) \right]$$

Hence

$$\text{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)}$$

$$\geq \phi \left( - \frac{h'}{h} \right)^p \left[ 1 + \frac{p \eta^2}{2(p-1)c_0^2} + \left( \frac{pa}{(p-1)c_0} - \frac{p(2-p)}{(p-1)^2c_0} \right) \eta^3 + O(\eta^4) \right]$$
Yaotian Shen and Zhihui Chen

If we show
\[ \frac{ap}{(p-1)c_0} \geq \frac{p(2-p)}{(p-1)^2 c_0^2} + O(\eta) \quad (3.2) \]
then we obtain
\[ \text{div } T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \geq \phi \left( -\frac{h'}{h} \right)^p \left[ 1 + \frac{p\eta^2}{2(p-1)c_0^2} \right] \quad (3.3) \]

If $1 < p < 2$, we assume that $\eta$ is small for the case (A$_1$). Since
\[ h_1 = \frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)} \]
and $\Omega \subset B_{D_0}(0)$ is bounded, we can choose $D_0$ large enough such that $h_1^{-1}(D_0)$ is small enough. Then $\eta = h_1^{-1}$ is small. Hence, we have (3.2) for a big enough.

The same argument gives (3.2) for the case (A$_2$).

If $p \geq 2$, we choose $a = 0$, then
\[ (1 + c_0^{-1}\eta)^{\frac{p}{p-1}} = 1 + \frac{p}{(p-1)c_0} \eta + \frac{p}{2(p-1)^2 c_0^2} \eta^2 + \frac{p(2-p)}{6(p-1)^3 c_0^3} (1 + c_0^{-1}\eta)^{\frac{3-2p}{p-1}} \eta^3 \]
for some $\xi \in (0, \eta)$, without any smallness assumption. Since $2 - p \leq 0$, we have
\[ (1 + c_0^{-1}\eta)^{\frac{p}{p-1}} \leq 1 + \frac{p}{(p-1)c_0} \eta + \frac{p}{2(p-1)^2 c_0^2} \eta^2 \]
Hence we prove (3.3).

Let $u \in C_0^\infty(\Omega)$. For $\epsilon > 0$, it follows from integration by parts that
\[ \int_{\Omega \setminus B_\epsilon(0)} |u|^p \text{div } T \, dx = -p \int_{\Omega \setminus B_\epsilon(0)} (T \cdot \nabla u)|u|^{p-2}u \, dx - \int_{\partial B_\epsilon(0)} |u|^p T \cdot \nabla r \, dS \]
Note that
\[ \phi \left( -\frac{h'}{h} \right)^p = r^{-(N-1)} \left( \int_r^a (\phi r^{N-1})^{-1/(p-1)} \, dr \right)^{-(p-1)} = r^{-(N-1)} h^{-p/(p-1)^2} (r) \]
then
\[ \left| \int_{\partial B_\epsilon(0)} |u|^p T \cdot \nabla r \, dS \right| \leq \int_{\partial B_\epsilon(0)} |u|^p r^{-(N-1)} h^{-p/(p-1)^2} (\epsilon) \, dS \]
which tends to 0 as $\epsilon \to 0$ since $h^{-1}(0) = 0$. Hence we obtain
\[ \int_{\Omega} |u|^p \text{div } T \, dx = -p \int_{\Omega} (T \cdot \nabla u)|u|^{p-2}u \, dx \]
By Hölder’s inequality and Young’s inequality, we have

\[
\int_{\Omega} |u|^p \text{div} T \, dx \leq p \left( \int_{\Omega} \phi |\nabla u|^p \, dx \right)^{1/p} \left( \int_{\Omega} |T \phi^{-1/p} p/(p-1)|u|^p \, dx \right)^{(p-1)/p}
\]

\[
\leq \int_{\Omega} \phi |\nabla u|^p \, dx + (p-1) \int_{\Omega} |T \phi^{-1/p} p/(p-1)|u|^p \, dx
\]

that is,

\[
\int_{\Omega} \phi |\nabla u|^p \, dx \geq \int_{\Omega} (\text{div} T - (p-1)|T \phi^{-1/p} p/(p-1)|) |u|^p \, dx
\]

This complete the proof by (3.3).

\[\Box\]

**Proof of Theorem 1.2 (2).** We complete the proof by four steps.

**Step 1.** Let \( \theta \in C_0^\infty(B_\delta) \) be such that \( 0 \leq \theta \leq 1 \) in \( B_\delta \) and \( \theta = 1 \) in \( B_{\delta/2} \), where \( B_\delta \) denotes the ball of radius \( \delta \) centered at the origin. We fix small positive parameters \( \alpha_0, \alpha_1 \) and define the functions

\[ w(x) = h^{1-\alpha_0/(p-1)} h_1^{1-\alpha_1/p} \]

and

\[ u(x) = \theta(x) w(x) \]

Let (A1) or (A2) happen. Hence \( u \in W_0^{1,p}(\Omega, \phi) \). To prove the proposition we shall estimate the corresponding Rayleigh quotient of \( u \) in the limit of the order \( \alpha_0 \to 0, \alpha_1 \to 0 \).

It is easily seen that

\[ \nabla w = \frac{p}{(p-1)c_0} h^{-\alpha_0/(p-1)} h_1^{-1+\alpha_1} \left( \frac{(p-1)c_0}{p} + \frac{\eta}{p} \right) \nabla r \]

where \( Y_1 = h_1^{-1} \) and \( \eta = -\alpha_0 + (1 - \alpha_1)Y_1 \).

Now \( \nabla u = \theta \nabla w + w \nabla \theta \) and hence, using the elementary inequality

\[ |a + b|^p \leq |a|^p + c_p(|a|^{p-1}|b| + |b|^p), \quad a, b \in \mathbb{R}^N \]
for $p > 1$, we obtain
\[
\int_{\Omega} \phi |\nabla u|^p \, dx \leq \int_{\Omega} \phi |\nabla w|^p \, dx + c_p \int_{\Omega} \phi |\nabla \theta |^p |w|^{p-1} \, dx + c_p \int_{\Omega} \phi |\nabla \theta |^p |w| \, dx
\]
(3.4)
\[
=: I_1 + I_2 + I_3
\]
(3.5)

We claim that
\[I_2, I_3 = O(1)\quad \text{uniformly as} \quad \alpha_0, \alpha_1 \text{ tend to zero.} \quad (3.6)\]

Let us give the proof for $I_2$. In fact,
\[
I_2 \leq C \int_{B_\delta} \phi h^{1-\frac{\alpha_0}{(p-1)c_0}} |h'|^{p-1} Y_{1}^{1+\alpha_1} [(p-1)c_0 + \alpha_0 + (1 - \alpha_1)Y_{1}]^{p-1} \]
\[
\cdot h^{1-\frac{\alpha_0}{(p-1)c_0}} Y_{1}^{1+\alpha_1} \, dx
\]
\[
\leq C \int_{B_\delta} \phi h^{1-\frac{\alpha_0}{(p-1)c_0}} |h'|^{p-1} Y_{1}^{1+\alpha_1} [(p-1)c_0 + \alpha_0 + (1 - \alpha_1)Y_{1}]^{p-1} \, dx
\]

It follows from the definition of $h$ (1.11) that
\[
\phi |h'|^{p-1} h = C r^{1-N} \quad (3.7)
\]
hence
\[
I_2 \leq C \int_{B_\delta} r^{1-N} h^{1-\frac{\alpha_0}{(p-1)c_0}} Y_{1}^{1+\alpha_1} [(p-1)c_0 + \alpha_0 + (1 - \alpha_1)Y_{1}]^{p-1} \, dx
\]

Then the boundedness of $h^{-1}$ together with the fact $Y_1(0) = 0$ implies that $I_2$ is uniformly bounded. The integral $I_3$ is treated similarly.

**Step 2.** Define
\[
A_0 = \int_{\Omega} \theta^p \phi h^{-\frac{\alpha_0 p}{(p-1)c_0}} (-h')^p Y_{1}^{-1+\alpha_1} \, dx
\]
\[
A_1 = \int_{\Omega} \theta^p \phi h^{-\frac{\alpha_0 p}{(p-1)c_0}} (-h')^p Y_{1}^{1+\alpha_1} \, dx
\]
\[
\Gamma_{01} = \int_{\Omega} \theta^p \phi h^{-\frac{\alpha_0 p}{(p-1)c_0}} (-h')^p Y_{1}^{\alpha_1} \, dx
\]
By Lemma 2.1, we have
\[ \phi h^{-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^p = \frac{(p-1)c_0}{p\alpha_0} \text{div}(\phi h^{1-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^{p-1} \nabla r) \]

Multiplying the above equality by \( \theta^p Y_1^{1+\alpha_1} \) and integrating over \( \Omega \), we obtain
\[
A_0 = \frac{(p-1)c_0}{p\alpha_0} \int_{\Omega} \theta^p Y_1^{1+\alpha_1} \text{div}(\phi h^{1-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^{p-1} \nabla r) \, dx
\]
\[
= \frac{(p-1)c_0}{p\alpha_0} \int_{\Omega} \phi h^{1-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^{p-1} \nabla (\theta^p Y_1^{1+\alpha_1}) \, dx
\]
\[
= \frac{(p-1)c_0}{p\alpha_0} \left( - \frac{p(1-\alpha_1)}{(p-1)c_0} \int_{\Omega} \theta^p \phi h^{1-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^{p-1} Y_1^{\alpha_1} \, dx \right.
\]
\[
+ \int_{\Omega} (\theta^p)' h^{1-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^{p-1} Y_1^{1+\alpha} \, dx \right)
\]
\[
= (1-\alpha_1)\Gamma_{01} + O(1)
\]

**Step 3.** We proceed to estimate \( I_1 \).
\[
I_1 = \int_{\Omega} \phi \theta^p |\nabla w|^p \, dx
\]
\[
\leq \left( \frac{p}{(p-1)c_0} \right)^p \int_{\Omega} \theta^p \phi h^{-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^p Y_1^{-1+\alpha_1} \left( \frac{(p-1)c_0}{p} + \frac{\eta}{p} \right)^p \, dx
\]
where \( \eta = -\alpha_0 + (1-\alpha_1)Y_1 \). Since \( \eta \) is small compared to \( (p-1)c_0/p \), we may use Taylor’s expansion to obtain
\[
\left( \frac{(p-1)c_0}{p} + \frac{\eta}{p} \right)^p \leq \left( \frac{(p-1)c_0}{p} \right)^p + \left( \frac{(p-1)c_0}{p} \right)^{p-1} \eta
\]
\[
+ \frac{p-1}{2p} \left( \frac{(p-1)c_0}{p} \right)^{p-2} \eta^2 + C\eta^3
\]
Using this inequality we can obtain
\[ I_1 \leq I_{10} + I_{11} + I_{12} + I_{13} \quad (3.8) \]
where
\[
I_{10} = \int_{\Omega} \theta^p \phi h^{-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^p Y_1^{-1+\alpha_1} \, dx = \int_{\Omega} \theta^p \phi h^{-\frac{\alpha p}{(p-1)\gamma_0}} Y_1^{-1+\alpha_1} \, dx
\]
\[
= \int_{\Omega} \theta^p \psi |w|^p \, dx = \int_{\Omega} \psi |w|^p \, dx \quad (3.9)
\]
\[
I_{12} = \frac{p}{2(p-1)c_0^2} \int_{\Omega} \theta^p \phi h^{-\frac{\alpha p}{(p-1)\gamma_0}} (-h')^p Y_1^{-1+\alpha_1} \eta^2 \, dx \quad (3.10)
\]
We shall prove that
\[ I_{11}, I_{13} = O(1) \] uniformly in \( \alpha_0, \alpha_1 \). \hfill (3.11)

Firstly,
\[
I_{11} = \frac{p}{(p-1)c_0} \left[ -\alpha_0 \int_\Omega \theta^p \phi (-h')^p h^{\frac{\alpha_0 p}{p-1}} Y_1^{-1+\alpha_1} \, dx \\
+ (1 - \alpha_1) \int_\Omega \theta^p \phi (-h')^p h^{\frac{-\alpha p}{p-1}} Y_1^{\alpha_1} \, dx \right] + O(1)
\]
\[
= \frac{p}{(p-1)c_0} ( -\alpha_0 A_0 + (1 - \alpha_1) \Gamma_{01} ) + O(1)
\]

Next we estimate \( I_{13} \).
\[
I_{13} \leq \alpha_0^3 \int_\Omega \theta^p \phi (-h')^p h^{\frac{-\alpha p}{p-1}} Y_1^{-1+\alpha_1} \, dx + C \int_\Omega \theta^p \phi (-h')^p h^{\frac{-\alpha p}{p-1}} Y_1^{2+\alpha_1} \, dx
\]
\[
=: I_{13}' + I_{13}''
\]

Since
\[
Y_1^{-1} = \frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)}
\]
we have
\[
I_{13}' \leq C \alpha_0^3 \int_0^\delta \left( \int_r^\infty (\phi r^{k-1})^{-1/(p-1)} \, dr \right)^{1-\alpha_0/c_0} \left( \ln \frac{h(r)}{h(D)} \right)^2 \, dr
\]
\[
\leq C \alpha_0^2 c_0 \int_0^\delta \left( \ln \frac{h(r)}{h(D)} \right)^2 \, dr \left( \int_r^\infty (\phi r^{k-1})^{-1/(p-1)} \, dr \right)^{-\alpha_0/c_0}
\]

Denote
\[
s = \left( \int_r^\infty (\phi r^{k-1})^{-1/(p-1)} \, dr \right)^{-\alpha_0/c_0}
\]

then we have
\[
I_{13}' \leq C \alpha_0^2 \int_0^\delta \left[ C - \frac{(p-1)c_0}{p\alpha_0} \ln s \right]^2 \, ds \leq O(1)
\]

The same argument gives \( I_{13}'' = O(1) \) uniformly in \( \alpha_0 \) and \( \alpha_1 \). Hence, by (3.4), (3.6), (3.8), (3.9) and (3.11), we conclude that
\[
\int_\Omega \phi |\nabla u|^p \, dx - \int_\Omega \psi |u|^p \, dx \leq I_{12} + O(1)
\]
\hfill (3.12)
uniformly in \( \alpha_0 \) and \( \alpha_1 \).

**Step 4.** We proceed to estimate \( I_{12} \) and complete the proof.

\[
I_{12} = \frac{p}{2(p-1)c_0^2} \int_{\Omega} \theta^p \phi h^{-\frac{\alpha_0 p}{\sigma-1/c_0}} (-h')^p Y_{1}^{-1+\alpha_1} \left( \alpha_0^2 + (1 - \alpha_1)^2 Y_{1}^2 - 2\alpha_0(1 - \alpha_1)Y_{1} \right) \, dx
\]
\[
= \frac{p}{2(p-1)c_0^2} \left( \alpha_0^2 A_{0} - 2\alpha_0(1 - \alpha_1)\Gamma_{01} + (1 - \alpha_1)^2 A_{1} \right)
\]
\[
= \frac{p}{2(p-1)c_0^2} A_{1} + O(1)
\]

if \( \alpha_0 \) and \( \alpha_1 \) tend to 0. Because

\[
\phi(-h')^p h^{-\frac{\alpha_0 p}{\sigma-1/c_0}} = \left( c_0 \int_{r} (\phi_r N^{-1})^{-1/(p-1)} \, dr \right)^{-\alpha_0/c_0 - 1} \cdot c_0 \phi(\phi_r N^{-1})^{-1/(p-1)}
\]

we have

\[
A_{1} \geq C \int_{0}^{\delta/2} \left( \int_{r} (\phi_r N^{-1})^{-1/(p-1)} \, dr \right)^{-\alpha_0/c_0 - 1} \cdot c_0 (\phi_r N^{-1})^{-1/(p-1)} h_{1}^{-1-\alpha_0} \, dr
\]
\[
\geq C \left( \int_{0}^{\delta/2} (\phi_r N^{-1})^{-1/(p-1)} \, dr \right)^{-\alpha_0/c_0} \frac{\delta/2}{0}
\]
\[
= C \frac{c_0}{\alpha_0} \left( \int_{0}^{a} (\phi_r N^{-1})^{-1/(p-1)} \, dr \right)^{-\alpha_0/c_0} \rightarrow \infty
\]
as \( \alpha_0 \) tends to 0. Since

\[
\int_{\Omega} \psi h_{1}^{-2}|u|^p \, dx = \int_{\Omega} \phi \left( \frac{-h'}{h} \right)^p h_{1}^{-2} \theta^p p^{-\frac{\alpha_0 p}{\sigma-1/c_0}} h_{1}^{-1-\alpha_1} \, dx
\]
\[
= \int_{\Omega} \theta^p \phi(-h')^p h^{-\frac{\alpha_0 p}{\sigma-1/c_0}} h_{1}^{-1-\alpha_1} \, dx = A_{1}
\]

by (3.12) and (3.13), we have

\[
\frac{\int_{\Omega} \phi |\nabla u|^p - \psi |u|^p \, dx}{\int_{\Omega} \psi h_{1}^{-2}|u|^p \, dx} \leq \frac{p}{2(p-1)c_0^2} A_{1} + O(1) \rightarrow \frac{p}{2(p-1)c_0^2} A_{1}
\]
as \( \alpha_0 \) tends to 0. This completes the proof.

*The proof of Theorem 1.3.* i) Assume \( (A_1) \) holds. Consider the case of \( p \geq 2 \).
Let $u \in C_0^\infty(\Omega)$ and set $v = u(x)/h(r)$. Then by Lemma 2.1 (2.2), we have
\[
\int_\Omega \phi |\nabla u|^p \, dx = \int_\Omega \phi \left| \frac{vh'}{|x|} + h \nabla v \right|^p \, dx \\
\geq \int_\Omega \phi |v|^p |h'|^p \, dx - p \int_\Omega \phi |vh'|^{p-2}(vh') \frac{x}{|x|} h \nabla v \, dx \\
+ c(p) \int_\Omega \phi h^p \left| \nabla v \right|^p \, dx
\]

Note that
\[
\int_\Omega \phi |v|^p |h'|^p \, dx = \int_\Omega \psi |u|^p \, dx
\]
and for any $\epsilon > 0$, by Lemma 2.2, (3.7) and (A_1) (or (A_2)), we have
\[
- \int_{\Omega \setminus B_\epsilon(0)} \phi |vh'|^{p-2} \, dx \\
= \int_{\Omega \setminus B_\epsilon(0)} \phi h(-h')^{p-1} \left( \frac{x}{|x|} \right) h \nabla |v|^p \, dx \\
= \int_{\partial B_\epsilon(0)} \phi (-h')^{p-1} |v|^p \, dS - \int_{\Omega \setminus B_\epsilon(0)} |v|^p \text{div}(\phi h(-h')^{p-2} \nabla h) \\
= \int_{\partial B_\epsilon(0)} \phi (-h')^{p-1} |v|^p \, dS \to 0
\]
as $\epsilon \to 0$. Hence, we obtain
\[
I_{1,\phi}(u) = \int_\Omega (\phi |\nabla u|^p - \psi |u|^p) \, dx \geq c(p) \int_\Omega \phi h^p |\nabla v|^p \, dx \quad (3.14)
\]
Taking $C_1 > 0$ such that $C_1 h_1^{-2} \leq c(p)$, it follows from (2.2) of Lemma 2.1 that
\[
c(p) \int_\Omega \phi h^p |\nabla v|^p \, dx \geq C_1 \int_\Omega \phi h^p h_1^{-2} |\nabla v|^p \, dx \\
\geq C_1 \int_\Omega \phi h_1^{-2} \left[ \frac{\nabla h_1}{h_1} |u|^p - p \left( \frac{\nabla h}{h} u |u|^{p-1} \right) + c(p) |\nabla u|^p \right] \, dx \\
\geq C_1 \int_\Omega \phi h_1^{-2} \left[ (c(p) - \epsilon) |\nabla u|^p - ((p - 1)\epsilon^{-1/(p-1)} - 1) \left( - \frac{h'}{h} \right)^p |u|^p \right] \, dx
\]
Taking $\epsilon = c(p)/2$, then by Theorem 1.2, we obtain
\[
I_{1,\phi}(u) \geq C \int_\Omega \phi h_1^{-2} |\nabla u|^p \, dx
\]
Now let \( 1 < p < 2 \). By using Lemma 2.1, (2.1) and arguments analogous to the case of \( p \geq 2 \), we have

\[
\int_\Omega (\phi |\nabla u|^p - \psi |u|^p) \, dx \geq c(p) \int_\Omega \frac{\phi |\nabla u - \frac{\nabla h}{h} u|^p}{(|\nabla u| + |u| |\frac{h'}{h}|)^{2-p}} \, dx 
\]

\[
\geq c(p) \int_\Omega \frac{\phi h_1^{-2} |\nabla u - \frac{\nabla h}{h} u|^2}{(|\nabla u| + |u| |\frac{h'}{h}|)^{2-p}} \, dx
\]

By Hölder’s inequality and Corollary 2.4, we have

\[
\int_\Omega \phi h_1^{-2} |\nabla u - \frac{\nabla h}{h} u|^p \, dx \leq \left( \int_\Omega \phi h_1^{-2} |\nabla u - \frac{\nabla h}{h} u|^2 \, dx \right)^{p/2} \left( \int_\Omega \phi h_1^{-2} (|\nabla u| + |\frac{h'}{h}| |u|)^p \, dx \right)^{1-p/2}
\]

\[
\leq C(I_{1,\phi}(u))^{p/2} \left( \int_\Omega \phi h_1^{-2} |\nabla u|^p \, dx \right)^{1-p/2}
\]

Note that

\[
\int_\Omega \phi h_1^{-2} |\nabla u|^p \, dx \leq C \left( \int_\Omega \phi h_1^{-2} |\nabla u - \frac{\nabla h}{h} u|^p \, dx + \int_\Omega \phi h_1^{-2} \frac{\nabla h}{h} |u|^p \, dx \right)
\]

\[
\leq C(I_1,\phi(u))^{p/2} \left( \int_\Omega \phi h_1^{-2} |\nabla u|^p \, dx \right)^{1-p/2} + \int_\Omega \psi h_1^{-2} |u|^p \, dx
\]

\[
\leq C(I_1,\phi(u))^{p/2} \left( \int_\Omega \phi h_1^{-2} |\nabla u|^p \, dx \right)^{1-p/2} + I_{1,\phi}(u)
\]

By Young’s inequality, we obtain

\[
\int_\Omega \phi h_1^{-2} |\nabla u|^p \, dx \leq CI_{1,\phi}(u)
\]

One can prove the result for the case of \((A_2)\) by the analogous argument.

ii) Let \( w \) and \( u \) be as those defined in the proof of Theorem 1.2 (2), and let \( p > 2 \). First, it follows from (3.12) and (3.13) that

\[
\int_\Omega \phi |\nabla u|^p \, dx - \int_\Omega \psi |u|^p \, dx \leq \frac{\theta p}{2(p-1)c_0^2} A_1 + O(1)
\]

By (2.2) we have

\[
|\nabla u|^p = |\theta \nabla w + w \nabla \theta|^p \geq \theta^p |\nabla w|^p - p\theta^{p-1} |\nabla w|^{p-1} \nabla \theta |w + c(p)|w|^p |\nabla \theta|^p
\]
Hence
\[
\int_{\Omega} \phi h_1^{-2} g(x)|\nabla u|^p \, dx \geq \min_{x \in B_{\delta}(0)} g(x) \int_{B_{\delta}(0)} \phi h_1^{-2} \theta^p |\nabla w|^p \, dx + \\
c(p) \int_{B_{\delta}(0)} \phi h_1^{-2} g(x)|w|^p |\nabla \theta|^p \, dx - p \int_{B_{\delta}(0)} \phi h_1^{-2} g(x) \theta^{p-1} |\nabla \theta||\nabla w|^{p-1} |w| \, dx
\]

Analogues to the argument of Step 3 for the proof of Theorem 1.2 (2), we obtain
\[
\int_{B_{\delta}(0)} \phi h_1^{-2} \theta^p |\nabla w|^p \, dx = \int_{B_{\delta}(0)} \phi h_1^{-2} \theta^p h^{-\frac{\alpha p}{(p-1)p_0}} (-h')^p Y_1^{-1+\alpha_1} \left( \frac{(p-1)c_0}{p} + \frac{\eta}{p} \right)^p \, dx
\]
where \( \eta = -\alpha_0 + (1-\alpha_1)Y_1 \). Because of \( p > 2 \), we have
\[
\left( \frac{(p-1)c_0}{p} + \frac{\eta}{p} \right)^p = \left( \frac{(p-1)c_0 - \alpha_0}{p} + \frac{\eta + \alpha_0}{p} \right)^p \\
\geq \left( \frac{(p-1)c_0 - \alpha_0}{p} \right)^p + \left( \frac{(p-1)c_0 - \alpha_0}{p} \right)^{p-1} (\eta + \alpha_0)
\]
Hence
\[
\int_{B_{\delta}(0)} \phi h_1^{-2} \theta^p |\nabla w|^p \, dx \geq \left( \frac{(p-1)c_0 - \alpha_0}{p} \right)^p \int_{B_{\delta}(0)} \phi \theta^p h^{-\frac{\alpha p}{(p-1)p_0}} (-h')^p Y_1^{1+\alpha_1} \, dx \\
+ \left( \frac{(p-1)c_0 - \alpha_0}{p} \right)^{p-1} (1-\alpha_1) \int_{B_{\delta}(0)} \phi \theta^p h^{-\frac{\alpha p}{(p-1)p_0}} (-h')^p Y_1^{2+\alpha_1} \, dx
\]
\[
=: J_1 + J_2
\]
By Step 3 of the proof of Theorem 1.2 (2) we know
\[
J_1 = \left( \frac{(p-1)c_0 - \alpha_0}{p} \right)^p A_1, \quad J_2 = O(1)
\]
if \( \alpha_0, \alpha_1 \) tend to 0. Next, we will estimate
\[
J_3 := \int_{B_{\delta}(0)} \phi h_1^{-2} g(x)\theta^{p-1} |\nabla \theta||\nabla w|^{p-1} |w| \, dx
\]
In fact,
\[
J_3 \leq C \int_{B_{\delta}(0)} g(x)\phi h_1^{-2} h^{1-\frac{\alpha p}{(p-1)p_0}} (-h')^p Y_1^{-1+\alpha_1} [(p-1)c_0 - \alpha_0 + (1-\alpha_1)Y_1]^{p-1} \, dx \\
\leq \int_{B_{\delta}(0)} g(x)\phi h_1^{-2} h^{1-\frac{\alpha p}{(p-1)p_0}} (-h')^p \left[ ((p-1)c_0 - \alpha_0)^{p-1} Y_1^{1+\alpha_1} + (1-\alpha_1)^{p-1} Y_1^{p+\alpha_1} \right] \, dx
\]
It follows from (3.7) that

$$J_3 \leq \int_0^\delta \tilde{g}(r) h_1^{p-\alpha_0} \left[ ((p-1)c_0 - \alpha_0)^{p-1}Y_1^{1+\alpha_1} + (1 - \alpha_1)^{p-1}Y_1^{p+\alpha_1} \right] dr$$

Set

$$\tilde{g}(r) = \frac{1}{N\omega_N} \int_{|\omega|=1} g(r\omega) d\omega$$

and we may assume

$$|\tilde{g}(r)h_1^{-1}(r)| \leq C$$

Then we obtain

$$J_3 \leq C \int_0^\delta h^{-\alpha_0} \left[ ((p-1)c_0 - \alpha_0)^{p-1}Y_1^{\alpha_1} + (1 - \alpha_1)^{p-1}Y_1^{p-1+\alpha_1} \right] dr \leq C$$

Hence

$$\frac{\int_\Omega (\phi|\nabla u|^p - \psi|u|^p) dx}{\int_\Omega \phi h_1^{-2}g(x)|\nabla u|^p dx} \leq \frac{\frac{p}{2(p-1)c_0} A_1 + O(1)}{\min_{x \in B_1(0)} g(x) \left( \frac{(p-1)c_0 - \alpha_0}{p} \right)^p A_1 + O(1)} \to 0$$

as $\delta \to 0$ since $A_1 \to \infty$ as $\alpha_0, \alpha_1 \to 0$ and $g(x) \to \infty$ as $x \to 0$.

We can prove our result for the case of $1 < p < 2$ by the similar argument. \qed

References


Yaotian Shen  
Department of Mathematics  
South China University of Technology  
Guangzhou 510640, China  
E-mail: maytshen@scut.edu.cn

Zhihui Chen  
Department of Mathematics  
South China University of Technology  
Guangzhou 510640, China  
E-mail: mazhchen@scut.edu.cn