Modeling Oscillatory Components with The Homogeneous Spaces $BMO^{-\alpha}$ and $\dot{W}^{-\alpha,p}$

John B. Garnett, Peter W. Jones, Triet M. Le and Luminita A. Vese

Abstract: This paper is devoted to the decomposition of an image $f$ into $u + v$, with $u$ a piecewise-smooth or “cartoon” component, and $v$ an oscillatory component (texture or noise), in a variational approach. The cartoon component $u$ is modeled by a function of bounded variation, while $v$, usually represented by a square integrable function, is now being modeled by a more refined and weaker texture norm, as a distribution. Generalizing the idea of Y. Meyer [34], where $v \in F = \text{div}(BMO) = BMO^{-1}$, we model here the texture component by the action of the Riesz potentials on $v$ that belongs to $BMO$ or to $L^p$. In an earlier work [28], the authors proposed energy minimization models to approximate $(BV, F)$ decompositions explicitly expressing the texture as divergence of vector fields in $BMO$. In this paper, we consider an equivalent more isotropic norm of the space $F$ in terms of the Riesz potentials, and study models where the Riesz potentials of oscillatory components belong to $BMO$ or to $L^p$, $1 \leq p < \infty$ (thus we consider oscillatory components in $BMO^\alpha$ or in $\dot{W}^{\alpha,p}$, with $\alpha < 0$). Theoretical, experimental results and comparisons to validate the proposed methods are...
Keywords: image decomposition, texture modeling, bounded mean oscillation, bounded variation, Sobolev spaces, Riesz potentials.

1 Introduction and motivations

We assume that a given grayscale image \( f \) is defined on \( \mathbb{R}^n \) or on \( \Omega = [0,1]^n \subset \mathbb{R}^n \), with \( \Omega = (0,1)^n \). When \( f \) is defined on \( \Omega \), we assume that \( f \) is periodic and \( \Omega \) is the fundamental domain (or we can also assume that \( f \) is extended by zero outside \( \Omega \)).

An important problem in image analysis is the decomposition of \( f \) into \( u + v \), where \( u \) is piecewise-smooth containing the geometric components of \( f \) and \( v \) is oscillatory, typically texture or noise. A general variational method for decomposing \( f \in X_1 + X_2 \) into \( u + v \), with \( u \in X_1 \) and \( v \in X_2 \), can be defined by the minimization problem

\[
\inf_{(u,v)\in X_1 \times X_2} \{ F_1(u) + \lambda F_2(v) : f = u + v \},
\]

where \( F_1, F_2 \geq 0 \) are functionals and \( X_1, X_2 \) are spaces of functions or distributions such that \( F_1(u) < \infty \) and \( F_2(v) < \infty \), if and only if \( (u,v) \in X_1 \times X_2 \). The constant \( \lambda > 0 \) is a tuning parameter. A good model for (1) is given by a choice of \( X_1 \) and \( X_2 \) so that with the given desired properties of \( u \) and \( v \), we obtain \( F_1(u) << F_1(v) \) and \( F_2(v) << F_2(u) \).

In standard approaches, the space \( L^2 \) is used to model \( v \) when \( f \) denotes the image of a real scene, \( u \) is a piecewise-smooth approximation of \( f \) (made up of homogeneous regions with sharp boundaries), and \( v \) is a residual (additive Gaussian noise or small details). For example, in the Mumford and Shah model [37] for two dimensional image segmentation, \( f \in L^\infty(\Omega) \subset L^2(\Omega) \) is split into \( u \in SBV(\Omega) \) [4], a piecewise-smooth function with its discontinuity set \( J_u \) composed of a union of curves of total finite length, and \( v = f - u \in L^2(\Omega) \) representing noise or texture. The (non-convex) model in the weak formulation is [35]

\[
\inf_{(u,v)\in SBV(\Omega) \times L^2(\Omega)} \left\{ \int_\Omega |\nabla u|^2dx + \alpha \mathcal{H}^1(J_u) + \beta \|v\|_{L^2(\Omega)}^2 : f = u + v \right\},
\]
Modeling Oscillatory Components...

where $H^1$ denotes the 1-dimensional Hausdorff measure, and $\alpha, \beta > 0$ are tuning parameters. With the above notations, the first two terms in the energy from (2) compose $F_1(u)$, while the third term makes $F_2(v)$. A related decomposition is obtained by the total variation minimization model of Rudin, Osher, and Fatemi [41] for image denoising. The (convex) decomposition model is

$$\inf_{(u,v) \in BV(\Omega) \times L^2(\Omega)} \left\{ |u|_{BV(\Omega)} + \lambda \|v\|_{L^2(\Omega)}^2, \ f = u + v \right\}, \quad (3)$$

where $|u|_{BV(\Omega)} = \int_\Omega |Du|$ (the semi-norm on the space $BV$) [16], and $\lambda > 0$ is a tuning parameter. This model is strictly convex and is easily solved in practice. However, it has some limitations pointed out by several authors ([46], [47], [34] among others). If $f = \alpha \chi_D$ is a multiple of the characteristic function of a disk $D$ centered at the origin and of radius $R$, we would like the minimizer $u$ to be $f$ if $R$ is not too small. However, for any $R \geq \frac{1}{\lambda \alpha}$ and any finite $\lambda > 0$, we have [34]

$$u = (\alpha - \frac{1}{\lambda R}) \chi_D, \ v = \frac{1}{\lambda R} \chi_D.$$

Model (3) is of the form $|u|_{BV(\Omega)} + \lambda \|f - u\|_{L^p(\Omega)}^q$, $p \geq 1$, $q \geq 1$, and the loss of intensity property is always present when we have $q > 1$ while keeping the total variation. In particular, we no longer have an intensity loss if we substitute $\| \cdot \|_{L^2}$ in (3) with $\| \cdot \|_{L^2}$ or $\| \cdot \|_{L^1}$, which was proposed in the continuous case by Cheon, Paranjpye, Vese and Osher [11], and further analysis in the $L^1$ case was made by Chan and Esedoglu [10], among others (see also earlier works of S. Alliney [1] and [2] in the discrete one dimensional case).

We are interested in function spaces that give small penalties to oscillations. As noted in [34], oscillatory components do not have small norms in $L^2$ or $L^1$. Moreover, Alvarez, Gousseau and Morel [21], [3] argue that $BV$ is not a good choice to model natural images. To overcome these drawbacks, we can relax the condition on $F_1(u) = |u|_{BV}$ or on $F_2(v) = \|v\|_{L^p}$, for $p = 1$ or $p = 2$. One way is to use a non-convex regularization in $u$ (like in (2), [19], [9], [50], [31], etc.), that is weaker than $| \cdot |_{BV}$. Another way is to use weaker norms than the $L^p$ norm. Here we keep a convex $BV$ regularization, and consider weaker norms than the $L^p$ norm, following [34]. Mumford and Gidas [36] also show that, under some assumptions, natural images are drawn from probability distributions supported by generalized functions, and not by functions.
Y. Meyer [34] questions the model (3) and proposes more refined versions, using weaker norms of generalized functions to model \(v\), instead of the \(\| \cdot \|_{L^2}\).

Among the spaces proposed in [34] to better model the texture component, is the space \(F\) and the minimization model

\[
\inf_{(u,v) \in \text{BV} \times F} \left\{ |u|_{\text{BV}} + \lambda \|v\|_F, \quad f = u + v \right\},
\]

where \(F\) is defined below.

**Definition 1.** In two dimensions, the space \(F\) consists of distributions \(v\) which can be written as

\[
v = \text{div}(\vec{g}) \text{ in } D', \quad \vec{g} = (g_1, g_2) \in \text{BMO}^2,
\]

with

\[
\|v\|_F = \inf \left\{ \|g_1\|_{\text{BMO}} + \|g_2\|_{\text{BMO}} : \ v = \text{div}(\vec{g}) \text{ in } D', \ \vec{g} \in \text{BMO}^2 \right\}.
\]

The space \(\text{BMO}\) is defined below.

**Definition 2.** We say that \(f \in L^1_{\text{loc}}\) belongs to \(\text{BMO}\) [24], [43], if

\[
\|f\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| dx < \infty,
\]

where \(Q\) is a square (it is sufficient to consider squares with sides parallel with the axes), and \(f_Q = |Q|^{-1} \int_Q f(x) dx\) denotes the mean value of \(f\) over the square \(Q\).

An equivalent norm of \(\text{BMO}\) can be obtained by taking the supremum over dyadic squares and their 1/3 translations, as in the work [17] by the first two authors. For \(f \in \text{BMO}(\Omega)\), the supremum is over squares \(Q \subset \Omega\).

In [34], Y. Meyer also proposed two other function spaces to model the oscillatory component \(v\), denoted by \(G\) and \(E\), with \(u \in \text{BV} \subset L^2 \subset G \subset F \subset E\).

The space \(G\) is defined like \(F\) but having \(\vec{g}\) in \((L^\infty)^2\) instead of \((\text{BMO})^2\), while \(E = \mathring{B}_{\infty,\infty}^{-1} = \triangle(\mathring{B}_{\infty,\infty}^1)\) is a homogeneous Besov space of regularity index \(-1\).

Meyer’s \(G\)-model is approximated and studied in [51]-[52], [39], [6], [5], [8], [38], [22], [53], [12], [29], [27], among others. Meyer’s \(E\) model was studied and discussed in [7], [18] and [26].
In [28], the third and fourth authors proposed several methods to numerically compute the \( BMO \)-norm of a function defined on a bounded domain \( \Omega \), and approximate Meyer’s \( F \)-model (4) by the convex variational relaxed problem,

\[
\inf_{u \in BV(\Omega), \, \vec{g} \in BMO(\Omega)^2} \left\{ |u|_{BV(\Omega)} + \mu \| f - u - \text{div} \vec{g} \|_{L^2(\Omega)}^2 + \lambda \left[ \| g_1 \|_{BMO(\Omega)} + \| g_2 \|_{BMO(\Omega)} \right] \right\} . \tag{5}
\]

As \( \mu \to \infty \), this model approximates model (4). An equivalent model was also proposed in [28], by setting \( \vec{g} = \nabla g \), i.e. \( v = \Delta g \), and minimizing

\[
\inf \left\{ |u|_{BV(\Omega)} + \mu \| f - u - \Delta g \|_{L^2(\Omega)}^2 + \lambda \left[ \| g_x \|_{BMO(\Omega)} + \| g_y \|_{BMO(\Omega)} \right] \right\} : u \in BV(\Omega), g, \Delta g \in L^2(\Omega), \nabla g \in BMO(\Omega)^2 \}. \tag{6}
\]

Formulations (5) and (6) are still approximations to Meyer’s \( F \)-model. In these models, a given image \( f \) is decomposed into \( u + v + r \), where \( u \in BV(\Omega) \) is piecewise smooth, \( v = \text{div}(\vec{g}) \in F \) or \( v = \Delta g = \text{div}(\nabla g) \in F \) consists of oscillatory components, and \( r \) is a residual. Numerically, \( r \) is negligible. The significance of \( r \) is also discussed in [18].

Other related decomposition models using wavelets are by I. Daubechies and G. Teschke [14], R. Coifman and D. Donoho [13], J. L. Starck, M. Elad, and D. Donoho [42], F. Malgouyres [33], S. Lintner and F. Malgouyres [32], A. Haddad and Y. Meyer [22], A. Haddad, [23], or J. Gilles [20].

In this paper, we consider an equivalent norm for the space \( F \) in terms of the Riesz potentials, and study models where the action of the Riesz potentials on the oscillatory components belong to \( BMO \). In other words, we model the oscillatory component \( v \) by imposing that \( (-\Delta)^{\alpha/2} v \) belongs to \( BMO \), for some \( \alpha < 0 \), i.e. \( v \in BMO^\alpha \). If \( \alpha = -1 \), we recover the space \( F \), but now the equivalent norm is defined in an isotropic way and we can obtain exact decompositions (4), and equivalent decompositions as in (5) and (6).

As a byproduct and for comparison, we also consider models when \( (-\Delta)^{\alpha/2} v \in L^p \), \( 1 \leq p < \infty \), i.e. \( v \) belongs to the homogeneous potential Sobolev space \( W^{\alpha,p} \), for some \( \alpha < 0 \). The case \( 1 \leq p < \infty \) and \( \alpha = -1 \) reduces to the case from [51], [52]. The case \( p = 2 \) and \( \alpha = -1 \) reduces to the model from [39] in an equivalent
PDE formulation, while the more general case with \( \alpha < 0 \) and \( p = 2 \) reduces to
the models proposed by L. Lieu [29], [30], and also related with the proposal from [36].

As noted in [34] in more details, the space \( F = \dot{BMO}^{-1} \) has also been used
in an analysis of the Navier-Stokes equations by Koch and Tataru [25], where
\( \dot{BMO}^{-1} \) is defined through another isotropic equivalent norm, also recalled in
the next section.

We conclude this section by an example, motivating that oscillatory functions
(not captured in the \( BV \)–cartoon component, having large total variations), have
small \( \dot{BMO}^{\alpha} \) norms (\( \alpha < 0 \)) and thus are rather captured in the texture compo-
nent (therefore such spaces do not penalize oscillations). Consider for simplicity
the case \( \alpha = -1 \) (similar discussion for general \( \alpha < 0 \)). Let
\( v_m(x) = \cos(mx) \), then
\[
\|v_m\|_{G} = \frac{1}{m}.
\]
Since we have the embedding \( G \subset F \), this implies that
\( \|v_m\|_F \leq C\|v_m\|_G \to 0 \) as \( m \to \infty \). Thus, more oscilla-
tions give smaller \( \dot{BMO}^{-1} \) norm, or in other words, oscillations are encouraged
in the texture component \( v \) in a minimization model such as (4). By comparison,
if \( v \in L^p \), \( p \geq 1 \), then
\( \|v_m\|_{L^p} \to constant > 0 \), as \( m \to \infty \).

2 The homogeneous spaces \( \dot{BMO}^{\alpha} \) and \( \dot{W}^{\alpha,p} \)

In this section, we consider a general form of function spaces, and in the definitions
we make no distinction between periodic functions or functions defined on \( \mathbb{R}^n \).
We recall the definition of the Riesz potentials (or the fractional powers of the
Laplacian)
\[
I_{-\alpha} v(x) = (-\Delta)^{\alpha/2} v(x) = (2\pi|\xi|^{\alpha} \hat{v}(\xi))^{\vee}(x) = k_{\alpha} * v(x),
\]
with \( k_{\alpha}(x) = ((2\pi|\xi|)^{\alpha})^{\wedge}(x) \), where as usual, \( ^{\wedge} \) indicates the Fourier transform
and \( ^{\vee} \) indicates the inverse Fourier transform. Of special significance is the case
\( -n < \alpha < 0 \). Then we can write the Riesz potentials as integral operators,
\[
(I_{-\alpha} v)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} |x-y|^{-n-\alpha} v(y) dy,
\]
where \( \gamma(\alpha) \) is a normalization constant depending on the dimension \( n \) and on \( \alpha \).
We also recall the Riesz transforms of a function $f$ in two dimensions:

$$(R_j f)(\xi) = \frac{i \xi_j \hat{f}(\xi)}, \quad j = 1, 2,$$

having the property

$$(R_1)^2 + (R_2)^2 = -I,$$

where $I$ is the identity operator. We note that the Riesz operators $R_j$ are bounded in $BMO$ [43], [40]: there is a positive constant $C_0$ such that, for any $f \in BMO$, we have

$$\|R_j f\|_{BMO} \leq C_0 \|f\|_{BMO}.$$

Our main motivation of this work is the following lemma, which provides an isotropic equivalent norm for $F$, easier to be used in practice. This will also lead to generalizations.

**Lemma 1.** The norm $\|v\|_F$ is equivalent with the norm $\|I_1 v\|_{BMO} = \|(-\Delta)^{-1/2} v\|_{BMO}$.

**Proof.** Again, we note that the Riesz operators $R_j$ are bounded in $BMO$,

$$\|R_j f\|_{BMO} \leq C_0 \|f\|_{BMO},$$

for some positive constant $C_0$.

We have:

$$v = -((R_1)^2 + (R_2)^2)v = -(-\Delta)^{1/2}(-\Delta)^{-1/2}((R_1)^2 + (R_2)^2)v$$

$$= R_1(-\Delta)^{1/2}(-R_1(-\Delta)^{-1/2}v) + R_2(-\Delta)^{1/2}(-R_2(-\Delta)^{-1/2}v)$$

$$= R_1(-\Delta)^{1/2}g_1 + R_2(-\Delta)^{1/2}g_2 = \text{div}(g_1, g_2),$$

with $g_j = -R_j((-\Delta)^{-1/2} v)$.

Then $\|g_j\|_{BMO} = \| -R_j((-\Delta)^{-1/2} v)\|_{BMO} \leq C_0 \|(-\Delta)^{-1/2} v\|_{BMO}$.

Therefore,

$$\|v\|_F := \inf_{\vec{g} \in BMO \times BMO, \text{div}\vec{g} = v} \left[ \|g_1\|_{BMO} + \|g_2\|_{BMO} \right] \leq 2C_0 \|(-\Delta)^{-1/2} v\|_{BMO}.$$
For the converse inequality, suppose $v = \text{div}(g_1, g_2)$, with $g_1, g_2 \in BMO$. Then
\[
v = \text{div}(g_1, g_2) = (-\Delta)^{1/2}(R_1g_1 + R_2g_2),\]
therefore
\[
(-\Delta)^{-1/2}v = R_1g_1 + R_2g_2,
\]
and then
\[
\|(-\Delta)^{-1/2}v\|_{BMO} = \|R_1g_1 + R_2g_2\|_{BMO} \leq \|R_1g_1\|_{BMO} + \|R_2g_2\|_{BMO} \\
\leq C_0\|g_1\|_{BMO} + C_0\|g_2\|_{BMO} = C_0\left[\|g_1\|_{BMO} + \|g_2\|_{BMO}\right].
\]
We conclude that
\[
\|(-\Delta)^{-1/2}v\|_{BMO} \leq C_0\inf_{g \in BMO \times BMO, \text{div}g=v} \left[\|g_1\|_{BMO} + \|g_2\|_{BMO}\right] = C_0\|v\|_F,
\]
and therefore the two norms are equivalent, since we have obtained
\[
\frac{1}{2C_0}\|v\|_F \leq \|(-\Delta)^{-1/2}v\|_{BMO} \leq C_0\|v\|_F.
\]

Thus, for $v \in F$, the quantity $\|Iv\|_{BMO} = \|(-\Delta)^{-1/2}v\|_{BMO}$ provides an equivalent norm for $\|v\|_F$ introduced in Definition 1. This isotropic norm can be used as an alternative way to the models proposed and solved in [28]. Moreover, we are led to consider more general cases, when $v$ is modeled by the space $BMO^\alpha$, $\alpha < 0$, defined below.

**Definition 3.** (See Strichartz [44] and [45]) We say that a function (or distribution) $v$ belongs to the homogeneous space $BMO^\alpha = I^\alpha(BMO)$, $\alpha \in \mathbb{R}$, if
\[
\|v\|_{BMO^\alpha} := \|I^{-\alpha}v\|_{BMO} < \infty.
\]
Equipped with $\|\cdot\|_{BMO^\alpha}$, $BMO^\alpha$ becomes a Banach space.

Elements in $BMO$ or $BMO^\alpha$ that are different by a constant are identified. In other words, we can assume that $v$ has zero mean ($\int v(x)dx = 0$) if $v \in BMO$ or $v \in BMO^\alpha$. 
The space $BMO^{\alpha}$ coincides with the classical Triebel-Lizorkin homogeneous space $F^\alpha_{\infty,2}$ [48]. An equivalent norm for $BMO^{\alpha}$ can also be obtained, as in [25]: let $\Phi(x) = Ce^{-2\pi|x|^2}$, where $C$ is chosen so that $\int \Phi(x) \, dx = 1$. Define $\Phi_t(x) = t^{-n}\Phi\left(\frac{x}{t}\right)$, $x \in \mathbb{R}^n$. For each $v \in L^1_{loc}$, let $w_t(x) = \Phi\sqrt{t} * v(x)$. We have the following characterization of $BMO$ [25], [43].

**Definition 4.** We say that $v \in BMO$ if

$$
\|v\|_{BMO} := \sup_{x,R} \left( \frac{4\pi}{Q(x,R)} \int_{Q(x,R)} \int_0^R t |\nabla (\Phi_t * v)|^2 \, dt \, dy \right)^{1/2} = \sup_{x,R} \left( \frac{1}{Q(x,R)} \int_{Q(x,R)} \int_0^{R^2} |\nabla w_t|^2 \, dt \, dy \right)^{1/2}. \tag{7}
$$

$$
\approx \sup_{x,R} \left( \frac{1}{Q(x,R)} \int_{Q(x,R)} \int_0^{R^2} \left|(-\Delta)^{1/2} w_t\right|^2 \, dt \, dy \right)^{1/2} < \infty,
$$

where $Q(x,R)$ denotes a square centered at $x$ with side length $R$, and "\approx" denotes equivalent norms.

Similarly, we have the following characterization of $BMO^{\alpha}$, which could be another alternative approach to the work in [28].

**Definition 5.** We say that $v$ belongs to $BMO^{\alpha}$, $\alpha \in \mathbb{R}$, if

$$
\|I_{-\alpha}v\|_{BMO} = \sup_{x,R} \left( \frac{4\pi}{Q(x,R)} \int_{Q(x,R)} \int_0^R t |\nabla (\Phi_t * (I_{-\alpha}v))|^2 \, dt \, dy \right)^{1/2} = \sup_{x,R} \left( \frac{4\pi}{Q(x,R)} \int_{Q(x,R)} \int_0^R t |\nabla (\Phi_t * (I_{-\alpha}v))|^2 \, dt \, dy \right)^{1/2} \tag{8}
$$

$$
= \sup_{x,R} \left( \frac{1}{Q(x,R)} \int_{Q(x,R)} \int_0^{R^2} \left|(-\Delta)^{1/2} (I_{-\alpha}w_t)\right|^2 \, dt \, dy \right)^{1/2} \approx \sup_{x,R} \left( \frac{1}{Q(x,R)} \int_{Q(x,R)} \int_0^{R^2} \left|(-\Delta)^{1/2} (I_{-\alpha}w_t)\right|^2 \, dt \, dy \right)^{1/2} < \infty.
$$

Again, "\approx" denotes equivalent norms.
In the remaining part of this paper, we use Definition 3 for $BMO^\alpha$. For comparison, substituting $BMO$ in Definition 3 by $L^p$, $1 \leq p < \infty$, we arrive to the homogeneous potential Sobolev spaces, which we recall here.

**Definition 6.** We say that a function (or distribution) $v$ belongs to the homogeneous potential Sobolev space $\dot{W}^{\alpha,p}$, for $\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$, if

$$\|v\|_{\dot{W}^{\alpha,p}} := \|I_{-\alpha}v\|_{L^p} < \infty.$$ 

Equipped with $\| \cdot \|_{\dot{W}^{\alpha,p}}$, $\dot{W}^{\alpha,p}$ becomes a Banach space.

Note that if $g \in \dot{W}^{\alpha,p}$, $\alpha < 0$, then $\int_{\Omega} g(x) \, dx = 0$. Some useful properties of $\dot{BMO}^\alpha$ and $\dot{W}^{\alpha,p}$ are recalled below:

- $I_{-s}$ is an isometry from $\dot{BMO}^\alpha$ and $\dot{W}^{\alpha,p}$ to $\dot{BMO}^{\alpha-s}$ and $\dot{W}^{\alpha-s,p}$, respectively, for all $s, \alpha \in \mathbb{R}$.

- Let $\tau_\delta f(x) = f(\delta x)$, $\delta > 0$, $x \in \mathbb{R}^n$, be the dilation operator. We have

$$\|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$

$$\|\tau_\delta f\|_{\dot{BMO}^\alpha(\mathbb{R}^n)} = \delta^{\alpha} \|f\|_{\dot{BMO}^\alpha(\mathbb{R}^n)},$$

$$\|\tau_\delta f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = \delta^{-\frac{n}{p} + \alpha} \|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}.$$ 

From this dilation property, we see that if $\alpha < 0$, $\| \cdot \|_{\dot{W}^{\alpha,p}}$ provides a better separation among different oscillations compared to $\| \cdot \|_{L^p}$, and for the same $\alpha < 0$, $\| \cdot \|_{\dot{W}^{\alpha,p}}$ provides a better separation among different oscillations compared to $\| \cdot \|_{\dot{BMO}^\alpha}$ (in other words, taking the same function $f$ but with two different oscillating levels, $\tau_{\delta_1} f$ and $\tau_{\delta_2} f$, with $1 \leq \delta_1 < \delta_2$, the difference in their norms is larger using $\| \cdot \|_{\dot{W}^{\alpha,p}}$, due to larger exponents in absolute value: $\frac{n}{p} < |\frac{n}{p} + \alpha|$ and $|\alpha| < |\frac{n}{p} + \alpha|$, thus larger exponent in absolute value will distinguish better between two levels of oscillations). The experimental results in this paper will support these remarks.

In [18], the authors have numerically considered the case when the oscillatory component $v$ belongs to $\dot{B}_{p,\infty}^\alpha$, $\alpha < 0$, as a generalization of the space $E$ proposed by Y. Meyer. The following remark shows that $\dot{B}_{p,q}^\alpha$ and $\dot{W}^{\alpha,p}$ are in fact close [49].

**Remark 1.** If $\alpha \in \mathbb{R}$ and $p \geq 1$, then

$$\dot{B}_{p,1}^\alpha \subset \dot{W}^{\alpha,p} \subset \dot{B}_{p,\infty}^\alpha.$$  

(9)
3 Modeling oscillations with $\dot{BMO}^\alpha$ and $\dot{W}^{\alpha,p}$

Given an image $f$, we would like to decompose it into $u + v$, where $u \in BV$, and $v$ is an element of $\dot{BMO}^\alpha$ or $\dot{W}^{\alpha,p}$, for $\alpha < 0$ and $1 \leq p < \infty$. In other words, we consider modeling oscillatory component $v$ (of zero mean) as $\Delta g$, where $g \in \dot{BMO}^s$ or $\dot{W}^{s,p}$, for $s < 2$, $1 \leq p < \infty$, in the minimization problems for image decomposition

$$\inf_{u,g} \{ |u|_{BV} + \mu \|f - u - \Delta g\|_{L^2}^2 + \lambda \|g\|_{BMO^\alpha} \}, \text{ and}$$

$$\inf_{u,g} \{ |u|_{BV} + \mu \|f - u - \Delta g\|_{L^2}^2 + \lambda \|g\|_{\dot{W}^{s,p}} \}. \quad (10)$$

The model (10), when $s = 1$, is equivalent with the model (6). Since $v$ belongs to $\dot{BMO}^\alpha$ or $\dot{W}^{\alpha,p}$ with $\alpha = s - 2$, we will also consider the exact decomposition models,

$$\inf_u \{ |u|_{BV} + \lambda \|f - u\|_{BMO^\alpha} \}, \text{ and}$$

$$\inf_u \{ |u|_{BV} + \lambda \|f - u\|_{\dot{W}^{\alpha,p}} \}. \quad (12)$$

Thus, when $\alpha = -1$ in (12), we recover Meyer’s model (4). Theorems 1 and 2 from [18] can be exactly carried out here to show existence of minimizers for the above models (10), (11), (12) and (13).

We discuss next scaling properties of the proposed minimization models. Recall the dilating operator $\tau_\delta f(x) = f(\delta x)$, $\delta > 0$. We have

$$|\tau_\delta f|_{BV(\mathbb{R}^n)} = \delta^{-n+1} |f|_{BV(\mathbb{R}^n)}, \quad \|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-n/p} \|f\|_{L^p(\mathbb{R}^n)},$$

$$\|\tau_\delta f\|_{BMO^\alpha(\mathbb{R}^n)} = \delta^{\alpha} \|f\|_{BMO^\alpha(\mathbb{R}^n)}, \quad \|\tau_\delta f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = \delta^{-n/p + \alpha} \|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$$

Following [18], we would like to characterize the parameters $\mu$ and $\lambda$ in the proposed models (10), (11), (12) and (13) when the image $f$ is being dilated by a factor $\delta$ (zoom in when $0 < \delta < 1$ and zoom out when $\delta > 1$).

**Proposition 1.** Denote

$$\mathcal{J}_{f,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \lambda \|f - u\|_{BMO^\alpha(\mathbb{R}^n)}$$

For a fixed $f$ and $\lambda > 0$, let $(u_\lambda, v_\lambda = f - u_\lambda)$ be a minimizer for the energy $\mathcal{J}_{f,\lambda}$. Then for $\lambda' = \lambda \delta^{-n+1-\alpha}$, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ minimizes the energy $\mathcal{J}_{\tau_\delta f,\lambda'}$. 
Proof. Since \((u_\lambda, v_\lambda = f - u_\lambda)\) is a minimizer, this implies
\[
\mathcal{J}_{f,\lambda}(u_\lambda) = |u_\lambda|_{BV(\mathbb{R}^n)} + \lambda\|v_\lambda\|_{BMO^\alpha(\mathbb{R}^n)}
\]
is minimal. Applying \(\tau_\delta\) to \(f, u_\lambda\) and \(v_\lambda\) using \(\lambda'\), we have
\[
\mathcal{J}_{\tau_\delta f,\lambda'}(\tau_\delta u_\lambda) = |\tau_\delta u_\lambda|_{BV(\mathbb{R}^n)} + \lambda'\|\tau_\delta v_\lambda\|_{BMO^\alpha(\mathbb{R}^n)}
= \delta^{-n+1}|u_\lambda|_{BV(\mathbb{R}^n)} + \lambda'\delta^\alpha\|v_\lambda\|_{BMO^\alpha(\mathbb{R}^n)}.
\]
We have \(\delta^n\mathcal{J}_{\tau_\delta f,\lambda'}(\tau_\delta u_\lambda)\) is minimized when \(\lambda' = \lambda\delta^{-n+1}\). Therefore, \((\tau_\delta u_\lambda, \tau_\delta v_\lambda)\) is a minimizer for \(\mathcal{J}_{\tau_\delta f,\lambda'}\) with \(\lambda' = \lambda\delta^{-n+1}\).

Similarly, when \(\|\cdot\|_{BMO^\alpha}\) is replaced by \(\|\cdot\|_{W^{\alpha,p}}\), we have the following result.

**Proposition 2.** For a fixed \(f\) and \(\lambda > 0\), let \((u_\lambda, v_\lambda = f - u_\lambda)\) be a minimizer for the energy,
\[
\mathcal{K}_{f,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \lambda\|f - u\|_{W^{\alpha,p}(\mathbb{R}^n)}
\]
Then for \(\lambda' = \lambda\delta^{-n+1}(-n/p+\alpha)\), \((\tau_\delta u_\lambda, \tau_\delta v_\lambda)\) minimizes \(\mathcal{K}_{\tau_\delta f,\lambda'}\).

Using the same techniques, we obtain the following results for the models (10) and (11).

**Proposition 3.** Fix an \(f\), \(\mu > 0\), and \(\lambda > 0\).

1. Let \((u_{\mu,\lambda}, v_{\mu,\lambda})\) be a minimizer for the energy from (10), which can be rewritten as
\[
\mathcal{J}_{f,\mu,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \mu\|f - u - v\|_{L^2(\mathbb{R}^n)}^2 + \lambda\|v\|_{BMO^\alpha(\mathbb{R}^n)}
\]
Then for \(\mu' = \mu\delta\) and \(\lambda' = \lambda\delta^{-n+1}\), \((\tau_\delta u_{\mu,\lambda}, \tau_\delta v_{\mu,\lambda})\) minimizes \(\mathcal{J}_{\tau_\delta f,\mu',\lambda'}\).

2. Let \((u_{\mu,\lambda}, v_{\mu,\lambda})\) be a minimizer for the energy from (11), which can be rewritten as
\[
\mathcal{K}_{f,\mu,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \mu\|f - u - v\|_{L^2(\mathbb{R}^n)}^2 + \lambda\|v\|_{W^{\alpha,p}(\mathbb{R}^n)}
\]
Then for \(\mu' = \mu\delta\) and \(\lambda' = \lambda\delta^{-n+1}(-n/p+\alpha)\), \((\tau_\delta u_{\mu,\lambda}, \tau_\delta v_{\mu,\lambda})\) minimizes \(\mathcal{K}_{\tau_\delta f,\mu',\lambda'}\).
4 Characterization of minimizers

In this section, we would like to show some results regarding the characterization of minimizers for the exact decompositions (12) and (13) under some assumptions or minor modifications. These can be seen as extensions and generalizations of the results from Lemma 4, Thm. 3 (page 32), Proposition 4 (page 33) and Thm. 4 (page 4) from [34].

4.1 The case $|u|_{BV} + \lambda \| I_{-\alpha} (f - u) \|_{BMO}^2$

We have the following equivalent formulations of $BMO$ for different values of $p \in [1, \infty)$, see [43] for example. For $f \in L^2_{loc}$, we have

$$
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2},
$$

thus if $\sup_Q \left( \frac{1}{|Q|} |f(x) - f_Q|^2 \, dx \right)^{1/2} \leq C$, then $f \in BMO$. Conversely, if $f \in BMO$ according to Definition 2, then for any $p < \infty$, $f$ is in $L^p_{loc}$ and $\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \leq c_p \|f\|_{BMO}^p$, for all squares $Q$.

Thus consider the problem with $p = 2$ in the definition of the equivalent $BMO$ norm, and we substitute (12) by

$$
\inf_u \mathcal{F}(u),
$$

where

$$
\mathcal{F}(u) = |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_{\alpha} \ast (f - u) - (k_{\alpha} \ast (f - u))_Q|^2 \, dx,
$$

or

$$
\mathcal{F}(u) = |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_{\alpha} \ast f - (k_{\alpha} \ast f)_Q - (k_{\alpha} \ast u - (k_{\alpha} \ast u)_Q)\|_{L^2(Q)}^2.
$$

Denote $\langle f, g \rangle_{L^2(Q)} := \int_Q f g \, dx$. Consider the quantity $\|\cdot\|_{\alpha, *} \,$ (possibly attains $\infty$), defined as

$$
\|f\|_{\alpha, *} = \sup_{Q, h \in BV, |h|_{BV} \neq 0} \frac{\frac{1}{|Q|} \langle k_{\alpha} \ast f - (k_{\alpha} \ast f)_Q, k_{\alpha} \ast h - (k_{\alpha} \ast h)_Q \rangle_{L^2(Q)}}{|h|_{BV}},
$$
where the supremum is taken over all squares $\bar{Q}$ satisfying

$$\bar{Q} = \arg \max_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast f - (k_\alpha \ast f)_Q|^2 \, dx.$$  \hfill (14)$$

**Remark 2.** Note that, if the functions are not sufficiently smooth, there may not be a square $\bar{Q}$ realizing the maximum in (14). In such cases, the results presented below can still be verified, working with a sequence of maximizing squares $\bar{Q}_{\epsilon_n}$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast f - (k_\alpha \ast f)_Q|^2 \, dx = \lim_{\epsilon_n \to 0} \frac{1}{|\bar{Q}_{\epsilon_n}|} \int_{\bar{Q}_{\epsilon_n}} |k_\alpha \ast f - (k_\alpha \ast f)_{\bar{Q}_{\epsilon_n}}|^2 \, dx.$$  

**Definition 7.** Given an $\alpha \in \mathbb{R}$, we say $f$ satisfies property (P) if for any $h \in BV$ and any square $\bar{Q}$ satisfying (14), we have

$$\liminf_{\epsilon_n \to 0} \frac{1}{|\bar{Q}_{\epsilon_n}|} \langle k_\alpha \ast f - (k_\alpha \ast f)_{\bar{Q}_{\epsilon_n}}, k_\alpha \ast h - (k_\alpha \ast h)_{\bar{Q}_{\epsilon_n}} \rangle_{L^2(\bar{Q}_{\epsilon_n})} \geq \frac{1}{|\bar{Q}|} \langle k_\alpha \ast f - (k_\alpha \ast f)_{\bar{Q}}, k_\alpha \ast h - (k_\alpha \ast h)_{\bar{Q}} \rangle_{L^2(\bar{Q})}$$

for some sequence of squares $Q_{\epsilon_n}$ and of small parameters $\epsilon_n > 0$ converging to zero, such that

$$Q_{\epsilon_n} = \arg \max_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast (f - \epsilon_n h) - (k_\alpha \ast (f - \epsilon_n h))_Q|^2 \, dx.$$  

**Proposition 4.**

(i) If $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$, then $u = 0$ and $v = f$ is a minimizer.

(ii) If $u = 0$ and $v = f$ is a minimizer and if, in addition, $f$ satisfies property (P) from (15), then $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$.

**Proof.**

(i) Let $h \in BV$ such that

$$\mathcal{F}(h) = |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast (f - h) - (k_\alpha \ast (f - h))_Q|^2 \, dx < +\infty.$$  

Since $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$, we have for all $h \in BV$ and all squares $\bar{Q}$ satisfying

$$\bar{Q} = \arg \max_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast f - (k_\alpha \ast f)_Q|^2 \, dx,$$  \hfill (16)$$
that

\[-2\lambda \frac{1}{|Q|} \langle k_\alpha \ast f - (k_\alpha \ast f)_Q, k_\alpha \ast h - (k_\alpha \ast h)_Q \rangle_{L^2(Q)} \geq -|h|_{BV}.\]

Then

\[\mathcal{F}(h) = |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha \ast f - (k_\alpha \ast f)_Q - (k_\alpha \ast h - (k_\alpha \ast h)_Q)\|_{L^2(Q)}^2\]

\[= |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \left[ \|k_\alpha \ast f - (k_\alpha \ast f)_Q\|_{L^2(Q)}^2 - 2 \langle k_\alpha \ast f - (k_\alpha \ast f)_Q, k_\alpha \ast h - (k_\alpha \ast h)_Q \rangle_{L^2(Q)} + \|k_\alpha \ast h - (k_\alpha \ast h)_Q\|_{L^2(Q)}^2 \right].\]

With \(\bar{Q}\) defined as in (16), we have

\[\mathcal{F}(h) \geq |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha \ast f - (k_\alpha \ast f)_Q\|_{L^2(Q)}^2 + \frac{1}{|Q|} \|k_\alpha \ast h - (k_\alpha \ast h)_Q\|_{L^2(Q)}^2\]

\[\quad - 2\lambda \frac{1}{|Q|} \langle k_\alpha \ast f - (k_\alpha \ast f)_Q, k_\alpha \ast h - (k_\alpha \ast h)_Q \rangle_{L^2(Q)}\]

\[= |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha \ast f - (k_\alpha \ast f)_Q\|_{L^2(Q)}^2 + \frac{1}{|Q|} \|k_\alpha \ast h - (k_\alpha \ast h)_Q\|_{L^2(Q)}^2\]

\[\quad - 2\lambda \frac{1}{|Q|} \langle k_\alpha \ast f - (k_\alpha \ast f)_Q, k_\alpha \ast h - (k_\alpha \ast h)_Q \rangle_{L^2(Q)}\]

\[\geq F(0) + \frac{1}{|Q|} \|k_\alpha \ast h - (k_\alpha \ast h)_Q\|_{L^2(Q)}^2 \geq F(0).\]

Therefore, \(u = 0\) is a minimizer.

(ii) Suppose now \(u = 0\) and \(v = f\) is a minimizer and \(f\) satisfies property \((P)\) from (15). We have

\[|h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \int_Q |(k_\alpha \ast (f - h) - (k_\alpha \ast (f - h))_Q)|^2 dx\]

\[\geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast f - (k_\alpha \ast f)_Q|^2 dx.\]

Thus

\[|h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \left\{ \int_Q |k_\alpha \ast f - (k_\alpha \ast f)_Q|^2 dx\right.\]

\[\quad - 2 \langle k_\alpha \ast f - (k_\alpha \ast f)_Q, k_\alpha \ast h - (k_\alpha \ast h)_Q \rangle_{L^2(Q)}\]

\[\left. + \int_Q |k_\alpha \ast h - (k_\alpha \ast h)_Q|^2 dx \right\} \geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha \ast f - (k_\alpha \ast f)_Q|^2 dx.\]

(17)
Let \( \hat{Q} \) be defined as the square depending on \( f \) and \( h \) that achieves the maximum in \( \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha*(f-h) - (k_\alpha*(f-h))_Q|^2 \). Then we can rewrite (17) as
\[
|h|_{BV} + \frac{1}{|Q|} \int_Q |k_\alpha*f - (k_\alpha*f)_Q|^2 \, dx \\
- 2 \left< k_\alpha*f - (k_\alpha*f)_Q, k_\alpha*h - (k_\alpha*h)_Q \right>_{L^2(\hat{Q})} \\
+ \int_Q |k_\alpha*h - (k_\alpha*h)_Q|^2 \, dx \geq \frac{1}{|Q|} \int_Q |k_\alpha*f - (k_\alpha*f)_Q|^2 \, dx.
\]
This implies
\[
|h|_{BV} + \frac{1}{|Q|} \sup_Q \int_Q |k_\alpha*f - (k_\alpha*f)_Q|^2 \, dx \\
- 2 \frac{1}{|Q|} \left< k_\alpha*f - (k_\alpha*f)_Q, k_\alpha*h - (k_\alpha*h)_Q \right>_{L^2(\hat{Q})} \\
+ \frac{1}{|Q|} \int_Q |k_\alpha*h - (k_\alpha*h)_Q|^2 \, dx \geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha*f - (k_\alpha*f)_Q|^2 \, dx.
\]
(18)

Changing \( h \) into \( \epsilon h \) in (18), dividing both sides by \( \epsilon > 0 \), and taking \( \epsilon \to 0 \), we obtain that for any \( h \in BV \) and any \( \bar{Q} \) satisfying (14),
\[
\frac{1}{|\bar{Q}|} \left< k_\alpha*f - (k_\alpha*f)_\bar{Q}, k_\alpha*h - (k_\alpha*h)_\bar{Q} \right>_{L^2(\bar{Q})} \leq \frac{1}{2\lambda}.
\]
Therefore, \( \|f\|_{\alpha,*} \leq \frac{1}{2\lambda} \).

Proposition 5. Assume now \( \|f\|_{\alpha,*} > \frac{1}{2\lambda} \).

(i) Suppose \( u \) is a minimizer and \( f - u \) satisfies the property (P) from (15) (with equality if \( h = u \)). Then \( u \) satisfies
\[
\frac{1}{2\lambda} |u|_{BV} = \frac{1}{|\bar{Q}|} \left< k_\alpha*(f-u) - (k_\alpha*(f-u))_\bar{Q}, k_\alpha*u - (k_\alpha*u)_\bar{Q} \right>_{L^2(\bar{Q})} \\
and \|k_\alpha*(f-u)\|_* = \frac{1}{2\lambda},
\]
where \( \bar{Q} = \arg\max_Q \frac{1}{|Q|} \|k_\alpha*(f-u) - (k_\alpha*(f-u))_Q\|^2_{L^2(Q)} \).

(ii) If \( u \) satisfies the properties in (19), then \( u \) is a minimizer.
Proof.

(i) Assume that $u$ is a minimizer. Then, for any small $\epsilon$ and any $h \in BV$, we have

$$
|u + \epsilon h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f - (u + \epsilon h)) - (k_\alpha * (f - (u + \epsilon h)))_Q\|^2_{L^2(Q)}
\geq |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f - u) - (k_\alpha * (f - u))_Q\|^2_{L^2(Q)}.
$$

(20)

Let $\hat{Q}$ be the square that achieves the maximum in the left-hand-side of the above equation (20), which depends on $k_\alpha * (f - (u + \epsilon h))$. By triangle inequality we obtain

$$
|\epsilon||h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f - u) - (k_\alpha * (f - u))_Q\|^2_{L^2(Q)}
- 2\epsilon \left<k_\alpha * (f - u) - (k_\alpha * (f - u))_Q, k_\alpha * h - (k_\alpha * h)_Q\right>_{L^2(\hat{Q})}
+ \epsilon^2 \|k_\alpha * h - (k_\alpha * h)_Q\|^2_{L^2(\hat{Q})}
\geq |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f - u) - (k_\alpha * (f - u))_Q\|^2_{L^2(Q)}.
$$

Thus,

$$
|\epsilon||h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f - u) - (k_\alpha * (f - u))_Q\|^2_{L^2(Q)}
- 2\lambda \epsilon \frac{1}{|Q|} \left<k_\alpha * (f - u) - (k_\alpha * (f - u))_Q, k_\alpha * h - (k_\alpha * h)_Q\right>_{L^2(\hat{Q})}
+ \lambda \epsilon^2 \frac{1}{|Q|} \|k_\alpha * h - (k_\alpha * h)_Q\|^2_{L^2(\hat{Q})}
\geq \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f - u) - (k_\alpha * (f - u))_Q\|^2_{L^2(Q)}.
$$

Therefore,

$$
|\epsilon||h|_{BV} + \lambda \epsilon^2 \frac{1}{|Q|} \|k_\alpha * h - (k_\alpha * h)_Q\|^2_{L^2(Q)}
\geq 2\lambda \epsilon \frac{1}{|Q|} \left<k_\alpha * (f - u) - (k_\alpha * (f - u))_Q, k_\alpha * h - (k_\alpha * h)_Q\right>_{L^2(\hat{Q})}.
$$

(21)
Taking in (21) $\epsilon > 0$, dividing by $\epsilon$, and letting $\epsilon \to 0$, we have, for any $h \in BV$,

$$|h|_{BV} \geq 2\lambda \frac{1}{|Q|} \langle k_\alpha * (f - u) - (k_\alpha * (f - u))_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \rangle. \quad (22)$$

Similarly, with $h = u$ in (20), $-1 < \epsilon < 0$, dividing by $\epsilon$, and letting $\epsilon \to 0$, we get

$$|u|_{BV} \leq 2\lambda \frac{1}{|Q|} \langle k_\alpha * (f - u) - (k_\alpha * (f - u))_{\bar{Q}}, k_\alpha * u - (k_\alpha * u)_{\bar{Q}} \rangle. \quad (23)$$

Therefore, (22) and (23) imply (19).

(ii) Let $w \in BV$ arbitrary, and let $h = w - u \in BV$, or $w = u + h$. We have

$$\mathcal{F}(w) = |w|_{BV} + \lambda \sup_Q \frac{1}{|Q|} ||k_\alpha * (f - w) - (k_\alpha * (f - w))_Q||^2_{L^2(Q)}$$

$$= |u + h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} ||k_\alpha * (f - (u + h)) - (k_\alpha * (f - (u + h)))_Q||^2_{L^2(Q)}$$

$$= |u + h|_{BV} + \lambda \sup_Q \left\{ \frac{1}{|Q|} ||k_\alpha * (f - u) - (k_\alpha * (f - u))_Q||^2_{L^2(Q)} \right.$$  

$$- 2 \frac{1}{|Q|} \langle k_\alpha * (f - u) - (k_\alpha * (f - u))_Q, k_\alpha * h - (k_\alpha * h)_Q \rangle_{L^2(Q)}$$

$$+ \frac{1}{|Q|} ||k_\alpha * h - (k_\alpha * h)_Q||^2_{L^2(Q)} \right\}.$$  

$$= |u + h|_{BV} + \lambda \sup_Q \left\{ \frac{1}{|Q|} ||k_\alpha * (f - u) - (k_\alpha * (f - u))_Q||^2_{L^2(Q)} \right.$$  

$$- 2 \frac{1}{|Q|} \langle k_\alpha * (f - u) - (k_\alpha * (f - u))_Q, k_\alpha * h - (k_\alpha * h)_Q \rangle_{L^2(Q)}$$

$$+ \frac{1}{|Q|} ||k_\alpha * h - (k_\alpha * h)_Q||^2_{L^2(Q)} \right\}.$$  

Let $\bar{Q}$ be the square that achieves the supremum in

$$\sup_Q \frac{1}{|Q|} ||k_\alpha * (f - u) - (k_\alpha * (f - u))_Q||^2_{L^2(Q)}.$$

We have

$$|u + h|_{BV} \geq 2\lambda \frac{1}{|Q|} \langle k_\alpha * (f - u) - (k_\alpha * (f - u))_{\bar{Q}}, k_\alpha * (u + h) - (k_\alpha * (u + h))_{\bar{Q}} \rangle_{L^2(\bar{Q})}.$$
This implies
\[
\mathcal{F}(w) \geq 2\lambda \frac{1}{|Q|} \langle k_{\alpha} * (f - u) - (k_{\alpha} * (f - u))_Q, k_{\alpha} * u - (k_{\alpha} * u)_Q \rangle_{L^2(Q)} \\
+ 2\lambda \frac{1}{|Q|} \langle k_{\alpha} * (f - u) - (k_{\alpha} * (f - u))_Q, k_{\alpha} * h - (k_{\alpha} * h)_Q \rangle_{L^2(\bar{Q})} \\
+ \lambda \frac{1}{|Q|} \|k_{\alpha} * (f - u) - (k_{\alpha} * (f - u))_Q\|_{L^2(Q)}^2 \\
- 2\lambda \frac{1}{|Q|} \langle k_{\alpha} * (f - u) - (k_{\alpha} * (f - u))_Q, k_{\alpha} * h - (k_{\alpha} * h)_Q \rangle_{L^2(\bar{Q})} \\
+ \lambda \frac{1}{|Q|} \|k_{\alpha} * h - (k_{\alpha} * h)_Q\|_{L^2(\bar{Q})}^2 \\
= |u|_{BV} \lambda \frac{1}{|Q|} \|k_{\alpha} * (f - u) - (k_{\alpha} * (f - u))_Q\|_{L^2(\bar{Q})}^2 \\
+ \lambda \frac{1}{|Q|} \|k_{\alpha} * h - (k_{\alpha} * h)_Q\|_{L^2(\bar{Q})}^2 \geq \mathcal{F}(u).
\]

Therefore, \( u \) is a minimizer.

Property (P) from Definition 7 could hold (even with equality) for distributions \( f \), when \( k_{\alpha} \) is a sufficiently smoothing kernel. If \( k_{\alpha} \) is not sufficiently smooth, we can introduce a very small amount of smoothing by additional convolution with another kernel, say the Poisson kernel. In other words, the quantity \( k_{\alpha} * f \) could be substituted by \( P_{\delta} * k_{\alpha} * f \), where \( P_{\delta} \) is the Poisson kernel with some small \( \delta > 0 \), thus making \( P_{\delta} * k_{\alpha} * f \) analytic.

The following counter-examples in one dimension show that property (P) (with equality or inequality) may not hold for instance for discontinuous functions \( f \) when \( \alpha = 0 \) (thus when \( k_{\alpha} * f = f \)).

Example 1. Consider on \( \mathbb{R} \) the intervals \( I_n = [2^{-n-1}, 2^{-n}] \), \( n \geq 0 \), and let \( c_n \) be the midpoint of \( I_n \). Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = 0 \) outside of \([0, 1] \), and

\[
f(x) = \begin{cases} 
+ (1 - 2^{-n}) & \text{if } x \in [2^{-n-1}, c_n], \quad (n \geq 1), \\
- (1 - 2^{-n}) & \text{if } x \in [c_n, 2^{-n}], \quad (n \geq 1), \\
+1 & \text{if } x \in [\frac{1}{2}, 1] \\
-1 & \text{if } x \in [\frac{3}{4}, 1].
\end{cases}
\]

Then \( \|f\|_{BMO} = 1 \) and \([\frac{1}{2}, 1]\) is the interval where the norm is attained. Now let \( h = -f \) on \([0, \frac{1}{2}] \) and \( h \equiv 0 \) otherwise. Then if \( \epsilon > 0 \), \( f - \epsilon h \) attains its BMO
norm on one of the intervals \([2^{-n-1}, 2^{-n}], n \geq 1\) (actually the norm increases to \(1 + \epsilon\) as \(n \to \infty\)).

But, for \(n \geq 1\),

\[
\frac{1}{2^n} \int_{I_n} (f - f_{I_n})(h - h_{I_n})\,dx = 1,
\]

while

\[
\frac{1}{2} \int_{[\frac{1}{2}, 1]} (f - f_{I_0})(h - h_{I_0})\,dx = 0,
\]

thus (15) with equality "\(=\)" instead of inequality "\(\geq\)" does not hold. A similar counter-example can be constructed in two dimensions.

The following counter-example shows that inequality "\(\geq\)" also may not hold in (15) for discontinuous functions.

**Example 2.** Similarly, let \(I_n = [2^{-n-1}, 2^{-n}], n \geq 0\), and \(c_n\) be the midpoint of the interval \(I_n\). Let \(J_n = [2^{-n-1}, c_n]\) and \(K_n = [c_n, 2^{-n}]\). On \(I_0\) let \(f = h = \chi_{J_0} - \chi_{K_0}\) and on \(I_n\) for \(n \geq 1\) let \(f = (1 - \frac{1}{n})(\chi_{J_n} - \chi_{K_n})\). Splitting each \(J_n\) and each \(K_n\) into two half intervals denoted \(A_n\) and \(B_n\), let \(h = \chi_{A_n} - \chi_{B_n}\). Then \(f\) and \(h\) have mean zero over all intervals \(I_n\). Again we assume that \(f\) and \(h\) are zero otherwise. We have \(Q = I_0\) and

\[
\frac{1}{|Q|} \int_{Q} f(x)h(x)\,dx = 1,
\]

but for \(n \geq 1\),

\[
\frac{1}{|I_n|} \int_{I_n} |f(x) - \epsilon h(x)|^2\,dx = (1 - \frac{1}{n})^2 + \epsilon^2
\]

and

\[
\frac{1}{|I_n|} \int_{I_n} f(x)h(x)\,dx = 0.
\]

The following example shows that, at least in one dimension, if \(f\) and \(h\) are sufficiently smooth (for example polynomials or analytic functions), then property (P) from Definition 7 holds with equality in (15).

**Example 3.** Let \(f\) and \(h\) be polynomials or analytic functions on a bounded interval \(I\) in \(\mathbb{R}\). Let \(Q = [x_0 - r, x_0 + r]\), be an arbitrary interval included in \(I\). Then the quantities \(\frac{1}{2r} \int_{Q} |f - f_Q|^2\,dx\), \(\frac{1}{2r} \langle f - f_Q, h - h_Q \rangle_{L^2(Q)}\), and \(\frac{1}{2r} \int_{Q} |h - h_Q|^2\,dx\) remain polynomials or analytic functions of \((x_0, r)\). Let \(P(\epsilon, x_0, r) = \)
\[ \frac{1}{2} \int_Q |(f - \epsilon h) - (f - \epsilon h)_{Q_0}|^2 dx, \] polynomial or analytic function in \((x_0, r)\) and quadratic polynomial in \(\epsilon\). If \((x_0^0, r^0)\) achieves the maximum of \(P(0, x_0, r)\), and if \((x_0^0, r^0)\), a bounded sequence, achieves the maximum of \(P(\epsilon, x_0, r)\), then there is a subsequence \((x_0^{\epsilon_n}, r^{\epsilon_n})\) and \(\epsilon_n \to 0\) such that \(\lim_{\epsilon_n \to 0} P(\epsilon_n, x_0^{\epsilon_n}, r^{\epsilon_n}) = P(0, x_0^0, r^0)\), thus property \((P)\) is satisfied in this case.

### 4.2 The case \(|u|_{BV} + \lambda \|f - u\|_{\dot{W}^{\alpha,p}}\)

Consider here the minimization

\[
\inf_u \{ F(u) = |u|_{BV} + \lambda \|f - u\|_{\dot{W}^{\alpha,p}} \}, \tag{25}
\]

for some \(\alpha < 0\), and \(1 \leq p < \infty\). Thus \(F(u)\) is the sum of two non-differentiable functionals at the origin. Assume that we “regularize” the second term (this is often done in practice, for valid computational calculations) by smoothing at the origin the \(L^p\) norm; thus substituting \(\|f - u\|_{L^p}\) by \(R_\delta(f - u) = \left\{ \int \sqrt{\delta^2 + |f - u|^2} dx \right\}^{1/p}\), for very small \(\delta > 0\).

Therefore, substitute the problem (25) by the regularized functional

\[
\inf_u \{ F_\delta(u) = |u|_{BV} + \lambda R_\delta(I_\alpha(f - u)) \}. \tag{26}
\]

Let \(f \in V = \dot{W}^{\alpha,p}\), and let \(V'\) be the topological dual of \(V\). We have \(V' = \dot{W}^{-\alpha,p'}\), where \(p'\) is the conjugate of \(p\). Denote by \(\langle \cdot, \cdot \rangle\) the duality pairing for \(V\) and \(V'\).

Problem (26) can be seen as a particular case of a more general case, where \(R_\delta\) is a Gateaux-differentiable functional on the Banach space \(V\), with continuous Gateaux derivative. For (any) fixed \(f \in V\), we have \(R_\delta'(f) \in V'\) and \(\langle R_\delta'(f), -v \rangle = \lim_{\epsilon \to 0} \frac{R_\delta(f - \epsilon v) - R_\delta(f)}{\epsilon}\), for any \(v \in V\). For any \(f \in V\), define now the quantity \(\| \cdot \|_{\alpha,*} \) (in \([0, +\infty]\)) as

\[
\|f\|_{\alpha,*} = \sup_{h \in BV; \ |h|_{BV} \neq 0} \frac{\langle R_\delta'(k_{\alpha} \ast f), k_{\alpha} \ast h \rangle}{|h|_{BV}}.
\]

We also assume that for any \(f, h \in V\),

\[
R_\delta(f - \epsilon h) = R_\delta(f) + \epsilon \langle R_\delta'(f), -h \rangle + O(\epsilon^2)
\]
in a neighborhood of the origin. Using the notation \( g(\epsilon) = R_\delta(f - \epsilon h) \) for fixed \( f \) and \( h \), this is equivalent with

\[
g(\epsilon) = g(0) + \epsilon g'(0) + O(\epsilon^2),
\]

where \( g'(0) = \langle R'_\delta(f), -h \rangle \).

We have the following characterizations of minimizers for (26), a more general case than (25). Note that these are more general than the quadratic case considered in [34]. For the converse implications below, to show that some \( u \) is a minimizer, we need more conditions on \( R_\delta \) related to convexity. The functional

\[
R_\delta(f - u) = \left\{ \int \sqrt{\delta^2 + |f - u|^2} dx \right\}^{1/p},
\]

defined in the particular case of interest to us, satisfies the assumptions mentioned above and the additional ones that are given below.

**Proposition 6.**

(i) Assume that \( u = 0 \) is a minimizer of (26). Then \( \|f\|_{\alpha, *} \leq \frac{1}{\lambda} \).

(ii) Assume that \( \|f\|_{\alpha, *} \leq \frac{1}{\lambda} \), and assume that \( R''_\delta \) exists and it is a continuous bilinear form on \( V \), satisfying \( R''_\delta(v)(h, h) \geq 0 \), for any \( v, h \in V \). Moreover, we assume that in a neighborhood of the interval \([-1, 1]\) we have

\[
g(\epsilon) = g(0) + \epsilon g'(0) + \frac{\epsilon^2}{2} g''(\xi_\epsilon),
\]

with \( \xi_\epsilon \) between 0 and \( \epsilon \), and \( g''(\xi_\epsilon) = R''_\delta(f - \xi_\epsilon h)(-h, -h) \geq 0 \). Then \( u = 0 \) is a minimizer of (26).

**Proof.**

(i) For any \( \epsilon \in \mathbb{R} \) and any \( h \in BV \), we have

\[
|\epsilon h|_{BV} + \lambda R_\delta(k_\alpha \ast (f - \epsilon h)) \geq \lambda R_\delta(k_\alpha \ast f),
\]

\[
|\epsilon||h|_{BV} + \lambda \left[ R_\delta(f) + \epsilon \langle R'_\delta(k_\alpha \ast f), -k_\alpha \ast h \rangle + O(\epsilon^2) \right] \geq \lambda R_\delta(f),
\]

\[
|\epsilon||h|_{BV} + \lambda \epsilon \langle R'_\delta(k_\alpha \ast f), -k_\alpha \ast h \rangle + \lambda O(\epsilon^2) \geq 0.
\]

Taking \( \epsilon > 0 \), dividing by \( \epsilon \) and letting \( \epsilon \to 0 \), we obtain

\[
|h|_{BV} \geq \lambda \langle R'_\delta(k_\alpha \ast f), k_\alpha \ast h \rangle, \quad \text{thus} \quad \frac{1}{\lambda} \geq \|f\|_{\alpha, *}.\]
(ii) Conversely, take any $h \in BV$. Then using the assumptions, we have

\[
|h|_{BV} + \lambda R_\delta(k_\alpha * (f - h)) \geq \lambda \langle R'_\delta(k_\alpha * f), k_\alpha * h \rangle + \lambda R_\delta(k_\alpha * f) + \lambda \langle R'_\delta(k_\alpha * f), -k_\alpha * h \rangle + \frac{\lambda}{2} g''(\xi_1)
\]

\[
= \lambda R_\delta(k_\alpha * f) + \frac{\lambda}{2} g''(\xi_1) \geq \lambda R_\delta(k_\alpha * f).
\]

Therefore, $u = 0$ is a minimizer.

\[\Box\]

**Proposition 7.** Assume that $\|f\|_{\alpha,*} > \frac{1}{\lambda}$.

(i) If $u$ is a minimizer, then

\[
\frac{1}{\lambda} = \|f - u\|_{\alpha,*} \text{ and } \frac{1}{\lambda} |u|_{BV} = \langle R'_\delta(k_\alpha * (f - u)), k_\alpha * u \rangle.
\]

(ii) Suppose that $u \in BV$ satisfies

\[
\frac{1}{\lambda} = \|f - u\|_{\alpha,*} \text{ and } \frac{1}{\lambda} |u|_{BV} = \langle R'_\delta(k_\alpha * (f - u)), k_\alpha * u \rangle,
\]

and assume in addition the same conditions from Proposition 6 (ii) on the regularity and convexity of $R_\delta$. Then $u$ is a minimizer.

**Proof.** By the assumption and the previous result, $u = 0$ cannot be a minimizer.

(i) If $u \in BV$ is a minimizer, then

\[
|u + \epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - (u + \epsilon h))) \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)).
\]

Thus

\[
|u + \epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - u)) + \lambda \epsilon \langle R'_\delta(k_\alpha * (f - u)), -k_\alpha * h \rangle + O(\epsilon^2) \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)).
\]

(27)

By triangle inequality, we also obtain

\[
|u| + |\epsilon||h|_{BV} + \lambda R_\delta(k_\alpha * (f - u)) + \lambda \epsilon \langle R'_\delta(k_\alpha * (f - u)), -k_\alpha * h \rangle + O(\epsilon^2) \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)).
\]

After terms cancellation and division by $\epsilon > 0$, taking $\epsilon \to 0$, we obtain that for any $h \in BV$,

\[
|h|_{BV} \geq \lambda \langle R'_\delta(k_\alpha * (f - u)), k_\alpha * h \rangle,
\]
therefore \[
\frac{1}{\lambda} \geq \|f - u\|_{\alpha,*}.
\] (28)

Taking now \(h = u\) in (27), with \(-1 < \epsilon < 0\), after cancellations and division by \(\epsilon < 0\) and letting \(\epsilon \to 0\), we obtain
\[
|u|_{BV} \leq \lambda \langle R'_\delta(k_\alpha * (f - u), k_\alpha * u) \rangle.
\] (29)

Combining (28) and (29), we obtain the desired results,
\[
\frac{1}{\lambda} = \|f - u\|_{\alpha,*}, \quad \frac{1}{\lambda} |u|_{BV} = \langle R'_\delta(k_\alpha * (f - u), k_\alpha * u) \rangle.
\]

(ii) Conversely, by the assumptions and taking \(\epsilon = 1\), we have
\[
|u + h|_{BV} + \lambda R_\delta(k_\alpha * (f - (u + h))) = |u + h|_{BV} + \lambda R_\delta(k_\alpha * (f - u)) + \lambda \langle R'_\delta(k_\alpha * (f - u), -k_\alpha * h) \rangle + \frac{\lambda}{2} g''(\xi_1)
\]
\[
\geq \lambda \langle R'_\delta(k_\alpha * (f - u), k_\alpha * (u + h)) \rangle + \lambda R_\delta(k_\alpha * (f - u)) + \lambda \langle R'_\delta(k_\alpha * (f - u), -k_\alpha * h) \rangle + \frac{\lambda}{2} g''(\xi_1)
\]
\[
= |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)) + \frac{\lambda}{2} g''(\xi_1)
\]
\[
\geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)),
\]

thus \(u\) is a minimizer. \(\square\)

5 Numerical minimization algorithms

For numerical studies, we consider spaces of functions or distributions that are periodic and \(\Omega = [0, 1]^2\) is the fundamental domain in \(\mathbb{R}^2\). We give in this section the ingredients for minimizing in practice the proposed decomposition models from Section 3, in a gradient descent and purely PDE approach, based on Uzawa’s minmax algorithm [15]. We formally compute the associated Euler-Lagrange equations, which are then discretized and solved by finite differences. We do not have a convergence proof of our algorithms, but these are stable and well-behaved in practice.
5.1 Algorithms for \((BV, BMO^\alpha)\) decompositions

For \(\alpha < 0\), recall the minimization problem (12) for exact decompositions

\[
\inf_u \mathcal{E}(u) = |u|_{BV(\Omega)} + \lambda \|I_\alpha(f-u)\|_{BMO(\Omega)} = |u|_{BV(\Omega)} + \lambda \|k_\alpha * (f-u)\|_{BMO(\Omega)}
\]

(30)

where \(k_\alpha(x) = ((2\pi|\xi|)^\alpha )^\vee (x)\) (here the dimension is \(n = 2\)).

We show the steps to solve (30). Using the classical definition of the \(BMO\) norm, we re-write (30) as

\[
\inf_{u \in BV(\Omega)} \left\{ \mathcal{E}(u) = \int_\Omega |\nabla u| dx + \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * (f-u) - c_Q| dx \right\},
\]

where \(\alpha < 0\), \(Q\) is a square with sides parallel with the axes, and \(c_Q\) denotes a constant which depends on \(k_\alpha * (f-u)\) in \(Q\). Here we take \(c_Q\) to be the median of \(k_\alpha * (f-u)\) in \(Q\). The main steps of the algorithm are as follows (see [28]):

1. Start with an initial guess \(u^0\).

2. If \(u^n\) is computed, \(n \geq 0\), evaluate \(k_\alpha * (f-u^n)\) using the Fast Fourier Transform and find a square \(Q = Q^n\) that achieves the \(BMO\) norm of \(k_\alpha * (f-u)\) in \(\Omega\) (by one of the methods proposed in [28]; here, we use the dyadic squares and their \(1/3\) translations, as explained in [17]).

3. Fix \(Q\) the square obtained at the previous step, denote by \(\chi_Q\) the characteristic function of this square \(Q\), and minimize with respect to \(u = u^{n+1}\) the energy

\[
\mathcal{E}(u) = \int_\Omega |\nabla u| dx + \lambda \frac{1}{|Q|} \int_Q |k_\alpha * (f-u) - c_Q| \chi_Q dx.
\]

(31)

The associated Euler-Lagrange equation in \(u = u^{n+1}\) can be computed, and we obtain using gradient descent

\[
\frac{\partial u}{\partial t} = \frac{\lambda}{|Q|} k_\alpha \ast \text{sign} \left[ (k_\alpha * (f-u) - c_Q) \chi_Q \right] + \text{div} \left( \frac{\nabla u}{|\nabla u|} \right),
\]

(32)

with \(Q = Q^n\) and \(u = u^{n+1}\). Note that \(c_Q\) is the median of \(k_\alpha * (f-u)\) in \(Q\).
4. Repeat steps 2 and 3 using equation (32), until convergence (update \( u^{n+1} \) and \( Q^{n+1} \) each time and repeat).

Similarly, for the minimization problem (10), again with \( s < 2 \) (\( \alpha = s - 2 \)),

\[
\inf_{u,g} \left\{ A(u, g) = |u|_{BV(\Omega)} + \mu\|f - u - \Delta g\|_{L^2(\Omega)}^2 + \lambda\|I_{s} g\|_{BMO(\Omega)} \right\},
\]

re-written as

\[
\inf_{u,g} \left\{ A(u, g) = \int_{\Omega} |\nabla u|dx + \mu \int_{\Omega} |f - u - \Delta g|^2dx + \lambda \sup_{Q} \frac{1}{|Q|} \int_{Q} |k_s * g - c_Q|dx \right\},
\]

where \( c_Q \) is the median of \( k_s * g \) over the square \( Q \), the main steps are as follows.

1. Start with initial guess \( u^0, g^0 \).

2. If \( u^n \) and \( g^n \) are computed, \( n \geq 0 \), evaluate \( k_s * g^n \) using the Fast Fourier Transform and find a square \( Q = Q^n \) that achieves the \( BMO \) norm of \( k_s * g \) in \( \Omega \) (by one of the methods proposed in [28]).

3. Fix \( Q \) the square obtained at the previous step, denote by \( \chi_Q \) the characteristic function of this square \( Q \), and minimize with respect to \( u = u^{n+1} \) and \( g = g^{n+1} \) the energy

\[
A(u, g) = \int_{\Omega} |\nabla u|dx + \mu \int_{\Omega} |f - u - \Delta g|^2dx + \lambda \frac{1}{|Q|} \int_{Q} |k_s * g - c_Q|\chi_Qdx,
\]

by solving the associated Euler-Lagrange equations using gradient descent

\[
\frac{\partial u}{\partial t} = 2\mu(f - u - \Delta g) + \text{div}\left( \frac{\nabla u}{|\nabla u|} \right),
\]

\[
\frac{\partial g}{\partial t} = -\frac{\lambda}{|Q|} k_s * \text{sign} [(k_s * g - c_Q) \chi_Q] + 2\mu \Delta (f - u - \Delta g)
\]

with \( Q = Q^n \) and \( u = u^{n+1}, g = g^{n+1} \). Note that \( c_Q \) is the median of \( k_s * g \) in \( Q \).

4. Repeat steps 2 and 3 using equation (32), until convergence (update \( u^{n+1}, g^{n+1} \) and \( Q^{n+1} \) each time and repeat).
5.2 Algorithms for \((BV, W^{\alpha,p})\) decompositions

For \(\alpha < 0\), recall the minimization problem (13)

\[
\inf_u \mathcal{E}(u) = |u|_{BV(\Omega)} + \lambda \|I_\alpha(f - u)\|_{L^p(\Omega)} = |u|_{BV(\Omega)} + \lambda \|k_\alpha \ast (f - u)\|_{L^p(\Omega)},
\]

(34)

which is again minimized using Euler-Lagrange equation and gradient descent, as follows. Solve to steady state

\[
\frac{\partial u}{\partial t} = \lambda \|k_\alpha \ast (f - u)\|_{L^p(\Omega)}^{1-p} k_\alpha \ast \left[|k_\alpha \ast (f - u)|^{p-2} k_\alpha \ast (f - u)\right] + \text{div}\left(\frac{\nabla u}{|\nabla u|}\right),
\]

computing the convolutions using the Fast Fourier Transform.

Finally, for the minimization problem (11), recalled here with \(s < 2\) (\(\alpha = s - 2\)),

\[
\inf_{u,g} \left\{ A(u, g) = |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|_{L^2(\Omega)}^2 + \lambda \|I_s g\|_{L^p(\Omega)}\right\},
\]

(35)

we use again the associated Euler-Lagrange equations and gradient descent, formally written as

\[
\frac{\partial u}{\partial t} = 2\mu (f - u - \Delta g) + \text{div}\left(\frac{\nabla u}{|\nabla u|}\right),
\]

\[
\frac{\partial g}{\partial t} = -\lambda \|k_s \ast g\|_{L^p(\Omega)}^{1-p} k_s \ast \left[|k_s \ast g|^{p-2} k_s \ast g\right] + 2\mu \Delta (f - u - \Delta g).
\]

In practice, the above Euler-Lagrange equations are discretized using finite differences. The calculations are stable and the numerical energy decreases versus iterations.

6 Numerical results and comparisons

Figure 1 shows three Barbara test images, to be used in our experimental calculations.

Figure 2 shows a decomposition of \(f_1\) from Figure 1 using the Rudin-Osher-Fatemi model (3). Note the loss of intensity on the face area.
Figure 1: Test images to be decomposed.
Figure 2: A decomposition of $f_1$ from Figure 1 using the Rudin-Osher-Fatemi model (3).

Figure 3 shows a decomposition of $f_1$ from Figure 1 using the model (5) from [28]. Here the oscillatory component is modeled as $v = \text{div} (\mathbf{g})$, $\mathbf{g} \in (BMO)^2$. We obtain an improvement in the loss of intensity, however vertical and horizontal textures are still kept in $u$.

Figure 4 shows a decomposition of $f_1$ from Figure 1 using the model (6) from [28]. Here the oscillatory component is modeled as $v = \Delta g$, $\nabla g \in (BMO)^2$. The decomposition is now more isotropic, textures are well captured in $v$ including non-repeated patterns. This comes from the property of $BMO$.

Figure 5 shows a decomposition of $f_1$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$, $s = 0.2$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$. Now, mostly repeated patterns are captured in $v$.

Figure 6 shows a decomposition of $f_2$ from Figure 1 using the model (10). Here the oscillatory component is modeled as $v = \Delta g$, $g \in BMO^\alpha$ with $\alpha = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.001$. As remarked earlier, non-repeated patterns are also captured in $v$. We also show the numerical energy decrease versus iterations for this test.

Figures 7-8 show decompositions of $f_2$ and $f_3$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0,$
Figure 3: A decomposition of $f_1$ from Figure 1 using the model (5) from [28]. Here the oscillatory component is modeled as $v = \text{div} (\vec{g}), \vec{g} \in (BMO)^2$.

Figure 4: A decomposition of $f_1$ from Figure 1 using the model (6) from [28]. Here the oscillatory component is modeled as $v = \Delta g, \nabla g \in (BMO)^2$. 
and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

Figures 9-10 show decompositions of $f_2$ and $f_3$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0005$.

Figure 11 shows a decomposition of $f_2$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1.5$, and $p = 1$. The parameters used are: $\mu = 10$, and $\lambda = 0.00005$.

Figure 12 shows a decomposition of $f_2$ from Figure 1 using the model (12). Here the oscillatory component $v \in BMO^{-0.5}$, $\lambda = 200$. The numerical energy versus iterations is also shown, illustrating that the algorithm is stable and well behaved in practice.

Figure 13 shows a decomposition of $f_2$ from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.1$, $p = 1$, $\lambda = 1.25$.

Figure 14 shows a decomposition of $f_2$ from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 15$.

Figure 15 shows a decomposition of $f_2$ from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.6$, $p = 1$, $\lambda = 30$. 
Figure 6: A decomposition of $f_2$ from Figure 1 using the model (10). Here the oscillatory component is modeled as $v = \Delta g$, $g \in BMO^\alpha$ with $\alpha = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.001$. The numerical energy versus iterations is also shown.
Figure 7: A decomposition of $f_2$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

In conclusion, we have shown experimental results and comparisons for image decomposition $f = u + v$, with $u \in BV$ and $v \in BMO^\alpha$ or $v \in W^{\alpha,p}$, for some $\alpha < 0$. The case of Sobolev spaces gives very good cartoon-texture separation.

Acknowledgments

The authors would like to thank Yunho Kim for reading the manuscript and for pointing out several remarks and typos.

References


Figure 8: A decomposition of $f_3$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$. 
Figure 9: A decomposition of $f_2$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0005$.


Figure 10: A decomposition of $f_3$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in W_{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.00025$. 

\[ f - u + 100 \]
Figure 11: A decomposition of $f_2$ from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1.5$, and $p = 1$. The parameters used are: $\mu = 10$, and $\lambda = 5 \times 10^{-5}$.


Figure 12: A decomposition of $f_2$ using (12) with $p = 1$. Here the oscillatory component $v \in \dot{BMO}^\alpha$, $\alpha = -0.5$, $\lambda = 25$. We also show a plot of the numerical energy versus iterations for this test.
Figure 13: A decomposition of $f_2$ from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.1$, $p = 1$, $\lambda = 1.25$.

Figure 14: A decomposition of $f_2$ from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 15$. 
Figure 15: A decomposition of $f_2$ from Figure 1 using the model (13). Here the oscillatory component $v \in W^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 30$.


John B. Garnett
Department of Mathematics,
University of California,
Los Angeles, CA 90095-1555,
U.S.A.
E-mail: jbg@math.ucla.edu

Peter W. Jones  
Department of Mathematics,  
Yale University,  
New Haven, CT 06511,  
U.S.A.  
E-mail: jones@math.yale.edu

Triet M. Le  
Department of Mathematics,  
Yale University,  
New Haven, CT 06511,  
U.S.A.  
E-mail: triet.le@yale.edu

Luminita A. Vese  
Department of Mathematics,  
University of California,  
Los Angeles, CA 90095-1555,  
U.S.A.  
E-mail: lvese@math.ucla.edu