Reshetnyak’s Theorem and The Inner Distortion

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Abstract: We prove that a quasilight mapping of finite distortion with locally \( n \)-integrable weak partials and locally integrable inner distortion is discrete and open.

Keywords: Mapping of finite distortion, quasiregular mapping, Reshetnyak’s theorem.

1. Introduction

We call \( f : \Omega \to \mathbb{R}^n \), \( n \geq 2 \), a mapping of finite distortion if \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \), \( J_f \in L^1_{\text{loc}}(\Omega) \), and if there exists a measurable function \( K : \Omega \to [1, \infty) \) such that

\[
|Df(x)|^n \leq K(x)J_f(x) \quad \text{a.e. } x \in \Omega.
\]

Here \( |Df(x)| \) and \( J_f(x) \) are the operator norm and the Jacobian determinant of \( Df(x) \), respectively. If \( K \in L^\infty(\Omega) \), \( f \) is called quasiregular, or a mapping of bounded distortion.

For a mapping of finite distortion \( f \), the outer and inner distortion functions \( K_O \) and \( K_I \) are defined as

\[
K_O(x) = \frac{|Df(x)|^n}{J_f(x)} \quad \text{and} \quad K_I(x) = \frac{|D^#f(x)|^n}{J_f(x)^{n-1}},
\]

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respectively, when \(0 < |Df(x)|, J_f(x) < \infty\), and \(K_O(x) = K_I(x) = 1\) otherwise. Here \(D^2f(x)\) is the adjoint matrix of \(Df(x)\). Then we have

\[
K_I^{1/(n-1)}(x) \leq K_O(x) \leq K_I^{n-1}(x) \quad \text{a.e. } x \in \Omega.
\]

In the late 1960s, Reshetnyak proved that a non-constant mapping of bounded distortion is always continuous, open and discrete. This theorem initiated the by now well-established theory of mappings of bounded distortion, see [13], [14], [6].

Recently, a lot of research has been done in order to find the sharp assumptions of Reshetnyak’s theorem in the class of mappings of finite distortion, cf. [3], [4], [5], [7], [8], [11]. In this note we continue this line of research by giving a new partial result towards a conjecture of Iwaniec and Šverák [7].

**Theorem 1.1.** Suppose that \(f : \Omega \to \mathbb{R}^n, n \geq 2\), is a quasilight mapping of finite distortion satisfying \(f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)\) and \(K_I \in L^{1}_{\text{loc}}(\Omega)\). Then \(f\) is discrete and open.

By definition, a mapping \(f\) is called quasilight if the components of every point-inverse \(f^{-1}(y)\) are compact. The Iwaniec-Šverák conjecture is Theorem 1.1 without the quasilightness assumption. In [7] the conjecture is proved for \(n = 2\). An example of Ball [2] shows that the integrability assumption on \(K_I\) cannot be relaxed in Theorem 1.1.

There are other partial results concerning the Iwaniec-Šverák conjecture, see [3], [4], [5] and [11]. The novelty in Theorem 1.1 lies in the fact that it only deals with the inner distortion; the previous results are proved under assumptions on the outer distortion function. In particular, Hencl and Malý [5] proved Theorem 1.1 assuming \(K_O \in L^{n-1}_{\text{loc}}(\Omega)\), and Manfredi and Villamor [11] without the quasilightness assumption when \(K_O \in L^p_{\text{loc}}(\Omega)\) for some \(p > n - 1\). It is clear that, when working with the inner distortion, one has to find methods different from those used in the above-mentioned works. We prove Theorem 1.1 by using the conformal modulus of \((n-1)\)-dimensional sets, the coarea formula, and elementary topological considerations. Also, we use several results concerning the theory of mappings of finite distortion. Another natural intermediate step towards the Iwaniec-Šverák conjecture would be the theorem of Manfredi and Villamor under the assumption \(K_I \in L^p_{\text{loc}}(\Omega)\) for some \(p > 1\) (instead of the assumption on \(K_O\)), which we cannot prove. For closely related results on the
global invertibility properties of Sobolev mappings, see [2, Theorem 2] and [15, Corollary 2].

2. Preliminaries

In this section we recall some known properties of mappings satisfying the assumptions of Theorem 1.1. First, let \( f : \Omega \rightarrow \mathbb{R}^n \) be a continuous map, and \( U \subset \subset \Omega \) open. Then the (local) topological degree \( \mu(y, f, U) \) is well-defined for every \( y \in \mathbb{R}^n \setminus f(\partial U) \), see [14, I.4]. We will use the following facts:

\[
(2.1) \quad \mu(y, f, U) = 0 \text{ if } y \not\in f(U),
\]

\[
(2.2) \quad \mu(y, f, U) = \mu(v, f, U)
\]

whenever \( y \) and \( v \) lie in the same component of \( \mathbb{R}^n \setminus f(\partial U) \), and

\[
(2.3) \quad \mu(y, f, U) = \sum_{i=1}^{k} \mu(y, f, U_i)
\]

if both sides are well-defined, and if \( U_1, \ldots, U_k \) are disjoint open sets satisfying

\[
U \cap f^{-1}(y) \subset \bigcup_{i=1}^{k} U_i \subset U.
\]

We call \( f \) sense-preserving if \( \mu(y, f, U) > 0 \) whenever \( y \in f(U) \setminus f(\partial U) \). Notice that if \( f \) is sense-preserving, then

\[
\mu(y, f, U) \leq \mu(y, f, V)
\]

whenever both sides are well-defined and \( U \subset V \).

We say that \( f \) satisfies condition \( N \) if the \( n \)-measure \( |f(E)| = 0 \) whenever \( |E| = 0 \). For mappings of finite distortion with locally \( n \)-integrable partials, we have

**Theorem 2.1** ([3, Theorem 1.3]). Suppose that \( f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n) \) is a mapping of finite distortion. Then

1. \( f \) has a continuous representative,
2. \( f \) is sense-preserving,
3. \( f \) satisfies condition \( N \),
4. \( f \) is differentiable almost everywhere in \( \Omega \).
Part 3. implies that the change of variables formula holds for $f$. In fact, if $U \subset \subset \Omega$ is open, we have
\begin{equation}
N(y, f, U) = \mu(y, f, U)
\end{equation}
for almost every $y \in \mathbb{R}^n \setminus f(\partial U)$, see [5, Proposition 2]. Here
$$N(y, f, U) = \text{card}\{f^{-1}(y) \cap U\}.$$ Since $K^{1/(n-1)} \leq K_I$ almost everywhere, [9, Theorem 1.2] implies

**Theorem 2.2.** Suppose that $f$ is as in Theorem 1.1. Then $J_f(x) > 0$ for almost every $x \in \Omega$. In particular, if $(A_i), A_i \subset \Omega$, is a decreasing sequence of measurable sets so that $|A_1| < \infty$ and $\cap_i A_i \subset f^{-1}(y)$ for some $y \in \mathbb{R}^n$, then
$$\int_{A_i} K_I \to 0 \quad \text{as } i \to \infty.$$ The following characterization of quasilightness will be useful in the sequel.

**Theorem 2.3** ([16, Theorem 3.1]). A mapping $f : \Omega \to \mathbb{R}^n$ is quasilight if and only if every point $x \in \Omega$ has a neighborhood $U \subset \subset \Omega$ such that $f(x) \notin f(\partial U)$.

We call a mapping $f$ light if every point-inverse $f^{-1}(y)$ is totally disconnected. Hence a light mapping is quasilight in particular.

**Lemma 2.4** ([14, VI Lemma 5.6]). If $f : \Omega \to \mathbb{R}^n$ is continuous, light and sense-preserving, then $f$ is discrete and open.

By combining Theorem 2.1 and Lemma 2.4, we see that in order to prove Theorem 1.1 it suffices to show that $f$ is light. We conclude this section with a topological lemma.

**Lemma 2.5.** Let $f$ be as in Theorem 1.1. Suppose that $V \subset \mathbb{R}^n$ is homeomorphic to $B(0,1)$ and $\overline{V}$ to $\overline{B}(0,1)$, and that $\emptyset \neq U \subset \subset \Omega$ is a component of $f^{-1}(V)$. Then $f(\partial U) = \partial V$, and $f(U) = V$.

**Proof.** First, $f(\partial U) \subset \partial V$ by the continuity of $f$. Hence, for every $a \in f(U)$, $\mu(a, f, U)$ is well-defined, and strictly positive by Theorem 2.1. By (2.1), there exists $b \in \mathbb{R}^n$ such that $\mu(b, f, U) = 0$. Hence, by (2.2), $f(\partial U)$ separates $f(U)$ and $b$, and so $f(\partial U) = \partial V$. Also, if there exists a point $p \in V \setminus f(U)$, then $\mu(p, f, U) = 0$. But $p$ and $f(U)$ lie in the same component of $\mathbb{R}^n \setminus f(\partial U) = \mathbb{R}^n \setminus \partial V$. 


Hence, by (2.2), \( \mu(p, f, U) = \mu(a, f, U) > 0 \) whenever \( a \in f(U) \). We conclude that \( f(U) = V \). \( \square \)

3. Preimages of radial segments

From now on we assume that \( f \) is as in Theorem 1.1. Recall from Section 2 that in order to prove Theorem 1.1 it suffices to show that \( f \) is light. We assume, in contrary, that there exists a point \( a \in \mathbb{R}^n \) such that some component of \( f^{-1}(a) \) has positive \( \mathcal{H}^1 \)-measure. Without loss of generality, \( a = 0 \in f(\Omega) \), and \( E \) is a component of \( f^{-1}(0) \) so that \( \mathcal{H}^1(E) > 0 \). Then Theorem 1.1 is proved if we can show that \( \mathcal{H}^1(E) \) has to be zero.

We denote the projection \((x_1, \ldots, x_n) \mapsto x_1\) by \( \text{pr} \). By scaling and rotating, if necessary, we may assume that \( \mathcal{H}^1(\text{pr}(E)) = 1 \). By Theorem 2.3, there exists a domain \( G \subset \subset \Omega \) so that \( E \subset G \), and a number \( M > 0 \) so that \( |f(x)| \geq M \) for every \( x \in \partial G \). Moreover, by Theorem 2.1, there exists \( m \in \mathbb{N} \) so that

\[
\mu(y, f, G) = m \quad \text{for every } y \in B(0, M).
\]

For \( 0 < R < M \), we denote the \( E \)-component of \( f^{-1}(B(0, R)) \) by \( E_R \). Then \( E_R \subset G \). We define radial segments

\[
I(R, \phi) = \{(t, \phi) : t \in (R/2, R)\}
\]

in polar coordinates, and denote \( A_R = B(0, R) \setminus \overline{B}(0, R/2) \), and \( U_R = E_R \cap f^{-1}(A_R) \). The first main ingredient in the proof of Theorem 1.1 is the following.

**Proposition 3.1.** There exists \( 0 < M_0 < M \), so that for each \( R < M_0 \) there exist \( \phi_R \in S(0, 1) \) and \( a_R \in \mathbb{R} \), so that if we denote

\[
L_R = (a_R - (4m)^{-1}, a_R + (4m)^{-1}),
\]

then

\[
L_R \subset \text{pr}(E) \quad \text{and} \quad \text{pr}^{-1}(L_R) \cap E_R \cap f^{-1}(I(R, \phi_R)) = \emptyset.
\]

**Proof.** For \( R < M \), define

\[
h_R : U_R \to S(0, 1), \quad h_R(x) = \frac{f(x)}{|f(x)|}.
\]
Then $h^{-1}_R (\phi) = f^{-1}(I(R, \phi)) \cap E_R$ for every $\phi \in S(0, 1)$. Also, we have
\[ |J_{n-1} h_R(x)| \leq \frac{|D^2 f(x)|}{|f(x)|^{n-1}} \text{ a.e. } x \in U_R, \]
for the $(n - 1)$-dimensional Jacobian of $h_R$. Then, the coarea formula (cf. [10]), and Hölder’s inequality yield
\[
\int_{S(0,1)} \mathcal{H}^1(h^{-1}_R (\phi)) \, d\mathcal{H}^{n-1}(\phi) = \int_{U_R} |J_{n-1} h_R| \leq \int_{U_R} \frac{|D^2 f|}{|f|^{n-1}}
\]
\[
= \left( \int_{U_R} \frac{K_I^{1/n} f^{(n-1)/n}}{|f|^{n-1}} \right) \leq \left( \int_{U_R} K_I \right)^{1/n} \left( \int_{U_R} \frac{J_f}{|f|^n} \right)^{(n-1)/n}.
\]

Since $\mu(y, f, E_R) \leq m$ for every $y \in B(0, R)$, and $U_R \subset E_R$, the change of variables formula gives
\[
\int_{U_R} J_f \frac{1}{|f|^n} \leq \int_{A_R} \frac{\mu(y, f, E_R)}{|y|^n} \, dy \leq m \omega_{n-1} \log 2.
\]
Moreover, by Theorem 2.2,
\[
\int_{U_R} K_I \to 0 \text{ as } R \to 0.
\]

Now, by combining (3.2), (3.3) and (3.4), we have: for every $\epsilon > 0$ there exists $k < M$ so that
\[
\int_{S(0,1)} \mathcal{H}^1(E_R \cap f^{-1}(I(R, \phi))) \, d\mathcal{H}^{n-1}(\phi) < \epsilon
\]
for every $R \leq k$. Moreover, by slightly changing the set $A_R$, we see that (3.5) also holds for $W_{R,\phi} = E_R \cap f^{-1}(I(R, \phi))$.

Let $R$ be as above. Next, we claim that, for each $\phi \in S(0, 1)$, $W_{R,\phi}$ consists of at most $m$ components. Fix $\phi$ and let $\{J_i\}, i = 1, \ldots, N$ be a finite set of preimage components of $I(R, \phi)$ in $E_R$. Denote by $I_\delta$ the closed $\delta$-neighborhood of $I(R, \phi)$. Then $I_\delta$ has $N_\delta$ different preimage components $\tilde{I}_j^\delta$ containing some $J_i$. When $\delta$ is small enough, $\tilde{I}_j^\delta \subset G$ for every $j = 1, \ldots, N_\delta$. Then, by Lemma 2.5, $f(\tilde{I}_j^\delta) = I_\delta$ for $\delta$ small enough. Moreover, for $\delta < \delta_0$ we have $N_\delta = N$. Then, if $y \in I(R, \phi)$, Theorem 2.1 and (2.3) yield
\[ N \leq \sum \mu(y, f, \text{int } \tilde{I}_j^\delta) \leq \mu(y, f, G) = m. \]
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This proves the claim.

Suppose that \( M_0 < M \) is small enough, so that (3.5) holds with \( \epsilon = \omega_{n-1}(100m)^{-1} \). Then, in particular, for every \( R \leq M_0 \) there exists \( \phi_R \in S(0,1) \) such that

\[
\mathcal{H}^1(\text{pr}(W_R,\phi_R)) < (100m)^{-1}.
\]

Moreover, we showed that \( W_R,\phi_R \) consists of at most \( m \) components. Now the proposition follows from our assumption \( \mathcal{H}^1(\text{pr}(E)) = 1 \). \( \square \)

4. Modulus estimates and the proof of Theorem 1.1

In this section we prove Theorem 1.1, except for an upper bound for the conformal modulus of certain \((n-1)\)-dimensional sets (Proposition 4.2). For a measurable function \( \omega \in L^1_{\text{loc}}(\Omega), \Omega \subset \mathbb{R}^n \), and a family \( \Lambda = \{V_i : i \in I\} \) of Borel sets, set

\[
M_\omega \Lambda = \inf_{\rho \in X(\Lambda)} \int_{\Omega} \omega \rho^{n/(n-1)},
\]

where \( X(\Lambda) \) is the set of all Borel functions \( \rho : \Omega \to [0,\infty] \) satisfying

\[
\int_{V_i} \rho \, d\mathcal{H}^{n-1} \geq 1
\]

for every \( V_i \in \Lambda \) with \( \mathcal{H}^{n-1}(V_i) > 0 \). If \( \omega = 1 \) almost everywhere in \( \Omega \), we denote \( M_\omega \) by \( M \).

Now fix \( R \) and \( a_R \) as in Proposition 3.1. Denote \( l = ((8m)^{-1},(4m)^{-1}) \),

\[
V_t^+ = E_R \cap \text{pr}^{-1}(\{a_R + t\}), \quad V_t^- = E_R \cap \text{pr}^{-1}(\{a_R - t\}), \quad V_t = V_t^+ \cup V_t^-,
\]

\[
Q^+_R = \{x \in V_t^- : t \in l\}, \quad Q^-_R = \{x \in V_t^+ : t \in l\},
\]

and

\[
\Lambda_R = \{V_t : t \in l\}.
\]

Lemma 4.1. We have

\[
(16m)^{-n/(n-1)} \left( \int_{E_R} K_I \right)^{-1/(n-1)} \leq M_{K_I^{-1/(n-1)}} \Lambda_R \leq m M f(\Lambda_R).
\]

Proof. Since \( f \in W^{1,n}(E_R,\mathbb{R}^n) \), the restrictions of \( f \) to the components \( G_t \) of \( V_t \) belong to \( W^{1,n}(G_t,\mathbb{R}^n) \) for almost every \( t \in l \). In particular, for those \( t \) the change of variables formula holds in \( V_t \), see [12]. Also, Theorems 2.1 and 2.2 show that \( \mathcal{H}^{n-1}(f(V_t)) > 0 \) for almost every \( t \in l \).
Now fix $\rho \in X(f(\Lambda_R))$. Then, for almost every $t \in \mathcal{L}$, the change of variables formula yields

\[(4.1) \quad \int_{V_t} (\rho \circ f)|D^2 f|d\mathcal{H}^{n-1} \geq \int_{fV_t} \rho d\mathcal{H}^{n-1} \geq 1,\]
i.e. the function $\rho' : E_R \to [0, \infty]$, defined as $\rho'(x) = (\rho \circ f)(x)|D^2 f(x)|$ for $x \in V_t$, $t \in \mathcal{L}$, when (4.1) holds, $\rho'(x) = \infty$ when $x \in V_t$, $t \in \mathcal{L}$, and (4.1) does not hold, and $\rho'(x) = 0$ otherwise, belongs to $X(\Lambda_R)$. Now, by using the change of variables formula in $E_R$, with the fact that $\mu(y, f, E_R) \leq m$ for every $y \in B(0, R)$,

we have

\[
\int_{E_R} (\rho')^{n/(n-1)} K_I^{-1/(n-1)} = \int_{E_R} (\rho \circ f)^{n/(n-1)}|D^2 f|^{n/(n-1)} K_I^{-1/(n-1)}
\]
\[
= \int_{E_R} (\rho \circ f)^{n/(n-1)} J_f
\]
\[
\leq \int_{R^n} \rho(y)^{n/(n-1)} \mu(y, f, E_R) dy \leq m \int_{R^n} \rho^{n/(n-1)}.\]

Since $\rho \in X(f(\Lambda_R))$ is arbitrary, the second inequality in the lemma follows.

To prove the first inequality, fix $g \in X(\Lambda_R)$. Then, for every $t \in \mathcal{L}$,

\[1 \leq \int_{V_t^+} g d\mathcal{H}^{n-1} + \int_{V_t^-} g d\mathcal{H}^{n-1}.\]

By Fubini’s theorem,

\[(8m)^{-1} \leq \int_{Q_R^+} g + \int_{Q_R^-} g,\]

so that one of the integrals, say the one over $Q_R^+$, is greater than $(16m)^{-1}$. Then, Hölder’s inequality yields

\[(4.2) \quad (16m)^{-1} \leq \int_{Q_R^+} g K_I^{-1/n} K_I^{1/n} \leq \left( \int_{Q_R^+} g^{n/(n-1)} K_I^{-1/(n-1)} \right)^{(n-1)/n} \left( \int_{Q_R^+} K_I \right)^{1/n}.\]

Since $g$ is arbitrary, (4.2) proves the first inequality in the lemma. \hfill \Box

In order to complete the proof of Theorem 1.1, we need an upper bound for $Mf(\Lambda_R)$.

**Proposition 4.2.**

\[Mf(\Lambda_R) \leq C,\]

where $C > 0$ only depends on $n$. 

We will prove Proposition 4.2 in Section 5. Assuming the proposition, Theorem 1.1 now follows: combining Lemma 4.1 with the proposition yields
\[(16m)^{-n/(n-1)}\left(\int_{E_R} K_I\right)^{-1/(n-1)} \leq mC,\]
where C does not depend on R. Thus,
\[\int_{E_R} K_I \geq T > 0,\]
with T independent of R. This contradicts Theorem 2.2, since
\[\bigcap_{R>0} E_R = E.\]
We conclude that \(\mathcal{H}^1(E) = 0\), as desired.

5. Proof of Proposition 4.2

We assume that \(n \geq 3\). For \(n = 2\) the proposition is trivial. The idea for the proof is to show, using Proposition 3.1, that the sets \(f(V_t)\) separate \(I(R, \phi_R)\) and another “large” set in \(A_R\). There are some technicalities, though, that slightly complicate matters.

Fix a point \(\xi \in \text{pr}^{-1}(a_R) \cap E\), and denote by \(W\) the \(\xi\)-component of \(\mathbb{R}^n \setminus (V_{(8m)^{-1}} \cup \partial E_R)\). Notice that, by the definition of \(V_t\),
\[(5.1) \quad \text{pr}(W) \subset (a_R - (8m)^{-1}, a_R + (8m)^{-1}).\]

**Lemma 5.1.** For almost every \(r \in (R/2, R)\) there exist \(q_r \in W\) and a neighborhood \(U_r \subset W\) of \(q_r\) so that \(|p_r| = |f(q_r)| = r\) and
\[f^{-1}(p_r) \cap U_r = \{q_r\}.\]

**Proof.** First, by Proposition 3.1, there exists a segment \(\alpha\) joining \(\partial E_R\) and \(\xi\) in \(W \cap \text{pr}^{-1}(a_R)\). Fix a small \(\epsilon > 0\). Then, for any \(x \in B^{n-1}(0, \epsilon)\), we can choose a segment \(\alpha_x\) as follows: if \(\tilde{\alpha}\) is the line spanned by \(\alpha\), then \(\tilde{\alpha}_x = \tilde{\alpha} + x\), \(x \in B^{n-1}(0, \epsilon) \subset H\), where \(H \ni 0\) is the hyperplane orthogonal to \(\tilde{\alpha}\). Moreover, \(\alpha_x\) is a segment in \(\tilde{\alpha}_x\) joining \(\partial E_R\) and \(B(\xi, \epsilon)\) in \(W\). Choose \(\epsilon\) to be small enough, so that \(f(\alpha_x)\) connects \(S(0, R)\) and \(S(0, R/2)\) for every \(x \in B^{n-1}(0, \epsilon)\).

By the definition of a mapping of finite distortion, and Theorems 2.1 and 2.2, there exists \(x_0 \in B^{n-1}(0, \epsilon)\) so that
(1) $f$ is absolutely continuous on $\alpha_{x_0}$,
(2) $f$ is differentiable $\mathcal{H}^1$-almost everywhere on $\alpha_{x_0}$,
(3) $J_f > 0$ $\mathcal{H}^1$-almost everywhere on $\alpha_{x_0}$.

If $f$ is differentiable at $z \in \alpha_{x_0}$, and $J_f(z) > 0$, then, for every $\nu > 0$ small enough,
\[
f(z) \notin f(S(z, \nu)).
\]
Because this is true for almost every $z \in \alpha_{x_0}$, the absolute continuity of $f$ on $\alpha_{x_0}$ completes the proof. \hfill $\Box$

Denote by $D$ the exceptional set in Lemma 5.1. For a radius $r \in (R/2, R) \setminus D$, denote $\{\beta_r\} = S(0, r) \cap I(R, \phi_R)$. By (5.1), Lemma 5.1 and (5.3) below, $\beta_r \neq p_r$ for every $r$.

**Lemma 5.2.** Let $\kappa : [0, 1] \to S(0, r)$ be a one-to-one $C^\infty$-path such that $\kappa(0) = p_r$ and $\kappa(1) = \beta_r$. Then, for every $t \in ((8m)^{-1}, (4m)^{-1})$,
\[
\kappa((0, 1)) \cap f(V_t) \neq \emptyset.
\]

**Proof.** Recall that
\[
pr^{-1}((a_R - (4m)^{-1}, a_R + (4m)^{-1})) \cap E_R \cap f^{-1}(\beta_r) = \emptyset
\]
by Proposition 3.1. For $q_r$ and $U_r$ as in Lemma 5.1, denote by $\tilde{\kappa}$ the $q_r$-component of $f^{-1}(\kappa([0, 1]))$. By using Lemma 2.5 as below, we see that $\tilde{\kappa} \neq \{q_r\}$. Then, by (5.2), we find $s \in (0, 1)$, and a component $\kappa'$ of $f^{-1}(\kappa([s, 1]))$ so that $\kappa' \cap U_r \neq \emptyset$ and $\kappa' \subset \tilde{\kappa}$.

We assume that $\kappa' \cap V_t = \emptyset$. Since $f(\partial E_R) = S(0, R)$, we conclude that $\kappa'$ is compact. On the other hand, $\beta_r = \kappa(1) \notin f(\kappa')$ by (5.3). Thus there exists $t \in (s, 1)$ so that
\[
t = \max\{\tau : \kappa(\tau) \in f(\kappa')\}.
\]
Choose a point $x_t \in f^{-1}(\kappa(t)) \cap \kappa'$. By our assumption on $\kappa'$, the $x_t$-component of $f^{-1}(\kappa(t))$ does not intersect $V_t$. Then there exists a ball $B = B(\kappa(t), \epsilon)$ so that the $x_t$-component $U_t$ of $f^{-1}(B)$ does not intersect $V_t$. By Lemma 2.5 $f(U_t) = B$, and since $\kappa$ is $C^\infty$, applying Lemma 2.5 to the $\epsilon$-neighborhoods of $\kappa((t - \delta, t + \delta))$ for small enough $\delta$, and the $x_t$-components of their preimages, shows that actually $\kappa([t, t + \delta]) \subset f(\kappa')$, contradicting (5.4). The proof is complete. $\Box$
Lemma 5.3. For every \( r \in (R/2, R) \setminus D \), there exists a Borel function \( \rho_r : S(0, r) \to [0, \infty) \) so that, whenever \( t \in ((8m)^{-1}, (4m)^{-1}) \),

\[
\int_{S(0, r) \cap f(V_t)} \rho_r \, d\mathcal{H}^{n-2} \geq C_1/r,
\]
and

\[
\int_{S(0, r)} \rho_r^{n/(n-1)} \, d\mathcal{H}^{n-1} \leq C_2/r,
\]

where the constants \( C_1, C_2 > 0 \) only depend on \( n \).

Proof. We first map \( S(0, r) \) onto \( S(e_n/2, 1/2) \) by a map \( T \) which is a composition of scaling, translation and rotation, so that \( T(\beta_r) = e_n \). Then, if \( \rho : S(e_n/2, 1/2) \to [0, \infty] \) satisfies

\[
\int_{(T \circ f)(V_t)} \rho \, d\mathcal{H}^{n-2} \geq C_1(n)
\]

for all \( t \in ((8m)^{-1}, (4m)^{-1}) \), and

\[
\int_{S(e_n/2, 1/2)} \rho^{n/(n-1)} \, d\mathcal{H}^{n-1} \leq C_2(n),
\]

then the function \( \rho_r = r^{1-n}(\rho \circ T) \) satisfies (5.5) and (5.6). Hence it suffices to show (5.7) and (5.8).

If we map \( S(e_n/2, 1/2) \) onto \( \mathbb{R}^{n-1} \) by the stereographic projection \( h \),

\[
h(x) = e_n + (x - e_n)/|x - e_n|^2,
\]

then \( e_n = T(\beta_r) \) gets mapped to \( \infty \). We denote

\[
a = (h \circ T)(p_r) \in \mathbb{R}^{n-1}.
\]

We define \( \rho : \mathbb{R}^{n-1} \to [0, \infty] \),

\[
\rho(x) = |x - a|^{2-n}(1 + |x|^2)^{n-2},
\]

and denote \( Y_t = (h \circ T \circ f)(V_t) \). Then we have to show that

\[
\int_{Y_t} \rho(x) \, d\mathcal{H}^{n-2}(x) = \int_{Y_t} |x - a|^{2-n} \, d\mathcal{H}^{n-2}(x) \geq C_1(n)
\]

for all \( t \in ((8m)^{-1}, (4m)^{-1}) \), and

\[
\int_{\mathbb{R}^{n-1}} \rho^{n/(n-1)}(x) \, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} |x - a|^{1-n+1/(n-1)} \, d\mathcal{H}^{n-1}(x) \leq C_2(n).
\]
By Lemma 5.2, for every $\alpha \in S^{n-2}(0,1)$, the half-line
\[ I_\alpha = \{ a + \alpha t : t > 0 \} \]
intersects $Y_t$. For $i \in \mathbb{Z}$, denote $A_i = B(a, 2^i) \setminus B(a, 2^{i-1})$, and
\[ \Phi_i = \{ \alpha \in S^{n-2}(0,1) : I_\alpha \cap A_i \cap Y_t \neq \emptyset \} . \]

Then a projection argument shows that
\begin{equation}
\tag{5.11}
\int_{Y_t \cap A_i} |x - a|^{2-n} \, d\mathcal{H}^{n-2}(x) \geq C(n)\mathcal{H}^{n-2}(\Phi_i).
\end{equation}

Since $\sum_i \mathcal{H}^{n-2}(\Phi_i) = \omega_{n-2}$, (5.9) follows by summing over $i$.

In order to prove (5.10), we first consider the case $|a| > 1$. We divide $\mathbb{R}^{n-1}$ to $N_1 = B^{n-1}(a,|a|/2)$, $N_2 = B^{n-1}(0,|a|/2)$ and $N_3 = \mathbb{R}^{n-1} \setminus (N_1 \cup N_2)$. Then
\begin{align*}
\int_{N_1} |x - a|^{1-n+1/(n-1)} \, d\mathcal{H}^{n-1}(x) &\leq C|a|^{-\frac{2}{n-1}} \int_{N_1} |x - a|^{1-n+1/(n-1)} \, d\mathcal{H}^{n-1}(x) \\
&\leq C|a|^{-\frac{1}{n-1}},
\end{align*}
and, since $10|x| \geq |x - a|$ for $x \in N_3$,
\begin{align*}
\int_{N_3} |x - a|^{1-n+1/(n-1)} \, d\mathcal{H}^{n-1}(x) &\leq C \int_{N_3} |x - a|^{1-n+1/(n-1)} \, d\mathcal{H}^{n-1}(x) \\
&\leq C|a|^{-\frac{1}{n-1}}.
\end{align*}

Combining the integrals proves (5.10) in the case $|a| > 1$. The case $|a| \leq 1$ is similar, but now it suffices to consider the division $\tilde{N}_1 = B^{n-1}(a,3)$, $\tilde{N}_2 = \mathbb{R}^{n-1} \setminus \tilde{N}_1$.

Define $\rho : A_R \to [0,\infty]$, $\rho(x) = \rho_{|x|}(x)$, where $\rho_r$ is as in Lemma 5.3 for $r \notin D$, and $\rho_r = 0$ otherwise. Since the restrictions of $f$ to the components $G^j_t$ of $V_t$ belong to $W^{1,n}(G^j_t, \mathbb{R}^n)$ for almost every $t$, $f(V_t)$ is countably $(n-1)$-rectifiable.
for \( t \in ((8^m)^{-1}, (4^m)^{-1}) \setminus Q \), where \( \mathcal{H}^4(Q) = 0 \). Then, Lemma 5.3, and the coarea formula for rectifiable sets, cf. [1, Theorem 2.93 and Remark 2.94], yield
\[
\int_{f(V_t)} \rho \, d\mathcal{H}^{n-1} \geq C(n) \int_{R/2}^R \int_{f(V_t) \cap S(0,r)} \rho \, d\mathcal{H}^{n-2} \, dr \geq C(n)
\]
for every \( t \in ((8^m)^{-1}, (4^m)^{-1}) \setminus Q \). Also, by Lemma 5.3,
\[
\int_{A_R} \rho^{n/(n-1)} = \int_{R/2}^R \int_{S(0,r)} \rho^{n/(n-1)} \, d\mathcal{H}^{n-1} \, dr \leq C(n) \int_{R/2}^R \frac{dr}{r} \leq C(n).
\]
By Theorem 2.1, \( M\{f(V_t) : t \in Q\} = 0 \). The proof of Proposition 4.2 is complete.

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