The Atiyah Conjecture and Artinian Rings

Peter A. Linnell and Thomas Schick

Abstract: Let $G$ be a group such that its finite subgroups have bounded order, let $d$ denote the lowest common multiple of the orders of the finite subgroups of $G$, and let $K$ be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Let $\mathcal{U}(G)$ denote the algebra of unbounded operators affiliated to the group von Neumann algebra $\mathcal{N}(G)$, and let $\mathcal{D}(KG, \mathcal{U}(G))$ denote the division closure of $KG$ in $\mathcal{U}(G)$; thus $\mathcal{D}(KG, \mathcal{U}(G))$ is the smallest subring of $\mathcal{U}(G)$ containing $KG$ that is closed under taking inverses. Suppose $n$ is a positive integer, and $\alpha \in M_n(KG)$. Then $\alpha$ induces a bounded linear map $\alpha : \ell^2(G)^n \to \ell^2(G)^n$, and $\ker \alpha$ has a well-defined von Neumann dimension $\text{dim}_{\mathcal{N}(G)}(\ker \alpha)$. This is a nonnegative real number, and one version of the Atiyah conjecture states that $d \text{dim}_{\mathcal{N}(G)}(\ker \alpha) \in \mathbb{Z}$. Assuming this conjecture, we shall prove that if $G$ has no nontrivial finite normal subgroup, then $\mathcal{D}(KG, \mathcal{U}(G))$ is a $d \times d$ matrix ring over a skew field. We shall also consider the case when $G$ has a nontrivial finite normal subgroup, and other subrings of $\mathcal{U}(G)$ that contain $KG$.

Keywords: Atiyah conjecture, group von Neumann algebra.

1. Introduction

In this paper $\mathbb{N}$ will denote the positive integers $\{1, 2, \ldots \}$, all rings will have a 1, subrings will have the same 1, and if $n \in \mathbb{N}$, then $M_n(R)$ will indicate the $n \times n$ matrices over the ring $R$ and $\text{GL}_n(R)$ the invertible matrices in $M_n(R)$.

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Let $G$ be a group, let $\ell^2(G)$ denote the Hilbert space with orthonormal basis the elements of $G$, and let $\mathcal{B}(\ell^2(G))$ denote the bounded linear operators on $\ell^2(G)$. Thus we can write elements $a \in \ell^2(G)$ in the form $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{C}$ and $\sum_{g \in G} |a_g|^2 < \infty$. Then $\mathbb{C}G$ acts faithfully on the left of $\ell^2(G)$ as bounded linear operators via the left regular representation, so we may consider $\mathbb{C}G$ as a subalgebra of $\mathcal{B}(\ell^2(G))$. The weak closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$ is the group von Neumann algebra $\mathcal{N}(G)$ of $G$. Also if $n \in \mathbb{N}$, then $M_n(\mathbb{C}G)$ acts as bounded linear operators on $\ell^2(G)^n$ and the weak closure of this ring in $\mathcal{B}(\ell^2(G)^n)$ is $M_n(\mathcal{N}(G))$. Let $1$ indicate the element of $\ell^2(G)$ which is $1$ at the identity of $G$ and zero elsewhere. Then the map $\theta : \mathcal{N}(G) \rightarrow \ell^2(G)$ is an injection, so we may regard $\mathcal{N}(G)$ as a subspace of $\ell^2(G)$. We can now define $tr : \mathcal{N}(G) \rightarrow \mathbb{C}$ by $tr(a) = a_1$. For $\alpha \in M_n(\mathcal{N}(G))$, we can extend this definition by setting $tr(\alpha) = \sum_{i=1}^n tr(\alpha_{ii})$, where $\alpha_{ij}$ are the entries of $\alpha$. A useful property is that if $\alpha$ is a positive operator, then $tr(\alpha) \geq 0$. Also we can use $tr$ to give any right $\mathcal{N}(G)$-module $M$ a well defined dimension $\dim_{\mathcal{N}(G)} M$, which in general is a non-negative real number or $\infty$ [10, §6.1]. If $e$ is a projection in $M_n(\mathcal{N}(G))$, then $\dim_{\mathcal{N}(G)} e M_n(\mathcal{N}(G)) = tr(e)$. Furthermore if $\alpha \in M_n(\mathcal{N}(G))$, so $\alpha$ is a Hilbert space map $\ell^2(G) \rightarrow \ell^2(G)^n$, then since $\ell^2(G)^n$ is a right $\mathcal{N}(G)$-module, $\dim_{\mathcal{N}(G)} \ker \alpha$ is well defined and is equal to $\dim_{\mathcal{N}(G)} \{ \beta \in \ell^2(G)^n \mid \alpha \beta = 0 \}$. Finally $\mathcal{N}(G)$ has an involution which sends an operator to its adjoint; if $a = \sum_{g \in G} a_g g$, then $a^* = \sum_{g \in G} \bar{a}_g g^{-1}$, where the bar indicates complex conjugation.

A ring $R$ is called regular, or sometimes von Neumann regular, if for every $x \in R$, there exists an idempotent $e \in R$ with $xR = eR$ [5, Theorem 1.1]. It is called finite, or directly finite, if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. Finally a $*$-regular ring $R$ is a regular ring with an involution $*$ with the property that $x \in R$ and $x^* x = 0$ implies $x = 0$. In a $*$-regular ring, given $x \in R$, there is a unique projection $e$ such that $xR = eR$; so $e = e^* = e^2$.

Let $\mathcal{U}(G)$ denote the algebra of unbounded operators on $\ell^2(G)$ affiliated to $\mathcal{N}(G)$ [10, §8]. Then the involution on $\mathcal{N}(G)$ extends to an involution on $\mathcal{U}(G)$, and $\mathcal{U}(G)$ is a finite $*$-regular algebra. Also if $M$ is a right $\mathcal{N}(G)$-module, then $\dim_{\mathcal{N}(G)} M = \dim_{\mathcal{N}(G)} M \otimes_{\mathcal{N}(G)} \mathcal{U}(G)$; in particular $\dim_{\mathcal{N}(G)} e \mathcal{U}(G) = tr(e)$.

For any subring $R$ of the ring $S$, we let $\mathcal{D}(R, S)$ denote the division closure of $R$ in $S$; that is the smallest subring of $S$ containing $R$ that is closed under taking inverses. In the case $G$ is a group and $K$ is a subfield of $\mathbb{C}$, we shall set
\(D(KG) = D(KG, U(G))\). For any group \(G\), let \(\text{lcm}(G)\) indicate the least common multiple of the orders of the finite subgroups of \(G\), and adopt the convention that \(\text{lcm}(G) = \infty\) if the orders of the finite subgroups of \(G\) are unbounded. One version of the strong Atiyah conjecture states that if \(G\) is a group with \(\text{lcm}(G) < \infty\), then the \(L^2\)-Betti numbers of every closed manifold with fundamental group \(G\) lie in the abelian group \(\frac{1}{\text{lcm}(G)} \mathbb{Z}\). This is equivalent to the conjecture that if \(n \in \mathbb{N}\), \(A \in \text{M}_n(\mathbb{Q}G)\) and \(\alpha : \ell^2(G)^n \to \ell^2(G)^n\) is the map induced by left multiplication by \(A\), then \(\text{lcm}(G) \dim_{\mathcal{N}(G)} M \otimes_{KG} \mathcal{N}(G) \in \mathbb{Z}\) [9, Lemma 2.2]. In this paper, we shall consider more generally the case when the coefficient ring is a subfield of \(\mathbb{C}\).

**Definition 1.1.** Let \(G\) be a group with \(\text{lcm}(G) < \infty\), and let \(K\) be a subfield of \(\mathbb{C}\). We say that the *strong Atiyah conjecture* holds for \(G\) over \(K\) if

\[
\text{lcm}(G) \dim_{\mathcal{N}(G)} M \otimes_{KG} \mathcal{N}(G) \in \mathbb{Z}
\]

for all \(\alpha \in \text{M}_n(KG)\).

This is equivalent to the conjecture that if \(M\) is a finitely presented \(KG\)-module, then \(\text{lcm}(G) \dim_{\mathcal{N}(G)} M \otimes_{KG} \mathcal{N}(G) \in \mathbb{Z}\) [10, Lemma 10.7]. Obviously if \(G\) satisfies the strong Atiyah conjecture over \(\mathbb{C}\), then \(G\) satisfies the strong Atiyah conjecture over \(K\) for all subfields \(K\) of \(\mathbb{C}\). The strong Atiyah conjecture over \(\mathbb{C}\) is known for large classes of groups; for example [6, Theorem 1.5] tells us that it is true if \(G\) has a normal free subgroup \(F\) such that \(G/F\) is an elementary amenable group. If \(K\) is the algebraic closure of \(\mathbb{Q}\) in \(\mathbb{C}\), it is known for even larger classes of groups, for example [4, Theorem 1.4] for groups which are residually torsion-free elementary amenable. The following result is well known; see for example [13, Lemma 3].

**Proposition 1.2.** Let \(G\) be a torsion-free group (i.e. \(\text{lcm}(G) = 1\)) and let \(K\) be a subfield of \(\mathbb{C}\). Then \(G\) satisfies the strong Atiyah conjecture over \(K\) if and only if \(D(KG)\) is a skew field.

The purpose of this paper is to generalize Proposition 1.2. We will denote the finite conjugate subgroup of the group \(G\) by \(\Delta(G)\), and the torsion subgroup of \(\Delta(G)\) by \(\Delta^+(G)\) (this is a subgroup, compare [12, Lemma 19.3]). We shall prove

**Theorem 1.3.** Let \(G\) be a group with \(d := \text{lcm}(G) < \infty\) and \(\Delta^+(G) = 1\), and let \(K\) be a subfield of \(\mathbb{C}\) that is closed under complex conjugation. Then \(G\) satisfies the strong Atiyah conjecture over \(K\) if and only if \(D(KG)\) is a \(d \times d\) matrix ring over a skew field.
It seems plausible that if $K$ is a subfield of $\mathbb{C}$ which is closed under complex conjugation and $G$ is a group with $\text{lcm}(G) < \infty$ which satisfies the strong Atiyah conjecture over $K$, then $D(KG)$ is a semisimple Artinian ring. However we cannot prove this, though we are able to prove a slightly weaker result, and to state this we require the following definition.

**Definition 1.4.** Let $R$ be a subring of the ring $S$. The extended division closure, $\mathcal{E}(R, S)$, of $R$ in $S$ is the smallest subring of $S$ containing $R$ with the properties

(a) If $x \in \mathcal{E}(R, S)$ and $x^{-1} \in S$, then $x \in \mathcal{E}(R, S)$.

(b) If $x \in \mathcal{E}(R, S)$ and $xS = eS$ where $e$ is a central idempotent of $S$, then $e \in \mathcal{E}(R, S)$.

Obviously $\mathcal{E}(R, S) \supseteq D(R, S)$. Note that if $\{R_i\}$ is a collection of subrings of $S$ satisfying 1.4(a) and 1.4(b) above, then $\bigcap_i R_i$ is also a subring of $S$ satisfying 1.4(a) and 1.4(b), consequently $\mathcal{E}(R, S)$ is a well defined subring of $S$ containing $R$. Also if $G$ is a group and $K$ is a subfield of $\mathbb{C}$, then we write $\mathcal{E}(KG)$ for $\mathcal{E}(KG, \mathcal{U}(G))$. Observe that, if $G$ is torsion free and if the strong Atiyah conjecture holds for $G$ over $K$, then $D(KG)$ is a division ring, hence $xD(KG) = D(KG)$ for every $0 \neq x \in D(KG)$ and consequently $\mathcal{E}(KG) = D(KG)$ in this case. We are tempted to conjecture that this is always the case. We hope to show in a later paper that this should follow from a suitable version of the Atiyah conjecture.

We shall prove

**Theorem 1.5.** Let $G$ be a group with $\text{lcm}(G) < \infty$, and let $K$ be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Suppose that $G$ satisfies the strong Atiyah conjecture over $K$. Then $\mathcal{E}(KG)$ is a semisimple Artinian ring.

Theorem 1.5 follows immediately from the more general Theorem 2.7 in Section 2. Thus in particular if $K$ is a subfield of $\mathbb{C}$ that is closed under complex conjugation and $G$ is a group with $\text{lcm}(G) < \infty$ which satisfies the strong Atiyah conjecture over $K$, then $KG$ can be embedded in a semisimple Artinian ring. In fact we can remove the hypothesis that $K$ is closed under complex conjugation to obtain

**Corollary 1.6.** Let $G$ be a group with $\text{lcm}(G) < \infty$ and let $K$ be a subfield of $\mathbb{C}$. Suppose that $G$ satisfies the strong Atiyah conjecture over $K$. Then $KG$ can be embedded in a semisimple Artinian ring.
In Section 3 we will show, somewhat unrelated to the rest of the paper, that $KG$ can be embedded in a least subring of $U(G)$ that is $\ast$-regular.

2. Proofs

Let $R$ be a subring of the ring $S$ and let $C = \{ e \in S \mid e$ is a central idempotent of $S$ and $eS = rS$ for some $r \in R \}$. Then we define

$$C(R, S) = \sum_{e \in C} eR,$$

a subring of $S$. In the case $S = U(G)$, we write $C(R)$ for $C(R, U(G))$. For each ordinal $\alpha$, define $E_\alpha(R, S)$ as follows:

$$\begin{align*}
\bullet & \quad E_0(R, S) = R; \\
\bullet & \quad E_{\alpha+1}(R, S) = D(C(E_\alpha(R, S), S)); \\
\bullet & \quad E_\alpha(R, S) = \bigcup_{\beta < \alpha} E_\beta(R, S) \text{ if } \alpha \text{ is a limit ordinal.}
\end{align*}$$

Then $E(R, S) = \bigcup_\alpha E_\alpha(R, S)$. Also in the case $R = KG$ where $G$ is a group and $K$ is a subfield of $\mathbb{C}$, we shall write $E_\alpha(KG)$ for $E_\alpha(KG, U(G))$. If $A \subseteq \mathbb{R}$, then $\langle A \rangle$ will indicate the additive subgroup of $\mathbb{R}$ generated by $A$.

**Lemma 2.1.** Let $G$ be a group, let $R$ be a subring of $U(G)$, let $n \in \mathbb{N}$, and let $x \in R$. Suppose that $xU(G) = eU(G)$ where $e$ is a central idempotent of $U(G)$. Then $\langle \dim_{N(G)} \beta U(G)^n \mid \beta \in M_n(R) \rangle = \langle \dim_{N(G)} \alpha U(G)^n \mid \alpha \in M_n(R + eR) \rangle$.

**Proof.** Set $E = eI_n$, the diagonal matrix in $M_n(R + eR)$ that has $e$'s on the main diagonal and zeros elsewhere. Then $E$ is a central idempotent in $M_n(U(G))$. Obviously

$$\langle \dim_{N(G)} \beta U(G)^n \mid \beta \in M_n(R) \rangle \subseteq \langle \dim_{N(G)} \alpha U(G)^n \mid \alpha \in M_n(R + eR) \rangle,$$

so we need to prove the reverse inclusion. Let $\alpha \in M_n(R + eR)$ and write $\alpha = \beta + E\gamma$ where $\beta, \gamma \in M_n(R)$. Then we have

$$\dim_{N(G)} \alpha U(G)^n = \dim_{N(G)} (\beta + \gamma) EU(G)^n + \dim_{N(G)} \beta (1 - E) U(G)^n.$$

Since $\dim_{N(G)} \beta (1 - E) U(G)^n = \dim_{N(G)} \beta U(G)^n - \dim_{N(G)} \beta EU(G)^n$, it suffices to prove that

$$\dim_{N(G)} E \beta U(G)^n \in \langle \dim_{N(G)} \delta U(G)^n \mid \delta \in M_n(R) \rangle$$

for all $\beta \in M_n(R)$. But $E \beta U(G)^n = \beta (xI_n) U(G)^n$ and the result follows. \(\square\)
Lemma 2.1 immediately gives the following corollary.

**Corollary 2.2.** Let $G$ be a group, let $R$ be a subring of $U(G)$, and let $n \in \mathbb{N}$. Then $\langle \dim_N(G) \alpha U(G)^n \mid \alpha \in M_n(R) \rangle = \langle \dim_N(G) \alpha U(G)^n \mid \alpha \in M_n(C(R)) \rangle$.

**Proof.** Let $e_1, \ldots, e_m$ be central idempotents of $U(G)$ such that for each $i$, there exists $\alpha_i \in R$ with $e_i U(G) = \alpha_i U(G)$. Then by induction on $m$, Lemma 2.1 tells us that the result is true if $\alpha \in M_n(R + e_1 R + \cdots + e_m R)$. Since $M_n(C(R))$ is the union of $M_n(R + e_1 R + \cdots + e_m R)$, the result is proven. □

**Lemma 2.3.** Let $R$ be a subring of the ring $S$, let $n \in \mathbb{N}$, and let $A \in M_n(D(R, S))$. Then there exist $0 \leq m \in \mathbb{Z}$ and $X, Y \in GL_{m+n}(S)$ such that $X \text{diag}(A, I_m) Y \in M_m+n(R)$.

**Proof.** This follows from [3, Proposition 7.1.3 and Exercise 7.1.4] and [7, Proposition 3.4]. □

**Lemma 2.4.** Let $G$ be a group and let $K$ be a subfield of $\mathbb{C}$. Then

\[
\langle \dim_N(G) x U(G)^n \mid x \in M_n(KG), \ n \in \mathbb{N} \rangle = \langle \dim_N(G) x U(G)^n \mid x \in M_n(E(KG)), \ n \in \mathbb{N} \rangle.
\]

**Proof.** Obviously

\[
\langle \dim_N(G) x U(G)^n \mid x \in M_n(KG), \ n \in \mathbb{N} \rangle \subseteq \langle \dim_N(G) x U(G)^n \mid x \in M_n(E(KG)), \ n \in \mathbb{N} \rangle.
\]

We shall prove the reverse inclusion by transfinite induction. So let $n \in \mathbb{N}$ and $x \in M_n(E(KG))$. Then we may choose the least ordinal $\alpha$ such that $x \in M_n(E_\alpha(KG))$. Clearly $\alpha$ is not a limit ordinal, and the result is true if $\alpha = 0$, so we may write $\alpha = \beta + 1$ for some ordinal $\beta$ and assume that the result is true for all $y \in M_n(E_\beta(KG))$. By Corollary 2.2, the result is true for all $y \in M_n(C(E_\beta(KG)))$ and now the result follows from Lemma 2.3. □

The following result from [8] will be crucial for our work here. Because of this, and because we use a slightly different formulation, we state it here.
Lemma 2.5. [8, Lemma 2] Let $G$ be a group, let $n \in \mathbb{N}$, and let $\alpha_1, \ldots, \alpha_n \in \mathcal{U}(G)$. Then $(\sum_{j=1}^{n} \alpha_j \alpha_j^*) \mathcal{U}(G) \supseteq \alpha_1 \mathcal{U}(G)$. It then also follows that

\[
(\sum_{j=1}^{n} \alpha_j \alpha_j^*) \mathcal{U}(G) = \sum_{j=1}^{n} \alpha_j \alpha_j^* \mathcal{U}(G) = \sum_{j=1}^{n} \alpha_j \mathcal{U}(G).
\]

Proof. The case $n = 2$ in the first statement is [8, Lemma 2]; the proof there can easily be modified to give the case for general $n$. Alternatively we can argue as follows: given $\alpha, \beta \in \mathcal{U}(G)$, there exists $\gamma \in \mathcal{U}(G)$ such that $\alpha \alpha^* + \beta \beta^* = \gamma \gamma^*$, by [2, Proposition 3 of Section 53 on p. 239 and Remark 1 of Section 55 on p. 249], because $\mathcal{N}(G)$ is an $AW^*$-algebra. Also it is obvious that $\alpha \mathcal{U}(G) \supseteq \alpha \alpha^* \mathcal{U}(G)$ and hence $\alpha \mathcal{U}(G) = \alpha \alpha^* \mathcal{U}(G)$. Using these facts and induction on $n$, we obtain the first statement of the lemma. Replacing 1 with $j$ in the right hand side, we see that

\[
\alpha_j \mathcal{U}(G) = \alpha_j \alpha_j^* \mathcal{U}(G) \subseteq \left(\sum_{j=1}^{n} \alpha_j \alpha_j^*\right) \mathcal{U}(G),
\]

and equality (2.6) follows. \hfill \qed

Theorem 2.7. Let $G$ be a group and let $K$ be a subfield of $\mathbb{C}$ which is closed under complex conjugation. Suppose there is an $\ell \in \mathbb{N}$ such that $\ell \dim_{\mathcal{N}(G)} \alpha \mathcal{U}(G)^n \in \mathbb{Z}$ for all $\alpha \in \mathfrak{M}_n(KG)$ and for all $n \in \mathbb{N}$. Then $\mathcal{E}(KG)$ is a semisimple Artinian ring.

Proof. First observe that Lemma 2.4 tells us that

\[
\ell \dim_{\mathcal{N}(G)} \alpha \mathcal{U}(G) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{E}(KG).
\]

Next note that the hypothesis tells us that $\mathcal{E}(KG)$ has at most $\ell$ primitive central idempotents. Indeed if $e_1, \ldots, e_{\ell+1}$ are (nonzero distinct) primitive central idempotents, then $e_i e_j = 0$ for $i \neq j$ and we see that the sum $\bigoplus_{i=1}^{\ell+1} e_i \mathcal{U}(G)$ is direct. But

\[
\dim_{\mathcal{N}(G)} \bigoplus_{i=1}^{\ell+1} e_i \mathcal{U}(G) = \sum_{i=1}^{\ell+1} \dim_{\mathcal{N}(G)} e_i \mathcal{U}(G) \geq (\ell + 1)/\ell > 1
\]

by (2.8), and we have a contradiction. Thus $\mathcal{E}(KG)$ has $n$ primitive central idempotents $e_1, \ldots, e_n$ for some $n \in \mathbb{N}$, $n \leq \ell$. For each $i$, $1 \leq i \leq n$, choose $0 \neq \alpha_i \in e_i \mathcal{E}(KG)$ such that $\dim_{\mathcal{N}(G)} \alpha_i \mathcal{U}(G)$ is minimal.
Fix $m \in \{1, 2, \ldots, n\}$. Since $\ell \dim_{\mathcal{N}(G)} \alpha \mathcal{U}(G) \in \mathbb{Z}$ for all $\alpha \in \mathcal{E}(KG)$ by (2.8), we may choose $g_1, \ldots, g_r \in G$ with $\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}) \mathcal{U}(G)$ maximal. Note that if $g_{r+1} \in G$, then

$$\left(\sum_{i=1}^{r+1} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G) \supseteq \sum_{i=1}^{r} g_i \alpha_m \mathcal{U}(G) \supseteq \left(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G)$$

by Lemma 2.5, hence

$$\dim_{\mathcal{N}(G)}\left(\sum_{i=1}^{r+1} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G) \geq \dim_{\mathcal{N}(G)}\left(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G)$$

and by maximality of $\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}) \mathcal{U}(G)$, we see that

$$\dim_{\mathcal{N}(G)}\left(\sum_{i=1}^{r+1} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G) = \dim_{\mathcal{N}(G)}\left(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G).$$

It follows that

$$\left(\sum_{i=1}^{r+1} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G) = \left(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G)$$

and we deduce from Lemma 2.5 that $g \alpha_m \mathcal{U}(G) \subseteq \left(\sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1}\right) \mathcal{U}(G)$ for all $g \in G$. Let $f \in \mathcal{U}(G)$ be the unique projection such that

$$f \mathcal{U}(G) = \sum_{i=1}^{r} g_i \alpha_m \alpha_m^* g_i^{-1} \mathcal{U}(G).$$

Then $gfg^{-1} \mathcal{U}(G) = \sum g_i \alpha_m \alpha_m^* g_i^{-1} \mathcal{U}(G) \subseteq \sum g_i \alpha_m \mathcal{U}(G) \subseteq f \mathcal{U}(G)$ for all $g \in G$, thus $gfg^{-1} \mathcal{U}(G) = f \mathcal{U}(G)$ and we deduce that $gfg^{-1} \mathcal{U}(G) = f \mathcal{U}(G)$ for all $g \in G$. Also $gfg^{-1}$ is also a projection, thus $gfg^{-1} = f$ for all $g \in G$ and we conclude that $f$ is a central projection in $\mathcal{E}(KG)$. Since $f \neq 0$, $f \mathcal{U}(G) \subseteq e_m \mathcal{U}(G)$ and $e_m$ is primitive, we conclude that $f = e_m$ and consequently $\sum_{i=1}^{r} g_i \alpha_m \mathcal{U}(G) = e_m \mathcal{U}(G)$. By omitting some of the terms in this sum if necessary, we may assume that

$$\sum_{1 \leq i \leq r, \; i \neq s} g_i \alpha_m \mathcal{U}(G) \neq e_m \mathcal{U}(G)$$

for all $s$ such that $1 \leq s \leq r$. We make the following observation:

$$\text{If } 0 \neq x \in g_s \alpha_m \mathcal{E}(KG), \text{ then } x \mathcal{U}(G) = g_s \alpha_m \mathcal{U}(G),$$

where $1 \leq s \leq r$. This is because $0 \neq x \mathcal{U}(G) \subseteq g_s \alpha_m \mathcal{U}(G)$ and by minimality of $\dim_{\mathcal{N}(G)} \alpha_m \mathcal{U}(G)$, we see that $\dim_{\mathcal{N}(G)} x \mathcal{U}(G) = \dim_{\mathcal{N}(G)} g_s \alpha_m \mathcal{U}(G)$ and consequently $x \mathcal{U}(G) = g_s \alpha_m \mathcal{U}(G)$.
We claim that $e_mE(KG) = \bigoplus_{i=1}^{r} g_i \alpha_m E(KG)$. Set $\sigma = (\sum_{i=1}^{r} g_i \alpha_m \alpha_i^* g_i^{-1})$. Since $\sigma \mathcal{U}(G) = e_m \mathcal{U}(G)$, we see that $(\sigma + (1 - e_m)) \mathcal{U}(G) \supseteq \sigma \mathcal{U}(G) + (1 - e_m) \mathcal{U}(G) = e_m \mathcal{U}(G) + (1 - e_m) \mathcal{U}(G) = \mathcal{U}(G)$.

Therefore, $\sigma + 1 - e_m$ is invertible in $\mathcal{U}(G)$ and hence $\sigma + 1 - e_m$ is invertible in $E(KG)$. Thus

$$e_m \sigma E(KG) = e_m (\sigma + 1 - e_m) E(KG) = e_m E(KG).$$

Moreover, $\sigma E(KG) \subseteq e_m E(KG)$ and therefore $e_m \sigma E(KG) = \sigma E(KG)$, hence

$$e_m E(KG) = \sigma E(KG) = \sum_{i=1}^{r} g_i \alpha_m E(KG).$$

If this sum is not direct, then for some $s$ with $1 \leq s \leq r$, we have $g_s \alpha_m E(KG) \cap \sum_{i \neq s} g_i \alpha_m E(KG) \neq 0$, and without loss of generality we may assume that $s = 1$.

So let $0 \neq x \in g_1 \alpha_m E(KG) \cap \sum_{i=2}^{r} g_i \alpha_m E(KG)$. Then $0 \neq x \mathcal{U}(G) \subseteq g_1 \alpha_m \mathcal{U}(G)$ and (2.10) shows that $x \mathcal{U}(G) = g_1 \alpha_m \mathcal{U}(G)$. It follows that $g_1 \alpha_m \mathcal{U}(G) \subseteq \sum_{i=2}^{r} g_i \alpha_m \mathcal{U}(G)$, consequently

$$\sum_{i=2}^{r} g_i \alpha_m \mathcal{U}(G) = e_m \mathcal{U}(G),$$

which contradicts (2.9) and our claim is established.

Now we show that $g_1 \alpha_m E(KG)$ is an irreducible $E(KG)$-module. Suppose $0 \neq x \in g_1 \alpha_m E(KG)$. Then $x \mathcal{U}(G) = g_1 \alpha_m \mathcal{U}(G)$ by (2.10) and using Lemma 2.5, we see as before that $xx^* + \sum_{i=2}^{r} g_i \alpha_i \alpha_i^* g_i^{-1} + 1 - e_m$ is a unit in $\mathcal{U}(G)$ and hence is also a unit in $E(KG)$. This proves that $x E(KG) = g_1 \alpha_m E(KG)$ and we deduce that $E(KG)$ is a finite direct sum of irreducible $E(KG)$-modules. It follows that $E(KG)$ is a semisimple Artinian ring.

Proof of Corollary 1.6. Set $k = K \cap \mathbb{R}$, the maximal real subfield of $K$. If $k = K$, then the result is obvious from Theorem 1.5, so we may assume that $k \neq K$. In this case the degree of $K$ over $k$ will be 2 and we may write $K = k(\alpha)$ where $\alpha^2 \in k$. Clearly $G$ satisfies the strong Atiyah over $k$, so we may embed $kG$ into a semisimple Artinian ring $A$ by Theorem 1.5. Now we may embed $KG$ into $M_2(kG)$ in the standard way, which we now describe. Let $F$ denote the free right $kG$-module with basis $\{e, f\}$. Then if $w \in KG$, we may write $w = u + \alpha v$ where $u, v \in kG$, and we define a right $kG$-module map of $F$ by $we = eu + fv$ and
\[wf = e\alpha^2v + fu.\] It is easily checked that \(KG\) acts on the left of \(F\) by right \(kG\)-modules maps, and it follows that we have embedded \(KG\) into \(M_2(kG)\). We deduce that \(KG\) embeds into \(M_2(A)\). Since \(A\) is semisimple Artinian, this matrix ring is also semisimple Artinian [11, Proposition 3.5.10 on p. 85], and the result follows. \(\Box\)

**Proposition 2.11.** Let \(G\) be a group with \(\Delta(G)\) finite and let \(K\) be a subfield of \(\mathbb{C}\) which is closed under complex conjugation and contains all \(|\Delta(G)|\)-th roots of unity, e.g. \(K = \mathbb{C}\) or \(K\) is the algebraic closure of \(\mathbb{Q}\) in \(\mathbb{C}\). Then \(E(KG) = D(KG)\).

**Proof.** If \(e\) is a central idempotent in \(U(G)\), then \(e \in C_G\), in particular \(e \in KG\). The result follows. \(\Box\)

The following result is well known, but we include a proof.

**Lemma 2.12.** Let \(G\) be a group, let \(e\) be a projection in \(N(G)\), and let \(\alpha \in N(G)\). Then \(\text{tr}(e\alpha^*e) \leq \text{tr}(\alpha e^*e)\).

**Proof.** Since \(\text{tr}(xy) = \text{tr}(yx)\) for all \(x, y \in N(G)\), we see that \(\text{tr}(e\alpha\alpha^*(1 - e)) = \text{tr}((1 - e)\alpha\alpha^*e) = 0\). Therefore \(\text{tr}(\alpha e^*e) = \text{tr}(e\alpha\alpha^*e) + \text{tr}((1 - e)\alpha\alpha^*(1 - e))\). Since \(\text{tr}((1 - e)\alpha\alpha^*(1 - e)) \geq 0\), the result follows. \(\Box\)

**Lemma 2.13.** Let \(G\) be a group, and let \((\alpha_n)\) be a sequence in \(N(G)\) converging strongly to \(\alpha\). Suppose that \(\ker \alpha = 0\). Then \(\dim_N(\ker \alpha_n)\) converges to 0.

**Proof.** By the principle of uniform boundedness, \(\|\alpha_n\|\) is bounded. Also by multiplying everything by a unitary operator if necessary, we may assume that \(\alpha\) is positive. Then \(\alpha_n - \alpha\) converges strongly to 0 and \((\alpha_n - \alpha)^*\) is bounded, hence \((\alpha_n - \alpha)^*(\alpha_n - \alpha)\) converges strongly to 0 and in particular \(\lim_{n \to \infty} \text{tr}((\alpha_n - \alpha)^*(\alpha_n - \alpha)) = 0\). Let \(e_n \in N(G)\) denote the projection of \(\ell^2(G)\) onto \(\ker \alpha_n\). Then \(e_n\alpha_n^* = \alpha_n e_n = 0\) and using Lemma 2.12, we obtain
\[
\text{tr}((\alpha_n - \alpha)^*(\alpha_n - \alpha)) \geq \text{tr}(e_n(\alpha_n - \alpha)^*(\alpha_n - \alpha)e_n) = \text{tr}(e_n\alpha^*\alpha e_n) \geq 0.
\]
Thus \(\lim_{n \to \infty} \text{tr}(e_n\alpha^*\alpha e_n) = 0\). Suppose by way of contradiction that \(\lim_{n \to \infty} \dim_N(\ker \alpha_n) \neq 0\). Then by taking a subsequence if necessary, we may assume that \(\dim_N(\ker \alpha_n) > \epsilon\) for some \(\epsilon > 0\), for all \(n \in \mathbb{N}\).
considering the spectral family associated to $\alpha^*\alpha$ [10, Definition 1.68], there is a closed $\alpha^*\alpha$-invariant $\mathcal{N}(G)$-submodule $X$ of $\ell^2(G)$ and a $\delta > 0$ such that \(\dim_{\mathcal{N}(G)}(X) > 1 - \epsilon/2\) and $\alpha^*\alpha > \delta$ on $X$. Because \(\dim_{\mathcal{N}(G)}(X) > 1 - \epsilon/2\) and \(\dim_{\mathcal{N}(G)}(\ker \alpha_n) > \epsilon\), we find that \(\dim_{\mathcal{N}(G)}(X \cap \ker \alpha_n) > \epsilon/2\) (use [10, Theorem 6.7]). Let $f_n$ denote the projection of $\ell^2(G)$ onto $X \cap \ker \alpha_n$, so $\operatorname{tr}(f_n) > \epsilon/2$. Since $\alpha^*\alpha > \delta$ on $X \cap \ker \alpha_n$, $f_n\alpha^*\alpha f_n \geq \delta f_n$, and because of positivity of $\operatorname{tr}$ we see that $\operatorname{tr}(f_n\alpha^*\alpha f_n) \geq \operatorname{tr}(\delta f_n) > \delta\epsilon/2$. Therefore $\operatorname{tr}(e_n\alpha^*\alpha e_n) > \epsilon\delta/2$ by Lemma 2.12, which shows that $\operatorname{tr}(e_n\alpha^*\alpha e_n)$ does not converge to 0, and the result follows. \(\square\)

**Proposition 2.14.** Let $G$ be a group with $\Delta^+(G) = 1$ and let $K$ be a subfield of $\mathbb{C}$ that is closed under complex conjugation. Assume that $\operatorname{lcm}(G) = d \in \mathbb{N}$ and that $G$ satisfies the strong Atiyah conjecture over $K$. Then $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field.

**Proof.** Let $p$ be a prime, let $q$ be the largest power of $p$ that divides $d$, and let $H \leq G$ with $|H| = q$ (so $H$ is a “Sylow” $p$-subgroup of $G$). Set $e = \frac{1}{q} \sum_{h \in H} h$, a projection in $\mathbb{Q}H$. We shall use the center valued von Neumann dimension $\dim^*$, as defined in [10, Definition 9.12]. Since $\Delta^+(G) = 1$, we see that $\dim^*(e \mathcal{U}(G)) = 1/q$ and $\dim^*(\mathcal{U}(G)) = (q-1)/q$. Therefore by [10, Theorem 9.13(1)],

$$
(1 - e)\mathcal{U}(G) \cong e\mathcal{U}(G)^{q-1}
$$

and we deduce that there exist orthogonal projections $e = e_1, e_2, \ldots, e_q \in \mathcal{U}(G)$ (so $e_i e_j = 0$ for $i \neq j$) such that $\sum_{i=1}^q e_i = 1$ and $e_i \mathcal{U}(G) \cong e\mathcal{U}(G)$ for all $i$. By [2, Exercise 13.15A, p. 76], there exist similarities (that is self adjoint unitaries) $u_i \in \mathcal{U}(G)$ with $u_1 = 1$ such that $e_i = u_i e u_i$. There is a countable subgroup $F$ of $G$ such that $u_i \in \mathcal{N}(F)$ for all $i$. By the Kaplansky density theorem [1, Corollary, p. 8] for each $i$ ($1 \leq i \leq q$) there exists a sequence $u_{ij} \in KF$ such that $u_{ij} \to u_i$ as $j \to \infty$ in the strong operator topology in $\mathcal{N}(F)$ with $u_{ij} = 1$ for all $j$. Set $v_j = \sum_{i=1}^q u_{ij} e u_i$. Then $v_j \to \sum_{i=1}^q e_i = 1$ strongly, hence for $1 \leq i \leq q$,

$$
\lim_{j \to \infty} \dim_{\mathcal{N}(F)}(v_j \mathcal{U}(F)) = \lim_{j \to \infty} \dim_{\mathcal{N}(F)}(u_{ij} \mathcal{U}(F)) = 1
$$

by Lemma 2.13. Now $\dim_{\mathcal{N}(F)}(x \mathcal{U}(F)) = \dim_{\mathcal{N}(G)}(x \mathcal{U}(G))$ for all $x \in \mathcal{U}(F)$, consequently

$$
\lim_{j \to \infty} \dim_{\mathcal{N}(G)}(v_j \mathcal{U}(G)) = \lim_{j \to \infty} \dim_{\mathcal{N}(G)}(u_{ij} \mathcal{U}(G)) = 1 \quad \text{for } 1 \leq i \leq q,
$$

and since by assumption $G$ satisfies the strong Atiyah conjecture over $K$, there exists $n \in \mathbb{N}$ such that $\dim_{\mathcal{N}(G)}(v_j \mathcal{U}(G)) = \dim_{\mathcal{N}(G)}(u_{ij} \mathcal{U}(G)) = 1$ for $1 \leq i \leq q$.
for all $j \geq n$, in particular $\dim_{\mathcal{N}(G)}(v_n \mathcal{U}(G)) = \dim_{\mathcal{N}(G)}(u_{in} \mathcal{U}(G)) = 1$ and we conclude that $v_n$ and $u_{in}$ $(1 \leq i \leq q)$ are units in $\mathcal{U}(G)$. Therefore $v_n$ and $u_{in}$ $(1 \leq i \leq q)$ are units in $\mathcal{D}(KG)$ and we deduce that $\sum_{i=1}^{q} u_{in} e \mathcal{D}(KG) = \mathcal{D}(KG)$, because

$$D(KG) = v_n D(KG) = \sum_{i=1}^{q} (u_{in} e u_{in}) D(KG) \subseteq \sum_{i=1}^{q} u_{in} e D(KG) \subseteq D(KG).$$

Since $\dim_{\mathcal{N}(G)} e \mathcal{U}(G) = 1/q$, we see that $\bigoplus_{i=1}^{q} u_{in} e \mathcal{U}(G) = \mathcal{U}(G)$, a direct sum, and we deduce that

$$(2.15) \quad \bigoplus_{i=1}^{q} u_{in} e \mathcal{D}(KG) = \mathcal{D}(KG),$$

also a direct sum.

Now suppose that $\varepsilon$ is a central idempotent in $\mathcal{C}(\mathcal{D}(KG))$. We want to prove that $\varepsilon = 0$ or 1, so assume otherwise. Now $\varepsilon u_{in} e \mathcal{U}(G) \cong \varepsilon e \mathcal{U}(G)$ for all $i$, which implies that $\dim_{\mathcal{N}(G)}(\varepsilon e \mathcal{U}(G)) = q \dim_{\mathcal{N}(G)}(\varepsilon e \mathcal{U}(G))$. Moreover, because of the Atiyah conjecture, $d \dim_{\mathcal{N}(G)}(\varepsilon e \mathcal{U}(G)) \in \mathbb{Z}$. These two observations together imply that $d \dim_{\mathcal{N}(G)}(\varepsilon e \mathcal{U}(G)) \in \mathbb{Q}$. Since this is true for all primes $p$, it follows that $\dim_{\mathcal{N}(G)} \varepsilon e \mathcal{U}(G) \in \mathbb{Z}$, so 0 and 1 are the only central idempotents of $\mathcal{C}(\mathcal{D}(KG))$.

Summing up, we have shown that $\mathcal{C}(\mathcal{D}(KG))$ contains no nontrivial central idempotents. Using Theorem 2.7, we see that $\mathcal{D}(KG)$ is a semisimple Artinian ring with no nontrivial central idempotents. Thus $\mathcal{D}(KG)$ is an $l \times l$ matrix ring over a division ring for some $l \in \mathbb{N}$. In particular, $\mathcal{D}(KG)$ is the direct sum of $l$ mutually isomorphic $\mathcal{D}(KG)$-submodules, so if $f$ is a primitive idempotent in $\mathcal{D}(KG)$, we see that $\dim_{\mathcal{N}(G)}(f \mathcal{U}(G)) = 1/l$. Furthermore Lemma 2.3 (or Lemma 2.4) show that $l \mid d$. On the other hand (2.15) shows that $q \mid l$, for all primes $p$, so $d \mid l$ and the result follows.

**Proof of Theorem 1.3.** If $G$ satisfies the strong Atiyah conjecture over $K$, then $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field by Proposition 2.14. Conversely suppose $\mathcal{D}(KG)$ is a $d \times d$ matrix ring over a skew field $F$. We need to show that if $M$ is a finitely presented $\mathcal{K} \mathcal{U}(G)$-module, then $\text{lcm}(G) \dim_{\mathcal{N}(G)} M \otimes_{\mathcal{K} \mathcal{U}(G)} \mathcal{U}(G) \in \mathbb{Z}$. However

$$M \otimes_{\mathcal{K} \mathcal{U}(G)} \mathcal{U}(G) \cong M \otimes_{\mathcal{K} \mathcal{U}(G)} \text{M}_d(F) \otimes_{\text{M}_d(F)} \mathcal{U}(G),$$

consequently $(M \otimes_{\mathcal{K} \mathcal{U}(G)} \mathcal{U}(G))^d$ is a finitely generated free $\mathcal{U}(G)$-module and we conclude that $\text{lcm}(G) \dim_{\mathcal{N}(G)} M \otimes_{\mathcal{K} \mathcal{U}(G)} \mathcal{U}(G) \in \mathbb{Z}$ as required. \qed
3. Embeddings in $\ast$-regular rings

There are other closures of group rings $KG$ in $\mathcal{U}(G)$ which may be useful, especially when lcm$(G) = \infty$. In general the intersection of regular subrings of a von Neumann regular ring is not regular [5, Example 1.10], however we do have the following result.

**Proposition 3.1.** Let $G$ be a group and let $\{R_i \mid i \in I\}$ be a collection of $\ast$-regular subrings of $\mathcal{U}(G)$. Then $\bigcap_{i \in I} R_i$ is also a $\ast$-regular subring of $\mathcal{U}(G)$.

**Proof.** Set $S = \bigcap_{i \in I} R_i$. Obviously $S$ is a $\ast$-subring of $\mathcal{U}(G)$; we need to show that $S$ is $\ast$-regular, that is given $s \in S$, there is a projection $e \in S$ such that $sS = eS$. We note that $\mathcal{D}(R_i, \mathcal{U}(G)) = R_i$ for all $i$. Indeed if $x \in R_i$ and $x$ is invertible in $\mathcal{U}(G)$, then $xR_i = eR_i$ where $e$ is a projection in $R_i$, consequently $x\mathcal{U}(G) = e\mathcal{U}(G)$ and since $x$ is invertible in $\mathcal{U}(G)$, we must have $e = 1$ and we deduce that $xR_i = R_i$. Similarly $R_ix = R_i$ and thus $x$ is invertible in $R_i$, so $\mathcal{D}(R_i, \mathcal{U}(G)) = R_i$ as asserted. Since $R_i$ is $\ast$-regular, for each $i \in I$, there is a projection $e_i \in R_i$ such that $e_iR_i = sR_i$. We now have $e_i\mathcal{U}(G) = e_j\mathcal{U}(G)$ for all $i, j$ and we deduce that $e_i = e_j$ for all $i, j \in I$, so there exists $f \in S$ such that $f = e_i$ for all $i$. Since $f\mathcal{U}(G) = s\mathcal{U}(G)$, we see that $fs = s$, so $s \in fS$ and hence $sS \subseteq fS$. Thus the result will be proven if we can show that $ss^*S \supseteq fS$. By Lemma 2.5,

$$(ss^* + 1 - f)\mathcal{U}(G) \supseteq (1 - f)\mathcal{U}(G) + s\mathcal{U}(G) = (1 - f)\mathcal{U}(G) + f\mathcal{U}(G) = \mathcal{U}(G)$$

and we see that $ss^* + 1 - f$ is a unit in $\mathcal{U}(G)$. Let $t \in \mathcal{U}(G)$ be the inverse of $ss^* + 1 - f$, so

$$(3.2) \quad (ss^* + 1 - f)t = 1.$$  

Since $\mathcal{D}(R_i, \mathcal{U}(G)) = R_i$ for all $i$, we deduce that $t \in R_i$ for all $i$ and hence $t \in S$. Moreover $fs = s$ and $f(1 - f) = 0$, so if we multiply (3.2) on the left by $f$, we obtain $ss^*t = f$ and the result is proven. \qed

Thus if $K$ is a subfield of $\mathbb{C}$ that is closed under complex conjugation and $G$ is any group, then there is a least subring of $\mathcal{U}(G)$ containing $KG$ that is $\ast$-regular.
Peter A. Linnell and Thomas Schick

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Peter A. Linnell
Department of Mathematics
Virginia Tech
Blacksburg, VA 24061-0123
USA
E-mail: plinnell@math.vt.edu
http://www.math.vt.edu/people/plinnell/

Thomas Schick
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3
D-37073 Göttingen
Germany
Email: schick@uni-math.gwdg.de
http://www.uni-math.gwdg.de/schick/