Mappings of Bounded Distortion Between Complex Manifolds

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Abstract: We obtain Liouville type theorems for holomorphic mappings with bounded s-distortion between $\mathbb{C}^n$ and positively curved Kähler manifolds.

Keywords: bounded distortion, Ricci curvature, Liouville theorem.

1 Introduction

The classic Liouville theorem states that every bounded holomorphic function on the entire complex plane is constant. H.Grötzsch observed that the classic Liouville theorem can be extended to quasi-conformal mappings. A smooth mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called quasi-conformal, if it is orientation preserving, and locally it is a diffeomorphism such that

$$|df|^n \leq KJ(f), \quad a.e. \quad (1.1)$$

for some positive constant $K \geq 1$, where $|df|$ is the operator norm of the Jacobian matrix $df$ and $J(f)$ is the determinant of $df$. It is well-known that every holomorphic mapping $f : \mathbb{C} \to \mathbb{C}$ is quasi-conformal with $K = 1$. More
precisely, $|df|^2 = J(f)$. However, in higher dimensional case, holomorphic mappings are not necessarily quasi-conformal. For example, the holomorphic mapping $f : D = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2| > \frac{1}{2}\} \to \mathbb{C}^2$ with $f(z_1, z_2) = (z_1, z_2^2)$ is not quasi-conformal, but it satisfies $|df|^2 = J(f)$. Here $2 < \dim_{\mathbb{R}} \mathbb{C}^2 = 4$. For more details, one can see Example 2.2 and Example 2.14. Hence, we can consider smooth mappings with bounded $s$-distortion,

$$|df|^s \leq K J(f) \quad (1.2)$$

for some $s \in (0, \infty)$.

In this paper, we consider holomorphic mappings with bounded $s$-distortion between complex manifolds and obtain Liouville type theorem for such mappings. Let’s recall several classic Liouville type theorem for holomorphic mappings and quasi-conformal mappings between manifolds.

**Theorem 1.1** (Yau’s Schwarz Lemma). Let $M$ be a complete Kähler manifold with Ricci curvature bounded from below by $K_1$. Let $N$ be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant $K_2$. Then if there is a non-constant holomorphic mapping $f$ from $M$ into $N$, we have $K_1 \leq 0$ and

$$f^* \left( dS_N^2 \right) \leq \frac{K_1}{K_2} dS_M^2$$

In particular, if $K_1 \geq 0$, every holomorphic mapping from $M$ into $N$ is constant.

For more details about Schwarz Lemma and related Liouville type theorem, we refer the reader to Ahlfors([1]), Yau([20]), Kobayashi([14]), Chen-Yang([4]), Tossati([19]) and reference therein.

The study of the Schwarz Lemma and Liouville type theorem for non-holomorphic quasi-conformal(quasi-regular) mapping was started from Kiernan([12]) in our knowledge. From then, there are many mathematicians study the harmonic mappings with various bounded distortion, for example, Chern, Goldberg, Har’El, Ishihara, Petridis, Shu([3],[6], [7], [8],[9],[17]), etc. We summarize their works wildly in the following:

**Theorem 1.2** (Generalized Schwarz Lemma). Let $M, N$ be complete Riemannian manifolds. Suppose the Ricci curvature of $M$ is bounded below by $-K_1$ and the
sectional curvature of $N$ is bounded from above by $-K_2$ where $K_1, K_2 > 0$. If $f: M \rightarrow N$ is a harmonic $K$-quasi-regular mapping, then

$$f^*(ds_N^2) \leq C \frac{K_1}{K_2} ds_M^2$$

where $C$ is a positive constant depending on $K$ and the dimension of the manifolds. In particular, if $f: \mathbb{R}^m \rightarrow N$ is a harmonic $K$-quasi-regular mapping, then $f$ is a constant.

The common conditions in the Schwarz Lemma and Liouville type theorem are

(1) The target should be negatively curved;

(2) The mapping should satisfy certain bounded distortion condition.

In this paper, we consider the Liouville type theorem for positively curved targets instead of negatively curved ones. By a geometric interpretation of inequality (1.2), we obtain:

**Main Theorem** Let $(N, h)$ be a complete Kähler manifold of complex dimension $n$, and $f: (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \rightarrow (N, h)$ be holomorphic. If $f$ is a mapping with bounded $2s$-distortion and $N$ has the curvature property $(Q_s)$, then $f$ is constant.

In fact, the curvature condition $(Q_s)$ has a geometric explanation. It is equivalent to the Griffiths positivity of the (formal) vector bundle $G = T^*N \otimes K_N^{-1/s}$. For any compact Kähler manifold with $c_1(M) > 0$, the anti-canonical line bundle $K_N'$ is a positive line bundle, so there exists some small $s \in (0, \infty)$ such that the vector bundle $G$ is Griffiths positive, i.e., the curvature condition $Q_s$ can be satisfied automatically (Theorem 2.10).

**Corollary 1.3.** Let $M$ be a compact Kähler manifold with $c_1(M) > 0$. Then there exist a Kähler metric $\omega$ and some $s_0 \in (0, \infty)$ such that any holomorphic mapping $f: \mathbb{C}^n \rightarrow (M, \omega)$ with bounded $s$-distortion, $s \in (0, s_0)$, is constant.

In particular, for $\mathbb{P}^n$, we obtain
Corollary 1.4. If $f : \mathbb{C}^n \longrightarrow \mathbb{P}^n$ is a holomorphic mapping with bounded $s$-distortion, $0 < s < n + 1$, with respect to the canonical metrics, then $f$ is a constant.

There do exist holomorphic mappings of bounded $s$-distortion between $\mathbb{C}^n$ and $\mathbb{P}^n$ for some $s$. For example, the canonical map $f : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \longrightarrow (\mathbb{P}^n, \omega_{FS})$

$$f(z_1, \cdots, z_n) = [1, z_1, \cdots, z_n]$$

is a holomorphic mapping with bounded $(2n + 2)$-distortion and it fails to be a mapping of bounded $s$-distortion for any $s \in (0, 2n + 2)$. In particular, it is not quasiconformal.

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2 Mappings of bounded $s$-distortion between manifolds

In the paper [18], the authors consider a generalized version of mappings with bounded distortion.

Definition 2.1. A smooth mapping $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has bounded $s$-distortion $(0 < s < \infty)$ if it is a constant or a local diffeomorphism and

$$|df|^s \leq KJ(f), \quad \text{for} \quad x \in \mathbb{R}^n$$

for some positive constant $K$.

It is obvious that, mappings of bounded $n$-distortion are quasiconformal. But in general, a holomorphic mapping can be bounded $s$-distortion for some $s$ but not quasiconformal.

Example 2.2. If $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, f(z_1, z_2) = (z_1, z_2^2)$ and $z_1 = x + \sqrt{-1}y, z_2 = s + \sqrt{-1}t$, then $(df)^t \cdot df$ is a $4 \times 4$ diagonal matrix with diagonal entries $1, 1, 4(s^2 + t^2), 4(s^2 + t^2)$. Therefore

$$J(f) = 4(s^2 + t^2), \quad |df| = \max\{1, 2\sqrt{s^2 + t^2}\}$$
There is no positive constant $K$ such that

$$|df|^4 \leq K J(f)$$

holds on $\mathbb{C}^2$. In fact, if $s^2 + t^2 > 1$,

$$\frac{|df|^4}{J(f)} = 4(s^2 + t^2)$$

which is undounded on $\mathbb{C}^2$. However, on $D = \{(z_1, z_2) \in \mathbb{C}^n \mid |z_2| > \frac{1}{2}\}$,

$$\frac{|df|^2}{J(f)} = 1$$

Hence, $f : D \to \mathbb{C}^2, f(z_1, z_2) = (z_1, z_2^2)$ is a holomorphic mapping of bounded 2-distortion, but it is not a mapping of bounded 4-distortion.

Now we go to define mappings of bounded distortion between manifolds. Let $f : (M, g) \to (N, h)$ be a smooth mapping between oriented $n$-dimensional Riemannian manifolds. In the local coordinates $(x^\alpha)$ and $(y^i)$ on $M$ and $N$ respectively, we set

$$J(x, f) = \frac{f^* dh}{dv} = \sqrt{\frac{\det(h_{ij}(f(x))}{\det(g_{\alpha\beta}(x))}} \det \left( \frac{\partial f^i}{\partial x^\alpha} \right)$$

where $f^i = y^i \circ f$. The pointwise operator norm of $df$ with respect to the metrics $g$ and $h$ is given by

$$|df(x)|^2 = \max_{X \neq 0} \frac{|f_* X|^2}{|X|^2} = \max_{X \neq 0} \frac{\sum_{i,j,\alpha,\beta} h_{ij} f^i_\alpha f^j_\beta X^\alpha X^\beta}{\sum_{\alpha,\beta} g_{\alpha\beta} X^\alpha X^\beta}$$

where $f^i_\alpha = \frac{\partial f^i}{\partial x^\alpha}$ and $X = X^\alpha \frac{\partial}{\partial x^\alpha}$. Here and henceforth we sometimes adopt the Einstein convention for summation. It is obvious that $|df(x)|^2$ is the maximal eigenvalue of the positive definite matrix $A = (A_{\alpha\beta})$ with respect to the metric $g$ where $A_{\alpha\beta} = \sum_{i,j} h_{ij} f^i_\alpha f^j_\beta$. So $J(x, f)$ and $|df(x)|$ are well defined and do not depend on the local coordinates.
Definition 2.3. A smooth mapping $f : (M, g) \rightarrow (N, h)$ between oriented $n$-dimensional manifolds is said to have bounded $s$-distortion ($0 < s < \infty$) with respect to the metrics $g$ and $h$ if it is a constant or a local diffeomorphism with

$$|df(x)|^s \leq KJ(x, f) \quad (2.4)$$

for some positive constant $K$.

Now we recall some notations on Kähler manifolds. Let $\{z^\alpha\}_{\alpha=1}^n$ be the local holomorphic coordinates on the Kähler manifold $(M, g)$, then the metric $g$ is locally represented by a Hermitian positive matrix $(g_{\alpha\bar{\beta}})$, that is

$$g_{\alpha\bar{\beta}} = g \left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right)$$

If $\nabla$ is the complexified Levi-Civita connection on $M$, the curvature of $\nabla$ is locally given by

$$R_{\alpha\beta\gamma\delta} = -\frac{\partial^2}{\partial z^\gamma \partial \bar{z}^\delta} g_{\alpha\beta} + g^\lambda\mu \frac{\partial g_{\alpha\lambda}}{\partial z^\gamma} \frac{\partial g_{\beta\mu}}{\partial \bar{z}^\delta}$$

The Ricci curvature of $\nabla$ is

$$Ric(g) = \sqrt{-1} R_{\alpha\beta} dz^\alpha \wedge \bar{d}z^\beta$$

where

$$R_{\alpha\beta} = g^{\gamma\delta} R_{\alpha\beta\gamma\delta} = -\frac{\partial^2 \log \det(g_{\lambda\mu})}{\partial z^\alpha \partial \bar{z}^\beta}$$

For more basic notations of complex geometry, we refer the reader to the book [5].

Let $f : (M, g) \rightarrow (N, h)$ be a holomorphic mapping between Kähler manifolds with complex dimension $n$. In the local holomorphic coordinates $w^\alpha$ and $z^i$ on $M$ and $N$ respectively,

$$df = f^i dw^\alpha \otimes \frac{\partial}{\partial z^i} \in \Gamma (M, T^{1,0} M \otimes f^*(T^{1,0} N)) \quad (2.5)$$

where $f^i = z^i \circ f$ and $f^i_\alpha = \frac{\partial f^i}{\partial w^\alpha}$ since $f$ is holomorphic. The operator norm $|df|^2$ is

$$|df|^2 = \max_{X \neq 0} \frac{|f_*X|^2}{|X|^2_g} = \max_{X \neq 0} \frac{\sum_{i,j,\alpha,\beta} h_{ij} f^i_\alpha \bar{f}^j_\beta X^\alpha \bar{X}^\beta}{\sum_{\alpha,\beta} g_{\alpha\beta} X^\alpha \bar{X}^\beta} \quad (2.6)$$
for any $X = X^n \frac{\partial}{\partial w^n} \in \Gamma(M, T^{1,0}M)$. On the other hand, the Riemannian volume and the Kähler metric is related by

$$dV = \frac{\omega^n}{n!}$$

therefore by formula (2.2),

$$J(f) = \frac{f^*\omega^n_h}{\omega^n_g} = \frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})} \left| \det \left( f^1_{\alpha} \right) \right|^2$$

(2.7)

The real dimension of the manifolds $M$ and $N$ is $2n$. That is, a bounded $2n$-distortion mapping is quasiconformal.

The curvatures of manifolds and $J(f)$ are related by the following formula (see [2]).

**Lemma 2.4.** Let $(M, g)$ and $(N, h)$ be two complete Kähler manifolds of complex dimension $n$. Let $f : (M, g) \rightarrow (N, h)$ be a holomorphic mapping, then at a point $J(f) \neq 0$, the following formula holds

$$Ric(\omega_g) - f^*Ric(\omega_h) = \frac{-1}{2} \partial\bar{\partial} \log J(f)$$

(2.8)

**Proof.** If $\varphi$ is a holomorphic function without zero point,

$$\partial\bar{\partial} \log |\varphi|^2 = \frac{|\varphi|^2 \partial \varphi \wedge \bar{\partial} \varphi - \bar{\varphi} \partial \varphi \wedge \bar{\varphi} \partial \varphi}{|\varphi|^4} = 0$$

Since $\det(f^1_{\alpha})$ is holomorphic, by formula (2.7), we obtain

$$\sqrt{-1} \frac{\partial\bar{\partial}}{2} \log J(f) = \sqrt{-1} \frac{\partial\bar{\partial}}{2} \log \frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})}$$

$$= Ric(\omega_g) - f^*(Ric(\omega_h))$$

\[\Box\]

**Lemma 2.5.** Let $(N, h, \omega_h)$ be a complete Kähler manifold. If $X$ is a holomorphic vector field of $N$, then on a small neighborhood of point $P$ such that $X_P \neq 0$,

$$T = \sqrt{-1} \left( \frac{\partial^2 \log |X^2_h|}{\partial z^k \partial \bar{z}^j} + \sum_{i,j} R_{ijk\ell} \frac{X^i X^j}{|X^2_h|} \right) dz^k \wedge d\bar{z}^\ell$$

(2.9)
is a semi-positive $$(1,1)$$ form where $$R_{ijkl}$$ is the curvature of $$(N,h)$$ given by

$$R_{ijkl} = -\frac{\partial h_{ij}}{\partial z^k} \frac{\partial h_{kl}}{\partial z^l}$$

Proof. We can choose the normal coordinates $$(z^1, \cdots, z^n)$$ centered at the fixed point $$P \in N$$, i.e., $$h_{ij}(P) = \delta_{ij}$$ and $$\frac{\partial h_{ij}}{\partial z^k}(P) = \frac{\partial h_{ij}}{\partial \bar{z}^l}(P) = 0$$. More precisely, on the small neighborhood of $$P$$,

$$h_{ij}(z) = \delta_{ij} - R_{ijkl} z^k \bar{z}^l + O(|z|^3)$$

Now we assume $$X = X^i \frac{\partial}{\partial z^i}, \ |X|_h^2 = \sum_{i,j} h_{ij} X^i \bar{X}^j$$. At point $$P$$,

$$\partial \bar{\partial} |X|_h^2 = \sum_{i,j,k,l} \left( \frac{\partial^2 h_{ij}}{\partial z^k \partial \bar{z}^l} X^i \bar{X}^j + h_{ij} \frac{\partial X^i}{\partial z^k} \bar{\partial} \frac{\partial X^j}{\partial \bar{z}^l} \right) dz^k \wedge d\bar{z}^l$$

that is,

$$\frac{\partial^2 |X|_h^2}{\partial z^k \partial \bar{z}^l} = - \sum_{i,j} R_{ijkl} X^i \bar{X}^j + \sum_i \frac{\partial X^i}{\partial z^k} \frac{\partial X^i}{\partial \bar{z}^l}$$

At the fixed point $$P$$,

$$\frac{\partial^2 \log |X|_h^2}{\partial z^k \partial \bar{z}^l} = - \sum_{i,j} R_{ijkl} X^i \bar{X}^j \frac{|X|_h^2}{|X|_h^2} + \frac{|X|_h^2}{|X|_h^4} \left( \sum_i \frac{\partial X^i}{\partial z^k} \frac{\partial X^i}{\partial \bar{z}^l} \right) - \left( \sum_j X^j \frac{\partial X^j}{\partial z^k} \right) \left( \sum_i X^i \frac{\partial X^i}{\partial \bar{z}^l} \right)$$

If we set

$$T_{kl} = \frac{\partial^2 \log |X|_h^2}{\partial z^k \partial \bar{z}^l} + \sum_{i,j} R_{ijkl} \frac{X^i \bar{X}^j}{|X|_h^2}$$

then by Schwarz inequality,

$$\sum_{k,l} T_{kl} v_k \bar{v}_l = \frac{|X|_h^2}{|X|_h^4} \left( \sum_i \frac{\partial X^i}{\partial z^k} v_k \frac{\partial X^i}{\partial \bar{z}^l} \bar{v}_l \right) - \left( \sum_j X^j \frac{\partial X^j}{\partial z^k} \bar{v}_l \right) \left( \sum_i X^i \frac{\partial X^i}{\partial \bar{z}^l} v_k \right) \geq 0$$

for any $$v \in \mathbb{C}^n$$. □
Definition 2.6. Let \((N, h)\) be a complete Kähler manifold. We say that \((N, h)\) has the curvature property \((Q_s)\) if for any \(u, v \in \mathbb{C}^n - \{0\}\),
\[
\frac{1}{s} \sum_{i,j,k,l} R_{ij}^k h_{kl} u^i \bar{\pi}^j v^k \bar{\nu}^\ell - \sum_{i,j,k,l} R_{ij}^k u^i \bar{\pi}^j v^k \bar{\nu}^\ell > 0
\tag{2.10}
\]
for some constant \(s \in (0, \infty)\).

Remark 2.7. We have a geometric explanation of the curvature formula (2.10). If \(E\) and \(F\) are two holomorphic vector bundles with connections \(\nabla^E\) and \(\nabla^F\), then the curvature of the induced connection on \(E \otimes F\) is
\[
R = R^E \otimes \text{Id}_F + \text{Id}_E \otimes R^F
\]
where \(R^E\) and \(R^F\) are the curvatures of \(E\) and \(F\) respectively. Apply this to the (formal) vector bundle \(G = T^* N \otimes K_N^{1/s}\) we get the curvature of it, which is locally given by
\[
\tilde{R}_{ijkl} = \frac{1}{s} R_{ij}^k h_{kl} - R_{ij}^k
\]
where \(K_N\) is the canonical line bundle of the manifold \(N\). The positivity condition in the definition is nothing but the Griffiths positivity of the vector bundle \(G\), i.e.
\[
\tilde{R}_{ijkl} u^i \bar{\pi}^j v^k \bar{\nu}^\ell > 0
\]

Example 2.8. Let \((\mathbb{P}^n, \omega_{FS})\) be the complex projective space with the Fubini-Study metric. Locally, it can be written as
\[
\omega_{FS} = \frac{\sqrt{-1}}{2} g_{ij} dz^i \wedge d\bar{z}^j
\]
It is well-known that
\[
R_{ijkl} = g_{ij} g_{kl} + g_{ik} g_{jl}, \quad R_{ij} = g^{kl} R_{ijkl} = (n + 1) g_{ij}
\]
It is obvious that, for nonzero \(X\),
\[
\frac{1}{s} R_{ij} |X|^2_j - R_{ijkl} X^k \bar{X}^\ell = \frac{n + 1 - s}{s} |X|^2_j g_{ij} - g_{ik} g_{jl} X^k \bar{X}^\ell
\]
Then basic linear algebra shows that the above matrix is Hermitian positive if and only if
\[
\frac{n + 1 - s}{s} - 1 > 0 \iff 0 < s < \frac{n + 1}{2}
\]
Corollary 2.9. \((\mathbb{P}^n, \omega)\) has the property \((Q_s)\) with \(0 < s < \frac{n+1}{2}\).

Formally, we have

\[ T^* \mathbb{P}^n \otimes (K_{\mathbb{P}^n})^\frac{1}{s} = T^* \mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n} \left( \frac{n+1}{s} \right) \]

which is positive if and only if \(\frac{n+1}{s} > 2\).

More generally, we have

Theorem 2.10. If \(M\) is a compact Kähler manifold with \(c_1(M) > 0\), then there exist a Kähler metric \(\omega\) and some \(s_0 \in (0, \infty)\) such that \((M, \omega)\) satisfies the curvature condition \((Q_s)\) for any \(s \in (0, s_0)\).

Proof. By Yau’s solution of Calabi conjecture([21]), if \(c_1(M) > 0\), there exists a Kähler metric such that

\[ \text{Ric}(\omega) > 0 \]

Since the manifold is compact, there exists \(\varepsilon > 0\) such that

\[ \text{Ric}(\omega) \geq \varepsilon \omega \]

It is obvious that the holomorphic bisectional curvature \(R_{ijkl}\) is also bounded, that is, there exists \(\varepsilon_1 > 0\) such that

\[ R_{ijkl}X^kX^l \xi^i \xi^j \leq \varepsilon_1 |X|^2 |\xi|^2 \]

Hence, there exists \(s_0 \in (0, \infty)\) such that

\[ \frac{|X|^2}{s_0} R_{ijkl} \xi^i \xi^j - R_{ijkl}X^kX^l \xi^i \xi^j \geq \varepsilon - \frac{s_0 \varepsilon_1}{s_0} |X|^2 |\xi|^2 > 0 \]

for any nonzero \(X = \sum X^i \frac{\partial}{\partial z^i}\) and \(\xi = \sum \xi^j \frac{\partial}{\partial z^j}\). The constant \(s_0\) depends on the manifold \(M\) and \(\omega\).

\[ \square \]

Theorem 2.11. Let \((N, h)\) be a complete Kähler manifold of complex dimension \(n\), and \(f : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \rightarrow (N, h)\) be holomorphic. If \(f\) is a mapping with bounded \(2s\)-distortion and \(N\) has the property \((Q_s)\), then \(f\) is constant.
Proof. Let \( \omega_h \) be the corresponding Kähler form of \( h \). On the local holomorphic coordinates \((z^1, \cdots, z^n)\) of \( N \), we can write
\[
\omega_h = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij} dz^i \wedge d\bar{z}^j
\]
If \( f \) is not constant, then for any nontrivial constant holomorphic vector field \( Y = Y^\alpha \frac{\partial}{\partial w^\alpha} \) on \( \mathbb{C}^n \), \( |f_* Y|_h^2 \) is nonzero everywhere by the local diffeomorphism property of mappings of bounded distortion. We claim that
\[
L = \frac{\sqrt{-1}}{2} L_{\gamma \delta} dw^\gamma \wedge d\bar{w}^\delta := \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log |f_* Y|^2 + \frac{1}{s} f^*(\text{Ric}(\omega_h))
\]
is a semi-positive \((1,1)\)-form on \( \mathbb{C}^n \) if \( N \) has curvature property \((Q_s)\).

In fact, we have
\[
f_* Y = \sum_{i,\alpha} \frac{\partial f^i}{\partial w^\alpha} Y^\alpha \frac{\partial}{\partial z^i}, \quad |f_* Y|_h^2 = \sum_{i,j,\alpha,\beta} h_{ij}(f) \frac{\partial f^i}{\partial w^\alpha} \frac{\partial f^j}{\partial w^\beta} Y^\alpha Y^\beta
\]
By a similar computation as Lemma 2.5,
\[
\frac{\partial^2 \log |f_* Y|^2}{\partial w^\alpha \partial \bar{w}^\beta} = -R_{ijk\ell} \cdot \frac{\partial f^k}{\partial w^i} \frac{\partial f^\ell}{\partial w^j} \frac{\partial f^i}{\partial w^\alpha} \frac{\partial f^j}{\partial w^\beta} Y^\alpha Y^\beta + W_{\gamma \delta}
\]
where \((W_{\gamma \delta})\) is a semi-positive Hermitian matrix by Schwarz inequality. In the sense of Hermitian positivity, we obtain
\[
L_{\gamma \delta} \geq \frac{1}{s} R_{ij} \frac{\partial f^i}{\partial w^\alpha} \frac{\partial f^j}{\partial w^\alpha} - R_{ijk\ell} \frac{\partial f^k}{\partial w^i} \frac{\partial f^\ell}{\partial w^j} \frac{\partial f^i}{\partial w^\alpha} \frac{\partial f^j}{\partial w^\beta} \frac{1}{|f_* Y|_h^2} \geq 0
\]
where the last step follows by the the curvature condition \((Q_s)\).

On the other hand, by Lemma 2.4,
\[
\frac{1}{s} f^*(\text{Ric}(\omega_h)) = \frac{1}{s} \text{Ric}(\omega_{\mathbb{C}^n}) - \frac{\sqrt{-1}}{2s} \partial \bar{\partial} \log J(f) = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log (J(f)^{1/2})
\]
By formula (2.11), we obtain
\[
L = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \left( \frac{|f_* Y|_h^2}{J(f)^{1/2}} \right) \geq 0
\]
The weight function
\[ \Phi = \frac{|f\ast Y|^2}{J(f)^2}, \tag{2.16} \]
is plurisubharmonic on \( \mathbb{C}^n \). By the definition (2.6) of \(|df|\),
\[ |f\ast Y|^2 \leq |df|^2 |Y|^2, \tag{2.17} \]
Then we get
\[ \Phi \leq \frac{|df|^2 |Y|^2}{J^{\frac{1}{2}}(f)} \leq K \frac{1}{2} |Y|^2 = K \frac{1}{2} \sum_{\alpha} |Y^\alpha|^2 \]
for some positive constant \( K \) by the definition of bounded \( 2s \)-distortion mapping. Since \( Y \) is a constant vector field on \( \mathbb{C}^n \), we obtain that \( \Phi \) is a plurisubharmonic function bounded from above and so \( \Phi \) is constant on \( \mathbb{C}^n \). That is \( L = 0 \). By formula (2.13), we know
\[ \frac{\partial f^i}{\partial w^\gamma} = 0 \]
for any \( i \) and \( \gamma \). Finally, we obtain \( f \) is anti-holomorphic, and so it is constant. \( \Box \)

**Corollary 2.12.** Let \( M \) be a compact Kähler manifold with \( c_1(M) > 0 \). Then there exist a Kähler metric \( \omega \) and some \( s_0 \in (0, \infty) \) such that any holomorphic mapping \( f: \mathbb{C}^n \rightarrow (M, \omega) \) with bounded \( s \)-distortion, \( s \in (0, s_0) \), is constant.

In particular, for \( \mathbb{P}^n \) we obtain

**Corollary 2.13.** If \( f: \mathbb{C}^n \rightarrow \mathbb{P}^n \) is a holomorphic mapping with bounded \( s \)-distortion, \( 0 < s < n + 1 \), with respect to the canonical metrics, then \( f \) is a constant.

**Proof.** It follows easily from Corollary 2.9. \( \Box \)

**Example 2.14.** Let \( f: (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \rightarrow (\mathbb{P}^n, \omega_{FS}) \) be the canonical map
\[ f(z_1, \cdots, z_n) = [1, z_1, \cdots, z_n] \]
Then \( f \) is a holomorphic mapping with bounded \( (2n + 2) \)-distortion and it fails to be a mapping of bounded \( s \)-distortion for any \( s \in (0, 2n + 2) \). In particular, it is not quasiconformal.
In fact, on the chart $U_0 = \{[z_0, \cdots, z_n] \mid z_0 = 1\}$,

$$\omega_{FS} = \frac{-1}{2} \partial \bar{\partial} \log \left(1 + \sum_{i=1}^{n} |z_i|^2\right) = \frac{-1}{2} \cdot h_{ij} \cdot dz_i \wedge d\bar{z}_j$$

(2.18)

where

$$h_{ij} = \frac{1 + \sum |z_i|^2 \delta_{ij} - z_i \bar{z}_j}{(1 + \sum |z_i|^2)^2}$$

It is obvious that $(h_{ij})$ has two different eigenvalues, $\lambda_{max} = (1 + \sum |z_i|^2)^{-1}$ with multiplicity $(n - 1)$ and $\lambda_{min} = (1 + \sum |z_i|^2)^{-2}$ with multiplicity 1. Therefore,

$$\det(h_{ij}) = \left(1 + \sum_{i=1}^{n} |z_i|^2\right)^{-n-1}$$

(2.19)

By formula (2.7),

$$J(f) = f^*\left(\omega_{FS}^n\right) = \left(1 + \sum_{i=1}^{n} |z_i|^2\right)^{-n-1}$$

(2.20)

On the other hand, on the chart $U_0$, $f(z_1, \cdots, z_n) = (z_1, \cdots, z_n)$ and $df = I$. By formula (2.6), $|df|^2$ is the maximal eigenvalue $\lambda_{max} = (1 + \sum_{i=1}^{n} |z_i|)^{-1}$ of the Hermitian positive matrix $(h_{ij})$. Therefore

$$|df| = \left(1 + \sum_{i=1}^{n} |z_i|^2\right)^{-\frac{1}{2}}$$

(2.21)

By the expression of $J(f)$ and $|df|$, we obtain

$$|df|^2(n+1) = J(f)$$

(2.22)

For any $0 < s < 2(n+1)$, the ratio

$$\frac{|df|^s}{J(f)} = \left(1 + \sum_{i=1}^{n} |z_i|\right)^{n+1-\frac{s}{2}}$$

(2.23)

is unbounded on $\mathbb{C}^n$. In summary, $f : \mathbb{C}^n \longrightarrow \mathbb{P}^n$ is a holomorphic mapping of bounded $(2n + 2)$-distortion and it fails to be a mapping of bounded $s$-distortion for any $s \in (0, 2n + 2)$. 
References


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