Moduli Spaces of Semistable Sheaves of Dimension 1 on $\mathbb{P}^2$

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Abstract: Let $M(d, \chi)$ be the moduli space of semistable sheaves of rank 0, Euler characteristic $\chi$ and first Chern class $dH$ ($d > 0$), with $H$ the hyperplane class in $\mathbb{P}^2$. We give a description of $M(d, \chi)$, viewing each sheaf as a class of matrices with entries in $\bigoplus_{i\geq 0} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. We show that there is a big open subset of $M(d, 1)$ isomorphic to a projective bundle over an open subset of a Hilbert scheme of points on $\mathbb{P}^2$. Finally we compute the classes of $M(4,1), M(5,1)$ and $M(5,2)$ in the Grothendieck ring of varieties, especially we conclude that $M(5,1)$ and $M(5,2)$ are of the same class.

Keywords: Moduli spaces, 1-dimensional sheaves on surfaces, projective plane, BPS states of weight 0.

1. Introduction.

Moduli spaces $M$ of semistable sheaves of dimension 1 on surfaces are very interesting and many people have studied on them. On $K3$ or abelian surfaces, for a large number of $M$, Yoshioka has given explicitly the deformation classes of them in [9]. Le Potier studied a lot on $M$ for $\mathbb{P}^2$ such as their Picard groups and rationalities in [5]. Drézet and Maican studied sheaves of dimension 1 on $\mathbb{P}^2$ with multiplicity 4, 5 and 6, via their locally free resolutions (see [2],[7] and [8]). But except few trivial cases, the classes of $M$ for $\mathbb{P}^2$ in the Grothendieck group of varieties are not known.

Let $M(d, \chi)$ be the moduli space of semistable sheaves of rank 0, first Chern class $dH$ ($d > 0$) and Euler characteristic $\chi$ on $\mathbb{P}^2$, $M(d, \chi) \simeq M(d, \chi')$ if $\chi \equiv \pm \chi' \pmod{d}$. There is a map $\pi : M(d, \chi) \to |dH|$ sending each sheaf to its support. Fibers of $\pi$ over integral curves are isomorphic to their (compactified) Jacobians. But fibers of $\pi$ over non-integral curves are not well understood.
In this paper we build a 1-1 correspondence between pure sheaves of dimension 1 on \( \mathbb{P}^2 \) and pairs \((E, f)\) with \( E \) direct sums of line bundles on \( \mathbb{P}^2 \) and \( f : E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow E \) injective, then after putting a stability condition on these pairs we can view \( M(d, \chi) \) as the moduli space of semistable pairs \((E, f)\). From this point of view, we somehow avoid studying fibers of \( \pi \) over non-integral curves. However for a general \( d \), the moduli space is still very complicated. We are only able to describe a big open set of \( M(d, \chi) \) with \( \chi = 1 \). We have the following proposition which is a generalization of Proposition 3.3.1 in [2] to all multiplicities.

**Proposition 1.1 (Proposition 4.5).** There is an open subset \( W^d \subset M(d, 1) \) with \( M(d, 1) - W^d \) of codimension \( \geq 2 \), and \( W^d \simeq \mathbb{P}(V^d) \) where \( V^d \) is a vector bundle of rank \( 3d \) over \( \text{Hilb}^{[(d-1)(d-2)]/2} (\mathbb{P}^2) \), \( \text{Hilb}^{[n]} (\mathbb{P}^2) \) the Hilbert scheme of \( n \)-points on \( \mathbb{P}^2 \) and \( \Omega^{[n]}_k \) the closed subscheme of \( \text{Hilb}^{[n]} (\mathbb{P}^2) \) parametrizing \( n \)-points lying on a curve of class \( kH \).

Denote by \([X]\) the class of a variety \( X \) in the Grothendieck ring of varieties. For \( d \leq 5 \) and \( \text{g.c.d.}(d, \chi) = 1 \), we compute \([M(d, \chi)]\) and get the following three theorems, with \( L := [A^1] \) the class of the affine line.

**Theorem 1.2 (Theorem 5.1).** For \( d \leq 3 \), \( M(d, 1) = W^d \). Moreover \( W^d \simeq |dH| \simeq \mathbb{P}^{3d-1} \) for \( d = 1, 2 \); \( W^3 \simeq C_3 \) with \( C_3 \) the universal curve in \( \mathbb{P}^2 \times |3H| \).

**Theorem 1.3 (Theorem 5.2).** \([M(4, 1)] = \sum_{i=0}^{17} b_{2i}L^i\) and

\[
\begin{align*}
b_0 &= b_{34} = 1, & b_2 &= b_{32} = 2, & b_4 &= b_{30} = 6, \\
b_6 &= b_{28} = 10, & b_8 &= b_{26} = 14, & b_{10} &= b_{24} = 15, \\
b_{12} &= b_{14} = b_{16} = b_{18} = b_{20} = b_{22} = 16.
\end{align*}
\]

In particular the Euler number \( e(M(4, 1)) \) of the moduli space is 192.

**Theorem 1.4 (Theorem 6.1).** \([M(5, 1)] = [M(5, 2)] = \sum_{i=0}^{26} b_{2i}L^i\) and

\[
\begin{align*}
b_0 &= b_{52} = 1, & b_2 &= b_{50} = 2, & b_4 &= b_{48} = 6, \\
b_6 &= b_{46} = 13, & b_8 &= b_{44} = 26, & b_{10} &= b_{42} = 45, \\
b_{12} &= b_{40} = 68, & b_{14} &= b_{38} = 87, & b_{16} &= b_{36} = 100, \\
b_{18} &= b_{34} = 107, & b_{20} &= b_{32} = 111, & b_{22} &= b_{30} = 112, \\
b_{24} &= b_{26} = b_{28} = 113.
\end{align*}
\]
In particular the Euler number of both moduli spaces is 1695.

**Remark 1.5.** The Euler numbers $e(M(d, \chi))$ have been computed in [4] partially using physics arguments for $M(d, \chi)$ smooth. They have $e(M(d, \chi)) = (-1)^{\dim(M(d, \chi))} n_0^d$ with $n_0^d$ so-called BPS states of weight 0 for the local $\mathbb{P}^2$ (see Equation (4.2) and Table 4 in Section 8.3 in [4]). We see that our result accords with theirs for $d \leq 5$.

**Remark 1.6.** Theorem 1.3 and Theorem 1.4 give the motive decompositions of $M(d, r)$ for $d = 4, 5$, $r$ coprime to $d$. But according to the result in [11] that these moduli spaces admit affine pavings, we also get cell decompositions of them.

The structure of the paper is arranged as follows. In Section 2, we construct a 1-1 correspondence between pure sheaves of dimension 1 and pairs $(E, f)$. The stability condition of $(E, f)$ is given in Section 3. In Section 4 we study the big open set $W^d$ in $M(d, 1)$ and prove Proposition 1.1. Theorem 1.2 and Theorem 1.3 are proved in Section 5 while Theorem 1.4 is proved in the last section—Section 6. We have Appendix A and B where we prove some technical lemmas used in Section 6.

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## 2. Pure sheaves of dimension 1 on $\mathbb{P}^2$.

From now on except otherwise stated, a pair $(E, f)$ on $\mathbb{P}^2$ always satisfies the following two conditions:

\begin{equation}
(2.1) \quad (1) E \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i) \text{ i.e. } E \text{ is a direct sum of line bundles on } \mathbb{P}^2;
\end{equation}

\begin{equation}
(2.2) \quad (2) f \in \text{Hom}(E \otimes \mathcal{O}_{\mathbb{P}^2}(-1), E) \text{ and moreover } f \text{ is injective}.
\end{equation}
Definition 2.1. We say two pairs \((E, f)\) and \((E', f')\) are isomorphic if \(E \cong E'\) and there exist two isomorphisms \(\varphi\) and \(\phi\) from \(E\) to \(E'\) such that the following diagram commutes

\[
\begin{array}{ccc}
E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f} & E \\
\varphi \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)} & \downarrow & \phi \\
E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f'} & E'.
\end{array}
\]

Define two sets as follows

\[A := \{\text{Isomorphism classes of pure sheaves of dimension 1}\};\]
\[B := \{\text{Isomorphism classes of pairs } (E, f)\}.\]

We have a set-map \(\theta\) from \(B\) to \(A\) sending each pair to its cokernel. We want to prove that \(\theta\) is bijective. First we have the following lemma.

Lemma 2.2. Let \(F\) be a sheaf of rank 0 and first Chern class \(dH\) \((d > 0)\) on \(\mathbb{P}^2\), then \(F\) is pure of dimension 1 if and only if \(F\) lies in the following exact sequence with \(E_F\) a direct sum of line bundles.

\[
0 \to E_F \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to E_F \to F \to 0.
\]

Proof. The “if” part is obvious: \(F\) in (2.4) is of rank 0 and has a locally free resolution of length 1, hence \(F\) is pure of dimension 1. To show the “only if”, it is enough to construct the sequence (2.4) for every pure sheaf \(F\). We first follow the construction given by Le Potier in Proposition 3.10 in [5].

Denote by \(\text{Supp}(F)\) the support of \(F\). Since \(F\) is a torsion sheaf, we can take a point \(x \in \mathbb{P}^2 - \text{Supp}(F)\). Let \(U := \mathbb{P}^2 - \{x\}\), then \(U\) is isomorphic to the total space of \(\mathcal{O}_{\mathbb{P}^1}(1)\) on \(\mathbb{P}^1\) with a projection \(p : U \to \mathbb{P}^1\). \(F\) is a sheaf of \(\mathcal{O}_U\)-modules. \(F\) is pure, hence \(p_* F\) is pure and hence of form \(\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(n_i)\). \(p_* F\) has a structure of \(p_* \mathcal{O}_U\)-module which gives a morphism \(f_1 : p_* F \to p_* F \otimes \mathcal{O}_{\mathbb{P}^1}(1)\). Let \(E_F := p^*(p_* F)\). Pull \(f_1\) back to \(U\) and define the following morphism

\[
(2.5) \quad \tilde{f} := (p^* f_1 - \lambda \text{id}_{E_F^*}) \otimes p^* \text{id}_{\mathcal{O}_{\mathbb{P}^1}(-1)} : \tilde{E}_F \otimes p^* \mathcal{O}_{\mathbb{P}^1}(-1) \to \tilde{E}_F,
\]

where \(\lambda\) is the canonical section of \(p^* \mathcal{O}_{\mathbb{P}^1}(1)\). \(\tilde{f}\) is injective and the cokernel is the sheaf \(F\).
On the other hand, \( \mathbb{P}^2 - U = \{ x \} \) is of codimension 2 and \( \widetilde{E}_F \) is a direct sum of line bundles on \( U \), hence both \( \tilde{f} \) and \( \widetilde{E}_F \) can be extended to the whole \( \mathbb{P}^2 \) and we get a resolution of \( F \) on \( \mathbb{P}^2 \) as in (2.4) and \( E_F \simeq j_* \widetilde{E}_F \simeq \bigoplus_i O_{\mathbb{P}^2}(n_i) \) with \( j: U \to \mathbb{P}^2 \) the open immersion. Hence the lemma. \( \square \)

**Remark 2.3.** The form of \( E \) in \((E, f)\) is determined by \( h^0(F(n)) \) with a finite number of \( n \), where \( F = \text{coker}(f) \) and \( F(n) := F \otimes O_{\mathbb{P}^2}(n) \). Moreover \( E_1 \simeq E_2 \) iff \( h^0(F_1(n)) = h^0(F_2(n)) \) for all \( n \).

Lemma 2.2 implies that \( \theta \) is surjective, then we have the injectivity by the following lemma.

**Lemma 2.4.** Take any two exact sequences

\[
0 \longrightarrow E_1 \otimes O_{\mathbb{P}^2}(-1) \xrightarrow{f_1} E_1 \xrightarrow{g_1} F_1 \longrightarrow 0
\]

\[
0 \longrightarrow E_2 \otimes O_{\mathbb{P}^2}(-1) \xrightarrow{f_2} E_2 \xrightarrow{g_2} F_2 \longrightarrow 0,
\]

with \( E_i \) direct sums of line bundles, then \( F_1 \simeq F_2 \) if and only if \( (E_1, f_1) \simeq (E_2, f_2) \).

**Proof.** We only need to show the “only if”. \( E_1 \simeq E_2 \) if \( F_1 \simeq F_2 \) by Remark 2.3. We then want to construct the commutative diagram (2.3). \( f_i \) can be represented by square matrices with entries in \( \bigoplus_{i \geq 0} H^0(O_{\mathbb{P}^2}(i)) \). After some invertible transformation, we can ask \( f_i \) to have the following form

\[
f_i = \begin{pmatrix} I_i & 0 \\ 0 & T_i \end{pmatrix},
\]

with \( I_i \) the identity matrix and \( T_i \) a square matrix with entries in \( \bigoplus_{i \geq 1} H^0(O_{\mathbb{P}^2}(i)) \). Hence we can write \( E_i \simeq K_i \oplus M_i \) and \( E_i \otimes O_{\mathbb{P}^2}(-1) \simeq K_i \oplus N_i \), such that \( f_i \) splits into the direct sum of an identity on \( K_i \) and a morphism \( t_i : N_i \to M_i \) represented by \( T_i \). We then have the following exact sequence which is a minimal free resolution of \( F_i \) (see [3] Page 5 Definition)

\[
0 \longrightarrow N_i \xrightarrow{t_i} M_i \xrightarrow{g_i|_{M_i}} F_i \longrightarrow 0.
\]
Because of the uniqueness of the minimal free resolution (see [3] Page 6 Theorem 1.6), we have the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow N_1 \xrightarrow{t_1} M_1 \xrightarrow{g_1|\alpha_1} F_1 \rightarrow 0 \\
\downarrow \sim \downarrow \sim \downarrow \sim \\
0 \rightarrow N_2 \xrightarrow{t_2} M_2 \xrightarrow{g_2|\alpha_2} F_2 \rightarrow 0.
\end{array}
\]

Hence we have \( K_1 \simeq K_2 \) because \( E_1 \simeq E_2 \) and \( M_1 \simeq M_2 \). We define a map \( \phi : E_1 \to E_2 \) to be \( I_{K_1} \oplus \alpha \) with \( I_{K_1} \) an isomorphism from \( K_1 \) to \( K_2 \), and similarly we define the other map \( \varphi \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)} : E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \) to be \( I_{K_1} \oplus \beta \). Then we have the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f_1} E_1 \xrightarrow{g_1} F_1 \rightarrow 0 \\
\downarrow \sim \downarrow \sim \downarrow \sim \\
0 \rightarrow E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f_2} E_2 \xrightarrow{g_2} F_2 \rightarrow 0.
\end{array}
\]

This finishes the proof of the lemma. \( \square \)

We finally get the following proposition.

**Proposition 2.5.** There is a 1-1 correspondence between isomorphism classes of pure sheaves of dimension 1 and isomorphism classes of pairs \((E,f)\).

### 3. The stability condition.

We put a stability condition on our pairs \((E,f)\), so that the map \( \theta \) induces a bijection from semistable pairs to semistable sheaves. Given a pair \((E,f)\) and its image \( F \) via \( \theta \), we write down the exact sequence

\[
0 \rightarrow E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f} E \xrightarrow{g} F \rightarrow 0.
\]

Recall that the slope of a torsion free sheaf \( E \), \( \mu(E) \), is defined as follows

\[
\mu(E) := \frac{deg(E)}{rank(E)};
\]

and for a sheaf \( F \) of dimension 1 we have

\[
\mu(F) := \frac{\chi(F)}{deg(F)}.
\]
We then have \( \mu(E) + 1 = \mu(F) \) for \( E, F \) in the sequence (3.1).

**Definition 3.1.** We say a pair \((E, f)\) is (semi)stable if for any sub-sheaf \( E' \subseteq E \) and \( E' \) a direct sum of line bundles such that \( f^{-1}(E') \cong E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \), we have \( \mu(E')(\leq) < \mu(E) \).

**Lemma 3.2.** \( \theta \) induces a bijection from semistable pairs to semistable sheaves.

**Proof.** Look at the sequence (3.1). To prove the lemma, we only need to prove that \( \forall F' \subseteq F, \exists E' \) a direct sum of line bundles such that \( \theta^{-1}(f|_{f^{-1}(E')}) \). Keep notations the same as in the proof of Lemma 2.2, and we see that \( p_*F' \) is a direct sum of line bundles on \( \mathbb{P}^1 \) and \( p_*F' \subset p_*F \). By following the construction in the proof of Lemma 2.2, we get \( E' \cong j_*p^*_p p_*F' \) a direct sum of line bundles and moreover \( f^{-1}(E') \cong E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \), hence the lemma. \( \Box \)

**Remark 3.3.** By Remark 2.3, we see that \( E_1 \) might not isomorphic to \( E_2 \) while \( F_1 \) is only S-equivalent to \( F_2 \).

From now on let \( H \) be the hyperplane class in \( \mathbb{P}^2 \) and \( u_{d, \chi} \) the class in the Grothendieck group of coherent sheaves on \( \mathbb{P}^2 \), which is of rank 0, first Chern class \( dH \) and Euler characteristic \( \chi \). Denote by \( M(d, \chi) \) the moduli space parametrizing semistable sheaves of class \( u_{d, \chi} \). Then \( M(d, \chi) \) is irreducible (see [5] Theorem 3.1) and the stable locus \( M(d, \chi)^s \) is smooth. \( M(d, \chi) \cong M(d, \chi') \) if \( \chi \equiv \pm \chi' \pmod{d} \). \( M(d, \chi) = M(d, \chi)^s \) if and only if \( g.c.d.(d, \chi) = 1 \). Hence \( M(d, 1) = M(d, 1)^s \) and the moduli space is smooth of dimension \( d^2 + 1 \) for all \( d \geq 1 \). Moreover there is a universal sheaf on \( M(d, 1) \times \mathbb{P}^2 \) by Theorem 3.19 in [5].

Let \( g.c.d(d, \chi) = 1 \), then we can assign every point \( F \) in \( M(d, \chi) \) uniquely to a pair \( (E, f) \) such that \( \text{rank}(E) = d \) and \( c_1(E) = (\chi - d)H \). We view every point in \( M(d, \chi) \) as a pair \( (E, f) \) and stratify \( M(d, \chi) \) by the form of \( E \), then every stratum is a constructible set in \( M(d, \chi) \).

We write down the following lemma for future use.

**Lemma 3.4.** Let \( (E, f) \) be a semistable pair. Let \( D', D'' \) be two direct summands of \( E \) such that \( D' \cong D'' \) and \( f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'' \). Then we have \( \mu(D') \leq \mu(E) \). In particular \( E \) must have the form \( \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^2}(n + i)^{\oplus a_i} \) with \( n \) some integer and \( a_i > 0 \) for all \( 1 \leq i \leq k \).
Proof. Since $D' \simeq D''$ and they are both direct summands of $E$, $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$ if and only if $f^{-1}(D'') = D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \simeq D'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence the first statement.

Write $E \simeq \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(n_i) \oplus a_i$ with $a_i > 0$ and $n_1 > n_2 > \ldots > n_k$, then we want to show that $n_i - n_{i+1} = 1$ for all $1 \leq i \leq k - 1$. Assume $\exists i_0$ such that $n_{i_0} - n_{i_0+1} > 2$, then $f\left((\bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i) \oplus a_i) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\right) \subset \bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i) \oplus a_i$ and $\mu(\bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i) \oplus a_i) > \mu(E)$, which is a contradiction. Hence the statement.

4. A big open subset in $M(d, 1)$.

We want to give a concrete description of an open subset in $M(d, 1)$, where the pairs $(E, f)$ satisfy that $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$. We first have the following lemma.

Lemma 4.1. If $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$, then $(E, f)$ is stable if and only if for any two direct summands $D', D''$ of $E$ such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.

Proof. The lemma is equivalent to Claim 4.2 in [6]. We also prove it here. Because of Lemma 3.4, we only need to prove the “if”. By direct observation we see that if $E' \subset E$ is a direct sum of line bundles such that $\mu(E') > \mu(E)$, then $E'$ is a direct summand of $E$. Hence the lemma.

Define $\tilde{W}^d := \{(E, f) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1} \} \subset M(d, 1)$, we then have the following lemma.

Lemma 4.2. $M(d, 1) - \tilde{W}^d$ is of codimension $\geq 2$ in $M(d, 1)$.

Proof. For any point $x \in \mathbb{P}^2$, denote $Y_x$ to be the open subset of $M(d, 1)$ where the pair $(E, f)$ satisfies that $x \notin \text{Supp}(\text{coker}(f))$. $M(d, 1)$ can be covered by finitely many $Y_x$. According to Proposition 3.14 in [5], $Y_x \cap (M(d, 1) - \tilde{W}^d)$ is of codimension $\geq 2$ in $Y_x$, hence the lemma.

Now we look at a pair $(E, f)$ in $\tilde{W}^d$. $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \not\subset \mathcal{O}_{\mathbb{P}^2}$ by Lemma 4.1, hence the restriction $f_{\text{restr}} : \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$ is nonzero. Therefore we can ask $f$ to identify $\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with a summand $\mathcal{O}_{\mathbb{P}^2}(-1)$
and then $f$ can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ A & 0 & B \end{pmatrix},$$

where $A$ is a $(d - 1) \times 1$ matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and $B$ a $(d - 1) \times (d - 2)$ matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. $B$ provides a morphism $f_B : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d - 1} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 2}$. By Lemma 4.1 the stability condition is equivalent to the following condition

**Condition 4.3.** For any $1 < d' < d$, $f_B(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d' - 1}) \not\subset \mathcal{O}_{\mathbb{P}^2}^{\oplus d' - 2}$.

Let $f_{B^t} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d - 2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 1}$ be a morphism represented by the transform of $B, B^t$. Then Condition 4.3 is equivalent to the following condition

**Condition 4.4.** For any $1 < d'' \leq d - 2$, $f_{B^t}^{-1}(\mathcal{O}_{\mathbb{P}^2}^{\oplus d''}) \not\subset \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d''}$.

We have the following diagram

$$(4.1) \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d - 2} \xrightarrow{f_{B^t}} \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 1} \xrightarrow{f_t} Q_f \to 0.$$  

The injectivity of $f_{B^t}$ is because of the injectivity of $f$. Let $F := \text{coker}(f)$ and $F^\vee := \text{Ext}^1(F, \mathcal{O}_{\mathbb{P}^2})$, then $F^\vee \simeq F$ and moreover $F$ and $F^\vee$ are determined by each other (see [10] Lemma A.0.13). We write down a commutative diagram as follows

$$(4.2) \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d - 2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d - 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-2) \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 1} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 1} \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 1} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d - 1} \to 0.$$
We see that the isomorphism class of $\mathcal{F}$ is determined by the pair $(Q_f, \sigma_f)$, hence so is the isomorphism class of $F$.

Define $W^d := \{(E, f) \in \tilde{W}^d | Q_f \text{ is torsion free}\}$. Then we have the following proposition.

**Proposition 4.5.** $W^d \simeq \mathbb{P}(\mathcal{V}^d)$, where $\mathcal{V}^d$ is a vector bundle of rank $3d$ over $N_0^d := \text{Hilb}^{\frac{(d-1)(d-2)}{2}}(\mathbb{P}^2) - \text{Hilb}^{[\frac{(d-1)(d-2)}{2}]}(\mathbb{P}^2)$ the Hilbert scheme of $n$-points on $\mathbb{P}^2$ and $\Omega_k$ the closed subscheme of $\text{Hilb}^{[n]}(\mathbb{P}^2)$ parametrizing $n$-points lying on a curve of class $kH$.

**Proof.** Condition 4.4 is satisfied automatically for $Q_f$ torsion free. Hence $W^d$ consists of all the pairs $(Q_f, \sigma_f)$ in diagram (4.1) with $Q_f$ torsion free.

Define $\bar{d} := \frac{(d-1)(d-2)}{2}$. If $Q_f$ in diagram (4.1) is torsion free, then by direct calculation we know that $Q_f \simeq I_{\bar{d}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-2)$, with $I_n$ the ideal sheaf of a 0-dimensional subscheme of length $n$ on $\mathbb{P}^2$.

The following lemma shows that $N_0^d$ parametrizes all the torsion free $Q_f$ in diagram (4.1).

**Lemma 4.6.** $I_{\bar{d}}(d-2) := I_{\bar{d}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-2)$ has the following resolution

\begin{equation}
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} \rightarrow I_{\bar{d}}(d-2) \rightarrow 0.
\end{equation}

if and only if $H^0(I_{\bar{d}}(d-3)) = 0$.

**Proof.** The lemma is equivalent to Proposition 4.5 in [1] and it is also a straightforward consequence of Corollary 3.9 in [3] Page 38 and Proposition 3.1 in [3] Page 32. 

Up to scalars, $\sigma_f$ can be viewed as an element in $\mathbb{P}H^0(Q_f(2))$ which is exactly $\text{det}(f)$.

Because $N_0^d$ is open in $Hilb^d(\mathbb{P}^2)$, on $\mathbb{P}^2 \times N_0^d$ we have a universal sheaf $\mathcal{I}_{\bar{d}}$ which restricted to the fiber over each point $[I_{\bar{d}}] \in N_0^d$ is the ideal sheaf $I_{\bar{d}}$. We have the diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{I}_{\bar{d}} & \rightarrow & \mathbb{P}^2 \times N_0^d \\
\downarrow q & & \downarrow p \\
\mathbb{P}^2 & \rightarrow & N_0^d.
\end{array}
\end{equation}

$R^i p_*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d)) = 0$ for $i \geq 1$ and $p_*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d))$ is locally free of rank $3d$. Define $\mathcal{V}^d := p_*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d))$. There is a 1-1 correspondence
between points in \( \mathbb{P}(\mathcal{V}^d) \) and isomorphism classes of \((Q_f, \sigma_f)\) with \(Q_f\) torsion free. To prove the proposition, it is enough to construct a family \( \mathcal{F} \) of stable sheaves of class \( u_{d,1} \) over \( \mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d) \).

We have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d) & \xrightarrow{id_{\mathcal{V}^d} \times \pi} & \mathbb{P}^2 \times N_0^d \\
p \downarrow & & \downarrow p \\
\mathbb{P}(\mathcal{V}^d) & \xrightarrow{\pi} & N_0^d \\
\end{array}
\]

Denote by \(\mathcal{O}_\pi(1)\) the relative polarization on \(\mathbb{P}(\mathcal{V}^d)\) over \(N_0^d\). We have the following exact sequence on \(\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)\)

\[
0 \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)} \to \bar{p}^* \mathcal{O}_\pi(1) \otimes (id_{\mathbb{P}^2} \times \pi)^*(I_d \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d)) \to \mathcal{F}^\vee \to 0.
\]

We see that fiberwise (4.6) is the first vertical exact sequence from the right hand side in (4.2) tensored by \(\mathcal{O}_{\mathbb{P}^2}(2)\). Hence \(\mathcal{F}^\vee\) is a family of stable sheaves of class \(u_{d,1}^\vee\). We get \(\mathcal{F}\) by taking the dual. Hence the proposition.

We now have a concrete description of \(W^d\).

**Proposition 4.7.** \(M(d, 1) - W^d\) is of codimension \(\geq 2\) in \(M(d, 1)\).

**Proof.** By Lemma 4.2, we only need to show that \(\tilde{W}^d - W^d\) is of codimension at least 2 in \(M(d, 1)\). Denote by \(|dH|\) the linear system of divisors of class \(dH\), then non-integral curves form a closed subset of codimension \(\geq 2\) in \(|dH|\).

Therefore by Proposition 2.8 and Lemma 3.2 in [5], we know that stable sheaves with non-integral supports form a closed subset of codimension \(\geq 2\) in \(M(d, 1)\). We then want to show that if \(Q_f\) in (4.1) is not torsion free, then \(Supp(F) = Supp(coker(f))\) is non-integral.

Denote by \(T_f\) the torsion of \(Q_f\). Since \(Q_f\) has a free resolution of length 1, \(T_f\) must be a pure sheaf supported on a curve in \(|d'H|\). Look back to the diagram (4.2), the map \(\delta\) restricted to \(T_f\) gives a nonzero element in \(\text{Hom}(T_f, F^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-2))\). If \(d' < d\), then \(Supp(F^\vee) = Supp(F)\) can not be integral. Now assume \(d' = d\) and we look at the following exact sequence

\[
0 \to T_f \to Q_f \to Q_f^\vee \to 0.
\]

The torsion free sheaf \(Q_f^\vee\) has the form \(I_n(m) := I_n \otimes \mathcal{O}_{\mathbb{P}^2}(m)\) such that \(m + d' = \text{deg}(Q_f) = d - 2\). On the other hand, the surjective morphism \(f_q\)
induces a surjective morphism from $\mathcal{O}_{\mathbb{P}_2}^{\oplus d-1}$ to $Q^f$, hence $m \geq 1$ and thus $d' < d - 2 < d$ which is a contradiction. This finishes the proof.

\[
5. \quad M(d, 1) \text{ with } d \leq 4.
\]

In this section we study $M(d, 1)$ with $d \leq 4$. Notice that for $d \leq 4$, up to isomorphism $M(d, 1)$ is the only moduli space such that there is no strictly semistable locus, since $M(d, \chi) \simeq M(d, \chi')$ if $\chi \equiv \pm \chi' \pmod{d}$.

For $d \leq 3$, $M(d, 1)$ is very easy to understand and the following theorem is already known by Theorem 3.5 and Theorem 5.1 in [5]. But however using our new description we give another proof. Recall that we have defined a big open subset $W^d \subset M(d, 1)$ in the previous section.

**Theorem 5.1.** For $d \leq 3$, $M(d, 1) = W^d$. Moreover $W^d \simeq |dH| \simeq \mathbb{P}^{3d-1}$ for $d = 1, 2$; $W^3 \simeq \mathcal{C}_3$ with $\mathcal{C}_3$ the universal curve in $\mathbb{P}^2 \times |3H|$.

**Proof.** By Lemma 3.4 we see that for $d \leq 3$, the sheaf $E$ in a stable pair $(E, f)$ can only have the form $\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$. From the proof of Proposition 4.7 we see that the torsion of $Q_f$ can only be supported on a curve of degree no bigger than $d - 3$, hence $Q_f$ is always torsion free for $d \leq 3$. Hence the first statement. By direct observation, we get the form of $W^d$ for $d \leq 3$. This finishes the proof. \qed

Denote by $[X]$ the class of a variety $X$ in the Grothendieck ring of varieties. Define $L := [\mathbb{A}^1]$ with $\mathbb{A}^1$ the affine line. We have the following theorem.

**Theorem 5.2.** $[M(4, 1)] = \sum_{i=0}^{17} b_i L^i$ and

\[
\begin{align*}
  b_0 &= b_{34} = 1; \quad b_2 = b_{32} = 2; \quad b_4 = b_{30} = 6; \\
  b_6 &= b_{28} = 10; \quad b_8 = b_{26} = 14; \quad b_{10} = b_{24} = 15; \\
  b_{12} &= b_{14} = b_{16} = b_{18} = b_{20} = b_{22} = 16.
\end{align*}
\]

In particular the Euler number $e(M(4, 1))$ of the moduli space is $192$.

To prove Theorem 5.2, we first define two strata as follows.

\[
\begin{align*}
  M_1 := & \quad \{(E, f) \in M(4, 1)|E \simeq \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 3}\}; \\
  M_2 := & \quad \{(E, f) \in M(4, 1)|E \simeq \mathcal{O}_{\mathbb{P}_2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2}(-2)\}.
\end{align*}
\]

According to Lemma 3.4, $M(4, 1) = M_1 \sqcup M_2$. 

Lemma 5.3. A pair \((E, f)\) with \(\text{rank}(E) = 4\) and \(\text{deg}(E) = -3\) is stable if and only if for any two direct summands \(D', D''\) of \(E\) such that \(D' \cong D''\) and \(f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''\), we have \(\mu(D') < \mu(E)\).

Proof. We only need to prove the lemma for \(E \cong \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\). We want to show that if \(\exists E' \subset E\), \(E'\) is a direct sum of line bundles with \(\mu(E') > \mu(E)\) and \(f^{-1}(E') \cong E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\), then \(\exists D, D' \subset E\) two direct summands with \(D \cong D'\) and \(\mu(D) > \mu(E)\), such that \(f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'\). With no loss of generality, we assume that \(E'\) has the form \(\bigoplus \mathcal{O}_{\mathbb{P}^2}(n_i) \oplus a_i\) with \(a_i > 0\) and \(n_i - n_{i+1} = 1\).

Let \(E' \cong E'' \subset E\) and \(E''\) is not a direct summand of \(E\). Then \(E''\) has to be one of the following two cases:

1. \(E'' \subset \mathcal{O}_{\mathbb{P}^2}^2\) and \(E'' \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)\);
2. \(E'' \subset \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)\) and \(E'' \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\).

Let \(E''\) be in case (1). If \(E'\) is a direct summand of \(E\), then \(f^{-1}(E') \neq E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\) because by Nakayama’s lemma \(E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \subset f^{-1}(E') \Rightarrow \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \subset f^{-1}(E')\) which contradicts the injectivity.

If \(E' = E''\) and \(f^{-1}(E') = E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\), then again by Nakayama’s lemma we have \(f(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\). Hence we get \(D = D' = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\).

If \(E' = E''\) and \(f^{-1}(E')\) is a direct summand of \(E \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\) isomorphic to \(E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\), then we have \(D = \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) = D'\) and \(f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'\), since there is no nonzero morphism from \(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)\) to \(\mathcal{O}_{\mathbb{P}^2}(-2)\). Hence case (1) is done.

Case (2) is analogous and this finishes the proof of the lemma. \(\Box\)

For a pair \((E, f) \in M_2\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
    b_1 & b_2 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    a_1 & a_2 & 0 & 0
\end{pmatrix}
\]

(5.1)

where \(b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))\) and \(a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))\). The injectivity of \(f\) implies that \(\text{det}(f) = b_1a_2 - b_2a_1 \neq 0\). Moreover by Lemma 5.3 \((E, f)\) is stable if and only if \(kb_1 \neq k'b_2\) for any \((k, k') \in \mathbb{C}^2 - \{0\}\).

Lemma 5.4. \([M_2] = [\mathbb{P}^2 \times \mathbb{P}^{13}].\)
Proof. We have the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \stackrel{(b_1,b_2)}{\rightarrow} & \mathcal{O}_{\mathbb{P}^2}^\oplus & \stackrel{f_r}{\rightarrow} & R_f & \rightarrow & 0 \\
 & & & & & \nearrow & \omega_f := f_r \circ (a_1,a_2) & & \\
& & & & \mathcal{O}_{\mathbb{P}^2}(-3) & & & &
\end{array}
\]

Since \(kb_1 \neq k'b_2\) for any \((k,k') \in \mathbb{C}^2 - \{0\}\), we have \(R_f \simeq I_1(1)\) and every \(I_1(1)\) can be put in sequence (5.2). Hence \(M_2\) consists of all the isomorphism classes of pairs \((R_f, \omega_f)\) with \(R_f \simeq I_1(1)\). Hence \(M_2\) is isomorphic to a projective bundle over \(\text{Hilb}^1(\mathbb{P}^2)\) with fibers isomorphic to \(\mathbb{P}(H^0(I_1(4))) \simeq \mathbb{P}^{13}\). Hence the lemma. □

The big open subset \(W^4\) defined in the previous section is contained in \(M_1\). We have \([W^4] = [(\text{Hilb}^{[3]}(\mathbb{P}^2) - \Omega_1^{[3]}) \times \mathbb{P}^{11}]\) by Proposition 4.5.

Lemma 5.5. \(\Omega_1^{[3]} \simeq C_1^{[3]}\) with \(C_1^{[3]}\) the relative Hilbert scheme of 3-points on the universal family \(C_1 \subset \mathbb{P}^2 \times |H|\), and hence \([\Omega_1^{[3]}] = [\mathbb{P}^2 \times \mathbb{P}^3]\).

Proof. We have a natural map \(\xi : C_1^{[3]} \rightarrow \Omega_1^{[3]}\). \(\xi\) is an isomorphism because there is at most one curve in \(|H|\) passing through any 3 points. \(C_1 \rightarrow |H|\) is a \(\mathbb{P}^1\)-bundle, hence the map \(p : C_1^{[3]} \rightarrow |H|\) is a projective bundle with fibers isomorphic to \((\mathbb{P}^1)^{[3]} \simeq \mathbb{P}^3\), therefore \([\Omega_1^{[3]}] = [C_1^{[3]}] = [\mathbb{P}^2 \times \mathbb{P}^3]\). □

Now we want to compute \([M_1 - W^4]\). Look back to diagram (4.1), we want to see what \(Q_f\) will be if it is not torsion free for \(d = 4\). We know that the torsion of \(Q_f\) can only be supported on a curve of degree no bigger than \(d - 3 = 1\) (See the proof of Proposition 4.7). We write down the following exact sequence

\[
0 \rightarrow T_f \rightarrow Q_f \rightarrow Q^{tf}_f \rightarrow 0,
\]

with \(T_f\) the torsion of \(Q_f\) and \(Q^{tf}_f\) a torsion free sheaf of rank 1.

Since \(T_f\) is supported on a curve in \(|H|\) and \(h^0(T_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \leq h^0(Q_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 0, T_f \simeq \mathcal{O}_H(t) \simeq \mathcal{O}_{\mathbb{P}^1}(t)\) with \(t \leq 0\). Let \(Q^{tf}_f \simeq I_n(m)\) with \(m > 0, n \geq 0\). Then we have \(m = 1\) and \(n - t = 1\) by direct calculation.
If $t = 0, n = 1$, then we have the following commutative diagram

\begin{equation}
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_H \rightarrow 0
\end{equation}

which contradicts Condition 4.4. Hence we have $t = -1, n = 0$ and $Q_f$ lies in the following exact sequence

\begin{equation}
0 \rightarrow \mathcal{O}_H(-1) \rightarrow Q_f \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0.
\end{equation}

$\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{C}$, so for a fixed projective line $\mathbb{P}^1$ of class $H$, if (5.4) does not split, $Q_f$ is unique up to isomorphism.

**Lemma 5.6.** $Q_f$ in (5.4) also lies in the following exact sequence (5.5) if and only if (5.4) does not split.

\begin{equation}
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^\oplus 2 \rightarrow \mathcal{O}_{\mathbb{P}^2}^\oplus 3 \rightarrow Q_f \rightarrow 0.
\end{equation}

**Proof.** If $Q_f \cong \mathcal{O}_H(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, it certainly can not lie in (5.5). If the sequence (5.4) does not split, then $Q_f$ is unique, so we only need to construct the sequence (5.5) with $Q_f$ contains $\mathcal{O}_H(-1)$ as its torsion. Write $\mathbb{P}^2 = \text{Proj}\mathbb{C}[x_0, x_1, x_2]$. With no loss of generality we assume that $\mathcal{O}_H(-1)$ is supported on $\{x_0 = 0\}$, then the following matrix represents a morphism $f_{B'} : \mathcal{O}_{\mathbb{P}^2}(-1)^\oplus 2 \rightarrow \mathcal{O}_{\mathbb{P}^2}^\oplus 3$ such that $\text{coker}(f_{B'})$ contains $\mathcal{O}_{\{x_0 = 0\}}(-1)$ as its torsion.

\begin{equation}
f_{B'} := \begin{pmatrix}
x_0 & 0 \\
x_1 & x_2 \\
0 & x_0
\end{pmatrix}.
\end{equation}

This finishes the proof. $\square$

**Remark 5.7.** $f_{B'}$ defined in (5.6) also satisfies the stability condition i.e. Condition 4.4.
Lemma 5.8. Decompose $|H|$ into cells and write $|H| = \bigcup_{i=0}^2 A^i$. Then $A^i$ parametrizes isomorphism classes of $Q_f$ such that there are pairs $(Q_f, \sigma_f) \in M_1 - W^4$ and $T_f$ are supported on curves in $A^i \subset |H|$.

Proof. Lemma 5.6 implies that there is a 1-1 correspondence between isomorphism classes of $Q_f$ and points in $|H|$. We need to decompose $|H|$ into cells so that we have a universal family over $\mathbb{P}^2 \times A^i$ for each $i$.

We have the following diagram

$$
\begin{array}{ccc}
C_1 & \longrightarrow & \mathbb{P}^2 \times |H| \\
\downarrow q & & \downarrow p \\
\mathbb{P}^2 & \longrightarrow & |H|
\end{array}
$$

with $C_1$ the universal curve of degree 1.

$\operatorname{Ext}^i(O_{\mathbb{P}^2}(1), O_H(-1)) = 0$ for all $i \neq 1$ and $\operatorname{Ext}^1(O_{\mathbb{P}^2}(1), O_H(-1)) \simeq \mathbb{C}$, therefore $L := \operatorname{Ext}^1_p(q^*O_{\mathbb{P}^2}(1), O_{C_i} \otimes q^*O_{\mathbb{P}^2}(-1))$ is a line bundle on $|H|$. Moreover $ch(L) = -ch(R^*p_\ast \circ R^*\operatorname{Hom}(q^*O_{\mathbb{P}^2}(1), O_{C_i} \otimes q^*O_{\mathbb{P}^2}(-1)))$, so by Grothendieck-Hirzbruch-Riemann-Roch Theorem we can compute and get that $c_1(L) = c_1(O_{|H|}(1))$ and hence $L \simeq O_{|H|}(1)$.

$L$ has a nowhere vanishing global section on each $A^i$, in other words, we have an exact sequence on $\mathbb{P}^2 \times A^i$

$$
0 \rightarrow O_{C_i} \otimes q^*O_{\mathbb{P}^2}(-1)|_{\mathbb{P}^2 \times A^i} \rightarrow Q^i \rightarrow q^*O_{\mathbb{P}^2}(1)|_{\mathbb{P}^2 \times A^i} \rightarrow 0,
$$

such that restricted on the fiber over any point $y \in A^i$ it does not split. Hence $Q^i$ is the family we want and hence the lemma. \qed

We rewrite diagram (4.1) for $d = 4$ as the following diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & O_{\mathbb{P}^2}(-1)^{\oplus 2} & \longrightarrow & \mathbb{P}^3 \longrightarrow & Q_f & \longrightarrow & 0 \\
& & f_{\ast t} & & f_q & & \\
& & f_{\ast t} & & \sigma_f = f_q \circ f_{\ast t} & & \\
& \downarrow & O_{\mathbb{P}^2}(-2) & & O_{\mathbb{P}^2}^3 & &
\end{array}
$$

Denote $\mathbb{P}(p_\ast(Q \otimes q^*O_{\mathbb{P}^2}(2)))$ to be the union of the projective bundles $\mathbb{P}(p_\ast(Q^i \otimes q^*O_{\mathbb{P}^2}(2)|_{\mathbb{P}^2 \times A^i}))$ over $A^i$ with fibers isomorphic to $\mathbb{P}H^0(Q_f(2)) \simeq \mathbb{P}^{11}$. Analogously, isomorphism classes of pairs $(Q_f, \sigma_f)$ can be parametrized by $\mathbb{P}(p_\ast(Q \otimes q^*O_{\mathbb{P}^2}(2)))$. However $\mathbb{P}(p_\ast(Q \otimes q^*O_{\mathbb{P}^2}(2))) \not\subseteq M(4, 1)$. Look
back to diagram (4.2), it is easy to see that

\[(5.10) \quad \mathbb{P}( p_*( \mathcal{Q} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(2))) \cap M(4, 1) = \{[(Q_f, \sigma_f)] | \text{Im}(\sigma_f) \not\subset T_f \}, \]

with \(\text{Im}(\sigma_f)\) the image of \(\sigma_f\) and \(T_f\) the torsion of \(Q_f\).

The complement of (5.10) in \(\mathbb{P}( p_*( \mathcal{Q} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(2)))\) is the union of the projective bundles \(\mathbb{P}( p_*( \mathcal{O}_{\mathbb{C}_1} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^2 \times \mathbb{A}^1}))\) over \(\mathbb{A}^1\) with fibers isomorphic to \(\mathbb{P}H^0(\mathcal{O}_{\mathbb{H}(1)}) \simeq \mathbb{P}^1\). The following lemma is a straightforward consequence.

**Lemma 5.9.** \([M_1 - W^4] = [\mathbb{P}^2 \times \mathbb{P}^{11} - \mathbb{P}^2 \times \mathbb{P}^1]\).

**Proof of Theorem 5.2.** By Lemma 5.4, Lemma 5.5 and Lemma 5.9, we have

\([M(4, 1)] = [(\text{Hilb}^3(\mathbb{P}^2) - \mathbb{P}^2 \times \mathbb{P}^3) \times \mathbb{P}^{11} + \mathbb{P}^2 \times (\mathbb{P}^{11} - \mathbb{P}^1) + \mathbb{P}^2 \times \mathbb{P}^{13}],\]

which leads to the theorem by direct calculation.

\[\square\]

6. \(M(5, 1)\) and \(M(5, 2)\).

Up to isomorphism \(M(5, 1)\) and \(M(5, 2)\) are the only two moduli spaces with \(d = 5\) such that there is no strictly semistable locus. In this section we prove the following theorem.

**Theorem 6.1.** \([M(5, 1)] = [M(5, 2)] = \sum_{i=0}^{26} b_{2i} \mathbb{L}^i\) and

\[
\begin{align*}
&b_0 = b_{52} = 1, \quad b_2 = b_{50} = 2, \quad b_4 = b_{48} = 6, \\
&b_6 = b_{46} = 13, \quad b_8 = b_{44} = 26, \quad b_{10} = b_{42} = 45, \\
&b_{12} = b_{40} = 68, \quad b_{14} = b_{38} = 87, \quad b_{16} = b_{36} = 100, \\
&b_{18} = b_{34} = 107, \quad b_{20} = b_{32} = 111, \quad b_{22} = b_{30} = 112, \\
&b_{24} = b_{26} = b_{28} = 113.
\end{align*}
\]

In particular the Euler number of both moduli spaces is 1695.

\[\Diamond\] **Computation for** \([M(5, 1)]\)

According to Lemma 3.4 we first stratify \(M(5, 1)\) into three strata defined as follows.

- \(M_1 := \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 4}\};\)
- \(M_2 := \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\};\)
- \(M_3 := \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}\}.\)
Lemma 6.2. A pair \((E, f)\) with \(\text{rank}(E) = 5\) and \(\text{deg}(E) = -4\) is stable if and only if for any two direct summands \(D', D''\) of \(E\) such that \(D' \cong D''\) and \(f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''\), we have \(\mu(D') < \mu(E)\).

Proof. See Appendix A. \qed

For a pair \((E, f) \in M_3\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
a_1 & 0 & 0 & 0 & b_1 \\
a_2 & 0 & 0 & 0 & b_2
\end{pmatrix},
\]

where \(b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))\) and \(a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(4))\). The injectivity of \(f\) implies that \(\text{det}(f) = b_2a_1 - b_1a_2 \neq 0\). Moreover by Lemma 6.2 \((E, f)\) is stable if and only if \(kb_1 \neq k'b_2\) for any \((k, k') \in \mathbb{C}^2 - \{0\}\).

Lemma 6.3. \([M_3] = [\mathbb{P}^2 \times \mathbb{P}^{19}]\).

Proof. The proof is analogous to that of Lemma 5.4. \(M_3\) is isomorphic to a projective bundle over \(\text{Hilb}^1(\mathbb{P}^2)\) with fibers isomorphic to \(\mathbb{P}(H^0(I_1(5))) \cong \mathbb{P}^{19}\). Hence the lemma. \qed

We stratify \(M_2\) into two strata as follows.

\[
M^s_2 := \{[(E, f)] \in M_2 | f|_{\mathcal{O}_{\mathbb{P}^2}^\oplus \otimes \mathcal{O}_{\mathbb{P}^2}(-1)^\oplus} \text{ is surjective onto } \mathcal{O}_{\mathbb{P}^2}(-1) \};
\]

\[
M^c_2 := M_2 - M^s_2.
\]

For a pair \((E, f) \in M^s_2\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
b_1 & b_2 & 0 & 0 & 0 \\
a_1 & a_2 & 0 & 0 & 0
\end{pmatrix},
\]

where \(b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))\) and \(a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))\). The injectivity of \(f\) implies that \(\text{det}(f) = b_1a_2 - b_2a_1 \neq 0\). Moreover by Lemma 6.2 \((E, f)\) is stable if and only if \(kb_1 \neq k'b_2\) for any \((k, k') \in \mathbb{C}^2 - \{0\}\).

Lemma 6.4. \([M^s_2] = [\text{Gr}(2, 6) \times \mathbb{P}^{16} - \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2] \) with \(\text{Gr}(2, 6)\) the Grassmannian parametrizing 2-dimensional linear subspaces of \(\mathbb{C}^6\).
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Proof. We have the following diagram

\[(6.3) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{(b_1, b_2)} \mathcal{O}^{\oplus 2}_{\mathbb{P}^2} \xrightarrow{f_r} R_f \longrightarrow 0.\]

Since $kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}$, the isomorphism classes of $R_f$ 1-1 correspond to points in $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(2))) = Gr(2, 6)$. Denote by $\mathcal{G}$ the tautological bundle on $Gr(2, 6)$. Then on $Gr(2, 6) \times \mathbb{P}^2$ we have the following exact sequence.

\[(6.4) \quad 0 \rightarrow q^*\mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow p^*\mathcal{G}^\vee \rightarrow \mathcal{R} \rightarrow 0,\]

with $p$ and $q$ the projections to $Gr(2, 6)$ and $\mathbb{P}^2$ respectively. $\mathcal{R}$ restricted to the fiber over $[(b_1, b_2)] \in Gr(2, 6)$ is $R_f$. Hence isomorphism classes of $(R_f, \omega_f)$ are parametrized by the projective bundle $\mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)))$ over $Gr(2, 6)$ with fibers isomorphic to $\mathbb{P}^{16}$.

However $M^s_2 \subseteq \mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)))$ and the complement of $M^s_2$ in $\mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)))$ consists of all $(R_f, \omega_f)$ such that the images of $\omega_f$ are contained in the torsions of $R_f$.

If $b_1$ is prime to $b_2$, then $R_f$ is torsion free. If $b_1$ is not prime to $b_2$, then $R_f$ lies in the following exact sequence.

\[(6.5) \quad 0 \rightarrow \mathcal{O}_H(-1) \rightarrow R_f \rightarrow I_1(1) \rightarrow 0,\]

with $H$ a hyperplane in $\mathbb{P}^2$. The closed subset $|H| \times Hilb^{[1]}(\mathbb{P}^2) \hookrightarrow Gr(2, 6)$ parametrizes all the $R_f$ that are not torsion free.

We write down the following diagram.

$$
\begin{array}{c}
\mathbb{P}^2 \\
p_1 \\
|H| \times Hilb^{[1]}(\mathbb{P}^2) \\
|H| \times Hilb^{[1]}(\mathbb{P}^2)
\end{array}
\xleftarrow{\text{p}} \xrightarrow{p}
\begin{array}{c}
n_1 \\
C_1 \\
|H| \times \mathbb{P}^2
\end{array}

Those $(R_f, \omega_f)$ not in $M^s_2$ are parametrized by the projective bundle $\mathbb{P}(p_*(p_1^*\mathcal{O}_{C_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ over $|H| \times Hilb^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_H(2))) \simeq \mathbb{P}^2$.

Hence we have $M^s_2 \simeq \mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3))) - \mathbb{P}(p_*(p_1^*\mathcal{O}_{C_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$, hence the lemma. \qed
For a pair \((E, f) \in M^c_2\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
    b_1 & b_2 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    a_1 & a_2 & 0 & b_3 & 0 \\
    0 & 0 & 0 & 0 & 1 \\
    e_1 & e_2 & 0 & a_3 & 0
\end{pmatrix}
\]

where \(b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))\), \(a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))\) and \(e_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))\). \(\det(f) \neq 0\). Lemma 6.2 implies that \((E, f)\) is stable if and only if \(kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}\) and \(k''a_3 \neq b \cdot b_3, \forall (k'', b) \in \mathbb{C} \times H^0(\mathcal{O}_{\mathbb{P}^2}(1)) - \{(0, 0)\}\).

**Lemma 6.5.** \([M^c_2] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \text{Hilb}^{[2]}(\mathbb{P}^2) \times \mathbb{P}^{17} - \text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]\).

**Proof.** We first write down the following two exact sequences.

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^p \rightarrow R_f \rightarrow 0
\]

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^p \rightarrow S_f \rightarrow 0
\]

Because of the stability condition, we see that both \(R_f\) and \(S_f\) are torsion free and hence \(R_f \simeq I_1(1)\) and \(S_f \simeq I_2(2)\). On the other hand, any \(I_1(1)\) \((I_2(2))\) can be put in the sequence (6.7) ((6.8)).

We write down a commutative diagram as follows.

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \\
& \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) \\
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \\
& \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) \\
0 & \rightarrow & R_f \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \\
& \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) \\
0 & \rightarrow & R_f \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \\
& \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) & \downarrow \mathcal{O}_{\mathbb{P}^2}^p(-1) \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]
We have another commutative diagram

\[
\begin{array}{c}
\mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{(e_1,a_1) \oplus (e_2,a_2)} (\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}) \oplus (e_1,a_1) \oplus (e_2,a_2) \\
\downarrow f_r \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)} \oplus f_r \\
R_f \oplus R_f(1) \xrightarrow{id_{R_f} \otimes f_r} R_f \otimes S_f.
\end{array}
\]

Isomorphism classes of \((E, f) \in M^s_2\) are parametrized by \((R_f, S_f, \omega_f)\) with \(\omega_f : \mathcal{O}_{\mathbb{P}^2}(-2) \to R_f \otimes S_f\) the composed map in (6.10). We write down the following diagram.

\[
\begin{array}{ccc}
\mathbb{P}^2 & \overset{q}{\longrightarrow} & \text{Hilb}^1[1](\mathbb{P}^2) \times \text{Hilb}^2[2](\mathbb{P}^2) \times \mathbb{P}^2 \\
\downarrow p_1 & & \downarrow p_2 \\
\text{Hilb}^1[1](\mathbb{P}^2) \times \mathbb{P}^2 & \xrightarrow{p} & \text{Hilb}^1[2](\mathbb{P}^2) \times \text{Hilb}^2[2](\mathbb{P}^2) & \times \mathbb{P}^2
\end{array}
\]

Denote by \(\mathcal{I}_1 (\mathcal{I}_2)\) the universal family of ideal sheaves on \(\text{Hilb}^1[1](\mathbb{P}^2) \times \mathbb{P}^2 (\text{Hilb}^2[2](\mathbb{P}^2) \times \mathbb{P}^2)\). Isomorphism classes of \((R_f, S_f, \omega_f)\) are parametrized by the projective bundle \(\mathbb{P}(p_*(p_1^*\mathcal{I}_1 \otimes p_2^*\mathcal{I}_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(5)))\) over \(\text{Hilb}^1[1](\mathbb{P}^2) \times \text{Hilb}^2[2](\mathbb{P}^2)\) with fibers isomorphic to \(\mathbb{P}(H^0(\mathcal{I}_1 \otimes \mathcal{I}_2 \otimes \mathcal{O}_{\mathbb{P}^2}(5))) \simeq \mathbb{P}^{17}\).

There are still points in \(\mathbb{P}(p_*(p_1^*\mathcal{I}_1 \otimes p_2^*\mathcal{I}_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(5)))\) that we must exclude. They are points \((R_f, S_f, \omega_f)\) such that the images of \(\omega_f\) are contained in the torsions of \(R_f \otimes S_f\).

We write down the following exact sequence.

\[
(6.11) \quad 0 \to I_1 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_x \to 0,
\]

with \(\mathcal{O}_x\) the skyscraper sheaf supported at a single point \(x\). Tensor (6.11) by \(I_2\), and we get

\[
0 \to Tor^1(\mathcal{O}_x, I_2) \to I_1 \otimes I_2 \to I_2 \to I_2 \otimes \mathcal{O}_x \to 0.
\]

We see that the torsion of \(I_1 \otimes I_2\) is isomorphic to \(Tor^1(\mathcal{O}_x, I_2)\). Tensor (6.8) by \(\mathcal{O}_x\) and we get

\[
0 \xrightarrow{0 \to Tor^1(\mathcal{O}_x, I_2(2)) \xrightarrow{(a_3, b_3)}} \mathcal{O}_x \xrightarrow{(a_3, b_3)} \mathcal{O}_x \otimes \mathcal{O}_x \xrightarrow{I_2(2) \otimes \mathcal{O}_x} 0.
\]

Hence we see that the torsion of \(R_f \otimes S_f\) is either zero or isomorphic to \(\mathcal{O}_x\). The later implies that \(R_f \simeq I_{\{x\}}(1)\) and \(Tor^1(\mathcal{O}_x, I_2) \neq 0 \Leftrightarrow (a_3, b_3)_x = 0\).
We then want to parametrize all \((I_1, I_2)\) such that \(I_1 \otimes I_2\) contain torsion. We first write down the following diagram

\[
\begin{array}{c}
P^2 \xleftarrow{q} P^2 \times \text{Hilb}^{[1]}(P^2) \times |H| \\
\downarrow{q_1} \quad \downarrow{p_3} \\
C_1 \hookrightarrow P^2 \times |H| \quad \text{Hilb}^{[1]}(P^2) \times |H| \hookrightarrow \mathbb{P}(\mathcal{V}^1),
\end{array}
\]

where \(\mathcal{V}^1\) is the rank 2 vector bundle on \(\text{Hilb}^{[1]}(P^2)\) defined as \(s_*(I_1 \otimes t^*O_{P^2}(1))\) with \(s\) and \(t\) the projection from \(\text{Hilb}^{[1]}(P^2) \times P^2\) to \(\text{Hilb}^{[1]}(P^2)\) and \(P^2\) respectively. Let \(Z\) be defined by the following Cartesian diagram

\[
\begin{array}{c}
\mathbb{P}(\mathcal{V}^1) \xrightarrow{p_3} \mathbb{P}(p_3^*(q_1^*(O_{C_1}) \otimes q^*O_{P^2}(1))) \\
\downarrow \quad \downarrow \\
\text{Hilb}^{[1]}(P^2) \times |H| 
\end{array}
\]

\(\mathbb{P}(\mathcal{V}^1)\) parametrizes all the pairs \(([x], [C]) \in \text{Hilb}^{[1]}(P^2) \times |H|\) such that \(x \in C\), and \(Z\) parametrizes \(([x_1], [(C, x_2)]) \in \text{Hilb}^{[1]}(P^2) \times C_1\) such that \(x_1 \in C\). Define \(\nu : Z \to \text{Hilb}^{[1]}(P^2) \times \text{Hilb}^{[2]}(P^2)\) such that \(\nu([x_1], [(C, x_2)]) = ([x_1], [x_1, x_2, C])\). It is easy to see that \(\nu\) is an embedding with its image exactly the set of points \((I_1, I_2)\) such that \(I_1 \otimes I_2\) have torsion.

\[p_*(p_1^*I_1 \otimes p_2^*I_2 \otimes p^*O_Z) \simeq p_*(p_1^*I_1 \otimes p_2^*I_2 \otimes O_Z)\]

by the flatness of \(p\).

\[p_1^*I_1 \otimes p_2^*I_2 \otimes p^*O_Z\]

contains \(p_1^*O_{Z_1}\) as its torsion where \(Z_1\) is the universal subscheme in \(\text{Hilb}^{[1]}(P^2) \times P^2\). Hence we can embed \(Z\) into \(\mathbb{P}(p_*(p_1^*I_1 \otimes p_2^*I_2 \otimes q^*O_{P^2}(5)))\) by taking the non-zero constant section of \(p_1^*O_{Z_1}\).

Hence we have \(M^5_5 \simeq \mathbb{P}(p_*(p_1^*I_1 \otimes p_2^*I_2 \otimes q^*O_{P^2}(5))) - Z\). The lemma follows since \([Z] = [\text{Hilb}^{[1]}(P^2) \times P^1 \times P^1]\).

Finally let \((E, f) \in M_1 - W^5\). Rewrite (4.1) for \(d = 5\) as follows

\[
\begin{array}{c}
0 \to O_{P^2}(-1) \otimes 3 \xrightarrow{f_{At}^*} O_{P^2}^3 \xrightarrow{f_s} Q_f \to 0 \\
\downarrow{f_{At}} \quad \quad \downarrow{\sigma_f = f_s \circ f_{At}} \\
O_{P^2}(-2)
\end{array}
\]

Notice that the torsion \(T_f\) of \(Q_f\) contains neither \(O_H\) nor \(O_{2H}(x)\) as a subsheaf with \(x\) a single point on the curve, otherwise we will have a diagram similar to diagram (5.3) which contradicts Condition 4.4. Also \(H^0(T_f \otimes O_{P^2}(-1)) = 0\). Hence we see that if \(c_1(T_f) = 2H\), then \(\chi(T_f) = 1\) and \(Q_f^T \simeq\)
Moduli Spaces of Semistable Sheaves of Dimension 1 on \( \mathbb{P}^2 \). Moreover since \( T_f \) does not contain \( \mathcal{O}_H(n) \) with \( n \geq 0 \), \( T_f \) is stable and hence by Theorem 5.1 there is only one stable sheaf for each curve in \( |2H| \). Hence \( T_f \simeq \mathcal{O}_{2H} \).

We have the following commutative diagram.

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \rightarrow K \\
& \simeq & j & \rightarrow T_f \\
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \rightarrow Q_f \\
& & f_i & \rightarrow & 0 \\
& & f_{eq} & \rightarrow & Q_f^{tf} \\
& & & \rightarrow & 0 \\
& & & \rightarrow & 0
\end{array}
\]

We stratify \( M_1 - W^5 \) into three strata as follows.

\[
\begin{align*}
\Pi_1 & := \{[(E, f)] \in M_1 - W^5 | T_f \simeq \mathcal{O}_{2H}, Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(1)\}; \\
\Pi_2 & := \{[(E, f)] \in M_1 - W^5 | T_f \simeq \mathcal{O}_H(-1), Q_f^{tf} \simeq I_2(2)\}; \\
\Pi_3 & := \{[(E, f)] \in M_1 - W^5 | T_f \simeq \mathcal{O}_H(-2), Q_f^{tf} \simeq I_1(2)\}.
\end{align*}
\]

A priori there is the fourth possibility that \( T_f \simeq \mathcal{O}_H(-3), Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2) \), we will explain why this case is excluded later in the computation for \([\Pi_3]\).

**Lemma 6.6.** \([\Pi_1] = [\mathbb{P}^5 \times \mathbb{P}^{14} - \mathbb{P}^5 \times \mathbb{P}^4]\).

**Proof.** Notice that \( \text{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H) = 0 \) for all \( i \neq 1 \) and \( \text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{2H}) \simeq \mathbb{C} \), and the proof is analogous to that of Lemma 5.9. \(\square\)

Let \((E, f) \in \Pi_2\). Since \( Q_f^{tf} \simeq I_2(2) \), we have the following exact sequence

\[
(6.16) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a,b)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{g} I_2(2) \rightarrow 0,
\]

with \( b \in H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \), \( a \in H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \) and \( b \) prime to \( a \). Take \( \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)) \) on (6.16) and we get

\[
(6.17) \quad 0 \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)) \rightarrow \text{Ext}^1(I_2(2), \mathcal{O}_H(-1)) \xrightarrow{\delta} \text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_H(-1)) \rightarrow 0,
\]
Lemma 6.7. \( Q_f \) with \( T_f \simeq \mathcal{O}_H(-1) \) and \( Q^{tf}_f \simeq I_2(2) \) lies in diagram (6.15) if and only if the image of the following exact sequence via \( \tilde{g} \) is not zero
\[
(6.18) \quad 0 \to \mathcal{O}_H(-1) \to Q_f \to I_2(2) \to 0,
\]
i.e. (6.18) is not contained in the image of \( \text{Hom}(\mathcal{O}_{p^2}(-1), \mathcal{O}_H(-1)) \).

Proof. The map \( \tilde{g} \) gives the following commutative diagram
\[
(6.19) \quad \begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_{p^2}(-1) & \xrightarrow{\simeq} & \mathcal{O}_{p^2}(-1) \\
\downarrow & & \downarrow^{(a,b)} \\
0 & \to & \mathcal{O}_H(-1) & \to & \tilde{Q}_f & \to & \mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2} & \to & 0 \quad (*) \\
\downarrow^{\simeq} & & \downarrow^{\delta} & & \downarrow^{g} \\
0 & \to & \mathcal{O}_H(-1) & \to & Q_f & \to & I_2(2) & \to & 0 \quad (**) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]
where the sequence \((*)\) is the image of the sequence \((**)\) via \( \tilde{g} \) and \( \tilde{Q}_f \) is the Cartesian product of \( Q_f \) and \( \mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2} \) over \( I_2(2) \).

From (6.19) we see that \( \text{Hom}(\mathcal{O}_{p^2}, \tilde{Q}_f) \simeq \text{Hom}(\mathcal{O}_{p^2}, \mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2}) \) and \( \text{Hom}(\mathcal{O}_{p^2}, Q_f) \simeq \text{Hom}(\mathcal{O}_{p^2}, I_2(2)) \). Moreover the map \( f_{tq} \) in (6.15) factors through a surjective map \( s : \mathcal{O}_{p^2}^{\oplus 4} \to \mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2} \) since \( \text{Hom}(\mathcal{O}_{p^2}^{\oplus 4}, \mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2}) \to \text{Hom}(\mathcal{O}_{p^2}^{\oplus 4}, I_2(2)) \), and \( s \) lifts to a map \( \tilde{s} : \mathcal{O}_{p^2}^{\oplus 4} \to \tilde{Q}_f \) such that \( f_q = \delta \circ \tilde{s} \).

If the sequence \((*)\) in (6.19) splits, then \( \tilde{Q}_f \simeq \mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2} \oplus \mathcal{O}_H(-1) \) and \( \delta \circ \tilde{s} \) can not be surjective. Hence \((*)\) does not split.

On the other hand, \( \text{Ext}^1(\mathcal{O}_{p^2}(1) \oplus \mathcal{O}_{p^2}, \mathcal{O}_H(-1)) \simeq \mathbb{C} \), hence \( \tilde{Q}_f \) is unique up to isomorphism if \((*)\) does not split. We see that in this case \( \tilde{Q}_f \simeq \mathcal{O}_{p^2} \oplus Q^1_f \) with \( Q^1_f \) lying in the following non-splitting sequence
\[
0 \to \mathcal{O}_H(-1) \to Q^1_f \to \mathcal{O}_{p^2}(1) \to 0.
\]
By Lemma 5.6 \( Q^1_f \) lies in the following sequence
\[
0 \to \mathcal{O}_{p^2}(-1)^{\oplus 2} \to \mathcal{O}_{p^2}^{\oplus 3} \xrightarrow{f_{tq}} Q^1_f \to 0.
\]
We then have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \to & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \\
\downarrow & & \downarrow & & \cong & \\
0 & \to & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \to & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \mathcal{O}_f & \to & 0 & \\
\downarrow & & \downarrow & & f_\sigma & \downarrow & \cong & \\
0 & \to & \mathcal{O}_{\mathbb{P}^2}(-1) & \to & \mathcal{O}^1_f \oplus \mathcal{O}_{\mathbb{P}^2} & \mathcal{O}_f & \to & 0 & \\
\end{array}
\]

Hence $Q_f$ lies in (6.19) and hence the lemma. □

**Lemma 6.8.** $[\Pi_2] = [\text{Hilb}^2(\mathbb{P}^2) \times |H| \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^{14} - \mathbb{P}^1)]$.

*Proof.* Lemma 6.7 implies that for fixed $\mathcal{O}_H(-1)$ and $I_2(2)$, isomorphism classes of $Q_f$ are parametrized by $\mathbb{P}(\text{Ext}^1(I_2(2), \mathcal{O}_H(-1))) - \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)))$. Hence isomorphism classes of all $Q_f$ are parametrized by the following scheme

\[
\mathbb{P}(\text{Ext}^1(I_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_C \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1))) - \text{Hilb}^2(\mathbb{P}^2) \times |H|;
\]

where $p$ and $q$ are projections from $\mathbb{P}^2 \times \text{Hilb}^2(\mathbb{P}^2) \times |H|$ to $\text{Hilb}^2(\mathbb{P}^2) \times |H|$ and $\mathbb{P}^2$ respectively, $I_2$ and $C_1$ are the pull back of the universal ideal sheaf and the universal curve to $\mathbb{P}^2 \times \text{Hilb}^2(\mathbb{P}^2)$ and $\mathbb{P}^2 \times |H|$ respectively. Notice that we embed $\text{Hilb}^2(\mathbb{P}^2) \times |H|$ into $\mathbb{P}(\text{Ext}^1(I_2 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_C \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)))$ by taking the nonzero constant section of the line bundle $\text{Hom}_p(q^*\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_C \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)) \cong p_*\mathcal{O}_C \cong \mathcal{O}_{\text{Hilb}^2(\mathbb{P}^2) \times |H|}$.

Analogously the space parametrizing $(Q_f, \sigma_f)$ is the difference of two projective bundles with fibers isomorphic to $\mathbb{P}(H^0(Q_f(2))) \cong \mathbb{P}^{14}$ and $\mathbb{P}(H^0(\mathcal{O}_H(1))) \cong \mathbb{P}^1$ respectively over the space parametrizing $Q_f$. Hence the lemma. □

Now we do the computation for $[\Pi_3]$ and we will also explain why the case that $T_f \cong \mathcal{O}_H(-3), Q_f^{T_f} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ is not included. Notice that the map $f_{tq}$ in (6.15) is not surjective on global sections. We first write down the
following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \rightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \Downarrow & & \\
0 & \rightarrow & G & \rightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4+n} & \rightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} & \rightarrow & 0 \\
\tau & \downarrow & \tau & \downarrow & \tau & \downarrow & \tau & \downarrow & \\
\mathcal{O}_{\mathbb{P}^2}^{\oplus n} & \rightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \\
0 & & 0 & & \\
\end{array}
\]

where \( n = 1 \) if \( Q^f_1 \simeq I_1(2) \) and \( n = 2 \) if \( Q^f_1 \simeq \mathcal{O}_{\mathbb{P}^2}(2) \).

From (6.21) we see that \( H^i(G(1 - i)) = 0 \) for \( i > 0 \), hence by Castelnuovo-Mumford regularity \( G(1) \) is globally generated. Therefore the map \( \tau \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(1)} : G(1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus n} \) must be surjective on global sections, since otherwise \( \tau \) is not surjective. Hence \( h^0(K(1)) = h^0(G(1)) - nh^0(\mathcal{O}_{\mathbb{P}^2}(1)) \). So if \( n = 2, Q^f_1 \simeq \mathcal{O}_{\mathbb{P}^2}(2) \), then we have \( h^0(K(1)) = 2 \) which implies that \( K \) can not contain \( \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \) as a subsheaf. Hence we only have \( n = 1 \) and \( Q^f_1 \simeq I_1(2) \).

\( T_f \simeq \mathcal{O}_H(-2) \) hence \( \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), T_f) = 0 \), the inclusion \( \iota \) in (6.15) is unique up to isomorphisms of \( \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \) for a fixed \( K \). Hence \( f_{B'} \) is determined by the inclusion \( j \) and hence is determined by \( f_{tq} \). Parametrizing \( f_{B'} \) is equivalent to parametrizing the surjective map \( f_{tq} \), hence equivalent to parametrizing \( \tilde{\tau} \). We first assume \( Q^f_1 \simeq I_{[0, 0, 1]}(2) \), then \( g \) can be represented by a \( 1 \times 5 \) matrix \((x_0^2, x_0x_1, x_0x_2, x_1x_2, x_2^2)\). \( \tilde{\tau} \) can be represented by \( h := (h_0, h_1, h_2, h_3, h_4) \) with \( h_i \in \mathbb{C} \). We want to parametrize the class of \( h \) modulo scalars.

The sheaf \( G \) can be generated by 6 generators \( <\epsilon_0, \epsilon_1, \epsilon_2, \eta_0, \eta_1, \eta_2> \) in \( H^0(G(1)) \) with two syzygies \((x_0\epsilon_0 + x_1\epsilon_1 - x_2\epsilon_2 = 0, x_0\eta_0 + x_1\eta_1 - x_2\eta_2 = 0)\).

The map \( \tau \) is determined by \( \tau^0 := h^0(\tau \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(1)} : H^0(G(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \), and also \( \tau \) is induced by \( \tilde{\tau} \). Hence \( \tau^0 \) is determined by \( h \) and we can write down explicitly the images of \( \epsilon_i \) and \( \eta_i \) as follows

\[
\begin{align*}
\tau^0(\epsilon_0) &= h_2x_1 - h_1x_2; \\
\tau^0(\epsilon_1) &= h_0x_2 - h_2x_0; \\
\tau^0(\epsilon_2) &= h_0x_1 - h_1x_0; \\
\tau^0(\eta_0) &= h_3x_1 - h_4x_2; \\
\tau^0(\eta_1) &= h_1x_2 - h_3x_0; \\
\tau^0(\eta_2) &= h_1x_1 - h_4x_0.
\end{align*}
\]
We can get (6.21) if and only if $\tau^0$ is surjective. In other words, the following $3 \times 6$ matrix has rank 3.

$$\text{Mat}_\tau := \begin{pmatrix}
0 & -h_2 & -h_1 & 0 & -h_3 & -h_4 \\
h_2 & 0 & h_0 & h_3 & 0 & h_1 \\
-h_1 & h_0 & 0 & -h_4 & h_1 & 0
\end{pmatrix}$$

By direct computation we see that

$$\text{rank}(\text{Mat}_\tau) < 3 \iff h_1 h_2 - h_0 h_3 = h_1^2 - h_0 h_4 = h_1 h_3 - h_2 h_1 = 0.$$ 

Hence we know that $f_{tq}$ are parametrized by $P_{\tau} := \mathbb{P}^4 - \{h_1 h_2 - h_0 h_3 = h_1^2 - h_0 h_4 = h_1 h_3 - h_2 h_1 = 0\}$.

One can easily compute that $[P_\tau] = [\mathbb{P}^4 - (\mathbb{P}^1 + A^2 + A^1)]$. Moreover we can cover $\text{Hilb}^1(\mathbb{P}^2)$ by finitely many Zariski open subsets $U_i$ such that $(I_1(2), \tilde{\tau})$ with $[I_1] \in U_i$ are parametrized by $U_i \times P_\tau$. For example, we can take $U_i$ such that $p_* (I_1(2))|_{U_i} \simeq \mathcal{O}^5_{\mathbb{P}}$, where $p$ and $q$ are the projections from $\text{Hilb}^1(\mathbb{P}^2) \times \mathbb{P}^2$ to $\text{Hilb}^1(\mathbb{P}^2)$ and $\mathbb{P}^2$ respectively and $I_1$ the universal ideal sheaf.

$(I_1(2), \tilde{\tau})$ determines $Q_f$ and analogously we know that $(Q_f, \sigma_f)$ are parametrized by a difference of two projective bundles over the space parametrizing $Q_f$. Hence we have the following lemma as a direct consequence.

**Lemma 6.9.** $[\Pi_3] = [\text{Hilb}^1(\mathbb{P}^2) \times (\mathbb{P}^4 - (\mathbb{P}^1 + A^2 + A^1)) \times (\mathbb{P}^{14} - 1)]$.

We have already known that $[W^5] = \mathbb{P}^{14} \times [\text{Hilb}^6(\mathbb{P}^2) - \Omega^6_2]$. The proof of the following lemma is postponed to the appendix.

**Lemma 6.10.** $[\Omega^6_2] = \mathbb{L}^{11} + 3\mathbb{L}^{10} + 8\mathbb{L}^9 + 18\mathbb{L}^8 + 30\mathbb{L}^7 + 39\mathbb{L}^6 + 38\mathbb{L}^5 + 28\mathbb{L}^4 + 15\mathbb{L}^3 + 6\mathbb{L}^2 + 2\mathbb{L}^1 + 1$.

**Proof.** See Appendix B.

**Proof of Theorem 6.1 for $M(5,1)$**. We have

$$[M(5,1)] = [M_3] + [M_5^2] + [M_5^2] + \sum_{i=1}^{3} [\Pi_i] + ([\text{Hilb}^6(\mathbb{P}^2)] - [\Omega^6_2]) \times [\mathbb{P}^{14}].$$

Combine Lemma 6.3, Lemma 6.4, Lemma 6.5, Lemma 6.6, Lemma 6.8, Lemma 6.9 and Lemma 6.10, and we get the result by direct computation.
\section{Computation for \([M(5,2)]\)}

We stratify \(M(5,2)\) into three strata defined as follows.

\[\begin{align*}
M_2 & := \{(E,f) \in M(5,1)| E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}\}; \\
M_3 & := \{(E,f) \in M(5,1)| E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}; \\
M_3' & := \{(E,f) \in M(5,1)| E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}.
\end{align*}\]

Here we use notation \(M_3'\) instead of \(M_4\) because we want to specify the lower index of the subspace to be \(h^0(F)\) with \(F\) any sheaf in it.

\textbf{Lemma 6.11.} A pair \((E,f)\) with \(\text{rank}(E) = 5\) and \(\text{deg}(E) = -3\) is stable if and only if for any two direct summands \(D', D''\) of \(E\) such that \(D' \simeq D''\) and \(f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''\), we have \(\mu(D') < \mu(E)\).

\textit{Proof.} See Appendix A. \qed

For a pair \((E,f) \in M_3'\), \(f\) can be represented by the following matrix

\[
(6.22) \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
d & 0 & 0 & b & 0 \\
c & 0 & 0 & a & 0
\end{pmatrix},
\]

where \(b \in H^0(\mathcal{O}_{\mathbb{P}^2}(1)), \ a \in H^0(\mathcal{O}_{\mathbb{P}^2}(2)), \ c \in H^0(\mathcal{O}_{\mathbb{P}^2}(4))\) and \(d \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))\). \(\det(f) = ad - bc \neq 0\) and by Lemma 6.2 \((E,f)\) is stable if and only if \(b\) is prime to \(a\).

\textbf{Lemma 6.12.} \([M_3'] = [\text{Hilb}^2(\mathbb{P}^2) \times \mathbb{P}^{18}]\).

\textit{Proof.}\ We have the following exact sequence

\[
(6.23) \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a,b)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \to I_2(2) \to 0,
\]

and for every \([I_2] \in \text{Hilb}^2(\mathbb{P}^2)\), \(I_2(2)\) lies in \((6.23)\). Hence analogous to Lemma 5.4, \(M_3'\) is isomorphic to a projective bundle over \(\text{Hilb}^2(\mathbb{P}^2)\) with fibers isomorphic to \(\mathbb{P}(H^0(I_2(5))) \simeq \mathbb{P}^{18}\). Hence the lemma. \qed
For a pair \((E, f) \in M_3\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
B & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
A & 0 & 0
\end{pmatrix},
\]

where \(A\) is a \(1 \times 3\) matrix with entries in \(H^0(\mathcal{O}_{\mathbb{P}^2}(3))\) and \(B\) a \(2 \times 3\) matrix with entries in \(H^0(\mathcal{O}_{\mathbb{P}^2}(1))\). The parametrizing space of \(B\) is of class \([\text{Hilb}^3(\mathbb{P}^2) - |H| \times \mathbb{P}^3 + |H|]\) by Lemma 5.5 and Lemma 5.8. We have the following lemma.

**Lemma 6.13.** \([M_3] = [(\text{Hilb}^3(\mathbb{P}^2) - |H| \times \mathbb{P}^3 + |H|) \times \mathbb{P}^{17} - |H| \times \mathbb{P}^2].\)

**Proof.** \(M_3\) is the union of a projective bundle over \(\text{Hilb}^3(\mathbb{P}^2) - \Omega_1^3\) with fiber isomorphic to \(\mathbb{P}(H^0(I_3(5))) \simeq \mathbb{P}^{17}\) and a difference of two projective bundles over \(|H|\) with fibers isomorphic to \(\mathbb{P}^{17}\) and \(\mathbb{P}(H^0(\mathcal{O}_H(2))) \simeq \mathbb{P}^2\) respectively. Hence the lemma. \(\square\)

We stratify \(M_2\) into two strata as follows.

\[
M_2^s := \left\{ [(E, f)] \in M_2 \left| \text{frs} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}} E \xrightarrow{(1)} \mathcal{O}_{\mathbb{P}^1} \right. \text{is injective.} \right\};
\]

\[
M_2^c := M_2 - M_2^s.
\]

For a pair \((E, f) \in M_2^s\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
A & 0 & 0 & B
\end{pmatrix},
\]

where \(A\) is a \(3 \times 2\) matrix with entries in \(H^0(\mathcal{O}_{\mathbb{P}^2}(2))\) and \(B\) a \(3 \times 1\) matrix with entries in \(H^0(\mathcal{O}_{\mathbb{P}^2}(1))\).

We stratify \(M_2^s\) into two strata as follows.

\[
\Xi_1 := \{ [(E, f)] \in M_2^s | B \simeq (x_0, x_1, x_2)^t \}; \quad \Xi_2 := M_2^s - \Xi_1.
\]
If $B \simeq (x_0, x_1, x_2)'$, then $(E, f)$ always satisfies the stability condition. We have the following diagram

\[
\begin{array}{c}
0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(x_0, x_1, x_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{f_0} E_0 \to 0,
\end{array}
\]

with $E_0$ a rank 2 bundle which is the dual of the kernel of the surjective map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{(x_0, x_1, x_2)} \mathcal{O}_{\mathbb{P}^2}(1)$. Isomorphism classes of $\xi_f$ are parametrized by $Gr(2, 15)$ since $h^0(E_0(2)) = 15$. Moreover $det(f) \neq 0 \iff$ the image of $\xi_f$ is a rank two subsheaf of $E_0 \ni \text{Im}(\xi_f)$ is not contained in a rank one subsheaf of $E_0$.

Assume $\text{Im}(\xi_f)$ is contained in a rank one subsheaf $E_1 \subset E_0$. Since $E_0$ is locally free, we ask $E_1$ to be a line bundle. Hence either $E_1 \simeq \mathcal{O}_{\mathbb{P}^2}$ or $E_1 \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. Notice that for any $n$ a map $\mathcal{O}_{\mathbb{P}^2}(n) \to E_0$ always factors through map $f_0$ in (6.24).

Since $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0) \simeq \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$, all inclusions $i : \mathcal{O}_{\mathbb{P}^2} \hookrightarrow E_0$ are parametrized by $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) - \{0\}$. Moreover $\forall i, i' \in H^0(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) - \{0\}, i \neq i'$, $\text{Im}(i) \cap \text{Im}(i') = \emptyset$. Hence all $\xi_f$ such that $\text{Im}(\xi_f)$ are contained in $\mathcal{O}_{\mathbb{P}^2} \simeq E_1 \subset E_0$ are parametrized by $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(2))) \times \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})) \simeq Gr(2, 6) \times \mathbb{P}^2$.

Let $i \in \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}) - \{0\}$. All inclusions $j : \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow E_0$ that do not factor through $i : \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^2}$ are parametrized by $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - i^*(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}))$, where $i^* : \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0) \hookrightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0)$ is the map induced by $i$. Moreover $\forall j, j' \in H^0(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - \{0\}, j \neq j'$, $\text{Im}(j) \cap \text{Im}(j') = \emptyset$, and $\forall i, i' \in H^0(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - \{0\}, i \neq i'$, $\text{Im}(i) \cap \text{Im}(i') = \emptyset$. Hence all $\xi_f$ such that $\text{Im}(\xi_f)$ are contained in $\mathcal{O}_{\mathbb{P}^2}(-1)$ but not $\mathcal{O}_{\mathbb{P}^2}$ in $E_0$ are parametrized by $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times (\mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0)) - \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0)) \times \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}))) \simeq \mathbb{P}^2 \times (\mathbb{P}^7 - \mathbb{P}^2 \times \mathbb{P}^2)$.

We have the following lemma as a direct consequence.

**Lemma 6.14.** $\Xi_1 = [Gr(2, 15) - Gr(2, 6) \times \mathbb{P}^2 - \mathbb{P}^2 \times (\mathbb{P}^7 - \mathbb{P}^2 \times \mathbb{P}^2)]$. 

For a pair \((E, f) \in \Xi_2\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
a_1 & a_2 & 0 & 0 & 0 \\
a_3 & a_4 & 0 & 0 & b_1 \\
a_5 & a_6 & 0 & 0 & b_2
\end{pmatrix},
\]

where \(b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))\) and \(a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))\). By Lemma 6.2 \((E, f)\) is stable if and only if \(kb_1 \neq k'b_2, ka_1 \neq k'a_2, \forall (k, k') \in \mathbb{C} - \{0\}\).

We write down the following two exact sequences.

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{(b_1, b_2)} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} f_r \longrightarrow R_f \longrightarrow 0 \\
0 & \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{(a_1, a_2)} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} f_s \longrightarrow S_f \longrightarrow 0
\end{align*}
\]

\(R_f \simeq I_1(1)\). Either \(S_f \simeq I_4(2)\) or \(S_f\) lies in the following exact sequence.

\[
0 \rightarrow \mathcal{O}_H(-1) \rightarrow S_f \rightarrow I_1(1) \rightarrow 0.
\]

Isomorphism classes of \((R_f, S_f)\) are parametrized by \(\text{Hilb}^{[1]}(\mathbb{P}^2) \times \text{Gr}(2, 6)\).

We have two commutative diagrams as follows.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \mathcal{O}_{\mathbb{P}^2}(-1)^{(a_1, a_2) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} f_r \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow S_f(-1) \longrightarrow 0 \\
0 & \mathcal{O}_{\mathbb{P}^2}(-2)^{(a_1, a_2)^{\oplus 2}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} f_s^{\oplus 2} \longrightarrow S_f^{\oplus 2} \longrightarrow 0 \\
0 & R_f \otimes \mathcal{O}_{\mathbb{P}^2}(-2)^{(a_1, a_2)} & R_f^{\oplus 2} f_r^{\oplus 2} \longrightarrow R_f^{\oplus 2} f_r \otimes S_f \longrightarrow 0 \\
\end{array}
\]
Isomorphism classes of \((E, f) \in \Xi_2\) are parametrized by \((R_f, S_f, \omega_f)\) with \(\omega_f : \mathcal{O}_{\mathbb{P}^2}(-2) \to R_f \otimes S_f\) the composed map in (6.30). Hence firstly we have a projective bundle over \(Hilb^{[1]}(\mathbb{P}^2) \times Gr(2, 6)\) with fibers isomorphic to \(\mathbb{P}(H^0(R_f \otimes S_f(2))) \simeq \mathbb{P}^{15}\), which contains \(\Xi_2\) as an open subset. The complement of \(\Xi_2\) in that projective bundle is the set of all \((R_f, S_f, \omega_f)\) such that \(\text{Im}(\omega_f)\) are contained in the torsions of \(R_f \otimes S_f\).

Torsion free \(S_f\) are parametrized by \(Gr(2, 6) - \mathbb{P}^2 \times \mathbb{P}^2\). For \(S_f\) torsion free, \(R_f \otimes S_f\) has torsion if and only if \((a_1, a_2)|_x = 0\) with \(R_f \simeq I_x(1)\), and the nonzero torsion must be isomorphic to \(\mathcal{O}_x\). Define \(V^1_i := p_*(I_1 \otimes q^*\mathcal{O}_{\mathbb{P}^2}(i))\) with \(I_1, p, q\) the same as before. \(V^1_1\) and \(V^2_1\) are two vector bundles of rank 2 and 5 respectively over \(Hilb^{[1]}(\mathbb{P}^2)\). Hence \((R_f, S_f, \omega_f)\) with \(S_f\) torsion free and \(\text{Im}(\omega_f)\) contained in the torsion of \(R_f \otimes S_f\) are parametrized by \(Gr(2, V^1_1) - Gr(2, V^1_2) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) \cup Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times \mathbb{P}(V^1_1)\), where \(Gr(2, V^1_1)\) is the relative Grassmannian of the vector bundle \(V^1_1\). And

\[
[Gr(2, V^1_1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) \cup Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times \mathbb{P}(V^1_1)]
\]

\[
= [Gr(2, V^1_1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) + [Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times \mathbb{P}(V^1_1)] - [\mathbb{P}(V^1_1)]
\]

\[
= [Hilb^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^2 + \mathbb{P}^2 \times \mathbb{P}^1 - \mathbb{P}^1)].
\]

Now let \(S_f\) lie in (6.28). Write \(R_f \simeq I_x(1)\) and \(I_y(1)\) the quotient of \(S_f\) in (6.28). If \(x \neq y\), then \(R_f \otimes I_y(1)\) is torsion free and in this case \((R_f, S_f, \omega_f)\) with \(\text{Im}(\omega_f)\) contained in the torsions of \(R_f \otimes S_f\) are parametrized by a projective bundle over \(Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2 \times \mathbb{P}^2 - Gr(2, V^1_1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))\) with fibers isomorphic to \(\mathbb{P}(H^0(I_1(1) \otimes \mathcal{O}_H(1))) \simeq \mathbb{P}^2\).

Finally we have a projective bundle over \(Gr(2, V^1_1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))\) with fibers isomorphic to \(\mathbb{P}(H^0(\mathcal{O}_x) \oplus H^0(I_1(1) \otimes \mathcal{O}_H(1))) \simeq \mathbb{P}^3\) parametrizing \((R_f, S_f, \omega_f)\) such that \(x = y\) and \(\text{Im}(\omega_f)\) are contained in the torsions of \(R_f \otimes S_f\). Hence we have the following lemma.

**Lemma 6.15.** \([\Xi_2] = [Hilb^{[1]}(\mathbb{P}^2) \times (Gr(2, 6) \times \mathbb{P}^{15} - Gr(2, 5) - \mathbb{P}^2 \times \mathbb{P}^2 - \mathbb{P}^2 \times \mathbb{P}^3 - \mathbb{P}^1 + \mathbb{P}^2 \times \mathbb{P}^2 + \mathbb{P}^1 \times \mathbb{P}^3 + \mathbb{P}^2)].\)
For a pair \((E, f) \in M_{2}^c\), \(f\) can be represented by the following matrix

\[
\begin{pmatrix}
  b_1 & b_2 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  A_1 & A_2 & 0 & B
\end{pmatrix},
\]

where \(b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))\), \(A_i\) is a \(3 \times 1\) matrix with entries in \(H^0(\mathcal{O}_{\mathbb{P}^2}(2))\) and \(B\) a \(3 \times 2\) matrix with entries in \(H^0(\mathcal{O}_{\mathbb{P}^2}(1))\). \((E, f)\) is stable, hence \(kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}\) and the parametrizing space \(M_B\) of \(B\) is of class \([\text{Hilb}^{[3]}(\mathbb{P}^2) - |H| \times \mathbb{P}^3 + |H|]\) by Lemma 5.5 and Lemma 5.8. We write down the following two exact sequences.

(6.31) \[
0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{f_B} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{f_r} R_f \to 0
\]

(6.32) \[
0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\langle b_1, b_2 \rangle} \xrightarrow{f_s} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{S_f} 0
\]

\(S_f \simeq I_1(1)\). Either \(R_f \simeq I_3(2)\) or \(R_f\) lies in the following exact sequence.

(6.33) \[
0 \to \mathcal{O}_H(-1) \to R_f \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0.
\]

Isomorphism classes of \((R_f, S_f)\) are parametrized by \(M_B \times \text{Hilb}^{[1]}(\mathbb{P}^2)\).

We have two commutative diagrams as follows.

(6.34)

\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \\
\downarrow f_B \oplus \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)} & & \downarrow f_B^{\oplus 4} \\
\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \to & S_f(-1)^{\oplus 2} & \to & 0 \\
\downarrow f_B^{\oplus 2} & & \downarrow \text{id}_{S_f} \oplus (b_1, b_2) \\
\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \to & \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} & \to & S_f^{\oplus 3} & \to & 0 \\
\downarrow f_s^{\oplus 2} & & \downarrow \text{id}_{S_f} \oplus f_r \\
R_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \to & R_f^{\oplus 2} & \to & R_f \otimes S_f & \to & 0
\end{array}
\]
Isomorphism classes of \((E,f) \in M_c^2\) are parametrized by \((R_f, S_f, \omega_f)\) with \(\omega_f : O_{\mathbb{P}^2}(-2) \to R_f \otimes S_f\) the composed map in (6.35). Hence we have a projective bundle over \(M_B \times \text{Hilb}^1(\mathbb{P}^2)\) with fibers isomorphic to \(\mathbb{P}(H^0(R_f \otimes S_f(2))) \simeq \mathbb{P}^{16}\), which contains \(M_2^c\) as an open subset.

We need to exclude all the points \((R_f, S_f, \omega_f)\) that \(\text{Im}(\omega_f)\) are contained in the torsions of \(R_f \otimes S_f\). Firstly let \(R_f\) lie in (6.33), then the torsion of \(R_f \otimes S_f\) is isomorphic to \(O_H(-1) \otimes I_1(1)\). These \((R_f, S_f, \omega_f)\) are parametrized by a projective bundle over \((\bigcup_{i=0}^2 \mathbb{A}^1) \times \text{Hilb}^1(\mathbb{P}^2)\) with fibers isomorphic to \(\mathbb{P}(H^0(O_H(1) \otimes I_1(1))) \simeq \mathbb{P}^2\).

Let \(R_f \simeq I_3(2)\). Denote \(S_f \simeq I_3(1)\). The torsion of \(R_f \otimes S_f\) is a linear subspace of \(O_x^{\mathbb{P}^2} \simeq \mathbb{C}^2\) which is the kernel of \(B^t|_x\). If \(x \in \text{Supp}(O_{\mathbb{P}^2}(2)/R_f)\), \(R_f \otimes S_f\) is torsion free. If \(\text{Supp}(O_{\mathbb{P}^2}(2)/R_f) = \{x, y, z\}\) with \(y, z \neq x\), for simplicity we let \(x = [0, 0, 1]\), \(y = [0, 1, 0]\) and \(z = [1, 0, 0]\) and the matrix \(B\) have the following form.

\[
\begin{pmatrix}
  x_1 & 0 \\
  x_0 & x_0 \\
  0 & x_2
\end{pmatrix}
\]

Hence for this case \(\text{Tor}(R_f \otimes S_f) \simeq O_x\).

Let \(R_f \simeq I_{\{x, 2y\}}(2)\), then \(B\) can be

\[
\begin{pmatrix}
  x_0 & 0 \\
  x_2 & x_0 \\
  0 & x_1
\end{pmatrix}
\]

Hence for this case \(\text{Tor}(R_f \otimes S_f) \simeq O_x\).

Let \(R_f \simeq I_{\{2x, y\}}(2)\), then \(B\) can be

\[
\begin{pmatrix}
  x_0 & 0 \\
  x_1 & x_0 \\
  0 & x_2
\end{pmatrix}
\]

Hence for this case \(\text{Tor}(R_f \otimes S_f) \simeq O_x\).
Let $R_f \simeq I_{\{3x\}}(2)$, then $B$ can be

$$
\begin{pmatrix}
x_0 & 0 \\
x_1 & x_0 \\
kx_2 & x_1
\end{pmatrix}, \text{ for any } k \in \mathbb{C}.
$$

Hence for this case $\text{Tor}(R_f \otimes S_f) \simeq \mathcal{O}_x$ if $k \neq 0$, $\text{Tor}(R_f \otimes S_f) \simeq \mathcal{O}_x^{\oplus 2}$ if $k = 0$.

The projective bundle $\mathbb{P}(\mathcal{V}_1)$ as defined before over $\text{Hilb}^{[1]}(\mathbb{P}^2)$ parametrizes all $(x, C)$ with $x$ a single point and $C$ a curve of degree 1 passing through $x$. Hence we have the universal family $\overline{\mathcal{C}}_1 \subset \mathbb{P}^2 \times \mathbb{P}(\mathcal{V}_1)$.

Denote $\mathcal{Z}_1$ to be the universal family of subschemes in $\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ and $\pi : \mathbb{P}(\mathcal{V}_1) \to \text{Hilb}^{[1]}(\mathbb{P}^2)$ the projection. Define $\overline{\mathcal{C}}_1^0 := \overline{\mathcal{C}}_1 - (\pi \times \text{id}_{\mathbb{P}^2})^* \mathcal{Z}_1$.

Denote $\mathbb{P}(\mathcal{V}_1)^{[2]}$ the relative Hilbert scheme of 2-points on $\mathbb{P}(\mathcal{V}_1)$ over $\text{Hilb}^{[1]}(\mathbb{P}^2)$. There is a natural embedding $\iota : \mathbb{P}(\mathcal{V}_1) \hookrightarrow \mathbb{P}(\mathcal{V}_1)^{[2]}$ sending every point to the double-point supported at it. We have the following diagram

$$
(6.36) \quad \overline{\mathcal{C}}_1^0 \times_{\mathbb{P}(\mathcal{V}_1)^{[2]}} \mathbb{P}(\mathcal{V}_1) - \pi^* \Delta(\mathbb{P}(\mathcal{V}_1)) \xrightarrow{\delta} \mathbb{P}(\mathcal{V}_1)^{[2]} - \iota(\mathbb{P}(\mathcal{V}_1)),
$$

with $\Delta$ the diagonal embedding and $\mathcal{X}$ defined to make (6.36) a Cartesian diagram. Notice that a priori $\mathcal{X}$ may not exist, but if it exists, it parametrizes isomorphism classes of $(R_f, S_f, \omega_f)$ with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{x,y,z\}}(2)$ for $\{x, y, z\} \in N_4^0$ i.e. $H^0(I_{\{x,y,z\}}(1)) = 0$, and $\text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f)$.

**Lemma 6.16.** $\mathcal{X}$ exists and $[\mathcal{X}] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^2 - \mathbb{P}^1) \times (\mathbb{P}^1 - 1)]$.

**Proof.** Take an affine cover of $\text{Hilb}^{[1]}(\mathbb{P}^2) = \bigcup_i U_i$ with $U_i \simeq \mathbb{A}^2$. It is enough to prove the lemma with $\text{Hilb}^{[1]}(\mathbb{P}^2)$ replaced by $U_i$. Denote by $\mathcal{Z}_1|_{U_i}$, $\mathbb{P}(\mathcal{V}_1)|_{U_i}$, $\mathbb{P}(\mathcal{V}_1)^{[2]}|_{U_i}$, $\overline{\mathcal{C}}_1|_{U_i}$, and $\overline{\mathcal{C}}_1^0|_{U_i}$ the pull back of these schemes via the open embedding $U_i \hookrightarrow \text{Hilb}^{[1]}(\mathbb{P}^2)$. Then we have that $\mathcal{Z}_1|_{U_i} \simeq U_i$, $\mathbb{P}(\mathcal{V}_1)|_{U_i} \simeq U_i \times \mathbb{P}^1$, $\mathbb{P}(\mathcal{V}_1)^{[2]}|_{U_i} \simeq U_i \times \mathbb{P}^2$, $\overline{\mathcal{C}}_1|_{U_i} \simeq U_i \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\overline{\mathcal{C}}_1^0|_{U_i} \simeq \ldots$
$U_i \times \mathbb{P}^1 \times \mathbb{A}^1$. Hence (6.36) becomes the following commutative diagram.

\[
\begin{array}{ccc}
U_i \times (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)) \times \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{\delta'} & \mathcal{X}_i \\
p_i & & p'_i \\
U_i \times (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)) & \xrightarrow{\delta_i} & U_i \times (\mathbb{P}^2 - \nu(\mathbb{P}^1)),
\end{array}
\]

with $\mathcal{X}_i \simeq U_i \times (\mathbb{P}^2 - \nu(\mathbb{P}^1)) \times \mathbb{A}^1 \times \mathbb{A}^1$. Hence the lemma.

Isomorphism classes of $(R_f, S_f)$ with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{2x,y\}}(2)$ for $H^0(I_{\{2x,y\}}(1)) = 0$ and $\text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f)$ are parametrized by $(\mathbb{P}(\mathcal{V}_1^1) \times \text{Hilb}^{[1]}(\mathbb{P}^2) \bar{\mathcal{C}}_1^0) - \Delta$, where $\Delta$ is defined by the following Cartesian diagram

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\pi} & \bar{\mathcal{C}}_1^0 \\
\Downarrow & & \Downarrow \\
\mathbb{P}(\mathcal{V}_1^1) & \xrightarrow{id} & \mathbb{P}(\mathcal{V}_1^1) \\
\Downarrow & & \Downarrow \\
\text{Hilb}^{[1]}(\mathbb{P}^2) & \xrightarrow{\pi} & \text{Hilb}^{[1]}(\mathbb{P}^2)
\end{array}
\]

**Lemma 6.17.** \([[(\mathbb{P}(\mathcal{V}_1^1) \times \text{Hilb}^{[1]}(\mathbb{P}^2) \bar{\mathcal{C}}_1^0) - \Delta] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)].\]

*Proof.* Take the affine cover $\text{Hilb}^{[1]}(\mathbb{P}^2) = \bigcup_i U_i$ with $U_i \simeq \mathbb{A}^2$. Replace $\text{Hilb}^{[1]}(\mathbb{P}^2)$ by $U_i$ and the lemma follows immediately.

The normal sheaf of $\bar{\mathcal{C}}_1$ in $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}_1^1)$ is locally free over $\bar{\mathcal{C}}_1^0$. We denote by $\mathcal{N}_C^0$ the total space of the normal bundle over $\bar{\mathcal{C}}_1^0$. Then isomorphism classes of $(R_f, S_f, \omega_f)$ with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{x,2y\}}(2)$ for $H^0(I_{\{x,2y\}}(1)) = 0$ and $\text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f)$ are parametrized by $\mathcal{N}_C^0$.

**Lemma 6.18.** \([\mathcal{N}_C^0] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)].\]

*Proof.* \([\mathcal{N}_C^0] = [\mathbb{A}^1 \times (\bar{\mathcal{C}}_1 - (\pi \times \text{id}_{\mathbb{P}^2})^* \mathcal{Z}_1])]$. Also we see \([\pi \times \text{id}_{\mathbb{P}^2})^* \mathcal{Z}_1] = [\mathbb{P}^1 \times \text{Hilb}^{[1]}(\mathbb{P}^2)]$ and $[\bar{\mathcal{C}}_1] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]$. Hence the lemma.

If $R_f \simeq I_{\{3x\}}(2)$ with $H^0(R_f(-1)) = 0$ and $\text{Tor}(R_f \otimes I_{\{x\}}) \simeq \mathcal{O}_x$, then $R_f$, viewed as an ideal of $\mathcal{O}_{\mathbb{P}^2,x} \simeq \mathbb{C}[[x_0,x_1]]$, is generated by $(kx_0 - x_1^2, m^3)$ with $m$ the maximal ideal in $\mathbb{C}[[x_0,x_1]]$ and $k \in \mathbb{C}^*$. Hence such $R_f$ are parametrized by $(x_0,k)$ for any fixed $x \in \mathbb{P}^2$. Hence isomorphism classes of these $(R_f, S_f, \omega_f)$ such that $\text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f)$ are parametrized by \(\mathbb{P}(\mathcal{V}_1^1) \times (\mathbb{A}^1 - \{0\}).\)
Finally, let \( S_f \simeq I_x(1) \), \( R_f \simeq I_{\{x\}}(2) \) with \( H^0(R_f(-1)) = 0 \) and \( \text{Tor}(R_f \otimes S_f) \simeq \mathcal{O}_{\mathbb{P}^2} \), then \( R_f \) is determined by \( x \) since \( R_f(-2) \simeq I_x^2 \). Hence isomorphism classes of these \((R_f, S_f, \omega_f)\) such that \( \text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f) \) are parametrized by \( \text{Hilb}^{[1]}([\mathbb{P}^2] \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}))) \simeq \text{Hilb}^{[1]}([\mathbb{P}^2] \times \mathbb{P}^1) \).

**Lemma 6.19.** \([M^5] = [M_B \times \text{Hilb}^{[1]}([\mathbb{P}^2] \times \mathbb{P}^{16} - \mathbb{P}^2 \times \mathbb{P}^2 - \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) - \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2 - \mathbb{P}^1)], \) with \([M_B] = [\text{Hilb}^{[3]}([\mathbb{P}^2]) \times \mathbb{P}^2 \times \mathbb{P}^2 + 1 + [H]]\).

**Proof.** \([M^5] = [M_B \times \text{Hilb}^{[1]}([\mathbb{P}^2] \times \mathbb{P}^{16} - \mathbb{P}^2 \times |H| \times \text{Hilb}^{[1]}([\mathbb{P}^2]) - \mathbb{P}(\mathcal{V}_1^1) \times (A^1 - 1) - \text{Hilb}^{[1]}([\mathbb{P}^2] \times \mathbb{P}^1 - \mathcal{X} - \mathbb{P}(\mathcal{V}_1^1) \times \text{Hilb}^{[1]}([\mathbb{P}^2]) \mathcal{C}_1^0 + \Delta - N_0^0].\)

By Lemma 6.16, Lemma 6.17 and Lemma 6.18, we get the lemma by direct computation. \(\square\)

**Proof of Theorem 6.1 for \( M(5, 2) \).** We have

\([M(5, 2)] = [M'_3] + [M_3] + [\Xi_1] + [\Xi_2] + [M^5].\)

Combine Lemma 6.12, Lemma 6.13, Lemma 6.14, Lemma 6.15 and Lemma 6.19, we get the result by direct computation. \(\square\)

**Appendix**

**Appendix A. Proofs of Lemma 6.2 and Lemma 6.11.**

**Lemma A.1 (Lemma 6.2).** A pair \((E, f)\) with \( \text{rank}(E) = 5 \) and \( \text{deg}(E) = -4 \) is stable if and only if for any two direct summands \( D', D'' \) of \( E \) such that \( D' \simeq D'' \) and \( f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'' \), we have \( \mu(D') < \mu(E) \).

**Proof.** We first prove the lemma for \( E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \). We want to show that if \( \exists E' \subset E \) a direct sum of line bundles with \( \mu(E') > \mu(E) \) and \( f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \), then \( \exists D, D' \subset E \) two direct summands with \( D \simeq D' \) and \( \mu(D) > \mu(E) \), such that \( f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D' \). With no loss of generality, we assume that \( E' \) has the form \( \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i} \) with \( a_i > 0 \) and \( n_i - n_{i+1} = 1 \).

Let \( E'' \simeq E' \subset E \) with \( E'' \) not a direct summand of \( E \). Then \( E'' \) has to be one of the following three cases:

1. \( E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \) and \( E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1); \)
2. \( E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \) and \( E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}; \)
3. \( E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \) and \( E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2). \)
By Nakayama’s lemma, we know that $E'' \otimes \mathcal{O}_{P^2}(-1)$ can’t be the preimage of any direct summand of $E$ and also $f^{-1}(E'') = E'' \otimes \mathcal{O}_{P^2}(-1) \Rightarrow f(D \otimes \mathcal{O}_{P^2}(-1)) \subset D$ with $D$ the smallest direct summand of $E$ containing $E''$.

So we assume that $f^{-1}(E'') = E' \otimes \mathcal{O}_{P^2}(-1)$ with $E'$ a direct summand of $E$ isomorphic to $E''$.

Let $E''$ be in case (1). By the assumption we have $f(E' \otimes \mathcal{O}_{P^2}(-1)) \subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$. On the other hand, write $E = \mathcal{O}_{P^2} \oplus E' \oplus \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-2)$, so for the other direct summand $\mathcal{O}_{P^2}$ we have $f(\mathcal{O}_{P^2} \otimes \mathcal{O}_{P^2}(-1)) \subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$. Hence $f((\mathcal{O}_{P^2} \otimes E') \otimes \mathcal{O}_{P^2}(-1)) \subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$, and hence we get $D = D' = E' \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$.

Case (2) is analogous to case (1).

Let $E''$ be in case (3). By the assumption we have $f(E' \otimes \mathcal{O}_{P^2}(-1)) \subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-1)$. Write $E = E' \oplus L$ with $L \simeq \mathcal{O}_{P^2}(-1)$. We can ask $f$ to identify $L \otimes \mathcal{O}_{P^2}(-1)$ with the summand $\mathcal{O}_{P^2}(-2)$ in $E$.

Denote by $f_o : E' \otimes \mathcal{O}_{P^2}(-1) \rightarrow E''$ the restriction of $f$. If $f_o((\mathcal{O}_{P^2} \otimes \mathcal{O}_{P^2}(-1)) \otimes \mathcal{O}_{P^2}(-1)) \subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$, then we have $D = D' = \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1) \subset E'$. If $f_o((\mathcal{O}_{P^2} \otimes \mathcal{O}_{P^2}(-1)) \otimes \mathcal{O}_{P^2}(-1)) \not\subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$, $f_o$ induces an isomorphism from the direct summand $\mathcal{O}_{P^2}(-1) \otimes \mathcal{O}_{P^2}(-1)$ of $E' \otimes \mathcal{O}_{P^2}(-1)$ to the direct summand $\mathcal{O}_{P^2}(-2)$ of $E''$. Hence we can ask $f_o$ to identify these two direct summands. Write $E' = \mathcal{O}_{P^2}(-1) \oplus L'$ with $L' \simeq \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-2)$, then we have $f_o(L' \otimes \mathcal{O}_{P^2}(-1)) \subset \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$. Moreover because $f$ identifies $L \otimes \mathcal{O}_{P^2}(-1)$ with the summand $\mathcal{O}_{P^2}(-2)$ in $E$, $f((L \oplus L') \otimes \mathcal{O}_{P^2}(-1))$ is contained in the direct summand $\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-2)$ of $E$, hence we have $D = L \oplus L'$ and $D'$ is the direct summand $\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-2)$ containing $f(D \otimes \mathcal{O}_{P^2}(-1))$.

This finishes the proof for $E \simeq \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-2)$.

Let $E \simeq \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-2)$. We have the following six possibilities for $E''$.

(4) $E'' \subset \mathcal{O}_{P^2}(-1)$ and $E'' \simeq \mathcal{O}_{P^2}$;
(5) $E'' \subset \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}$ and $E'' \simeq \mathcal{O}_{P^2}(-1)$;
(6) $E'' \subset \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$ and $E'' \simeq \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$;
(7) $E'' \subset \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$ and $E'' \simeq \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$;
(8) $E'' \subset \mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-1)$ and $E'' \simeq \mathcal{O}_{P^2} - \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-2)$.

Analogously $E'' \otimes \mathcal{O}_{P^2}(-1)$ can not be the preimage of any direct summand of $E$ and also $f^{-1}(E'') = E'' \otimes \mathcal{O}_{P^2}(-1) \Rightarrow f(D \otimes \mathcal{O}_{P^2}(-1)) \subset D$ with $D$ the smallest direct summand of $E$ containing $E''$. Let $E_3''$ and $E_4''$ be the bundles in case (6) and (7) respectively. $f^{-1}(E_3'') = E_3'' \otimes \mathcal{O}_{P^2}(-1) \Rightarrow$
f((\mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1)) \otimes \mathcal{O}_{p_2}(-1)) \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1), \text{ and } f^{-1}(E'') = E'' \otimes \mathcal{O}_{p_2}(-1) \Rightarrow f((\mathcal{O}_{p_2}(1) \otimes \mathcal{O}_{p_2}(-1))) \subset \mathcal{O}_{p_2}(1).

Hence we then assume \( E' \) a direct summand of \( E \) isomorphic to \( E'' \) and \( f^{-1}(E'') = E' \otimes \mathcal{O}_{p_2}(-1) \).

For case (4), by assumption we have \( f((\mathcal{O}_{p_2} \otimes \mathcal{O}_{p_2}(-1)) \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \) hence \( D = D' = \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \).

Bundles in case (5), case (8) and case (9) can not be direct summands of \( E \), hence these three cases are done.

For case (6), by assumption we have \( f((\mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1)) \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \) hence \( D = D' = \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \).

For case (7), by assumption we have \( f(\mathcal{O}_{p_2} \otimes \mathcal{O}_{p_2}(-1)) \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \)

This finishes the proof for the whole lemma. \( \Box \)

Lemma A.2 (Lemma 6.11). A pair \((E, f)\) with \(\text{rank}(E) = 5\) and \(\text{deg}(E) = -3\) is stable if and only if for any two direct summands \(D', D''\) of \(E\) such that \(D' \simeq D''\) and \(f(D' \otimes \mathcal{O}_{p_2}(-1)) \subset D''\), we have \(\mu(D') < \mu(E)\).

Proof. We use the same notations as in the proof of Lemma A.1, we list out all the possibilities of \(E''\) as follows.

Let \( E \simeq \mathcal{O}_{p_2}^2 \oplus \mathcal{O}_{p_2}(-1)^{\oplus 3} \).

(1) \( E'' \subset \mathcal{O}_{p_2}^2 \) and \( E'' \simeq \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \);

(2) \( E'' \subset \mathcal{O}_{p_2}^2 \) and \( E'' \simeq \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \);

(3) \( E'' \subset \mathcal{O}_{p_2}^2 \) and \( E'' \simeq \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \);

(4) \( E'' \subset \mathcal{O}_{p_2}^2 \oplus \mathcal{O}_{p_2}(-1) \) and \( E'' \simeq \mathcal{O}_{p_2}^2 \oplus \mathcal{O}_{p_2}(-1)^{\oplus 2} \);

(5) \( E'' \subset \mathcal{O}_{p_2}(1) \) and \( E'' \simeq \mathcal{O}_{p_2} \);

(6) \( E'' \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \) and \( E'' \simeq \mathcal{O}_{p_2}^2 \);

(7) \( E'' \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \) and \( E'' \simeq \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \);

(8) \( E'' \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2}(-1) \) and \( E'' \simeq \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \);

(9) \( E'' \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \) and \( E'' \simeq \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \);

(10) \( E'' \subset \mathcal{O}_{p_2}(1) \oplus \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1)^{\oplus 2} \) and \( E'' \simeq \mathcal{O}_{p_2}(1) \mathcal{O}_{p_2} \oplus \mathcal{O}_{p_2}(-1) \oplus \mathcal{O}_{p_2}(-2) \).

Cases (1) (5) (6) (7) (8) (9) are the same as cases (1) (4) (5) (6) (7) (8) in Lemma A.1 respectively. Case (10) is analogous to case (3) in Lemma A.1. Cases (2) (3) (4) are analogous to case (1). Hence the lemma. \( \Box \)
Appendix B. Proof of Lemma 6.10.

Lemma B.1 (Lemma 6.10). \[ [2_2^6] = L_1^{11} + 3L_1^{10} + 8L_1^9 + 18L_1^8 + 30L_1^7 + 39L_1^6 + 38L_1^5 + 28L_1^4 + 15L_1^3 + 6L_1^2 + 2L_1 + 1. \]

Proof. Denote by \( C_2 \) the universal curve in \( \mathbb{P}^2 \times |2H| \) and \( C_2^6 \) the relative Hilbert scheme of 6-points on \( C_2 \) over \( |2H| \). We have a surjective map \( \xi : C_2^6 \to \Omega_2^6 \). The fiber of \( \xi \) over \( I_{\{x_1,\ldots,x_6\}} \) consists of all curves passing through \( x_1,\ldots,x_6 \) and hence isomorphic to \( \mathbb{P}(H^0(I_{\{x_1,\ldots,x_6\}}(2))) \). Let \( S_n := \{ I_6 \in \Omega_2^6 | h^0(I_6(2)) = n + 1 \} \), then \( \Omega_2^6 = \coprod_{n=0}^2 S_n \). Define \( \mathcal{R}_n := \xi^{-1}(S_n) \), then \( \mathcal{R}_n \simeq \mathbb{P}(p_*(I(2)|_{\mathbb{P}^2 \times S_n})) \) is a projective bundle over \( S_n \) with fibers isomorphic to \( \mathbb{P}^n \).

Denote by \( C_2^o \) the family of integral curves in \( |2H| \). \( C_2^o \) is open in \( C_2 \).

Lemma B.2. \[ [C_2^6] = [(|2H| - \text{Sym}^2(|H|)) \times \mathbb{P}^6] \text{ with Sym}^2(|H|) \text{ the symmetric power of order 2 of } |H|. \]

Proof. The subspace \( |2H|^o \) in \( |2H| \) parametrizing integral curves is \( |2H| - \text{Sym}^2(|H|) \). \( C_2^o[6] \) is a projective bundle over \( |2H|^o \) with fibers isomorphic to \( \mathbb{P}^6 \). Hence the lemma. \( \square \)

Denote by \( C_2^R \to (\text{Sym}^2(|H|) - |H|) \) and \( C_2^N \to |H| \) the families of reducible curves and non-reduced curves in \( |2H| \) respectively. Let \( C_2^R \) be a reducible curve in \( |2H| \) and \( C_2^N \) a non-reduced curve. Denote \( R_n^R (\mathcal{R}_n^R) = \mathcal{R}_n \cap \text{Hilb}^6[6] (C_2^R) (C_2^R[6]) \) and \( R_n^N (\mathcal{R}_n^N) = \mathcal{R}_n \cap \text{Hilb}^6[6] (C_2^N) (C_2^N[6]) \). Then we have the following lemma

Lemma B.3. \[ [\mathcal{R}_n^N] = [R_n^N \times |H|], \text{ for } n = 0, 1, 2. \]

Proof. We can take an affine cover of \( |H| \), write \( |H| = \cup_j V_j \) such that \( C_2^N|_{V_j} \simeq V_j \times C_2^N \). Hence the lemma. \( \square \)

Denote \( m =< x, y > \) the maximal ideal of \( S := \mathbb{C}[[x,y]] \). To study \( R_n^N \) for \( n = 0, 1, 2 \), we write down a table for ideals in \( S/(x^2) \) as Table I.
Table I  Ideals $I$ of $S$ containing $(x^3)$

<table>
<thead>
<tr>
<th>Co-length of $I$</th>
<th>Ideal $I$</th>
<th>$I \cap (m^2 - m^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$m$</td>
<td>$C_x^2 \oplus C_{xy} \oplus Cy^2$</td>
</tr>
<tr>
<td>2</td>
<td>$m^2 + (kx + ky)S, (k, k') \neq 0$</td>
<td>$C_x^2 \oplus C_{xy} \oplus Cy^2$</td>
</tr>
<tr>
<td>3</td>
<td>$m^3 + (x + ky^2)S$</td>
<td>$C_x^2 \oplus C_{xy} \oplus Cy^2$</td>
</tr>
<tr>
<td>4</td>
<td>$x^2S + (kx^2 + k'xy)S + m^2, (k, k') \neq 0$</td>
<td>$C_x^2 \oplus (C(ky^2 + k'xy)$</td>
</tr>
<tr>
<td></td>
<td>$(x + ky^2 + k'xy)S + m^2, k \neq 0$</td>
<td>$C_x^2$</td>
</tr>
<tr>
<td></td>
<td>$(x + k'y)S + m^2$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
<tr>
<td>5</td>
<td>$x^2S + (xy + kx^2 + k'y)S + m^3, k \neq 0$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
<tr>
<td></td>
<td>$(x + ky^2 + k'y^2)S + m^3, k \neq 0$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
<tr>
<td></td>
<td>$(x + k'y^2)S + m^3$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
<tr>
<td>6</td>
<td>$x^2S + (kxy^2 + k'y^2)S + m^4, (k, k') \neq 0$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
<tr>
<td></td>
<td>$(x + ky^2 + k'y^2)S + m^4, (k, k') \neq 0$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
<tr>
<td></td>
<td>$(x + k'y^2)S + m^4$</td>
<td>$C_x^2 \oplus C_{xy}$</td>
</tr>
</tbody>
</table>

Let $C_r^N$ be the reduced curve supported on $C^N$, then $C_r^N \simeq \mathbb{P}^1$. Denote by $S^i$ the subset in $\text{Hilb}^{[6]}(C_r^N)$ consisting of $[I_i^r \cap I_{6-i}]$ with $[I_i^r] \in \text{Hilb}^{[i]}(C_r^N)$ and $i$ maximal for this expression. $S^i_n := S^i \cap R_n$. We then have

**Lemma B.4.** $S^i_n$ are empty except the following 9 terms:

- $[S^6_0] = [\mathbb{P}^6]$; $[S^4_1] = [\mathbb{P}^2 \times \mathbb{P}^1 \times A^1]$; $[S^3_2] = [\mathbb{P}^2 \times \mathbb{P}^1 \times A^2]$;
- $[S^5_0] = [\mathbb{P}^2 \times \mathbb{P}^1 \times A^3]$, $[S^2_2] = [\mathbb{P}^2 \times \mathbb{P}^2 \times A^2]$;
- $[S^4_1] = [\mathbb{P}^2 \times \mathbb{P}^1 \times A^4]$, $[S^3_2] = [\mathbb{P}^2 \times \mathbb{P}^2 \times A^3]$;
- $[S^5_0] = [\mathbb{P}^2 \times \mathbb{P}^1 \times (A^4 + 2A^3 + A^2) + \mathbb{P}^2 \times A^4 + \mathbb{P}^2(\mathbb{P}^3 - \mathbb{P}^1 \times \mathbb{P}^1) \times A^3]$,
- $[S^6_1] = [\mathbb{P}^2 \times \mathbb{P}^1 \times A^1]$.

We omit the proof of Lemma B.4 since it can be done by elementary analysis and computation case by case. Lemma B.4 together with Lemma B.6 gives $[R_n^N]$ for $n = 0, 1, 2$.

To compute $[R_n^R]$, we first define $\tilde{C}_2^R$ by the following Cartesian diagram.

$$
\begin{array}{c}
\tilde{C}_2^R \rightarrow \pi_1 \rightarrow C_2^R \\
\downarrow \quad \downarrow \\
\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2) \rightarrow \pi \rightarrow \text{Sym}^2(\mathbb{P}^2) - \mathbb{P}^2,
\end{array}
$$
where $\pi$ is the quotient of the free action of the order two permutation group $\sigma_2$. The action of $\sigma_2$ lifts to $\tilde{C}_2^{[6]}$ with $\tilde{C}_2^{[6]}$ the quotient. Recall that $R_n^R (\mathcal{R}_n^R) = R_n \cap Hilb^{[6]}(C^R) (\tilde{C}_2^{[2]}).$ Let $\tilde{\mathcal{R}}_n^R := \pi_2^{-1}(\mathcal{R}_n^R)$ with $\pi_2$ the lift of $\pi$. $\mathcal{R}_n^R$ is the quotient of $\tilde{\mathcal{R}}_n^R$ by the action of $\sigma_2$.

Lemma B.5. \( [\tilde{\mathcal{R}}_n^R] = [(\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)) \times R_n^R] \) for $n = 0, 1, 2$.

Proof. Analogous to Lemma B.3, we can take an affine cover of $\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)$ which trivializes $\tilde{C}_2^R$.

Denote by 0 the only singular point in $C^R$. $C^R - \{0\} = \mathbb{A}^1 \sqcup \mathbb{A}^1$. $\tilde{\mathcal{O}}_{C^R,0} \simeq S/(xy) = \mathbb{C}[[x,y]]/(xy).$ We make a table for ideals of $S/(xy)$ as Table II.

<table>
<thead>
<tr>
<th>Co-length of $I$</th>
<th>Ideal $I$</th>
<th>$I \cap (m^2 - m^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$m$</td>
<td>$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$</td>
</tr>
<tr>
<td>2</td>
<td>$m^2 + (kx + ky^2)S, (k,k') \neq 0$</td>
<td>$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$</td>
</tr>
<tr>
<td>3</td>
<td>$m^3 + (x + ky^2)S$</td>
<td>$\mathbb{C}x^2 \oplus \mathbb{C}xy$</td>
</tr>
<tr>
<td>4</td>
<td>$(x + ky^2)S + m^3, (k,k') \neq 0$</td>
<td>$\mathbb{C}xy \oplus \mathbb{C}(kx^2 + ky^2)$</td>
</tr>
<tr>
<td>5</td>
<td>$(y + kx^3)S + m^2$</td>
<td>$\mathbb{C}xy \oplus \mathbb{C}y^2$</td>
</tr>
<tr>
<td>6</td>
<td>$xyS + m^3$</td>
<td>$\mathbb{C}xy$</td>
</tr>
<tr>
<td></td>
<td>$xyS + (x^2 + ky^2)S + m^3, k \neq 0$</td>
<td>$\mathbb{C}xy$</td>
</tr>
<tr>
<td></td>
<td>$xyS + x^2S + m^4$</td>
<td>$\mathbb{C}xy \oplus \mathbb{C}y^2$</td>
</tr>
<tr>
<td></td>
<td>$xyS + (y^2 + kx^2)S + m^3, k \neq 0$</td>
<td>$\mathbb{C}xy$</td>
</tr>
<tr>
<td></td>
<td>$xyS + y^2S + m^3$</td>
<td>$\mathbb{C}x^2 \oplus \mathbb{C}xy$</td>
</tr>
<tr>
<td></td>
<td>$(x + ky^4)S + m^3$</td>
<td>$\mathbb{C}x^2 \oplus \mathbb{C}xy$</td>
</tr>
<tr>
<td></td>
<td>$(y + kx^4)S + m^3$</td>
<td>$\mathbb{C}x^2 \oplus \mathbb{C}xy$</td>
</tr>
</tbody>
</table>

\( \sigma_2 \) acts on $Hilb^{[6]}(C^R)$ by exchanging the two irreducible components of $C^R$. Write $Hilb^{[6]}(C^R) = H^x \sqcup H^s \sqcup H^y$ such that $\sigma_2(H^x) = H^y$ and $\sigma_2(H^s) = H^s$. Let $H_n^x = \mathcal{R}_n \cap H^x$ and analogously we have $H_n^y$ and $H_n^s$. Then $\sigma_2(H_n^x) = H_n^y, \sigma_2(H_n^s) = H_n^s$.

Lemma B.6. 1) \( [H_0^x] = [\mathbb{A}^6 + 2\mathbb{A}^5 + 3\mathbb{A}^4 + 3\mathbb{A}^3 + 2\mathbb{A}^2 - 1] \);
2) \([H^2_1] = [A^6 + 2A^5 + 2A^4 + 2A^3 + 2A^2 + 2A^1]\);
3) \([H^2_2] = [P^6];\)
4) \([H^0_6] = [A^6 + A^4 \times P^1 + A^2 \times P^1 + P^1];\)
5) \(H^s_1 = H^s_2 = \emptyset.\)

Proof. \(\forall I_6 \in \text{Hilb}^{[6]}(C^R), \exists 0 \leq i \leq 6, \text{ such that } I_6 = I_0^i \cap I_{6-i}\) with \([I_0^i] \in \text{Hilb}^i(\{0\})\) and \([I_{6-i}] \in \text{Hilb}^{[6-i]}(C^R - \{0\}) = \text{Hilb}^{[6-i]}(A^1 \sqcup A^1).\) Then the lemma can be proved by elementary analysis and computation case by case.

Lemma B.7. \([R^R_n] = [H^x_n \times (P^2 \times P^2 - \Delta(P^2))] + [H^s_n \times \text{Sym}^2(P^2) - P^2].\)

Proof. \(\sigma_2(H^x_n \times (P^2 \times P^2 - \Delta(P^2)) = H^y_n \times (P^2 \times P^2 - \Delta(P^2))\) and \(\sigma_2(H^s_n \times (P^2 \times P^2 - \Delta(P^2)) = H^s_n \times (P^2 \times P^2 - \Delta(P^2)).\) Hence the lemma.

Now combine Lemma B.2, Lemma B.3, Lemma B.4, Lemma B.5, Lemma B.6 and Lemma B.7, we get \([R_n]\) for \(n = 0, 1, 2.\) Since \([\Omega_2^{[6]}] = \sum_{n=0}^{2} [S_n]\) and \([S_n \times P^n] = [R_n],\) Lemma 6.10 follows after direct computation.

References


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