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STABLE BUNDLES REVISITED

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The topological classification of complex vector bundles over a Riemann surface is of course very simple: they are classified by one integer $c_1(E)$, corresponding to the first Chern class of E. On the other hand, a classification in the complex-analytic—or algebro-geometric—category leads to "continuous moduli" and subtle phenomena which have links with number theory, gauge theory, and conformal field theory. I will try to report briefly on some of these developments here.

The simplest instance of our problem occurs when $\dim E = 1$, that is, when we are dealing with complex analytic line bundles over M—a compact Riemann surface of the genus g. Because line bundles form a group under the tensor product $L, L' \rightsquigarrow L \otimes L'$, the set of isomorphism classes of line bundles $J(X)$ will in this case inherit an abelian group structure, and the first fundamental theorem of the subject, going back to Riemann, Abel, and Jacoby, asserts that $J(M)$ can be given the structure of a complex analytic abelian group. More precisely the first Chern class gives rise to a homomorphism $J(M) \to \mathbb{Z} \to 0$ whose kernel $J_0(M)$ is a complex analytic, and indeed *algebraic*, torus of $\dim_{\mathbb{C}} = g$:

$$0 \to J_0(M) \to J(M) \to \mathbb{Z} \to 0.$$

A proof of this exact sequence on the complex analytic level is actually quite easy, granted the basics of sheaf theory and Hodge theory. Indeed, the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{e^{2\pi i}} \mathbb{C} \to \mathbb{C}^* \to 0$$

induces an analogous sequence on the corresponding sheafs of germs of functions, and hence leads to an exact sequence in cohomology:

$$H^0(M; \underline{\mathbb{C}}^*) \to H^1(M; \underline{\mathbb{Z}}) \to H^1(M; \underline{\mathbb{C}}) \to H^1(M; \underline{\mathbb{C}}^*) \xrightarrow{\delta} H^2(M; \underline{\mathbb{Z}}).$$

But $H^1(X; \underline{\mathbb{C}}^*)$ is seen to be precisely $J(M)$, by passing from a line bundle to its transition functions, while by the Dolbaux resolution, $H^1(M; \underline{\mathbb{C}}) \simeq H^{0,1}(M)$, that is, the vector space of type $(0, 1)$ in $H^1(M; \mathbb{R})$.

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Finally, the δ in this sequence corresponds to c_1, so that the exactness of the cohomology sequence together with the obvious computation that $H^0(M;\underline{\mathbb{C}}) = H^0(M;\mathbb{C}^*) = \mathbb{C}$, leads to the description of $J_0(M)$ as the quotient

$$J_0(M) = H^{0,1}(M)/H^1(M;\mathbb{Z})$$

of a g-dimensional complex vector space by a lattice, i.e., a torus. The *algebraic* structure on $J_0(M)$ is much less trivial.

In any case, these classical results provide us with as beautiful an answer as we could hope for: this Jacobian $J(M)$ varies functorially with M, and its own function theory is intimately related to that of M. In particular, there is a natural map

$$M \xrightarrow{\iota} J_1(M)$$

which maps a point $p \in M$ to the isomorphism class of the line bundle L_p on M determined by p as a divisor on M. (In terms of a local coordinate z_p centered at p in M, L_p can also be described by the data: let $U = \{U_0, U_1\}$ be the cover of M consisting of a small disc, U_0, about p, and let $U_1 = M - p$. Let $g_{U_0, U_1} = \frac{1}{z_p}$, so that the assignment $f_0 = z_p$, $f_1 \equiv 1$ defines a holomorphic section of L_p with precisely one zero at p. Thus $c_1(L_p) = 1$, and so $p \to L_p$ takes values in the component $J_1(M)$ of bundles with $c_1 = 1$.) Iterating this map $(g-1)$ times leads to the diagram:

$$\begin{array}{ccc}
\underbrace{M \times \cdots \times M}_{(g-1)} & \longrightarrow & \underbrace{J_1 \times \cdots \times J_1}_{(g-1)} \xrightarrow{m} J_{g-1}(M) \\
\downarrow & & \nearrow_{\iota^{g-1}} \\
M^{(g-1)} & &
\end{array}$$

and so to a canonical map of the $M^{(g-1)}$, the $(g-1)$st *symmetric power* of M into $J_{g-1}(M)$. The image of this arrow is now of codimension 1 in $J_{g-1}(M)$ and so defines a divisor (the θ-divisor), and hence a line bundle $\mathscr{L}_\theta$, over $J_{g-1}(M)$.

In the terminology of our century the classical θ-functions of Jacoby of level k now appear as the space of holomorphic sections of the bundle $\mathscr{L}_\theta^k$ over J_{g-1}:

$$\theta\text{-functions of level } k \cong H^0(J_{g-1}(M);\mathscr{L}_\theta^k).$$

From the classical formulas one can then also compute the number of such θ functions to be k^g:

$$\dim H^0(J_{g-1}(M);\mathscr{L}_\theta^k) = k^g.$$

To summarize, for $n = 1$, the classical theory teaches us that the moduli space of line bundles is

A) a complex variety each of whose components is a torus of $\dim 2g$ over $\mathbb{R}$ so that its Poincaré series is given by

$$P_t\{J_k(M)\} = (1+t)^{2g},$$

and

B) the component $J_{g-1}(M)$ carries a natural line bundle $\mathscr{L}_\theta$, with

$$\dim H^0\{J_{g-1}(M); \mathscr{L}_\theta^k\} = k^g.$$

The primary difficulty of extending these results to higher-dimensional vector bundles is that the problem immediately becomes infinite dimensional. This comes about because bundles E of $\dim > 1$ will in general have many nontrivial automorphisms, and to properly deal with this situation the Mumford notion of stability is indispensable. The condition is as follows:

$$E \text{ is stable if and only if for any subbundle } F \subset E, \qquad (1)$$
$$\frac{c_1(F)}{\dim F} < \frac{c_1(E)}{\dim E}.$$

Let me illustrate the power of this condition, say for a bundle E of $\dim 2$. First, remark that E always admits a line subbundle $L \subset E$ and so fits into an exact sequence

$$0 \to L \to E \to Q \to 0$$

with L and Q line bundles.

Indeed, any rational section of E will determine such an L, and of course the Chern classes of the three constituents are related by

$$c_1(E) = c_1(L) + c_1(Q).$$

Now, if E is stable, we will have $c_1(L) < c_1(E)/2$ whence $c_1(Q) > c_1(E)/2$, and it follows that the sequence cannot split if E is stable!

Similarly a stable E can have no nontrivial automorphism. Indeed, consider an automorphism

$$\varphi : E \to E.$$

The characteristic polynomial of φ is clearly constant because M is compact. If the eigenvalues of φ differ, φ will split E into two line bundles, which is ruled out by stability. Hence the eigenvalues are equal to λ, say, so that $\psi = \varphi - \lambda 1$ will have to be a nilpotent endomorphism: $\psi^2 = 0$. Now if ψ is not identically 0, and has constant rank, then we have an exact sequence

$$0 \to L \to E \overset{\psi}{L} \to 0$$

where $L = \ker\psi$. But by stability $c_1(L) < c_1(E)/2$, which rules out this alternative. Hence $\psi \equiv \lambda \cdot 1$. q.e.d.

The nonconstant rank case is eliminated similarly.

In any case, armed with this concept, the structure of vector bundles now becomes tractable. First of all, Mumford shows that the isomorphism classes of stable bundles do admit a natural structure as a smooth algebraic variety $J^n(M)$, which varies functorially with M. The tangent space to $J^n(M)$ at E is given by

$$T_E(J^n(M) \simeq H^1(M; \operatorname{End} E).$$

By Riemann-Roch and stability $(\dim H^0 = 1!)$ we see that $J^n(M)$ has dimension $n^2(g-1)+1$. Just as for $J = J^1$, these spaces fall into components $J^n_k(M)$, $k = c_1(E)$ according to the topological type of E, but these will not be isomorphic in general. In fact,

$$J^n_k \text{ is compact} \Leftrightarrow n \text{ and } k \text{ are relatively prime.}$$

In short, for $n > 1$, the "true" analogues of the Jacobian $J(M)$ are the $J^n_r(M)$, with $(n, r) = 1$. When $(n, r) \neq 1$, one usually compactifies $J^n_r(M)$ by semistable bundles (replace $<$ by $\leq$) modulo a certain equivalence relation, to get a complete, but in general *singular*, variety, and in the sequel $J^n_r(M)$ will always denote this compactified version.

Actually this stability notion also helps to explain the structure of *all* bundles over M. For instance, suppose E has dimension two but is *not* semistable. Then one finds that E contains a *unique* line bundle $L_E \subset E$ of maximal c_1, so that we can attach to E a unique exact sequence

$$0 \to L_1 \to E \to L_2 \to 0.$$

If we set $\mu_i = c_1(L_i)$, then the bundles of this type are therefore naturally parametrized by the set

$$\mathscr{C}_\mu \colon J_{\mu_1}(M) \times J_{\mu_2}(M) \times H^1(M; L_2^* \otimes L_1), \qquad u_1 + u_2 = c_1(E); \mu_1 \geq \mu_2,$$

where the last factor measures the extension determined by E. But note that this is no longer a true parametrization of *isomorphism classes*. For, as is easily seen, two bundles whose extensions are multiples of each other, $\eta_1 = \lambda\eta_2$, $\lambda \neq 0$ in $H^1(M; L_2^* \otimes L_1)$, will lead to isomorphic bundles.

Still this seems to be the best one can do while staying in a *Hausdorff* framework, and it is therefore plausible to consider the disjoint union

$$\mathscr{M}_r^{(2)} = J_r^2(M) \coprod_\mu \mathscr{C}_\mu, \qquad \mu = (u_1, u_2) \quad \text{with } \mu_1 + \mu_2 = r, \mu_1 \geq \mu_2,$$

as some sort of covering of the set of all complex 2-dimensional bundles with $c_1 = r$ over M. In the algebraic category this data leads to what I believe is called the "stack of bundles" over M.

In any case, this stratification plays an essential role in the approaches that Harder [5], Harder & Narasimhan [6], and Atiyah & Bott [1] took to the problem of computing $H^*(J_r^n)$. The Harder-Narasimhan method is number-theoretical and has as its denouement the beautiful counting formula which is a function field analogue (A. Weil, Tamagawa) of a corresponding counting formula going back to Minkowski and C. L. Siegel in the number field case:

$$\sum_E \frac{1}{|\operatorname{Aut} E|} = \frac{1}{q-1} q^{(n^2-1)(g-1)} \zeta_M(2) \ldots \zeta_M(n).$$

Here on the left one is counting the isomorphism classes of vector bundles (of fixed determinant!) over a curve of genus g, defined over a finite field F_q, each vector bundle contributing the reciprocal of the number of automorphisms it has. On the right the answer is given in terms of the ζ-function of M:

$$\zeta_{M(s)} = \frac{\prod_1^{2q}(1-\omega_i q^{-s})}{(1-q^{-s})(1-q\cdot q^{-s})},$$

the $\omega_i(M)$ being intrinsically defined algebraic integers with $|\omega_i| = q^{1/2}$, and in terms of these $\{\omega_i\}$ the number of rational points on the Jacobian $J(M)$ of M is given by:

$$|J(M)| = \prod_1^{2g}(1-\omega_i).$$

Using these formulas it is an easy but beautiful computation to find the contribution of each stratum in $\mathscr{M}_r^{(2)}$ to the left-hand side of the Minkowski-Siegel-Weil formula. For $\mathscr{M}_1^{(2)}$ this then leads to the relation:

$$\frac{|J_1^2|}{q-1} + \frac{|J|^2}{(q-1)^2} \cdot \sum_{r=0}^{\infty} \frac{1}{q^{2r+2-g}} = \frac{1}{(q-1)} \cdot q^{3q-3} \zeta_M(2), \tag{2}$$

and this relation in turn enabled Harder to compute $|J_1^2|$ by subtraction, and hence, via the Weil conjectures, to obtain a hold on the Poincaré polynomial of J_1^2! Actually these computations were done *before* the solution of the Weyl conjectures by Deligne and Grothendieck, so that at that time they merely used this example to check these Weyl conjectures against the computations Newstead [10] had made for $P_t(J_1^2)$, using a topological method which will be explained below.

First, however, one parting comment on the "covering" $\mathcal{M}_r^{(2)}$. It is unfortunately not true that the vector spaces $H^1(X; L_2^* \otimes L_1)$ in C_u are constant in dimension: For large genus this vector space jumps as L_1 and L_2 vary over J_{u_1} and J_{u_2}. A proper algebro-geometric model of the "set of all bundles" should therefore, I think, be built out of the virtual bundles

$$V_u \xrightarrow{\pi_u} J_u \times J_u$$

with fiber $H^1(X; L_2^* \otimes L_1) - H^0(X; L_2^* \otimes L_1)$.

These bundles have a constant (even if negative!) dimension, and the disjoint union

$$\widehat{\mathcal{M}}_1^{(2)} = J_1^{(2)} \coprod V_1 \coprod V_2 \coprod \cdots$$

now precisely corresponds to the left-hand side of (2), if we agree to count the number of points of a virtual vector space, V, by $q^{\dim V}$.

Indeed, with this understanding, the expression

$$|J_1^2| + \sum_{i=1}^{\infty} |V_i|$$

precisely goes over into the left-hand side of (2), of course multiplied by $|J|$, because we have not fixed the determinant in the present discussion.

Now in topology it often happens that we are given a decomposition of a space Y, by strata of the form:

$$Y = Y_0 \coprod W_1 \coprod W_2 \cdots \coprod W_k \cdots$$

with the W_i vector bundles over Y_i, so that Y is finally put together by successively attaching the boundary sphere-bundle of W_i to what has already been built. Indeed, any smooth function f on a manifold Y with reasonable critical behavior induces such a decomposition. Here Y_0 is the absolute minimum of f, and W_i is the "negative bundle" of f over the critical set Y_i of "index i". Thus these W_i are spanned by the direction of steepest descent for f along Y_i.

What is novel in the algebraic framework about our $\mathcal{M}_1^{(2)}$ is therefore that the bundles V_μ are virtual, and actually get more and more negative as $\mu \to \infty$. In topology such a state of affairs is usually remedied by "suspending" to convert the virtual bundles into honest ones. In the present situation that would indicate that one has to suspend more and more as the strata are added—in short, one seems to be building something "down from ∞", a phenomenon often present in the heuristics of modern physics.

It is time to turn to the Yang-Mills version of this story. I first learned about the $J_r^n(M)$ when I visited India in 1976 and was taught the fundamental results of Seshadri, Narasimhan, and Ramanan personally by Ramanan, and became intrigued by the problem of computing the cohomology of the $J_r^n(M)$. Later that year I went to Oxford where Atiyah and Hitchin were in the midst of Yang-Mills theory on the four sphere. It was therefore natural for Atiyah and me to test this new mathematical toy on the problem I had brought along from India.

In short we started to explore the Yang-Mills theory of unitary bundles over a Riemann surface of genus g.

At first, the problem looks quite different. One now deals with a principal $U(n)$ bundle P over M, and considers the space $\mathscr{A}$, of connections over P. Technically such a connection A is then an equivariant 1-form A on P with values in the Lie algebra u_n of U_n whose curvature

$$F(A) = dA + A^2$$

is seen to descend to M and appears that as a 2-form with values in u_n:

$$F(A) \in \Omega^2(M; u_n).$$

The *size* of the curvature is now measured with the aid of Riemann structure on M via the formula

$$\mathrm{YM}(A) = \int_M \mathrm{tr}\{F(A) \wedge *F(A)\}.$$

A crucial property of YM is that it is *invariant under the group of automorphism* $\mathscr{G}(P)$, of P, which cover the identity on M, and it is *this* group which eventually furnishes us with the proper geometric and topological analogue of the right-hand side in the Minkowski-Siegel-Weil formula.

In fact, Atiyah and I found that the "proper" description of the homotopy type of this "stack of all bundles" is precisely that of the topological classifying space $B\mathscr{G}$ of the group $\mathscr{G}(P)$.

But first a word about the relation of our two subjects. What do complex structures on E have to do with connections on P? The link is the following one. A principal U_n bundle P has associated to it a canonical C^∞ Hermitian vector bundle E over M. On the other hand, a Riemann structure on M induces a complex structure on M. Finally a holomorphic structure on E compatible with the one on M together with the Hermitian structure defines a unique "connection of type 1-1" on P. Conversely a connection of type $(1, 1)$ on P defines a complex structure on $E(P)$, and under this correspondence the group of gauge transformations $\mathscr{G}$ mediates between equivalent complex structures on $E(P)$. Grosso Modo then, the space of complex structures on E should be the quotient space $\mathscr{A}/\mathscr{G}$. On the other hand, $\mathscr{G}$ does not act freely on $\mathscr{A}$, and so one has to take this quotient in the "equivariant sense". But now the contractibility of $\mathscr{A}$ implies that homotopy theoretically one is dealing precisely with $B\mathscr{G}(P)$.

In this topological framework of our basic question, the critical points of Yang-Mills are now easily determined: Certainly the absolute minimum of YM on $\mathscr{A}(P_0)$, where P_0 is the trivial U_n-bundle, occurs when $F(A) = 0$. The isomorphism classes of these are therefore precisely the *flat* U_n *bundles* over M, and these are parametrized by their holonomy—that is, by the conjugacy class of the homomorphism of $\pi_1(M)$ to U_n they induce. Thus

$$\operatorname{Min} \mathrm{YM}|_{c_1=0} = \operatorname{Hom}(\pi_1(M);\, U_n)^{U_n}.$$

But the cornerstone of the whole analytic theory of stable bundles was precisely the result of Seshadri [13] that the semistable bundles of trivial topological type correspond to this same space; that is,

$$J_0^n = \operatorname{Hom}(\pi_1(M);\, U_n)^{U_n}.$$

Armed with this knowledge it is then easy to see that the higher critical points of Yang-Mills, say in the two-dimensional case, correspond to splittings of the structure group from $U(2)$ to $U(1)\times U(1)$. Thus the critical sets in the Morse stratification for the Yang-Mills function $\mathrm{YM}\colon \mathscr{A} \to \mathbb{R}$ fall into components

$$Y_1^{(2)} = J_1^2 \coprod_{\mu} J_{\mu_1}^1 \times J_{\mu_2}^1, \qquad \mu_1 \geq \mu_2,\, u_1 + u_2 = 1,$$

which are clearly in 1-1 correspondence with the stratification of $\mathscr{M}_r^{(2)}$ discussed earlier.

Next, the "negative-bundles" W_μ over these critical sets are now found to be $H^1(X;\, L_1^* \otimes L_2)$, with L_1 and L_2 as before. Note that $L_1^* \otimes L_2$ is just the dual bundle to the extension bundle we found earlier. It is therefore strictly negative, and hence does *not* jump when L_1 and L_2 vary in J_{μ_1} and J_{μ_2}. To compute its dimension we again use $R-R$ which now yields

$$-\dim W_\mu = c_1(L_1^* \otimes L_2) - (g-1),$$

or, equivalently,

$$\dim W_\mu = c_1(L_1 \otimes L_2^*) + (g-1).$$

This compares to

$$\dim V_\mu = -c_1(L_1 \otimes L_2^*) + (g-1),$$

so that

$$\dim W_\mu + \dim V_\mu = 2(g-1),$$

and explains the dual relation of the two bundles we have been discussing.

In *truly* favorable situations for the Morse theory, the critical sets with their negative bundles determine the cohomology of the total space in the sense that for any coefficient field k the Poincaré polynomials $P_t(Y_i; k) = P_t(Y_i)$ of the critical sets Y_i determine $P_t(Y)$ according to the simple law

$$P_t(Y_i) + \sum_{i \geq l} t^{\dim W_i} P_t(Y_i) = P_t(Y).$$

(In general, these two sides are of course only related by the Morse inequalities.)

In the equivariant theory, there would be a corresponding formula with $P_t(Y_i)$ replaced by $GP_t(Y_i)$, the Poincaré series of the equivariant cohomology of Y_i. Recall now that the equivariant cohomology for a group G "counts" a point by the ordinary cohomology of its classifying space BG, and in general "counts" an orbit G/H, by the ordinary cohomology of BH. In favorable situations the equivariant contribution of Y_i, that is, $GP_t(Y_i)$ is therefore simply equal to $P_t(Y_j) \times P_t(BH)$ where H is the stabilizer of Y_j.

Now although $\mathscr{G}(P)$ and $\mathscr{A}(P)$ are infinite dimensional, the stabilizers of the action of $\mathscr{G}$ on $\mathscr{A}$ are easily determined:

The stable bundles J_1^2 are stabilized only by U_1, the center of $\mathscr{G}(P)$, whereas the $J_{\mu_1} \times J_{\mu_2}$ are all centralized by $U_1 \times U_1$.

Thus continuing in our assumption of the "most favorable state of affairs", these considerations lead to the formula:

$$\frac{P_t(J_1^2)}{1-t^2} + \sum \frac{(1+t)^{4g}}{(1-t^2)(1-t^2)} t^{2(2r+g)}$$

as the equivariant count of the strata in $\mathscr{A}$. (Recall here that $P_t(BU_1) = 1/(1-t^2)$.) On the other hand, the total equivariant cohomology of $\mathscr{A}$ is counted by $P_t(B\mathscr{G})$, which in the case under consideration is seen to be:

$$P_t(B\mathscr{G}) = \frac{\{(1+t)(1+t^3)\}^{2g}}{(1-t^2)^2(1-t^4)}.$$

In short, *if YM is a perfect equivariant Morse function*, on $\mathscr{A}$, then we should have the equality:

$$(3)\qquad \frac{P_t(J_1^2)}{1-t^2}+\sum\frac{(1+t)^{4g}}{(1-t^2)^2}t^{2(2r+g)}=\frac{((1+t)(1+t^3))^2}{(1-t^2)^2(1-t^4)}.$$

When Michael Atiyah and I reached this point in our tentative arguments, we were unaware of the Harder's computation, and it was only later that we realized how precise the analogy was, and how precisely the right-hand side of C. L. Siegel computes $P_t(B\mathscr{G})$ in the U_n case, if one lets $q \to t^{-2}$ and the $\omega_1 \to -1^{-1}$. (See [1, p. 596].)

In any case, after we checked that (3) agreed with the computations Newstead had made in this case for $g = 1, \cdots, 6$, we pushed on to prove that YM was indeed equivariantly perfect for the group U_n, so that one obtains an inductively defined procedure for counting $P_t(J_r^n)$. Over the rationals this result is then really quite equivalent to the Harder-Narasimhan procedure, but our method produces better results over the integers. In particular, the Yang-Mills approach shows that J_r^n, $(n, r) = 1$, has no torsion.

By the way, Newstead's computations are based on the Narasimhan-Seshadri theorem, which can be used to give the following explicit model for J_1^n:

Let $F_g = \{X_i, Y_i: i = 1 \dots g\}$ be the free group generated by X_i and Y_i, and let

$$\mathscr{E}_g = [X_i, Y_i]\dots[X_g, Y_g]$$

be the product of the indicated commutators, so that $\pi_1(M) \simeq F_g/[\mathscr{E}_g]$. With this understood, the Narasimhan-Seshadri result yields the identity:

$$J_r^n = \operatorname{Hom}(F_g; SU(n))^{SU(n)},$$
$$\varphi(\mathscr{E}_g) = \xi^r \qquad\qquad \xi = e^{2\pi i/n}.$$

Put differently, consider the map:

$$\underbrace{SU_n \times \cdots \times SU_n}_{2g} \xrightarrow{\mu} SU_n$$

sending $\{X_1, Y_1; \dots; X_gY_g\}$ to $\Pi([X_i, Y_i])$.

Then

$$J_r^n = (\mu^{-1}\xi^r)/SU_n,$$

where SU_n acts on all factors by conjugation.

This gives a no-nonsense C^∞ description of J_r^n which, however, is difficult to translate into direct cohomological information, except for

$n = 2$, where $SU_2 = S^3$, and μ is seen to have only two critical values. This fact, used with considerable ingenuity, then produces Newstead's original tables for $g = 1, \cdots, 6$.

It would be interesting to derive our basic induction formula directly from this picture, but in view of the infinite summation that underlies the other approaches I expect this to be unlikely.

Finally, let me remark that a truly Morse-theoretic approach to this problem is really only possible now, due to the work of K. Uhlenbeck and her student, Georgios D. Daskalopoulos. We based our consideration essentially on the stratification $\mathscr{M}_r^{(n)}$ which in the general case was studied in detail by Shatz [14].

Let me now turn to the question of extending the classical θ-functions to the nonabelian case. Interestingly enough the recent impetus for this development originates in consideration of modern physics, rather than number theory or algebraic geometry proper, and I think it is fair to say that a definitive algebro-geometric treatment of this whole subject is still lacking. On the more historical side, there can be no question that the basic reference to all we have been, and will be, talking about is Andre Weil [17].

The first question to be settled is of course the definition of the θ-divisor in the general case. The one-dimensional construction J outlined earlier has no known generalization, but there is one alternate definition of this classical θ which does extend. Indeed, consider a stable bundle E of Chern class k. Then by Riemann-Roch:

$$\dim H^0(M; E) - \dim H^1(M; E) = k - n(g-1).$$

Hence on the component $J^n_{n(g-1)}(M)$ this number vanishes, and in the classical case $(n = 1)$ it is well known that the θ-divisor can also be identified with the set of those $L \subset J_{g-1}(M)$ for which

$$\dim H^0(M; L) = \dim H^1(M; L) \neq 0.$$

On the other hand, this definition has the plausible extension to all dimensions:

$$E \in \theta \Rightarrow \dim H^0(M; E) = \dim H^1(M; E) \neq 0.$$

Unfortunately, however, $J^n_{n(g-1)}(M)$ is a singular space, and so it has to be checked that the "jumping" condition defining θ satisfies the nontrivial technical requirement of being a Cartier divisor. A theorem to this effect is, I believe, to be published by Seshadri and some of his collaborators. Of course the physicists, never much concerned with such technical hurdles, not only assume that this generalized θ-divisor exists, but even furnish us with a remarkable formula for the number of these generalized θ-functions over M. This is a formula due to E. Verlinde for $\dim H^0(J^n_{n(g-1)}(M); \mathscr{L}^k)$, where $\mathscr{L}$ is the line bundle associated to the θ as defined above.

Let me close this account by describing this remarkable "Verlinde formula" [16], but in a representation theoretic transcription, developed by Andras Szenes and me. We of course hope that ultimately this formulation, or one close to it, might lead to a more direct proof than the one that is at present available, say, via the difficult and very technical papers of T. Tsuchiya, K. Ueno, and T. Yamada [15].

In any case here is our version of the Verlinde formula, at least in the case $n = 2$. First of all one separates the nonabelian case from the abelian one, by fixing the determinant of E. Thus Verlinde is computing $\dim H^0(SJ^{(2)}_{n(g-1)}(M); \mathscr{L}^k)$, where SJ^n describes the fiber of the determinant map $J^n \to J$:

$$1 \to SJ^n \to J^n \overset{\det}{\to} J \to 1.$$

This step reduces the considerations from U_2 to SU_2 and we formulate Verlinde's answer purely in terms of the representative, or character, ring of $G = SU_2$. Recall then that as a ring over $\mathbb{C}$ this character ring $R(G)$ is freely generated by the standard representation of SU_2, say V, so that $R(G)$ can be thought of quite simply as the polynomial ring in one variable V:

$$R(G) = \mathbb{C}[V].$$

On the other hand, as the representative ring of a compact Lie group, $R(G)$ has a natural linear functional defined on it, which I write Φ, and which on any finite-dimensional G-module W simply counts the dimension of the G-invariant subspace W^G in W:

$$\Phi(W) = \dim W^G = \int_G \chi(W)\, dg.$$

Essentially, then, Φ is given by the Haar measure on G normalized to 1. The ring $R(G)$ has a natural involution, sending W to its dual G-module W^*, and our Φ is *positive* on $R(G)$ in the sense that

$$\Phi(WW^*) \geq 0.$$

Finally this Φ is seen to define a positive definite inner product on $R(G)$, by the rule

$$\langle W, W' \rangle = \Phi(W^* \otimes W'),$$

and, as is well known, the classes of the irreducible representations in $R(G)$ then form an *orthonormal basis* for $R(G)$.

For example, in the case under discussion, $G = SU_2$, we have $V^* = V$, so that * is the identity. On the other hand, $\Phi(V^k)$ is not so easy to compute, but is clearly determined recursively in terms of Clebsh-Jordan formulas. Indeed, if we write $V_0, V_1, \ldots$ for the irreducibles in ascending order: V_i of dimension $i+1$, then they are recursively given by

$$V_0 = 1; \qquad V_k \cdot V = V_{k+1} + V_{k-1},$$

so that the first four terms are:

$$\begin{aligned} V_0 &= 1 & V_3 &= V^3 - 2V \\ V_1 &= V & V_4 &= V^4 - 3V^2 + 1 \\ V_2 &= V^2 - 1 & &\text{etc.} \end{aligned}$$

Consequently, by Shur's lemma, $\Phi(V_k) = 0$ for $k > 0$, and so $\Phi(V) = 0$, $\Phi(V^2) = 1$, $\Phi(V^3) = 0$, and $\Phi(V^4) = 2$, etc.

In any case, consider now the ideal $I_k \subset R(G)$ generated by the irreducible V_k, and the corresponding quotient ring

$$R_k(G) = R(G)/I_k, \qquad I_k = \{V_k\}.$$

A first lemma is now that the *additive* subspace spanned by the irreducibles $V_0, \cdots, V_{k-1}$ spans a complement to I_k in $R(G)$:

$$R(G) = I_k \oplus \{V_0, \cdots, V_{k-1}\}.$$

Additively we therefore have

$$R_k(G) \cong \{V_0, \cdots, V_{k-1}\},$$

which enables us to push Φ (forward!) to a linear function Φ_* on $R_k(G)$. (Put more mundanely: if $x \in R_k(G)$, then $\Phi_*(x)$ is defined as the coefficient of V_0 when x is written in the additive base furnished by the natural projections $\underline{V}_i$ of V_i; $i = 0, \cdots, k-1$ to $R_k(G)$.)

With all this understood consider now the element

$$\underline{k} = \underline{V}_0 \otimes \underline{V}_0^* + \underline{V}_1 \otimes \underline{V}_1^* + \cdots + \underline{V}_{k-1} \otimes \underline{V}_{k-1}^*$$

in $R_k(G)$. In terms of this $\underline{k}$ Verlinde's formula is equivalent to

$$\dim H^0(SJ^n_{n(g-1)}(M; \mathscr{L}^k)) = \Phi_*(\underline{k}^g). \tag{4}$$

Remark. (1) I like this formulation, because it gives such a natural extension of the classical formula: $\dim H^0(J; \mathscr{L}^k) = k^g$. However, to bring it more into line with the physics literature, or say the concepts of topological field theory, the following quite equivalent way of writing the right-hand side of (4) is useful.

Equipped with Φ_* the algebra $R_k(G)$ becomes a finite-dimensional commutative H^*-algebra and as such is isomorphic to the algebra of functions on a finite measure space X_k, of cardinality k. Let us take this set to be the integers $0 \le i \le k-1$, and let $w(i)$ be the measure of the ith point, so that the Φ_*-induced norm on $R_k(G)$ corresponds to the integral:

$$f\cdot : \to \sum |f(i)|^2 \cdot \omega_i.$$

Now under the inner product induced on $R_k(G)$ by Φ_* the elements $\{\underline{V}_i\}$, $0 \le i \le k-1$, are seen to be an orthonormal basis for $R_k(G)$ so that the element

$$\underline{k} = \sum_{i=0}^{k} \underline{V}_i \otimes \underline{V}_i^*$$

is canonically defined in $R_k(G)$, and thus is *independent* of the orthonormal basis chosen. But the δ-functions

$$\delta_i(j) = \delta_{ij},$$

clearly, have norm $= \omega_i$, so that

$$e_i = \frac{\delta_i}{\sqrt{\omega_i}}$$

form an orthonormal system in terms of which

$$\underline{k} = \sum \frac{\delta_i}{\omega_i},$$

and on this basis, the idempotent property of the δ_i implies that

$$\underline{k}^g = \sum \frac{\delta_i}{\omega_i^g}.$$

Hence

$$\Phi_*(\underline{k}^g) = \sum_{i=0}^{k-1} \omega_i^{(1-g)}.$$

The right-hand side now takes on an even better form if we pass to the square roots of the ω_i:

(5) $$\Phi_*(\underline{k}^g) = \sum_{i=0}^{k-1} (\omega_i)^{1/2\cdot\chi(M)}.$$

In this form one is now also closer to the Verlinde formula in [16] and, as Szenes first remarked, the ω_i are now identified with the volumes of certain adjoint orbits in SU_2—or equivalently to the value of the Weyl measure on the maximal torus T of SU_2 on certain points of finite order in T.

(2) The element $\underline{k} = \sum V_i \otimes V_i^*$ is of course canonically defined in $R(G)$ for any finite group. (The sum here is over *all* the irreducibles of $R(G)$.) In this situation one has

$$\Phi(\underline{k}^g) = \sum w_i^{1/2\chi(M)},$$

where i ranges over the conjugacy classes of G, and the weight $\omega(x)$ of a given conjugacy class $x \in G$ is 1 over the number of elements in its stabilizer:

$$\omega(x) = 1/|s_x|.$$

Thus in the Verlinde formula the $R_k(G)$ plays an analogous role to $R(G)$ for a finite group. There are also connections to "quantum group" constructions, which, however, are not sufficiently clear to me to report on here.

In low genuses one can actually check this formula by independent procedures. For instance, when $g = 2$, Ramanan had shown long ago that

$$SJ_0^2 = \mathbb{C}P_3.$$

(This is the only case where the "even" SJ_0^2 has no singularities.) More generally for $SU(n)$ the Verlinde formula also checks the recent computations of Beauville, Narasimhan and Ramanan [3] for $k = 1$. Finally, Szenes has conjecturally extended (6) to the other components SJ_k^n as well, and there one can then check matters against Riemann-Roch computations. For instance, for $SJ_1^{(2)}$ he finds:

$$\dim H^0(SJ_1^{(2)}; \mathscr{L}^k) = \Phi_*((2\underline{k})^g \cdot V_{2k}). \tag{6}$$

On the right-hand side one is therefore dealing with the ring $R_{2k}(G)$, and on the left-hand side $\mathscr{L}$ is now defined as the generator of the Picard group of $SJ_2^{(2)}$. For low genus one can then compute the left-hand side using Riemann-Roch either from Ramanan's explicit description of $SJ_1^{(2)}(M)$ for hyperelliptic M, or from the cohomological data in [1].

Unfortunately the Verlindes' direct arguments in [16] are largely unintelligible to us. It is possible, though, to trace a path (a very long one) from the computation of [15] to the Verlinde formula, via the notion of parabolic bundles, due to Mehta and Seshadri [9], and an informal account of such a voyage can be found in the Oxford Notes of Atiyah-Hitchin-Segal [11]. The essential new ingredient which the physics-inspired literature adds to this subject is a sort of multiplicative Meyer-Vietoris theorem, which holds when two punctured surfaces M_1 and M_2 are sewed together. Here let me just point the way with the following comments.

Note that as a consequence of the orthogonality of the V_i, $i = 0, \cdots, k-1$, in $R_k(G)$ one has the recurrence relation

$$\sum_{\lambda=0}^{k-1} \Phi(k^{g_1} \cdot V_\lambda)\Phi(k^{g_2} V_\lambda^*) = \Phi(k^{g_1+g_2}). \tag{7}$$

Indeed, by the orthogonality we can compute the left-hand side of (9) using the idempotent basis

$$\text{Left-hand side} = \sum \Phi(k^{g_1} e_\lambda)\Phi(k^{g} \overline{e}_\lambda).$$

But $k^{g_1} e_\lambda = \delta_\lambda / \omega_\lambda^{g_1+1/2}$ whence $\Phi(k^{g_1} e_\lambda) = 1/\omega^{g_1-1/2}$ so that the λth term of (7) contributes $\omega_\lambda^{1/2-g_1} \cdot \omega_\lambda^{1/2-g_2}$. But then (5) becomes clear.

The recursion satisfied by Verlinde's answer is the dimensional consequence of the Künneth property, of the functor $M \to H^0(SJ_0^n(M); \mathscr{L}^k)$ expressed by the "fusion rules". Indeed, the choice of $p \in M$ and a representation, V_λ, $0 \le \lambda \le k$, determine a "parabolic structure" on M in the sense of Seshadri, and a corresponding space of semistable bundles, which I will notate $SJ_0^n(M; V_\lambda)$. This space again inherits a 0-divisor with corresponding line bundle $\mathscr{L}$, so that the spaces $H^0(SJ_0^n(M; V_\lambda); \mathscr{L}^k)$ become well defined and provide the constituents of the Künneth decomposition:

$$\sum_{\lambda=1}^{k-1} H^0(SJ_0^n(M_1; V_\lambda); \mathscr{L}^k) \otimes H^0(SJ_0^n(M_2; V_\lambda^*); \mathscr{L}^k) \simeq H^0(SJ_0^n(M_1 \vee M_2), \mathscr{L}^k). \tag{8}$$

How might such a formula fit into the more standard algebraic geometry picture? Presumably the answer is the following:

The assignment $M \to H^0(SJ_0^n(M); \mathscr{L}^k)$ defines a vector bundle over the moduli space of curves of a fixed genus g which can be endowed with a projectively *flat* connection. See, for instance, the very interesting papers of Axelrod, Della Pietra and Witten [2] on the one hand and Hitchin [7] and Ramadas [12] on the other. The holonomy of this bundle around paths linking the compactification divisor of the moduli space near the stable curve $M_1 \vee M_2$ should then hopefully induce the decomposition

(8), and this procedure might eventually lead to a better mathematical understanding of Verlinde's arguments. However at the moment we still seem far away from a direct and conceptual proof of their formula.

Finally, let me remark that in view of Riemann-Roch, the dimensions of the $H^0(J_r^n(M); \mathscr{L}^k)$ are given in terms of certain characteristic numbers of the varieties $J_r^n(M)$, and these in turn would be completely understood if we knew the homological position of $J_r^n(M) \subset B\mathscr{G}$, i.e., if we knew the homomorphism which fundamental cycle of $J_r^n(M)$ determines on $H^*(B\mathscr{G}_n)$, whose generators have direct geometric meaning. Computations of this sort would also be useful in the better understanding and computations of the Donaldson invariants for four-manifolds.

Added in proof. A recent Oxford preprint by M. Thaddeus entitled "Conformal field theory and the cohomology of the moduli space of stable bundles" carries out this program in the rank-2 case.

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