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A REPORT ON SOME RECENT PROGRESS ON NONLINEAR PROBLEMS IN GEOMETRY

RICHARD M. SCHOEN

Introduction

In this paper we describe some currently active research areas in the field of geometric partial differential equations. Our initial intention was to write a more comprehensive survey of progress in this general field, but in the end we have chosen a few specific topics of the many more we could have included in a survey paper. The topics which are presented here were chosen primarily because they are of current interest to the author, and in the case of several of them there is very recent progress to be reported. Thus this paper is more of a report on selected currently active research than any attempt at a survey of the field.

In the first section we discuss recent (unpublished) results on conformal deformation of Riemannian metrics to constant (or prescribed) scalar curvature. The major stress here is on large solutions rather than minima for the variational problem. In particular, we include a description of the author's results on the strong Morse inequalities for the Yamabe problem. We also include a brief discussion of a related result (joint with D. Zhang) for the prescribed curvature problem on the 3-sphere S^3 as well as a brief description of recent results on the boundary value problem. Finally we describe progress on complete constant scalar curvature metrics.

In §2 we introduce the Ricci flow equation of R. Hamilton and describe some of the current problems and recent progress on this important nonlinear evolution equation.

In §3 we discuss harmonic maps and some applications. This includes a description of a result of M. Micallef and J. D. Moore on manifolds of positive curvature. We also describe (and slightly amplify) recent work of F. Helein on regularity of weakly harmonic maps into special target manifolds. Finally we give a systematic description of our recent work on the behavior of harmonic maps from manifolds into nonpositively curved Riemannian simplicial complexes. We also briefly describe the application

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of this (joint with M. Gromov) to proving the arithmeticity of lattices in certain rank one Lie groups.

Finally, in §4 we describe the current status of existence and compactness questions for closed and complete constant (nonzero) mean curvature surfaces in three-dimensional Euclidean space. This includes a discussion of recent work of N. Kapouleas (as yet unpublished) which incorporates certain tori of H. Wente into his earlier constructions to produce closed surfaces of constant mean curvature for any genus.

1. Scalar curvature

One of the basic facts about the geometry of surfaces is the uniformization theorem which tells us that every Riemannian metric on a surface is conformally related to a metric of constant curvature. If this metric is geodesically complete and is normalized to have curvature ± 1, 0, it is then unique (in every case except the two-sphere which has a three-parameter family of curvature one metrics in the conformal class of the standard metric). For general manifolds M of dimension three or more there is no known method to construct canonical Riemannian metrics. The most natural determined system of equations for a Riemannian metric is the Einstein system $\mathrm{Ric}(g) = cg$, where $\mathrm{Ric}(g)$ denotes the Ricci tensor of g, and c is a constant necessarily given by $c = n^{-1}R$, $n = \dim M$ and $R =$ scalar curvature of g. The Ricci operator $g \to \mathrm{Ric}(g)$ plays the role of a nonlinear Laplace operator which commutes with the action of the diffeomorphism group on the space of metrics; for a recent account of many aspects of the Einstein equations see [7]. The Einstein equations arise as the Euler-Lagrange equations for a variational integral on the space of metrics. This integral is the Einstein-Hilbert energy which assigns to a Riemannian metric g the value $\mathbf{R}(g) = \int_M R_g \, d\mu_g$, where $d\mu_g$ denotes the Riemannian volume measure of g. The Einstein equations are then the differential equations satisfied by a stationary point of $R(\cdot)$ on the space of unit volume metrics. Analysis of the second variation of $R(\cdot)$ at an Einstein metric g_0 shows that for deformations of g_0 which preserve its conformal class, g_0 is a local minimum modulo a finite-dimensional index space, say of dimension I_c. On the other hand, for deformations which are orthogonal (in the L^2 sense) to both the conformal class and the diffeomorphism group, g_0 is a local maximum modulo a finite-dimensional index space, say of dimension I_T. A theorem of M. Obata (see [45] for detailed discussion) shows that in fact g_0 achieves the minimum of $R(\cdot)$

in its conformal class, and therefore $I_c = 0$. For constant curvature metrics it is easy to show that $I_T = 0$, and it appears that most Einstein metrics which can presently be constructed by analytic means (such as Kähler-Einstein metrics) have $I_T = 0$. A detailed discussion of the total scalar curvature variational problem is given in [52].

The variational problem obtained by restricting $R(\cdot)$ to unit volume Riemannian metrics in a given conformal class is called the Yamabe problem. The Euler-Lagrange equation for the Yamabe problem is the equation $R_g \equiv C$ for a constant $C = \int_M R_g \, d\mu_g$. The first attempt to produce a minimizer for this problem was given by Yamabe [67]. It was pointed out by N. Trudinger [62] that Yamabe's proof contained a serious gap, and this gap was filled for nonpositive conformal classes; i.e., conformal classes containing a metric with everywhere nonpositive scalar curvature. Several years later T. Aubin [4] proved the first result in the positive case. He showed that for $n \geq 6$ and for a nonconformally flat conformal class of metrics the problem has a minimizer. Aubin's method was local, exploiting the local conformal deviation of the given metric from the Euclidean metric. In conformally flat cases the Yamabe problem is clearly a global problem, and in the delicate low-dimensional cases where the equation contains relatively more information the problem is also global. The existence of a minimizer was finally shown in [51]. Explicit examples are given in [52] for which there are arbitrarily many solutions of the Yamabe problem with arbitrarily large energy; i.e., R_g large and unit volume. It is natural to ask whether there is a bound on the energy of solutions in a given conformal class, and whether in any sense the solutions obey a Morse theory as they would if the variational integral satisfied the Palais-Smale condition. We now describe our progress on these questions.

Suppose we fix a background matrix g_0 in our conformal class, and write our unknown metric in the form $g = u^{4/(n-2)} g_0$. It is then well known (see [52]) that the equation of constant scalar curvature becomes

$$Lu + cu^{(n+2)/(n-2)} = 0, \qquad u > 0, \tag{1.1}$$

where $Lu = \Delta_{g_0} u - c(n) R_{g_0} u$, $c(n) = \frac{n-2}{4(n-1)}$. The variational problem is the problem of extremizing $E(u) = -\int_M uLu \, d\mu_{g_0}$ subject to the constraint $\mathrm{Vol}(g) = \int_M u^{2n/(n-2)} \, d\mu_{g_0} = 1$, thus leading to the Euler-Lagrange equation (1.1). We introduce the so-called subcritical regularization of this problem by embedding it in a family of problems indexed by an exponent $p \in [1, \frac{n+2}{n-2}]$. These are the equations

$$Lu + cu^p = 0, \qquad u > 0. \tag{$(1.1)_p$}$$

The corresponding variational problem is to extremize $E(u)$ subject to the constraint $V_p(u) \equiv \int_M u^{p+1}\, d\mu_{g_0} = 1$. In this way we connect the eigenvalue problem for L (i.e., $p = 1$) to the Yamabe problem, $p = \frac{n+2}{n-2}$. The constant c in $(1.1)_p$ is dependent on the solution and not specified in advance. It is well known that the variational problem corresponding to $(1.1)_p$ for $p \in [1, \frac{n+2}{n-2})$ does satisfy the Palais-Smale condition, and thus a Morse theory exists for these problems. A consequence of the Morse theory is the strong Morse inequalities which are relations satisfied among the critical points. Rather than attempt to construct a gradient flow for the Yamabe problem, we instead concentrate on convergence of critical points as $p \uparrow \frac{n+2}{n-2}$. Our main analytic result can be stated very simply.

Theorem 1.1. *Assume (M, g_0) is not conformally diffeomorphic to the round n-sphere S^n. There is a constant Λ depending only on the background metric g_0 such that if u is any solution of $(1.1)_p$ for any $p \in [1, \frac{n+2}{n-2}]$, then $\|u\|_{C^2(M)} \le \Lambda$.*

This result implies that all solutions of $(1.1)_p$ have uniformly bounded C^2-norm, and hence there can be no divergence as $p \uparrow \frac{n+2}{n-2}$. The fact that S^n has a noncompact group of conformal transformations implies that the conclusion of Theorem 1.1 is false on S^n. The result of Theorem 1.1 has been announced in [54] together with a detailed proof in the locally conformally flat case.

Note that the constant c in $(1.1)_p$ is the energy of the solution. Thus if one wants to consider large energy solutions it is desirable to multiply u by a constant to normalize the equation. Thus we see that $(1.1)_p$ for $p \in (1, \frac{n+2}{n-2}]$ is equivalent to

$$(1.2)_p \qquad Lu + u^p = 0, \qquad u > 0,$$

where the volume $V_p(u)$ is uncontrolled.

As in the original proof of the existence of a minimizer for the Yamabe problem, there is both a local and a global component to Theorem 1.1. The following result is local in nature and may be viewed as a strong generalization of Aubin's Theorem [4].

Theorem 1.2. *Let g_0 be a smooth Riemannian metric defined in the unit n-ball B_1^n. Suppose there is a sequence of solutions $\{u_j\}$ of $(1.2)_{p_j}$, $p_j \in (1, \frac{n+2}{n-2}]$, such that for any $\varepsilon > 0$ there is a constant $C(\varepsilon)$ so that $\sup_{B_1 \setminus B_\varepsilon} u_j \le C(\varepsilon)$ and $\lim_{j\to\infty}(\sup_{B_1} u_j) = \infty$. It follows that the Weyl tensor $W(g_0)$ satisfies $\|W(g_0)\|(x) \le C|x|^m$ for some integer $m > \frac{n-6}{2}$.*

The previous theorem implies that there cannot be an isolated point of blow-up for a sequence of locally defined solutions of $(1.2)_p$ unless the background conformal structure osculates the flat structure up to high order (for large n). Theorem 1.2 is one of the components required for the proof of Theorem 1.1.

The existence theorem which follows from Theorem 1.1 gives the strong Morse inequalities for the Yamabe problem since these inequalities are well known for $(1.1)_p$ with $p \in [1, \frac{n+2}{n-2})$, and Theorem 1.1 shows that all critical points converge as $p \uparrow \frac{n+2}{n-2}$. In case all solutions of the Yamabe problem are nondegenerate, as will be true for a generic conformal class of Riemannian metrics, the existence result asserts that there are a finite number of solutions of the variational problem. Let C_μ for $\mu = 0, 1, 2, \dots$ be the number of solutions of Morse index μ. The following system of inequalities holds:

$$(1.3) \qquad (-1)^\lambda \leq \sum_{\mu=0}^{\lambda} (-1)^{\lambda-\mu} C_\mu, \qquad \lambda = 0, 1, 2, \dots.$$

Since there is a finite number of solutions in all, it follows from (1.3) that

$$(1.4) \qquad 1 = \sum_{\mu=0}^{\infty} (-1)^\mu C_\mu,$$

where the sum on the right is finite. Note that the system (1.3) of Morse inequalities is that satisfied by a proper Morse function on a finite-dimensional Euclidean space. This is quite natural since the space of unit volume metrics in a conformal class is a contractible space. The question arises as to whether higher index solutions of the Yamabe problem exist in typical conformal classes. Explicit examples of this are given in [52] and discussed in detail. A recent result of D. Pollack shows that every Riemannian metric of positive scalar curvature is C^1 close to a smooth metric for which there exists an arbitrarily large number of solutions of the Yamabe problem with arbitrarily high index. To say this more precisely, if g is a metric of positive scalar curvature, and S is any positive integer, then there exists a sequence of smooth metrics g_j converging in C^1 norm to g such that the conformal class of g_j contains at least S solutions of the Yamabe problem for each j. In other words, the set of smooth metrics for which there are arbitrarily many solutions of the Yamabe problem is dense in the C^1 norm on the space of metrics of positive scalar curvature.

It may be useful to give more geometric statements of the analytical results which we have described. The result of Theorem 1.1 for $p = \frac{n+2}{n-2}$ may be described in the following way. Let $\mathscr{S}$ denote the set of all metrics on M with constant scalar curvature equal to 1. Let $\mathscr{M}$ denote the space of all smooth Riemannian metrics on M. The space of smooth functions $C^\infty(M)$ acts on $\mathcal{M}$ by conformal multiplication; that is, if $v \in C^\infty(M)$ and $g \in \mathcal{M}$, then $e^{2v}g \in \mathcal{M}$. The orbit space $\mathcal{C}$ is thus the space of conformal classes of metrics on M. The natural map $\Pi\colon \mathcal{M} \to \mathcal{C}$ is defined, and it is true that at any nondegenerate metric $g \in \mathcal{S}$, the map Π defines a local diffeomorphism from a neighborhood of g in $\mathcal{S}$ onto a neighborhood of $\Pi(g)$ in $\mathcal{C}$ (see [7]). Thus $\mathcal{S}$ is an infinite-dimensional manifold on the complement of the set of degenerate solutions of the Yamabe problem. For more discussion of the singular structure of $\mathcal{S}$ see [7]. Theorem 1.1 then implies

Theorem 1.1′. *The projection map $\Pi\colon \mathcal{S} \to \mathcal{C}$ is a proper map with respect to $C^{k,\alpha}$ topologies for k large enough.*

We now describe some recent work on a related variational problem for which blow-up does occur in the subcritical regularization. This is a joint work with Dong Zhang (see [72]). Suppose $K(x) \in C^\infty(S^3)$ is a smooth positive function on S^3. The prescribed curvature problem is then the problem of finding a metric g conformal to the standard metric g_0 on S^3 such that the scalar curvature $R(g)$ is equal to K at each point. By writing $g = u^4 g_0$ for a positive function u on S^3, the desired equation for u is

$$(1.5) \qquad L_0 u + \tfrac{1}{8}Ku^5 = 0, \qquad L_0 u = \Delta_{g_0} u - \tfrac{3}{4}u.$$

As above (1.5) is equivalent to the Euler-Lagrange equation for the problem of extremizing $E(u) = -\int_{S^3} uL_0 u\, d\mu_{g_0}$ subject to the "volume" constraint $V_k(u) = \int_{S^3} Ku^6\, d\mu_{g_0} = 1$. We again consider the subcritical regularizations:

$$(1.5)_p \qquad L_0 u + \tfrac{1}{8}Ku^p = 0, \qquad p \in [1, 5].$$

For (1.5), work of Kazdan and Warner [35] shows that there may be no solution for certain choices of the function K. As a consequence we see that Theorem 1.1 must fail in this case. Assuming that the critical points of K are nondegenerate and $\Delta K \neq 0$ at each critical point, we let $\mathcal{G} \subseteq S^3$ be the following finite set of points: $\mathcal{G} = \{x \in S^3 \colon \nabla K(x) = 0,\ \Delta K(x) < 0\}$. For each $x \in \mathcal{G}$, let $i(x) \in \{1, 2, 3\}$ denote the Morse index of K at x. The following result then describes the precise nature of the blow-up occurring in the subcritical regularization.

Theorem 1.3. *Corresponding to each $x \in \mathcal{G}$, there is precisely one solution $u_{x,p}$ of $(1.5)_p$ for $p < 5$ which diverges as $p \uparrow 5$. This solution* *has Morse index* $3 - i(x)$, *and it diverges at the lowest possible energy level by concentration at* x.

It is possible to give a rather precise description of $u_{x,p}$ for $p < 5$ and p near 5. As a consequence of this theorem, we have a version of the strong Morse inequalities for the prescribed curvature problem on S^3. For $\mu = 0, 1, 2$ let $D_\mu = \#\{x \in \mathcal{G}: i(x) = 3 - \mu\}$, and for $\mu = 0, 1, 2, \ldots$ let $C_\mu = \#$ of solutions of (1.5) with Morse index μ. We then have

$$(1.6) \qquad (-1)^\lambda \le \sum_{\mu=0}^{\lambda} (-1)^{\lambda-\mu}(C_\mu + D_\mu), \qquad \lambda = 0, 1, 2, \ldots.$$

Note that we have assumed here that all critical points of (1.5) are nondegenerate. This holds for K generic. In particular, we see that the equality

$$1 = \sum_{\mu=0}^{\infty} (-1)^\mu C_\mu + \sum_{\mu=0}^{2} (-1)^\mu D_\mu$$

holds, and hence if we let $N(K) = |1 - \sum_{\mu=0}^{2} (-1)^\mu D_\mu|$ we see that the number of solutions of (1.5) is always bounded below by $N(K)$. (One can easily arrange $N(K)$ to be arbitrarily large by choosing K appropriately.) We also remark that without a genericity assumption one can interpret the previous equality as evaluating the degree of a Fredholm map. To put these results in historical context, we note that the first general existence theorem for the prescribed curvature problem on S^2 was due to J. Moser [43] who proved it for K satisfying $K(x) = K(-x)$ on S^2. The higher-dimensional version of Moser's theorem was considered by J. Escobar and the author [21]. Here a solution is constructed by minimization among symmetric metrics, and the result is proved for $n = 3, 4$. The result on S^3 that $N(K) > 0$ implies the existence of a solution is due to A. Bahri and J. M. Coron [5]. The Bahri-Coron result was proven on S^2 by A. Chang and P. Yang [12] and the analogue of our result by Z. C. Han [28]. The main analytic difficulties involved with higher energy blow-up do not occur for $n = 2$, because surfaces of positive curvature are much more restricted metrically than higher-dimensional manifolds of positive scalar curvature. (Certain other difficulties occur for $n = 2$ because the nonlinearity is exponential.) We also remark that N. Korevaar and the author have constructed examples for $n \ge 6$ where infinite energy blow-up occurs in $(1.5)_p$. This suggests that the analogue of our results for higher-dimensional spheres will be difficult to obtain. A partial result for $n = 4$ was obtained by Dong Zhang [72].

As mentioned above, the Yamabe problem has both a local and global component. The global considerations arise in ruling out small energy blow-up on manifolds of dimension 3, 4, and 5 (where Theorem 1.2 gives no information), and at points where the metric is conformally flat to a high order. In the case of conformally flat manifolds these global considerations arise as a special case of a general structure theorem. We now describe our work with S. T. Yau [57]. If (M^n, g_0) is a locally conformally flat Riemannian manifold of dimension $n \geq 3$, then the universal covering $\tilde{M}$ can be mapped conformally into S^n by the so-called developing map $\Phi: \tilde{M} \to S^n$. There is a group homomorphism $\rho: \pi_1(M) \to O(n+1, 1)$, the group of conformal automorphisms of S^n, and the map Φ is equivariant in the sense that $\Phi \circ \gamma = \rho(\gamma) \circ \Phi$. In general, the map Φ may be quite pathological; however, one of the main results of [57] is the following.

Theorem 1.4. *If (M, g_0) is complete, conformally flat with nonnegative scalar curvature, then $\Phi: \tilde{M} \to S^n$ is injective, and the universal covering of M is conformally diffeomorphic to a subdomain of S^n. Moreover, ρ injects onto a discrete subgroup of $O(n+1, 1)$.*

This leads naturally to the question of which domains in S^n have complete conformal metrics of nonnegative scalar curvature. It is shown that a necessary condition on Ω for the existence of such a metric is that the Hausdorff dimension of $S^n \backslash \Omega$ be at most $\frac{1}{2}(n-2)$. There is a more general injectivity criterion derived in [57] which we now describe. Suppose (M, g_0) is a complete Riemannian manifold and $\Phi: M^n \to S^n$ is a conformal immersion into S^n. Let $L_0 u = \Delta_0 u - \frac{n-2}{4(n-1)} R(g_0) u$ be the conformally invariant Laplacian on M. Given a point $x_0 \in M$, it can be shown that L_0 has a minimal positive fundamental solution with pole at x_0, which we denote by G_{x_0}. We then define $p(M, g_0)$ by

$$p(M, g_0) = \inf\{p \in \mathbf{R}^+ : G_{x_0} \in L^p(M \backslash B_1(X_0))\}.$$

Thus $p(M, g_0)$ is a measure of the decay at infinity of G_{x_0}. It is straightforward to see that $p(M, g_0)$ is independent of x_0, and is such that $D(M, g_0) \equiv \frac{n-2}{2} p(M, g_0) \in [0, n]$ for any manifold. Notice also that in case M is the universal covering space of a compact manifold, then $D(M, g_0)$ does not depend on the choice of metric in the quotient, but only the conformal structure; in fact, $D(M, g_0)$ is a quasi-isometry invariant for metrics in the same conformal class. The general injectivity theorem can then be stated.

Theorem 1.5. *Suppose that (M, g_0) is a complete Riemannian manifold, and $\Phi: M \to S^n$ is a conformal immersion, and assume that $|R(g_0)|$*

is bounded for $n = 3, 4$. *If* $D(M, g_0) < \frac{(n-2)^2}{n}$, *then* Φ *is injective and defines a conformal diffeomorphism of* M *with a subdomain of* S^n.

These injectivity theorems may be used to derive a general geometric statement about conformally flat manifolds of constant scalar curvature. The proof of the following result appears in [54].

Theorem 1.6. *If* (M, g_0) *is conformally flat and complete with* $R_{g_0} \equiv 1$, *then any embedded round (i.e., with totally umbilic boundary) ball in* M *is geodesically convex for the metric* g_0.

The proof of this result uses the injectivity theorems together with the Alexandrov reflection method. It is relatively straightforward to prove Theorem 1.1 for conformally flat M from Theorem 1.6, and the proof is discussed in [54].

Finally we discuss the results which have been obtained for the existence of complete conformally flat metrics with constant positive scalar curvature. If we consider simply connected manifolds, then by the injectivity theorem there is no loss of generality in restricting attention to domains Ω in S^n. In case $\Omega = S^n$, the theorem of M. Obata [45] states that all solutions are standard in the sense that they arise by pullback of the standard metric by a conformal transformation of S^n. When $\Omega = S^n - \{x_1\}$ for a point $x_1 \in S^n$, the methods of Gidas, Ni, and Nirenberg [23] imply that there is no such metric on Ω. An old question which has arisen in various contexts is the existence question for $\Omega = S^n - \{x_1, \cdots, x_k\}$ for a prescribed set of k points. When $k = 2$, it was shown by Caffarelli, Gidas, and Spruck [9] that all solutions become rotationally symmetric when the points are put in antipodal position on S^n. (This can be accomplished by a conformal transformation.) It is also shown in [9] that any weak solution of (1.1) on the punctured Euclidean ball is either extendable across the origin or is asymptotic to one of the family of distinct ODE solutions. A general existence theorem for global weak solutions of (1.1), in fact complete conformal metrics of constant scalar curvature, was derived in [53]. This produces solutions on large classes of domains Ω where $S^n - \Omega$ is a totally disconnected set of small Hausdorff dimension. In the case of prescribed singular points we show the following theorem in [53].

Theorem 1.7. *If* $x_1, \cdots, x_k \in S^n$ *with* $k \geq 2$, *then there exists a complete conformal metric with scalar curvature identically one on* $\Omega = S^n - \{x_1, \cdots, x_k\}$.

One of the main difficulties in handling (1.1) analytically is the high degree of instability which solutions tend to have. This difficulty arises already in the Yamabe problem, but it is far more serious when one deals with complete solutions. Here it typically happens that the linearized equation has zero embedded in its continuous spectrum. This problem is overcome in [53] by imposing an infinite number of conditions so that the operator may be inverted, and then varying the geometry of the approximate solution to compensate for these conditions. If one considers domains $\Omega = S^n - \Sigma^k$, where Σ^k is a smooth closed k-dimensional submanifold with $k < \frac{1}{2}(n-2)$, then it is unknown generally whether Ω carries a complete conformal metric with scalar curvature one. For $\Sigma^k = S^k$, a round subsphere, such solutions exist in abundance because $S^n - S^k$ is conformally diffeomorphic to $H^{k+1} \times S^{n-k-1}$, H^{k+1} being the hyperbolic space of dimension $k+1$. It has recently been shown by R. Mazzeo and N. Smale [38] that when Σ^k is a small perturbation of S^k many solutions do exist on $S^n - \Sigma^k$. Their proof involves a delicate linear analysis and perturbation argument. Their solutions have the simplest possible boundary asymptotic behavior. One of the interesting questions in general is to determine the possible boundary asymptotics for global solutions. For example in the case $\Omega = S^n - \{x_1, x_2, \cdots, x_k\}$ for $k \geq 3$ it is not known whether there is a solution on Ω which is asymptotic to the simplest possible ODE solution near one of the points. (The simplest ODE solution near $x = 0$ has the form $c \cdot |x|^{(n-2)/2}$ in stereographic coordinates.)

Finally we mention the Yamabe problem for manifolds with boundary. The most natural homogeneous geometric boundary condition is a mixed Dirichlet-Neuman condition which states that the mean curvature of the boundary in the new metric vanishes. This boundary condition is of the form

$$\frac{\partial u}{\partial \nu} + \frac{n-2}{2} H_{g_0} u = 0 \quad \text{on } \partial M,$$

where (M, g_0) is a Riemannian manifold with smooth compact boundary, H_{g_0} denotes the mean curvature of ∂M with respect to g_0, and ν denotes the outward unit (with respect to g_0) normal. Thus one constructs metrics (M, g) satisfying $R(g) \equiv 1$ on M and $H_g \equiv 0$ on ∂M. This problem has been solved by J. Escobar [20] in the sense that he has produced a minimizing solution of the variational problem. A variant on this problem might be to prescribe H_g to be constant on ∂M and R_g to be constant in M. Escobar has considered this in [19] in case $R_g \equiv 0$ in M and $H_g \equiv 1$ on ∂M. This amounts, for $\Omega \subseteq \mathbf{R}^n$, to finding positive harmonic functions in Ω satisfying a critical nonlinear boundary condition.

2. Evolution of Riemannian metrics

One of the most striking developments in geometric partial differential equations during the last decade has been the successful application of heat equation methods to discuss the evolution of Riemannian metrics on smooth manifolds. The central evolution equation which arises in this context is the Ricci flow equation introduced by R. Hamilton [25]. The idea here is that the Ricci curvature may be thought of as a second-order differential operator from the space of metrics to the symmetric quadratic tensors. It is well known that if one introduces harmonic coordinates locally on M for a metric g, then the Ricci curvature takes the form

$$R_{ij} = -\tfrac{1}{2}\Delta_g(g_{ij}) + Q_{ij}(g, \partial g),$$

where Q_{ij} is an expression which is a quadratic polynomial in the partial derivatives ∂g. Thus minus twice the Ricci operator becomes locally a quasilinear elliptic operator with diagonal leading order term which is the Laplacian in the unknown metric g. This suggests that the equation

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \tag{2.1}$$

should be an essentially parabolic equation for the evolving metric g. This is the Ricci flow equation. In order to globalize the above discussion concerning harmonic coordinates, we may introduce a fixed background metric $\hat{g}$ on M. The difference D of the Levi-Civita connections may then be expressed as a global first-order differential operator applied to the metric g. Precisely we have

$$D^i_{jk} = \Gamma^i_{jk} - \hat{\Gamma}^i_{jk} = \frac{1}{2}\sum_l g^{il}(g_{jl;k} + g_{kl;j} - g_{jk;l}), \tag{2.2}$$

where we have chosen a local basis, and "semicolon" is used to denote covariant differentiation with respect to the background metric $\hat{g}$. Similarly, the difference of the Ricci tensors of g and $\hat{g}$ is given by

$$R_{ij} - \hat{R}_{ij} = \sum_k \left\{ D^k_{ij}\,;\,k - D^k_{ki;j} + \sum_l (D^k_{kl}D^l_{ji} - D^k_{jl}D^l_{ki}) \right\}. \tag{2.3}$$

By direct calculation from (2.2) we find that the second-order terms of the right-hand side of (2.3) may be written

$$-\frac{1}{2}\sum_{k,l} g^{kl} g_{ij;kl} + \frac{1}{2}\sum_{k,l} g^{kl}(g_{il;jk} + g_{jl;ik} - g_{kl;ij}).$$

If we introduce the vector field $X = X(g)$ given by

$$X^p(g) = -\sum_{i,k,l} g^{pi} g^{kl} \left(g_{ik;l} - \frac{1}{2} g_{kl;i} \right),$$

we see that (2.3) may be written

$$\mathrm{Ric}(g) = \mathrm{Ric}(\hat{g}) - \tfrac{1}{2} F(g) - \tfrac{1}{2} \mathscr{L}_X g, \tag{2.4}$$

where $\mathscr{L}_X g$ denotes the Lie derivative of g with respect to X, and $F(g)$ is an elliptic operator of the form

$$F_{ij}(g) = \sum_{k,l} g^{kl} g_{ij;kl} + Q_{ij}(g, \nabla g),$$

where Q is a quadratic polynomial in the covariant derivatives ∇g of g (taken with respect to $\hat{g}$) with coefficients depending on g. To solve the initial value problem for (2.1) we first solve the quasilinear parabolic equation

$$\begin{aligned} \frac{\partial \overline{g}}{\partial t} &= F(\overline{g}) - 2\,\mathrm{Ric}(\hat{g}), \\ \overline{g}(0) &= g_0 \end{aligned} \tag{2.5}$$

for a given initial metric g_0. This has a local solution by standard parabolic theory. The corresponding time dependent vector field $X(t) = X(\overline{g}(t))$ is then given, so we may integrate the system of ordinary differential equations to determine a family of maps $\Phi_t : M \to M$:

$$\begin{aligned} \frac{d\Phi}{dt}(t, x) &= X(t, x), \\ \Phi(0, x) &= x. \end{aligned} \tag{2.6}$$

We then let $\Phi_t(x) = \Phi(t, x)$, and we have $\Phi_0 =$ Identity. Now define $g(t) = \Phi_t^* \overline{g}(t)$, and compute

$$\frac{\partial g}{\partial t} = \Phi_t^* \left(\frac{\partial \overline{g}}{\partial t} \right) + \mathscr{L}_{X(t)} g(t).$$

By (2.4) and (2.5) we obtain

$$\Phi_t^* \left(\frac{\partial \overline{g}}{\partial t} \right) = \Phi_t^*(-2\,\mathrm{Ric}(\overline{g}) - \mathscr{L}_X \overline{g}) = -2\,\mathrm{Ric}(g) - \mathscr{L}_X g.$$

Therefore $g(t)$ is a solution of the initial value problem for the Ricci flow equation (2.1) with initial metric $g(0) = g$. The local existence theorem was originally proven by R. Hamilton [25] using a less transparent argument. The proof given here is essentially that of D. DeTurck [16]. The philosophy of this argument is quite clear in that (2.1) is parabolic except for the difficulty arising from the diffeomorphism invariance of the Ricci operator. The time dependent family of diffeomorphisms Φ_t has the effect of fixing this gauge freedom.

The main geometric interest in (2.1) lies in the large time behavior of solutions, although the local existence theorem has also proven to be useful as we will see later. The simplest large time behavior which one might expect for a parabolic equation is that the solutions approach a time independent solution. The time independent solutions of (2.1) are Ricci flat metrics; however this class of solutions can be enlarged by additionally imposing a volume constraint on the solution. This leads to an equivalent equation, called the normalized Ricci flow,

$$\frac{\partial g}{\partial t} = -2\left(\mathrm{Ric}(g) - \frac{r}{n}g\right), \tag{2.7}$$

where $r = \mathrm{Vol}(g)^{-1}\int_M R(g)\,d\mu_g$. Equation (2.7) may be derived directly from (2.1) by setting $\overline{g} = (V_0/\mathrm{Vol}(g))^{2/n}g$, where $g(t)$ satisfies (2.1) and one makes an appropriate change of the time variable $\tau = \tau(t)$. One sees that $\overline{g}(\tau)$ is then a solution of (2.7), and $\mathrm{Vol}(\overline{g}(\tau)) \equiv V_0$ for all t. The time independent solutions of (2.7) are, of course, the Einstein metrics. Thus studying the large time behavior of (2.7) presents an analytic approach to solving the elliptic Einstein equation.

The first global result obtained via the Ricci flow was done by R. Hamilton [25] for compact (M, g_0) with $\dim M = 3$ and $\mathrm{Ric}(g_0) > 0$. This result was later simplified and extended to four dimensions by Hamilton [26] in which case the assumption on g_0 was that g_0 have positive curvature operator. We give a brief summary of the main ideas involved here. The main estimate of both papers is a strong pinching estimate on the eigenvalues of the curvature operator of the form

$$\max\{|\lambda_i - \lambda_j|\} \leq CR^{1-\varepsilon_0}$$

for a constant C depending only on the initial metric, and a fixed small positive number ε_0. This estimate implies that at points where the curvature is large (note that the scalar curvature provides an upper bound on the full curvature tensor) the eigenvalues are relatively very close together. Thus the crucial high curvature regions of the solution are becoming round. This pinching estimate together with the result that the flow preserves the positivity assumption made on g_0 are derived through the analysis of a quadratic first-order autonomous system of ordinary differential equations on the vector space of linear endomorphisms of the Lie algebra of $\mathrm{SO}(n)$.

This vector space may be considered as the space of curvature operators, and the dynamical system on this space is the ODE part of the parabolic evolution equation satisfied by the curvature matrix of g. The idea then is to look for a *convex* invariant set for this ODE system so that the parabolic system will preserve the same set. This is reasonable to expect because the heat equation should preserve convex constraints. Hamilton then constructs convex invariant sets by finding Lyapunov functions for the flow. A pinching estimate of the type described above then implies convergence of the normalized flow to a constant curvature metric. In the final part of this scheme, it is necessary to go from pointwise pinched curvature to globally pinched curvature. This can be done formally if $\dim M \geq 3$ by the twice contracted second Bianchi identity which states that

$$\frac{n-2}{2n} R_{ji} = \sum_j T_{ij;j}$$

in an orthonormal basis, where T denotes the trace-free Ricci tensor. Thus if T is small, one can expect R to be nearly constant so that it might be reasonable to derive global curvature pinching from pointwise pinching.

It is presently unknown whether the Ricci flow converges if $\dim M \geq 5$ and g_0 has positive curvature operator. While the positivity is preserved under the flow, the necessary pinching estimates have not been derived. The corresponding ODE system is more difficult to analyze because the Lie algebra is higher-dimensional and more complicated. The Micallef-Moore theorem which we will discuss in the next section does imply that a manifold of positive curvature operator is covered by a topological sphere.

The general method outlined above, because it relies on the twice contracted second Bianchi identity, also does not work on surfaces. Recently the Ricci flow has been understood on compact two-manifolds by somewhat different methods. This problem was discussed in [27] by R. Hamilton. The normalized Ricci flow on surfaces is the equation $\frac{\partial g}{\partial t} = (r-R)g$, where r denotes the average value of R, and R is twice the Gaussian curvature. Thus $r = (\text{Area}(g))^{-1} 4\pi\chi(M)$ by the Gauss-Bonnet theorem, and since the flow is normalized the area is constant in time. For surfaces of positive genus it is shown in [27] that the flow exists for all time and converges to the unique constant curvature metric in the conformal class of g_0. As one would expect, the most subtle case is $M = S^2$. In this case an interesting notion which arises and plays an important role in the arguments is the notion of a soliton (actually travelling wave) solution. This means a solution of the normalized Ricci flow which has the form $g_t = F_t^* g_0$ for some smooth curve of diffeomorphisms $F_t \colon M \to M$, $t \in [0, \infty)$, with $F_0 = \mathrm{id}$. Thus the geometry for a soliton solution remains constant; in fact the solution at every time is isometric to the solution at time zero. On the other hand, if the path F_t is divergent, then the solution g_t will diverge in the space of metrics. In [27] it is shown that if $R(g_0) > 0$ on S^2, then the Ricci flow converges to a constant curvature metric. In detail the proof is quite different from the higher-dimensional case and relies on a Harnack-type inequality and the decrease of an entropy integral in order to bound the geometry of g_t for all time. Finally to prove convergence of g_t it is shown that divergence would imply the existence of a soliton solution on S^2. An explicit argument however excludes this possibility. The Ricci flow analysis for smooth metrics on S^2 was later completed by B. Chow [13] who removed the assumption of positivity on the initial metric g_0. Thus even a dumbbell metric on S^2 will converge under the Ricci flow to a constant curvature metric. The Ricci flow on positively curved two-dimensional orbifolds has been studied by B. Chow and L. F. Wu [14]. The interesting feature of this problem is that constant curvature metrics need not exist. It is shown in this case that the solution of the Ricci flow converges asymptotically to a soliton solution.

The Ricci flow on complete manifolds has been studied by W. X. Shi [58]. He has established short time existence for complete initial metrics of bounded curvature, and in certain cases has obtained long time results. In particular, he has recently announced [59] the result that a complete noncompact Kähler manifold with positive bounded holomorphic sectional curvature, maximal volume growth, and weak decay on the average total scalar curvature on balls is biholomorphic to $\mathbf{C}^n$. The proof involves showing that the Ricci flow exists for all time and converges to a flat metric. In the Kähler case, it was shown by H. D. Cao [10] that for compact manifolds with $c_1 = 0$ or $c_1 < 0$ the Ricci flow converges to a Kähler-Einstein metric.

The Ricci flow has also been applied recently by M. Min-Oo [41] and R. Ye [70] to construct Einstein metrics on Ricci-pinched manifolds of negative scalar curvature. A consequence of the results of R. Ye [71] is the existence of an abundance of Einstein metrics on higher-dimensional manifolds $n \geq 5$ whose sectional curvatures are pinched near a negative constant. Finally, the Ricci flow has been used by D. Yang [68], [69] to convert integral to pointwise bounds on curvature, and hence to establish certain results of Riemannian geometry under integral rather than pointwise bounds on curvature.

3. Harmonic maps and applications

For a map $u: M^n \to X^k$, we can assign to u various energies if M and X have Riemannian metrics g, h respectively. This can be done by considering the pullback metric u^*h, and diagonalizing this with respect to g. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of u^*h are then the squares of the "principal stretches" of u, and the symmetric scalar functions of $\lambda_1, \cdots, \lambda_n$ are well defined functions on M, independent of basis or coordinate choice. The simplest of these scalar functions is the trace, $\mathrm{Tr}_g(u^*h) = \sum_{i=1}^n \lambda_i$, which we refer to as the energy density of u, and denote $e(u)$. The total energy of u is then given by $E(u) = \int_M e(u)\, d\mu_M$. The critical points of $E(u)$ on the space of maps are referred to as harmonic maps. The theory of harmonic maps includes the theory of geodesics (with constant speed parametrization) when $n = 1$, and the theory of harmonic functions when $X = \mathbf{R}$.

The theory for $n = 2$ has played an important role in minimal surface theory since the work of C. B. Morrey in the 1940's. It has only been relatively recently, beginning with the work of Sacks and Uhlenbeck, that the higher critical point theory for $n = 2$ has been developed. For $M = S^2$, the result which one can expect is a partial Morse theory for the energy functional. We state one result of this type which is a slight refinement of [48] due to Micallef-Moore [40].

Theorem 3.1. *Suppose $\pi_{p+2}(X) \neq 0$ for some integer $p \geq 0$. There exists a nonconstant harmonic map $u: S^2 \to X$ whose Morse index (for the energy functional) is at most p.*

This theorem has found a striking geometric application to the study of manifolds of nonnegative curvature in [40]. To describe these results, we consider the complexification of the tangent space $T_xX \otimes \mathbf{C}$ at a point x, and extend the curvature tensor to complex vectors by linearity. The complex sectional curvature of a two-dimensional subspace π of $T_xX \otimes \mathbf{C}$ is then defined by $K(\pi) = \langle R(v, w)v, w\rangle$, where $\{v, w\}$ is any unitary basis of π, and $\langle\cdot, \cdot\rangle$ denotes the Hermitian inner product on $T_xX \otimes \mathbf{C}$. A subspace π is said to be isotropic if every vector $v \in \pi$ has square zero; that is, $(v, v) = 0$, where $(\cdot, \cdot)$ denotes the complex linear extension of the real inner product to $T_xX \otimes \mathbf{C}$. We say that X has *positive isotropic sectional curvature* if $K(\pi) > 0$ for every isotropic two-plane π. If the dimension $k = \dim X$ is less than four, then every X has positive isotropic curvature because there are no isotropic two-dimensional subspaces π in this case. In general, the classical conditions of (pointwise) $\frac{1}{4}$-pinching and positive curvature operator are easily seen (see [40]) to imply positive isotropic sectional curvature. The main harmonic map theorem of [40] can then be stated.

Theorem 3.2. *Assume X has positive isotropic sectional curvature. Then any nonconstant harmonic map $u: S^2 \to X^k$ has Morse index at least $[[\frac{k}{2}]] - 1$, where $[[\cdot]]$ denotes the integer part of a number.*

Combining Theorems 3.1 and 3.2 we conclude that $\pi_p(X) = 0$ for $p = 2, 3, \cdots, [[\frac{k}{2}]]$ $(k \geq 4)$. An immediate consequence is that X is a homotopy sphere if $\pi_1(X) = 0$. Using known results from topology one concludes

Theorem 3.3. *A compact simply connected manifold with positive isotropic sectional curvature is homeomorphic to the sphere.*

It should be observed that the condition of positive isotropic sectional curvature does not imply thaι $\pi_1(X)$ is finite (in particular, it does not imply positive Ricci curvature). For example, in dimension four any locally conformally flat manifold with positive scalar curvature has positive isotropic sectional curvature. It is well known that the fundamental group of such a manifold can be a free group.

The present state of affairs with the regularity properties for harmonic maps is slightly better than at the time of our survey [50]. Recall that a regularity theory for minimizing maps exists [55]. There has recently been progress on the structure of the singular set for minima by R. Hardt, F. H. Lin, and L. Simon [29], [60]. For example, they show that the singular set for a minimizing map from $\Omega \subset \mathbf{R}^4$ to S^2 consists of curves and isolated points. Recall that a map $u \in H^1(M, X)$ is a weak solution if it satisfies the Euler-Lagrange equation in the distributional sense. A map u is *stationary* if it is a weak solution, and in addition satisfies the condition that its energy is critical with respect to variations of the type $u \circ F_t$, where $F_t: M \to M$ is a smooth path of diffeomorphisms of M fixing the boundary. It can be shown (see [50] for a more detailed discussion) that stationary maps satisfy the monotonicity property for the scale invariant energy in balls. This property plays an important role in the regularity theory for minimizing maps [55]. For $n = \dim M = 2$, the stationary property imlies the holomorphicity of the Hopf differential. It was shown by M. Grüter [24] that weakly conformal weak solutions $(n = 2)$ are regular. This was extended to stationary maps in [50]. For $n \geq 3$, no regularity theorem exists for stationary maps, although such a theory should be true. In the past few years it has been shown that the stationarity condition is much more restrictive than the weak solution property. For example, in [47] many weak solutions $(n = 3)$ which are not stationary are constructed. Note that smooth weak solutions are clearly stationary, so the distinction exists only for maps with singularities.

While the regularity properties of weak solution are still not understood, there has been a recent development by F. Helein [30], who has shown that for maps from a surface M^2 to the standard sphere S^k, weak solutions are in fact regular. This result exploits the special form of the equation for maps to the round sphere. We make some remarks concerning this phenomenon. These observations arose in a discussion with Rugang Ye. First there is a result about linear equations in two variables due to H. Wente [64]. In general, a solution u of $\Delta u = f$ with $f \in L^1_{\text{loc}}(\Omega)$, $\Omega \subset \mathbf{R}^2$, fails to be continuous. We may consider a special class of L^1 functions generated in the following way: Let $v: \Omega \to \mathbf{R}^{2q}$ be a map which is locally in H^1 and let $\omega = \sum_{j=1}^q dx_j \wedge dy_j$ be the standard symplectic form on $\mathbf{R}^{2q}$. We may then write $v^* w = f dx \wedge dy$ with $f \in L^1_{\text{loc}}(\Omega)$. Let $\mathscr{F}$ denote the space of functions f which arise in this way for some q; that is, the L^1_{loc} functions which can be expressed as sums of Jacobians of H^1 maps. Wente's theorem can be stated:

Theorem 3.4. *Let $\Omega \subset \mathbf{R}^2$ be a domain, and let u be a solution of $\Delta u = f$ in Ω. If $f \in \mathscr{F}$, then u is continuous on any compact subdomain of Ω.*

Now, let us consider maps $u: \Omega \to S^k$. The Euler-Lagrange equation for harmonic maps is

$$\Delta u^j = -|\nabla u|^2 u^j, \qquad j = 1, \cdots, k+1,$$

where

$$S^k = \{u \in \mathbf{R}^{k+1} : |u|^2 = 1\}, \qquad |\nabla u|^2 = \sum_{\alpha=1}^{2} \sum_{i=1}^{k+1} \left(\frac{\partial u^i}{\partial x^\alpha}\right)^2.$$

This equation can be rewritten as

$$\Delta u^j = \sum_{i=1}^{k+1} \sum_{\alpha=1}^{2} \frac{\partial u^i}{\partial x^\alpha} \left(\frac{\partial u^j}{\partial x^\alpha} u^i - \frac{\partial u^i}{\partial x^\alpha} u^j\right), \qquad j = 1, \cdots, k+1,$$

since

$$\sum_{i+1}^{k+1} \frac{\partial u^i}{\partial x^\alpha} u^i = \frac{1}{2} \frac{\partial}{\partial x^\alpha}(|u|^2) = 0.$$

Finally observe that for any i, j we have

$$\sum_{\alpha=1}^{2} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial u^j}{\partial x^\alpha} u^i - \frac{\partial u^i}{\partial x^\alpha} u^j\right) = 0,$$

so that the 1-form β_{ij} on Ω given by

$$\beta_{ij} = \sum_{\alpha=1}^{2} \left(\frac{\partial u^j}{\partial x^\alpha} u^i - \frac{\partial u^i}{\partial x^\alpha} u^j \right) dx^\alpha$$

is co-closed. Thus $*\beta_{ij} = -dh_{ij}$ for a function $h_{ij} \in H^1_{\text{loc}}(\Omega)$ (we may assume Ω is simply connected since this theory is local). Thus we may rewrite the harmonic map equation as

$$\Delta u^j = \sum_{i=1}^{k+1} *(du^i \wedge *\beta_{ij}) = \sum_{i=1}^{k+1} *(du^i \wedge dh_{ij}).$$

In particular, $\Delta u^j \in \mathscr{F}$ for $j = 1, \cdots, k+1$, and hence u is continuous. It then follows that u is smooth. Thus we get the following result.

Theorem 3.5 (F. Helein). *Weakly harmonic maps from surfaces into round spheres are regular.*

It is unknown whether the regularity theorem holds for arbitrary target spaces. We now make some observations which clarify the geometry of the situation above, and show that Theorem 3.5 does hold whenever X is a compact homogeneous Riemannian manifold. Recently F. Helein has extended his method to cover the case in which X is homogeneous. First we associate to any Killing vector field of X a conjugate function $v \in H^1_{\text{loc}}(\Omega)$ which satisfies an equation of the form $\Delta v = f$ with $f \in \mathscr{F}$. Thus v is continuous by Theorem 3.4. We then show that if the Killing vector fields span the tangent space of X at each point, then the map u is itself regular.

Let K be a Killing vector field on X, and let $K^\#$ denote the associated 1-form. We observe that $u^*(K^\#)$ is an $L^2_{\text{loc}}(\Omega)$ 1-form which is weakly co-closed whenever $u\colon \Omega \to X$ is a weakly harmonic map. To see this, observe that

$$u^*(K^\#) = \sum_{\alpha=1}^{2} \left\langle u_* \left(\frac{\partial}{\partial x\alpha} \right), K \right\rangle dx^\alpha,$$

so that if $\varphi \in C_0^\infty(\Omega)$, then

$$\int_\Omega \langle u^*(K^\#), d\varphi \rangle \, dx^1 \, dx^2 = \int_\Omega \sum_{\alpha=1}^{2} \left\langle u_* \left(\frac{\partial}{\partial x^\alpha} \right), K \right\rangle \frac{\partial \varphi}{\partial x_\alpha} \, dx^1 \, dx^2.$$

First note that if u is *any* map in $H^1_{\text{loc}}(\Omega, X)$, then u can be approximated locally in H^1 norm by smooth maps. Let $\{u_p\}$ be a smooth approximating sequence, and observe that because K is a Killing vector field we have

$$\int_\Omega \langle u_p^*(K^\#), d\varphi\rangle\, dx^1\, dx^2 = \int_\Omega \sum_{\alpha=1}^{2} \left\langle u_{p^*}\left(\frac{\partial}{\partial x^\alpha}\right), \nabla_{\partial/\partial x^\alpha}(\varphi K)\right\rangle dx^1\, dx^2,$$

where ∇ denotes the connection on the pullback of the tangent bundle of X by u_p. This is true because in an orthonormal basis we have

$$\sum_{\alpha=1}^{2} \left\langle u_{p^*}\left(\frac{\partial}{\partial x^\alpha}\right), \nabla_{\partial/\partial x^\alpha} K\right\rangle = \sum_{\alpha=1}^{2}\sum_{i,j=1}^{k} K_{i;j}\frac{\partial u_p^i}{\partial x^\alpha}\frac{\partial u_p^j}{\partial x^\alpha} = 0$$

since $K_{i;j}$ is antisymmetric. Thus we may let $p \to \infty$, and deduce

$$\int_\Omega \langle u^*(K^\#), d\varphi\rangle\, dx^1\, dx^2 = \int_\Omega \sum_{\partial=1}^{2} \left\langle u_*\left(\frac{\partial}{\partial x^\alpha}\right), \nabla_{\partial/\partial x^\alpha}(\varphi K)\right\rangle dx^1\, dx^1.$$

Since φK is an H_0^1 section of $u^*(TX)$, the fact that u is weakly harmonic implies that this vanishes for each $\varphi \in C_0^\infty(\Omega)$. Thus $u^*(K^\#)$ is weakly co-closed, and we may write $u^*(K^\#) = *dv$ for a function v which is unique up to an additive constant. Moreover, $v \in H^1_{\mathrm{loc}}(\Omega)$ and may be thought of as a conjugate function to u in the direction of the Killing field K. We now compute $\Delta v = *d*dv = *d(u^*(K^\#))$. We observe that if u is a smooth (not necessarily harmonic) map, then we have (in an orthonormal basis)

$$\begin{aligned} d(u^*(K^\#)) &= \sum_{i=1}^{k}\left[\frac{\partial}{\partial x^1}\left(\frac{\partial u^i}{\partial x^2}K^i\right) - \frac{\partial}{\partial x^2}\left(\frac{\partial u^i}{\partial x^1}K^i\right)\right] dx^1\wedge dx^2 \\ &= \left[\left\langle \frac{\partial u}{\partial x^2}, \nabla_{\partial/\partial x^1}K\right\rangle - \left\langle \frac{\partial}{\partial x^1}, \nabla_{\partial/\partial x^2}K\right\rangle\right] dx^1\wedge dx^2. \end{aligned}$$

If $X \subset \mathbf{R}^Q$ isometrically, then K may be extended as a smooth vector field on $\mathbf{R}^Q$, and

$$d(u^*(K^\#)) = -\sum_{i=1}^{Q} du^i \wedge d(K^i\circ u).$$

By approximation as above this holds weakly for any H^1 map u, and therefore $\Delta v \in \mathscr{F}$. Hence by Theorem 3.4, $v \in C^0(\Omega, X)$.

Now let $\{K_1, \cdots, K_s\}$ be a basis for the vector space of Killing vector fields on X. For each K_l we construct a conjugate function v_l as above, and these v_l satisfy equations $\Delta v_l = f_l \in \mathscr{F}$, $l = 1, \cdots, s$. Let $0 \in \Omega$ be any point, and $\sigma > 0$ with $B_\sigma(0) \Subset \Omega$. We then have the equation

$$\int_{B_\sigma(0)} |\nabla v_l|^2 \, dx^1 \, dx^2 = -\int_{B_\sigma(0)} (v_l - \overline{v}_l) f_l \, dx^1 \, dx^2 + \int_{\partial B_\sigma(0)} (v_l - \overline{v}_l) \frac{\partial v_l}{\partial r} \, ds$$

for any constant $\overline{v}_l$. Taking $\overline{v}_l$ to be the average of v_l on $\partial B_\sigma(0)$ and applying the Poincaré inequality on $\partial B_\sigma(0)$ as well as the Schwarz inequality we obtain

$$\int_{B_\sigma(0)} |\nabla v_l|^2 \, dx^1 \, dx^2 \le \int_{B_\sigma(0)} |v_l - \overline{v}_l| |f_l| \, dx^1 \, dx^2 + \sigma \int_{\partial B_\sigma(0)} |\nabla v_l|^2 \, ds.$$

Since v_l is continuous, if σ is chosen small enough, $\sup_{B_\sigma(0)} |v_l - \overline{v}_l|$ can be made smaller than any preassigned number. On the other hand, $|f_l| \le ce(u)$ for a constant c depending only on X. Thus for sufficiently small σ we have

$$\int_{B_\sigma(0)} |v_l - \overline{v}_l| |f_l| \, dx^1 \, dx^2 \le \varepsilon_0 \int_{B_\sigma(0)} e(u) \, dx^1 \, dx^2$$

for any $\varepsilon_0 > 0$. Summing on l then gives

$$\sum_{l=1}^{s} \int_{B_\sigma(0)} |\nabla v_l|^2 \, dx^1 \, dx^2 \le \varepsilon_0 s \int_{B_\sigma(0)} e(u) \, dx^1 \, dx^2 + c\sigma \int_{\partial B_\sigma(0)} e(u) \, ds,$$

where we have used the obvious inequality $\sum_{l=1}^{s} |\nabla v_l|^2 \le ce(u)$. Now if we assume that $\{K_1, \cdots, K_s\}$ span the tangent space of X at each point, then clearly for any vector $w \in T_x X$ and any $x \in X$ there is a constant $c > 0$ so that $\|w\|^2 \le c \sum_{l=1}^{s} \langle w, K_l \rangle^2$. Since $|\nabla v_l|^2 = \sum_{\alpha=1}^{2} \langle \frac{\partial u}{\partial x^\alpha}, K_l \rangle^2$, we see that

$$\int_{B_\sigma(0)} e(u) \, dx^1 \, dx^2 \le \varepsilon_0 c \int_{B_\sigma(0)} e(u) \, dx^1 \, dx^2 + c\sigma \int_{\partial B_\sigma(0)} e(u) \, ds,$$

and hence if $\varepsilon_0 c \le \frac{1}{2}$, then

$$\int_{B_\sigma(0)} e(u) \, dx^1 \, dx^2 \le 2c\sigma \frac{d}{d\sigma} \int_{B_\sigma(0)} e(u) \, ds.$$

This Morrey-type inequality then implies that $\int_{B_\sigma(0)} e(u) \, dx^1 \, dx^2$ decays like a power of r, and hence by Morrey's Lemma [42] the map u is Hölder continuous and therefore smooth. Hence we have established the following result of Helein.

Theorem 3.6. *If the Killing vector fields of X span the tangent space at each point (e.g., if X is homogeneous), then any weakly harmonic map from a surface to X is regular.*

There has been a good deal of activity on harmonic maps and related variational problems generated by the theory of liquid crystals. We will not discuss this direction here, but instead refer the interested reader to the survey paper of F. H. Lin [37] and the references therein.

One of the most interesting geometric applications of harmonic mapping theory has been to the study of rigidity questions for nonpositively curved manifolds. The Mostow rigidity theorem [44] was proven for Hermitian locally symmetric spaces by Y. T. Siu [61] by showing that certain harmonic maps are holomorphic. The existence theory of Eells and Sampson [17] asserts that maps into manifolds of nonpositive curvature can be deformed to harmonic maps. In certain cases the Bochner method can be used to show that harmonic maps satisfy more equations (such as the Cauchy-Riemann equations in Siu's case). The Bochner argument was improved in a useful way by J. Sampson [49] who showed that harmonic maps from a Kähler manifold to a manifold of negative curvature operator have rank at most two. These results were systematically analyzed by J. Carlson and D. Toledo [11] who studied the possible ranks of harmonic maps from Kähler manifolds into locally symmetric spaces. The Kähler assumption on the domain was removed by K. Corlette [15] who realized that one gets conditions on the second derivatives of a harmonic map whenever the domain manifold possesses a parallel form w of any degree. In case the stabilizer of the form $w(x) \in \bigwedge^p T_x^*M$ in $\mathrm{GL}(T_xM)$ is a compact group, it follows that the map is totally geodesic; that is, has vanishing second covariant derivatives. Corlette then used this to establish super-rigidity results for homomorphisms of lattices in the rank-1 groups $\mathrm{Sp}(n, 1)$ and F_4^{-20} into Lie groups (so-called Archimedian super-rigidity). Recall that super-rigidity had been conjectured by A. Selberg and proved by G. Margulis for semisimple algebraic groups of real rank at least two (see [73]). On the other hand it is known to fail for lattices in the rank-1 groups $\mathrm{SO}(n, 1)$ for all n and $\mathrm{SU}(n, 1)$ for $n = 2, 3, 4$. (See [73] for references to this literature.)

It was pointed out to the author by M. Gromov that the applications of harmonic maps to rigidity problems could be substantially amplified if one could allow the image space X to be a singular space such as a simplicial complex. In particular, one can study p-adic representations of a lattice by constructing maps into Bruhat-Tits buildings associated with simple algebraic p-adic groups. (See K. Brown [8] for a readable treatment of Euclidean buildings.) This program has been successful, and it can be shown that p-adic super-rigidity holds for lattices in $\mathrm{Sp}(n, 1)$ and F_4^{-20}. We now describe our results in this direction. Complete details will appear in a forthcoming paper with M. Gromov. To make plausible the idea that there might be a good theory of harmonic maps into certain types of singular spaces, recall that if X is a nonpositively curved manifold and $u: \Omega \to X$ is a harmonic map from an open subset $\Omega \subseteq M$, then there is a uniform Lipschitz estimate for any $\Omega_1 \Subset \Omega$ of the form

$$\sup_{\Omega_1} e(u) \leq c(M, \Omega_1) \int_\Omega e(u)\, d\mu_g,$$

where the constant c is independent of both u and the target space X. The traditional proof of this inequality is based on the Bochner formula

$$\tfrac{1}{2}\Delta e(u) = |\nabla du|^2 - \sum_{i,j=1}^{n} \langle \mathscr{R}^X(u_*(e_i \wedge e_j)), u_*(e_i \wedge e_j)\rangle + \mathrm{Ric}^m(du, du),$$

where $e_1, \cdots, e_n$ is a local orthonormal tangent frame on M and $\mathscr{R}^X$ denotes the curvature operator of X. Since the vectors $u_*(e_i \wedge e_j)$ are simple, we see that if X has nonpositive sectional curvature, then $e(u)$ is a subsolution of a linear operator of the form $\Delta_M + c$, where c depends only on a lower bound for the Ricci curvature of M. The pointwise bound on $e(u)$ then follows from standard mean-value type inequalities. The Lipschitz estimate suggests that there may be an existence theory for Lipschitz harmonic maps into singular spaces of nonpositive curvature. We now proceed to formulate such a result.

Let X be a Riemannian simplicial complex; that is, a (locally compact) simplicial complex the faces of which are endowed with Riemannian metrics extending smoothly to the closure such that lower-dimensional faces have the metric induced from inclusion in the closure of higher-dimensional faces. For example, a one-dimensional Riemannian simplicial complex is a graph with a length assigned to each edge. We assume for simplicity that X is embedded in a Euclidean space $\mathbf{R}^Q$ in such a way that each face has the induced Riemannian metric from $\mathbf{R}^Q$.

Assuming that X is connected, it is clear that any two points $x_0, x_1 \in X$ can be joined by a path $\gamma: [0, 1] \to X$ which is Lipschitz as a map to $\mathbf{R}^Q$. We can then define the Riemannian distance function $d(x_0, x_1)$ by

$$d(x_0, x_1) = \inf\{L(\gamma): \gamma \text{ a Lipschitz path from } x_0 \text{ to } x_1\}.$$

It is easy to see that $d(\cdot, \cdot)$ is a metric, and that (X, d) is a complete metric space. Moreover, the infimum is attained by a Lipschitz path. To see this, we introduce the energy of a path $E(\gamma)$, and observe that standard analytic methods produce an energy minimizing path γ from x_0 to x_1. Of course, as a map $\gamma: [0, 1] \to \mathbf{R}^Q$, γ is Hölder continuous with the

Hölder exponent $\frac{1}{2}$ by the fundamental theorem of calculus. We now claim that the function $\frac{d\gamma}{dt}$, the distributional derivative of γ, has constant length almost everywhere on $[0, 1]$. To see this, let $\zeta(t)$ be a smooth real-valued function with compact support in $(0, 1)$, and consider the path of maps $\gamma_s(t)$ given by $\gamma_s(t) = \gamma(t + s\zeta(t))$. By the minimizing property of γ we see directly that $E(\gamma) \le E(\gamma_s)$, and hence the function $s \mapsto E(\gamma_s)$ has a minimum at $s = 0$. We examine this function more carefully:

$$E(\gamma_s) = \int_0^1 \left\| \frac{d\gamma_s}{dt} \right\|^2 dt = \int_0^1 \left\| \frac{d\gamma_s}{dt} \right\|^2 (1 + s\zeta'(t))^{-1}\, d\tau,$$

where $\tau = t + s\zeta(t)$, s being a *fixed* small number. Now $\frac{d\gamma_s}{dt}(t) = \frac{d\gamma_s}{d\tau} \cdot (1 + s\zeta'(t))$ by the chain rule, and hence $E(\gamma_s) = \int_0^1 \|\frac{d\gamma}{d\tau}\|^2 (1 + s'(t))\, d\tau$. Since $\frac{d}{ds} E(\gamma_s) = 0$ at $s = 0$, we conclude

$$\int_0^1 \left\| \frac{d\gamma}{dt} \right\|^2 \zeta'(t)\, dt = 0$$

for every smooth function $\zeta(t)$ with compact support in $(0, 1)$. This implies that $\|\frac{d\gamma}{dt}\|$ is equal to a constant almost everywhere on $[0, 1]$. We must then have $\|\frac{d\gamma}{dt}\| = L(\gamma)$ a.e., and hence $E(\gamma) = L(\gamma)^2$. Now if γ_1 is *any* Lipschitz path from x_0 to x_1, we can parametrize γ_1 proportionate to arclength so that $\|\frac{d\gamma_1}{dt}\| = L(\gamma_1)$ a.e. Thus

$$L^2(\gamma_1) = E(\gamma_1) \ge E(\gamma) = L^2(\gamma),$$

and we see that γ realizes the distance between x_0 and x_1. In general, γ will be no smoother than Lipschitz; however, any open segment of γ which lies in a face of X is a curve of least length in that face, and hence a smooth geodesic.

In order to expect a theory of continuous harmonic mappings from higher-dimensional manifolds to X it will be necessary to assume that X has nonpositive curvature in a suitable sense. (Without this assumption harmonic maps will sometimes be discontinuous.) Let $y, x_0, x_1 \in \mathbf{R}^n$, and let $x(s)$ be the unit speed geodesic from x_0 to x_1 parametrized on $[0, l]$, $l = |x_0 - x_1|$. Notice that the function $f_0(s) = |x(s) - y|^2$ is determined by the conditions $f_0''(s) = 2$, $f_0(0) = |x_0 - y|^2$, $f_0(l) = |x_1 - y|^2$. We say that X has *nonpositive curvature* if for any three points $y, x_0, x_1 \in X$, the function $f(s) = d^2(x(s), y)$ satisfies $f(s) \le f_0(s)$, where $x(s)$, $s \in [0, l]$, $l = d(x_0, x_1)$, denotes a unit speed geodesic from x_0 to x_1, and f_0 is the function satisfying $f_0'' = 2$, $f_0(0) = d^2(x_0, y)$,

$f_0(l) = d^2(x_1, y)$. Thus the condition states that points of the side of a geodesic triangle opposite to y in the space X are at least as close to y as they would be in a Euclidean triangle with the same side lengths. The reader can consult [8] for an elementary discussion of nonpositively curved metric spaces. The following properties can be derived from the definition of nonpositive curvature. First, any two points in X can be joined by precisely one geodesic path. Second, if x_0, x_1 and y_0, y_1 are two pairs of points in X, and we parametrize the geodesic paths from x_0 to x_1 and from y_0 to y_1 by $x(t), y(t)$ for $t \in [0, 1]$, where t is proportional to arclength along each of the paths, then the function $g(t) = d(x(t), y(t))$ is a convex function of t. Note that this second property implies that geodesics from a point spread more quickly than Euclidean geodesics since we may take $x_0 = y_0$ to conclude that $d(x(t), y(t)) \geq (t/t')d(x(t'), y(t'))$ for $0 \leq t' \leq t \leq 1$. Finally, we observe that for any $\lambda \in [0, 1]$ and $y \in X$ we can define a map $R_{\lambda, y}: X \to X$ by setting

$$R_{\lambda, y}(x) = x(\lambda),$$

where $x(t)$, $t \in [0, 1]$, denotes the constant speed geodesic from y to x. By our previous discussion we see that each $R_{\lambda, y}$ is a Lipschitz map; in fact

$$d(R_{\lambda, y}(x_0), R_{\lambda, y}(x_1)) \leq \lambda d(x_0, x_1),$$

so the Lipschitz constant is at most λ. Moreover, the family of maps $R_{\lambda, y}$, $\lambda \in [0, 1]$, defines a deformation retraction of X to the point y, so that X is necessarily contractible.

The direct method enables us to construct an energy minimizing map $u: \Omega \to X$ with a given map φ specified on $\partial\Omega$ (say, for example, that φ is Lipschitz). Here Ω is a compact manifold with boundary. This map will be a priori in the space $H^1(\Omega, X)$ defined by

$$H^1(\Omega, X) = \{v \in H^1(\Omega, \mathbf{R}^Q): v(x) \in X \text{ a.e. } x \in \Omega\}.$$

We now discuss some important properties of energy minimizing maps. Since our discussion is local on the domain manifold, we will assume the domain metric is Euclidean. Generally, certain allowable error terms appear in the calculations. We will merely sketch the arguments here (full details will appear elsewhere), so we omit this complication. The first observation is that the usual monotonicity inequality for harmonic maps holds. This is derived by considering a variation of the map u of the following type: Let $\zeta(x)$ by a smooth function with compact support in Ω, and for $|\tau|$ small consider the diffeomorphism of Ω given by $f_\tau(x) = (1+\tau\zeta(x))x$. Consider the maps $u_\tau = u \circ F_\tau$. Then the function $\tau \mapsto E(u_\tau)$

has a minimum at $\tau = 0$, because $u = u_0$ is energy minimizing. To analyze this condition we perform a change of variables as we did in the geodesic case above; that is, let $y = F_\tau(x)$ and observe

$$E(u_\tau) = \int_\Omega \sum_{i,j,k} \frac{\partial y_j}{\partial x_i} \frac{\partial y_k}{\partial x_i} \left(\frac{\partial u}{\partial y_j} \cdot \frac{\partial u}{\partial y_k} \right) \det \left(\frac{\partial x_p}{\partial y_q} \right) dy.$$

In particular, $E(u_\tau)$ is a differentiable function of τ, and by direct calculation we have

$$\frac{d}{d\tau} E(u_\tau)|_{\tau=0}$$
$$= \int_\Omega \left[|\nabla u|^2 (2-n)\zeta - |\nabla u|^2 \sum_i x_i \frac{\partial \zeta}{\partial x_i} + 2 \sum_{i,j} \frac{\partial \zeta}{\partial x_i} x_j \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \right] dx.$$

Taking ζ to be an approximation to the characteristic function of a ball, say $B_\sigma(0) \subset \mathbf{R}^n$, we get (since u minimizes)

$$0 = (2-n) \int_{B_\sigma} |\nabla u|^2 \, dx + \sigma \int_{\partial B_\sigma} |\nabla u|^2 \, d\Sigma - 2\sigma \int_{\partial B_\sigma} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma. \tag{3.1}$$

We can gain additional information by using the fact that X has nonpositive curvature. Let $\zeta(x) \geq 0$ be a smooth nonnegative function with compact support in Ω, and for small $\tau \geq 0$ consider the variation of u given by $u_\tau = F_{1-\tau\zeta(x),y} \circ u$, where $F_{\lambda,y}\colon X \to X$ was constructed above, and $y \in X$ is any chosen point. The following calculation can be justified; we omit the details here but operate formally:

$$\frac{\partial u_\tau}{\partial x_i} = dF_{1-\tau\zeta,y} \left(\frac{\partial u}{\partial x_i} \right) - \tau \frac{\partial \zeta}{\partial x_i} \cdot \frac{\partial F_{1-\tau\zeta,y}}{\partial \lambda}.$$

From the contracting property of $F_{\lambda,y}$ it follows that

$$E(u_\tau) \leq \int_\Omega [(1-\tau\zeta)^2 |\nabla u|^2 - \tau \nabla\zeta \cdot \nabla(d^2(u(x), y))] \, dx + o(\tau).$$

Using the fact that $E(u) \leq E(u_\tau)$ we obtain

$$0 \leq -\tau \int_\Omega [2\zeta |\nabla u|^2 + \nabla \cdot \nabla(d^2(u(x), y))] \, dx + o(\tau).$$

Dividing by τ and letting τ go to zero we have

$$\int_\Omega [2\zeta |\nabla u|^2 + \nabla\zeta \cdot \nabla(d^2(u(x), y))] \, dx \leq 0$$

for any $\zeta \geq 0$ with compact support. Choosing ζ to approximate the characteristic function of $B_\sigma(0)$ yields

$$2\int_{B_\sigma} |\nabla u|^2\, dx \leq \int_{\partial B_\sigma} \frac{\partial}{\partial r}(d^2(u(x), y))\, d\Sigma. \tag{3.2}$$

If we let $E(\sigma) = \int_{B_\sigma} |\nabla u|^2\, dx$ and $I(\sigma) = \int_{\partial B_\sigma} d^2(u(x), y)\, d\Sigma$, we can compute logarithmic derivatives

$$\frac{I'(\sigma)}{I(\sigma)} = \frac{n-1}{\sigma} + (I(\sigma))^{-1} \int_{\partial B_\sigma} \frac{\partial}{\partial r}(d^2(u, y))\, d\Sigma,$$

whereas from (3.1) we obtain

$$\frac{E'(\sigma)}{E(\sigma)} = \frac{n-2}{\sigma} + 2(E(\sigma))^{-1} \int_{\partial B_\sigma} \left|\frac{\partial u}{\partial r}\right|^2\, d\Sigma.$$

Therefore

$$\frac{I'(\sigma)}{I(\sigma)} - \frac{E'(\sigma)}{E(\sigma)} = \frac{1}{\sigma} + (E(\sigma)I(\sigma))^{-1}\left[E(\sigma)\int_{\partial B_\sigma} \frac{\partial}{\partial r}(d^2(u, y))\, d\Sigma \right.$$
$$\left. -2I(\sigma)\int_{\partial B_\sigma} \left|\frac{\partial u}{\partial r}\right|^2\, d\Sigma\right],$$

which together with (3.2) implies the inequality

$$\frac{d}{d\sigma}\log\left(\frac{I(\sigma)}{\sigma E_\sigma}\right) \leq (E(\sigma)I(\sigma))^{-1}\left[\left(\int_{\partial B_\sigma} d(u, y)\frac{\partial}{\partial r}d(u, y)\right)^2 \right.$$
$$\left. - \left(\int_{\partial B_\sigma} d^2(u, y)\right)\left(\int_{\partial B_\sigma}\left|\frac{\partial u}{\partial r}\right|^2\right)\right]. \tag{3.3}$$

Since $|\frac{\partial}{\partial r}d(u, y)| \leq |\frac{\partial u}{\partial r}|$, by the Schwarz inequality we see that

$$\frac{d}{d\sigma}\left(\frac{\sigma E(\sigma)}{I(\sigma)}\right) \geq 0. \tag{3.4}$$

For any $x \in \Omega$, $\sigma > 0$, $y \in X$ we define $\operatorname{Ord}(x, \sigma, y)$ by

$$\operatorname{Ord}(x, \sigma, y) = \frac{\sigma\int_{B_\sigma(x)} |\nabla u|^2}{\int_{\partial B_\sigma(x)} d^2(u, y)\, d\Sigma}.$$

The reason for this notation is that for a harmonic function u,

$$\lim_{\sigma\downarrow 0} \operatorname{Ord}(x, \sigma, u(x)) = \operatorname{Order}_x(u - u(x));$$

that is, the order with which u attains its value $u(x)$ at x. In particular, for harmonic functions (or harmonic maps into smooth manifolds of nonpositive curvature) this limit is a positive integer.

Generally if $x \in \Omega$ and $\sigma > 0$, then the function

$$y \mapsto \int_{\partial B_\sigma(x)} d^2(u, y)\, d\Sigma$$

is a proper convex function on X and hence has a unique minimum point $y_{x,\sigma} \in X$. The function $y \mapsto \operatorname{Ord}(x, \sigma, y)$ thus has a unique maximum point at $y_{x,\sigma}$. We now define $\operatorname{Ord}(x)$ by

$$\operatorname{Ord}(x) = \lim_{\sigma \to 0} \operatorname{Ord}(x, \sigma, y_{x,\sigma})$$

observing that $\sigma \mapsto \operatorname{Ord}(x, \sigma, y_{x,\sigma})$ is a monotone increasing function of σ so that the limit exists. Moreover, for a fixed $\sigma > 0$, the function $x \mapsto \operatorname{Ord}(x, \sigma, y_{x,\sigma})$ is a continuous function, and hence it follows that the function $x \mapsto \operatorname{Ord}(x)$ is upper semicontinuous since it is the decreasing limit of a family of continuous functions. It can be seen that $\operatorname{Ord}(x) \geq 1$ for almost every $x \in \Omega$ and hence for *all* $x \in \Omega$ by upper semicontinuity. Suppose we are at a point $x_0 \in \Omega$ where $\alpha = \operatorname{Ord}(x_0)$. Fix $\sigma_0 > 0$ so that $B_{\sigma_0}(x_0) \Subset \Omega$, and let $\sigma_1 \in (0, \sigma_0)$. Because of the monotonicity of the ratio we have

$$\sigma \int_{B_\sigma(x_0)} |\nabla u|^2 \geq \alpha \int_{\partial B_\sigma} d^2(u, y_1)\, d\Sigma$$

for all $\sigma \in [\sigma_1, \sigma_0)$, where $y_1 = y_{x_0, \sigma_1}$. Combining this with (3.2) yields

$$\begin{aligned} \alpha I(\sigma) &\leq \frac{1}{2}\sigma \int_{\partial B_\sigma(x_0)} \frac{\partial}{\partial r}(d^2(u(x), y_1))\, d\Sigma(x) \\ &= \frac{1}{2}(\sigma I'(\sigma) - (n-1)I(\sigma)). \end{aligned}$$

Therefore $\frac{d}{d\sigma}(\sigma^{-2\alpha-(n-1)} I(\sigma)) \geq 0$ for $\sigma \in [\sigma_1, \sigma_0]$, and hence it follows that $\sigma_1^{-(n-1)} I(\sigma_1) \leq c\sigma_1^{2\alpha}$ (we have now fixed σ_0). Since the function $d^2(u(x), y_1)$ is a subharmonic function (as we saw above), by the mean-value inequality we obtain

$$\sup_{x \in B_{\sigma_{1/2}}(x_0)} d^2(u(x), y_1) \leq c\sigma_1^{2\alpha}.$$

In particular, it follows that $d(u(x), u(x_0)) \leq c|x - x_0|^\alpha$ for $x \in B_{\sigma_0}(x_0)$. As a consequence we conclude that u is Lipschitz in the interior of Ω.

Proposition 3.7. *A minimizing harmonic map from Ω to X is Lipschitz in the interior provided that X has nonpositive curvature.*

It is possible to make the transition from the Dirichlet problem to the homotopy problem (see [50]) to prove, for example, an extension of the Eells-Sampson theorem with singular image space. For the purpose of the statement assume that X is a Riemannian complex whose universal covering space $\tilde{X}$ has nonpositive curvature. Let $\mathscr{M}$ be a smooth compact Riemannian manifold, and $\varphi: M \to X$ be a given continuous map.

Proposition 3.8. *There is a Lipschitz harmonic map $u: M \to X$ which is freely homotopic to φ. Moreover, u minimizes energy among all Lipschitz maps homotopic to φ.*

In order to obtain more information than a Lipschitz bound on a harmonic mapping u, we make the observation that the ratio $\operatorname{Ord}(x, \sigma, y)$ is invariant under both rescalings of the domain and the range. To say this precisely, observe that if $\lambda, \mu > 0$, then we can consider a rescaled map $u_{\lambda,\mu}(x) = \mu u(\lambda x)$ which is a new harmonic map from $\lambda\Omega = \{\lambda x: x \in \Omega\}$ to the complex $\mu X = \{\mu v: v \in X\}$. Of course the centers of dilation in both Ω and $\mathbf{R}^Q$ are arbitrary. Notice that the complex μX again has nonpositive curvature since distances are multiplied by a constant factor. It is easily checked that $\operatorname{Ord}^u(x, \sigma, y)$ is equal to $\operatorname{Ord}^{u_{\lambda,\mu}}(\lambda x, \lambda\sigma, \mu y)$. In particular, by translation of coordinates assume that $0 \in \Omega$, $u(0) = 0 \in X \subset \mathbf{R}^Q$. For any small $\lambda > 0$, let μ be defined by

$$\mu = \max\{d(u(x), 0): |x| \le \lambda\}.$$

The map $u_{\lambda,\mu}$ then has the property that it maps the unit ball in $\mathbf{R}^n$ into the unit ball (but no smaller ball) centered at 0 in μX. Now let $\{\lambda_i\}$ be a sequence tending to zero, and let $u_i = u_{\lambda_i,\mu_i}$ be defined as above. The sequence $\{u_i\}$ has a uniform Lipschitz bound on compact subsets of $\mathbf{R}^n$, and since $d^2(u_i, 0)$ is subharmonic we have

$$\max\{d^2(u_i(x), 0): |x| = 1\} = 1$$

for every i. (Note that distance is computed here in $\mu_i X$.) Therefore a subsequence again denoted $\{u_i\}$ converges to a map $u_0: \mathbf{R}^n \to X_0$ which is a harmonic map from $\mathbf{R}^n$ to the tangent cone of X at 0 which we denote X_0. Note that X_0 is a geometric cone in $\mathbf{R}^Q$ so that the distance from a point $v \in X_0$ to 0 is simply $|v|$, the Euclidean norm of v. Because of the scale invariance of the order function one can deduce that $\operatorname{Ord}^{u_0}(0, \sigma, 0) = \operatorname{Ord}^u(0) \equiv \alpha$ for every $\sigma > 0$. Now since $R_{\lambda,0}(v) = \lambda v$, it can be seen that equality holds in (3.2), and hence we have

$$E(\sigma) = \frac{1}{2}\sigma^{n-1}\frac{d}{d\sigma}(\sigma^{1-n}I(\sigma)).$$

Using the fact that $E(\sigma) = \sigma^{-1}\alpha I(\sigma)$, we then conclude that $\sigma^{1-n}I(\sigma) = \sigma^{2\alpha}I(1)$. Since the right-hand side of (3.3) must be zero, it follows that $\frac{\partial u}{\partial r} = h(r)u$ for some function $h(r)$, so that $u_0(r, \xi) = g(r)u_0(1, \xi)$ for some Lipschitz function $g(r)$ with $g(1) = 1$. From the fact that $\sigma^{1-n}I(\sigma) = \sigma^{2\alpha}I(1)$ we deduce that $g(r) = r^\alpha$, and hence u_0 is homogeneous of degree $\alpha = \mathrm{Ord}(0)$.

In general, homogeneous harmonic maps can be quite complicated, however, there is one particularly simple class of such maps which we now describe. Suppose there is an isometric embedding $E: \mathbf{R}^m \to X_0$ which has totally geodesic image. Given a harmonic map $w: \mathbf{R}^n \to \mathbf{R}^m$, we can then construct a harmonic map $E \circ w: \mathbf{R}^n \to X_0$. If $E(0) = 0$, and w is a homogeneous map (hence given by spherical harmonics of some degree), then $E \circ w$ is also a homogeneous map. The following important lemma tells us that homogeneous maps of degree 1 are all described in this way.

Lemma 3.9. *Suppose $u_0: \mathbf{R}^n \to X_0$ is homogeneous of degree 1. There exists an isometric embedding $E: \mathbf{R}^m \to X_0$ for some $1 \le m \le n$ and a linear map $L: \mathbf{R}^n \to \mathbf{R}^m$ so that $u_0 = E \circ L$.*

The essential reason for the truth of this lemma is that the map u_0 has least possible order at 0, and we must have $\mathrm{Ord}(x) = 1$ for every $x \in \mathbf{R}^n$ since the function $\mathrm{Ord}(x)$ is upper semicontinuous. This tells us that the map u_0 is essentially homogeneous of degree 1 about every point, and in particular is totally geodesic. It can then be shown that the map u_0 is of the above type.

In order to use the Bochner method for harmonic maps into singular spaces it seems to be necessary to discuss higher differentiability properties of harmonic maps. We discuss now a result of this type. We first define what we mean by a smooth point of a harmonic map u. The key observation here is that if there is an isometric totally geodesic embedding E from a neighborhood U of a point p_0 in a smooth Riemannian manifold into X, then a harmonic map $w: B_\varepsilon(x_0) \to U$ gives rise to a harmonic map $u = E \circ w: B_\varepsilon(x_0) \to X$. Such a map deserves to be considered smooth. Thus for a general harmonic $u: \Omega \to X$ we make the following definition.

Definition 3.10. A point $x_0 \in \Omega$ is a regular point of u if u factors locally through an isometric totally geodesic (local) embedding of a smooth manifold into X.

Thus a point x_0 is regular if $u = E \circ w$ as above in a neighborhood of x_0. Let $\mathbf{R}(u)$ denote the regular set, and observe that $\mathbf{R}(u)$ is an open subset of the interior of Ω. Let $\mathcal{S}(u) = \Omega \backslash \mathbf{R}(u)$ denote the singular set of u.

Without further hypothesis on X there are unlikely to be many regular points. We now impose a further hypothesis on X which will imply that the set of regular points is very large. This hypothesis is satisfied in important examples such as buildings [8].

Hypothesis (H). Any two adjacent simplices of X are contained in the image of a totally geodesic isometric embedding of a Euclidean space into X.

One may think of Hypothesis (H) as providing us with sufficiently many totally geodesic submanifolds of X. Without some hypothesis there are likely to be very few of these, and as a result very few regular points of a harmonic map u into X.

We now state the first result which provides a local description of order one points for a harmonic map into a complex X satisfying (H).

Theorem 3.11. *Assume X satisfies* (H), *and $u: \Omega \to X$ is harmonic. There exists a positive number ε_0 depending only on n and X such that for any $x_0 \in \Omega$ we have $\mathrm{Ord}^u(x_0) \geq 1 + \varepsilon_0$ or $\mathrm{Ord}(x_0) = 1$. If $\mathrm{Ord}(x_0) = 1$, then there is a ball $B_{r_0}(x_0)$ for some $r_0 > 0$ such that $u(B_{r_0}(x_0))$ is contained in a totally geodesic subcomplex X_1 of X such that X is isometric to $\mathbf{R}^d \times X_2^{k-d}$ for some integer d with $1 \leq d \leq k$ and for a $(k-d)$-dimensional complex X_2 which also satisfies* (H). *Thus the map u may be written as $u = (u_0, u_1)$, where $u_0: B_{r_0}(x_0) \to \mathbf{R}^d$ is a smooth rank d harmonic map, and $u_1: B_{r_0}(x_0) \to X_1$ satisfies $\mathrm{Ord}^{u_1}(x_0) \geq 1 + \varepsilon_0$.*

We will refer to the integer d given in Theorem 3.11 as the rank of u at x_0. If $\mathrm{Ord}^u(x_0) > 1$, we will say that u has rank 0 at x_0. Note also that the upper semicontinuity together with the gap $(1, 1 + \varepsilon_0)$ omitted in the values of $\mathrm{Ord}(x)$ implies that the set $\mathcal{S}_1$ defined by

$$\mathcal{S}_1 = \{x \in \Omega : \mathrm{Ord}(x) > 1\}$$

is a relatively closed subset of Ω. Moreover, the set $\mathcal{S}_1$ can be shown to have Hausdorff dimension at most $n-2$ by an argument which goes back to H. Federer [22]. To illustrate the argument we show that the set $\mathcal{S}_1$ is discrete for $n = 2$, and we omit the remaining details for $n \geq 3$ as they will appear elsewhere. Assume on the contrary that $0 \in \mathcal{S}_1 \subset \Omega \subset \mathbf{R}^2$ is an accumulation point of $\mathcal{S}_1$. Let $\{x_j\} \subset \mathcal{S}_1$ be a sequence with $x_j \neq 0$ and $0 = \lim x_j$. Let $\lambda_j = |x_j|$, and construct the dilated sequence

$\{u_j\}$ as above. The u_j are then harmonic maps, a subsequence of which converges to a homogeneous map $u_0 \colon \mathbf{R}^2 \to X_0$ where we assume as above that $0 \in X$, and X_0 denotes the tangent cone to X at 0. Notice that by construction each of the u_j has a point on the unit circle $|x_j|^{-1}x_j$ at which the order is greater than $1+\varepsilon_0$. It then follows that u_0 has at least one point $x_0 \in S^1$ at which $\mathrm{Ord}^{u_0}(x_0) > 1$. Since u_0 is homogeneous, we have $\mathrm{Ord}^{u_0}(\lambda x_0) = \mathrm{Ord}^{u_0}(x_0)$ for $\lambda > 0$, and hence $\mathrm{Ord}^{u_0}(0) \geq \mathrm{Ord}^{u_0}(x_0)$. Further, since $\frac{d}{d\lambda}u(\lambda x_0) = 0$ for $\lambda \geq 0$ it follows that the ray $\{\lambda x_0 \colon \lambda \geq 0\}$ is mapped to 0 by u_0. We now rescale at the point x_0, and consider the maps u_{0i} given by

$$u_{0i}(x) = \mu_i u_0(x_0 + \lambda_i x),$$

where μ_i is chosen as above. A subsequence of u_{0i} then converges to a homogeneous harmonic map $u_1 \colon \mathbf{R}^2 \to X_0$. Since the map

$$u_{0i}\left(\lambda x + \frac{\lambda - 1}{\lambda i}x_0\right) = \lambda^\alpha u_{0i}(x)$$

for each i and every $\lambda > 0$, we may differentiate in λ and set $\lambda = 1$ to obtain $\nabla u_{0i} \cdot x_0 = 0(\lambda_i)$. Hence the limit u_1 satisfies $\nabla u_1 \cdot x_0 = 0$. If we take $x_0 = (1, 0)$, then u_1 is independent of the first coordinate x_1, and hence $u_1(s) = u_1(0, s)$ is a constant speed geodesic. But clearly it then follows that $\mathrm{Ord}^{u_1}(0) = 1$, a contradiction. This shows that $\mathscr{S}_1$ is a discrete set of points. Note that by Theorem 3.11 for $k = \dim X = 1$ we have $\mathscr{S} \subseteq \mathscr{S}_1$, and hence the Hausdorff dimension of $\mathscr{S}$ is at most $n-2$. Note also that for a regular point $x_0 \in \Omega$, there is a neighborhood of x_0 in which u may be represented by a smooth harmonic map into a Euclidean space. In particular, it follows that $\Delta e(u) = 2|\nabla\nabla u|^2$, where $|\nabla\nabla u|^2$ represents the sum of squares of the second derivatives of u. The following result will be important for the application of the Bochner method for maps into complexes.

Proposition 3.12. *The Hausdorff dimension of $\mathscr{S}$ is a most $n-2$. For any $\Omega_1 \Subset \Omega$, there is a sequence of Lipschitz functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathscr{S}$, $0 \leq \psi_i \leq 1$, and $\psi_i(x) \to 1$ for all $x \in \Omega_1 - \mathscr{S}$ such that*

$$\lim_{i\to\infty} \int_{\Omega_1} |\nabla\nabla u||\nabla\psi_i|\,dx = 0.$$

This result states that in an average sense the second derivatives of u blow-up more slowly than the reciprocal of the distance to $\mathscr{S}$. We outline the proof of this result. It is proven by induction on $k = \dim X$. For $k = 1$ we have already seen that $\mathscr{S} \subseteq \mathscr{S}_1$, and hence the first statement follows. To prove the second, let $\varepsilon > 0$ and $d > (n-2)$. Let Ω_2 be a compact subdomain of Ω with $\Omega_1 \Subset \Omega_2$, and choose a finite covering $\{B_{r_j}(x_j)\colon j = 1, \cdots, l\}$ of the compact set $\mathscr{S} \cap \overline{\Omega}_2$ satisfying $\sum_{j=1}^{l} r_j^d \leq \varepsilon$. Let φ_j be a Lipschitz function which is zero on $B_{r_j}(x_j)$ and identically one on $\mathbf{R}^n - B_{2r_j}(x_j)$ such that $|\nabla \varphi_j| \leq 2r_j^{-1}$. We assume also that $x_j \in \mathscr{S} \cap \overline{\Omega}_2$. Let φ be defined by $\varphi = \min\{\varphi_j\colon j = 1, \cdots, l\}$, and observe that φ vanishes in a neighborhood of $\mathscr{S} \cap \overline{\Omega}_2$, and $\varphi \equiv 1$ on $\mathbf{R}^n - \bigcup_{j=1}^{l} B_{2r_j}(x_j)$. Now let $\psi = 1 - \varphi^2(1-\varphi)^2$, and observe

$$\int_{\Omega_1} |\nabla\nabla u||\nabla \psi|\,dx \leq 2\left(\int_{\Omega_1} |\nabla\nabla u|^2 |\nabla u|^{-1} \varphi^2(1-\varphi)^2\,dx\right)^{1/2} \times \left(\int_{\Omega_1} |\nabla u||\nabla \varphi(1-\varphi)|^2\,dx\right)^{1/2}$$

by the Schwarz inequality. On the other hand, an elementary result for harmonic maps (see [56]) implies that on the regular set we have for a positive ε_n depending only on n

$$\Delta|\nabla u| \geq \varepsilon_n |\nabla\nabla u|^2 |\nabla u|^{-1}.$$

Let ζ be a smooth function with support in Ω_2 and $\zeta \equiv 1$ on Ω_1, and observe

$$\int_{\Omega_1} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2(1-\varphi)^2\,dx \leq \int_{\Omega} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2(1-\varphi)^2\zeta^2\,dx.$$

Then using the above differential inequality we obtain

$$\int_{\Omega} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2(1-\varphi)^2\zeta^2\,dx \leq 4\int_{\Omega} |\nabla u|\varphi(1-\varphi)\zeta\nabla|\nabla u|\cdot\nabla(\varphi(1-\varphi)\zeta),$$

which easily implies

$$\int_{\Omega} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2(1-\varphi)^2\zeta^2\,dx \leq c\int_{\Omega} |\nabla u||\nabla(\varphi(1-\varphi)\zeta)|^2\,dx.$$

Combining this with our previous bound yields

$$\int_{\Omega_1} |\nabla\nabla u||\nabla\psi|\,dx \leq c\int_{\Omega} |\nabla u|(\zeta^2|\nabla\varphi(1-\varphi)|^2 + \varphi^2(1-\varphi)^2|\nabla\zeta|^2)\,dx.$$

Therefore

$$\int_{\Omega_1} |\nabla\nabla u||\nabla\psi|\,dx \leq c\sum_{j=1}^{l} r_j^{-2}\int_{B_{2r_j}(x_j)} |\nabla u|\,dx.$$

Recall that we can estimate the decay of the energy on small balls in terms of the value of $\operatorname{Ord}(\cdot)$ at the center; in particular, since $\operatorname{Ord}(x_j) \geq 1 + \varepsilon_0$ we have

$$\int_{B_{2r_j}(x_j)} |\nabla u|^2 \, dx \leq c r_j^{n+2\varepsilon_0}.$$

Therefore

$$\int_{\Omega_1} |\nabla \nabla u| |\nabla \psi| \, dx \leq c \sum_{j=1}^{l} r_j^{n-2+\varepsilon_0} \leq c\varepsilon$$

provided $d \leq n - 2 + \varepsilon_0$. The desired result now follows by choosing a sequence $\varepsilon_i \to 0$ and a corresponding sequence $\{\psi_i\}$ of functions. This completes the proof for $k = 1$. To verify the inductive step, first we cover the set $(S \cap \mathcal{S}_1) \cap \overline{\Omega}_2$ with balls and construct a function ψ_0 as above. Next we cover the set $(\mathcal{S} - \bigcup_{j=1}^{l} B_{r_j}(x_j)) \cap \overline{\Omega}_2$ with balls $\{B_{r_p}(y_p) : p = 1, \cdots, q\}$ in which the map u can be written $u = (u_0, u_1)$ as in the conclusion of Theorem 3.11. We then use the inductive assumption to construct a function ψ_p vanishing in a neighborhood of $\mathcal{S} \cap \overline{B_{r_p}(y_p)}$ and identically one outside a slightly larger neighborhood with

$$\int_{B_{r_p}(y_p)} |\nabla \nabla u| |\nabla \psi_p| \, dx \leq \varepsilon 2^{-p}.$$

We finally set $\psi = \min\{\psi_0, \psi_1, \cdots, \psi_q\}$, and conclude

$$\int_{\Omega_1} |\nabla \nabla u| |\nabla \psi| \, dx \leq c\varepsilon.$$

Choosing a sequence of ε's and ψ's hence completes the argument.

The main application of these results is to apply the Bochner method for harmonic maps into complexes which satisfy (H). To illustrate this point we extend a vanishing theorem of K. Corlette [15] to this setting. Suppose that M is a quarternionic Kähler manifold which is compact without boundary, and that $u : M \to X$ is a harmonic map where the universal cover $\tilde{X}$ of X is a complex of nonpositive curvature satisfying (H). Let ω be the quaternionic Kähler 4-form on M, and notice that at a regular point of u we can form the exterior product $\omega \wedge du$ which is locally a 5-form with values in the pullback of the tangent bundle of a smooth flat manifold containing the image of u locally. The main vanishing result then comes from the conclusion that $\omega \wedge du$ is co-closed with respect to the exterior derivative arising from the pullback connection ∇. The local computation of [15] shows that

$$d_\nabla \delta_\nabla (\omega \wedge du) = 0$$

since in our case the image manifold is flat. Let ψ be a function constructed from Proposition 3.12 which vanishes in a neighborhood of $\mathscr{S}$ and is identically one outside a slightly larger neighborhood such that

$$\int_M |\nabla\nabla u||\nabla\psi|\, d\mu_M \leq \varepsilon$$

for any given $\varepsilon > 0$. We then form the invariant scalar function $\psi\langle \omega \wedge du, d_\nabla\delta_\nabla(\omega\wedge du)\rangle$ on M. Integrating by parts yields

$$\int_M \psi\|\delta_\nabla(\omega\wedge du)\|^2\, d\mu \leq c\int_M |\nabla u||\nabla\nabla u||\nabla\psi|\, d\mu.$$

Since $|\nabla u|$ is bounded, we have

$$\int_M \psi\|\delta_\nabla(\omega\wedge du)\|^2\, d\mu \leq c\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary we conclude that $\delta_\nabla(\omega\wedge du) \equiv 0$ on the regular set of u. As in [15] this implies that u is totally geodesic, and since the image is flat we would have a local parallel one-form on M if $du \not\equiv 0$. Therefore we conclude that $du \equiv 0$, so that u is constant. It is possible to apply this to study p-adic representations of lattices in $\mathrm{Sp}(n, 1)$ and F_4^{-20} with the image complex taken to be a Euclidean building (see [8]), and prove the arithmeticity of such lattices. This is carried out in a forthcoming joint work with M. Gromov.

We close this section with some historical remarks concerning certain aspects of the proof we have outlined. The monotone ratio which plays a central role originates for harmonic functions with the Three Circles Theorem of J. Hadamard. The statement for a harmonic function $u(x)$ is that the log of the average of u^2 on ∂B_r is a convex function of $\log r$. It is easily seen that the first logarithmic derivative of this function is given by

$$\left(\int_{\partial B_r} u^2\, d\Sigma\right)^{-1}\left(r\int_{B_r}|\nabla u|^2\, dx\right).$$

The fact that this is monotone increasing then expresses the convexity statement. This kind of ratio was used by F. J. Almgren [3] in his study of multivalued harmonic mappings, and also by F. H. Lin [37] for harmonic maps into the cone over a manifold (a problem which arose in the theory of liquid crystals). Also we remark that the more standard monotonicity formula (3.1), the monotonicity of $r^{2-n}E_r(u)$, plays an important role in the regularity theory for minimizing harmonic maps given by the author and K. Uhlenbeck [55]. As we have seen above the monotonicity of the ratio is obtained by combining this with an inequality (3.2) expressing the strong convexity of the distance function in a space of nonpositive curvature.

4. Complete constant mean curvature surfaces

In this section we summarize the striking progress which has been made in the past several years on the global question concerned with constant mean curvature surfaces in the Euclidean space $\mathbf{R}^3$. Let $\Sigma^2 \hookrightarrow \mathbf{R}^3$ be an immersed two-dimensional surface. If g, h denote the first and second fundamental forms of Σ, we then have the mean curvature H given by

$$H = \sum_{i,j=1}^{2} g^{ij} h_{ij}.$$

Thus H is the trace of the second fundamental form with respect to g. The equation of nonzero constant mean curvature may be written $H = 1$ since the sign of H is reversed by reversing the choice of unit normal, and the magnitude of H may be normalized by a homothety. The only obvious closed surface Σ of constant mean curvature is a sphere of the appropriate radius. Notice that a sphere lies in a nontrivial family of solutions with three parameters which may be taken as the coordinates of the center. The constant mean curvature equation is the Euler-Lagrange equation for the variational problem associated with the isoperimetric inequality; that is, a surface Σ has constant mean curvature if and only if Area(Σ) is extremal for variations of Σ, which preserve the enclosed three-dimensional volume. Thus the physical model described by constant mean curvature surfaces is that of a soap bubble where gravity is neglected. It is then a natural question to ask whether a mathematical soap bubble is necessarily round. This question was popularized by H. Hopf [31], and was sometimes referred to as a Hopf conjecture, although it is presently an unsolved problem to determine if Hopf actually conjectured an affirmative answer to this question or whether he simply proposed it as an interesting geometric problem. In any case the question has generated a wealth of deep and important mathematics.

The first result on this problem was given in 1853 by J. H. Jellet [32] who showed that a closed surface of constant mean curvature which is star-shaped with respect to some point is a round sphere. This result was greatly improved by H. Hopf [31] who showed that any surface Σ of constant mean curvature which is homeomorphic to S^2 is necessarily a round sphere. Hopf's method of proof has been important for a variety of problems, so we recall it. The main observation is that the second fundamental form h is a symmetric divergence free tensor on Σ since the Codazzi equations imply

$$\sum_{i,j=1}^{2} g^{ij} h_{ki;j} = H_{;k} = 0.$$

If we choose isothermal coordinates (x^1, x^2) locally on Σ, and set $z = x^1 + \sqrt{-1}x^2$, then the locally defined function $\varphi(z) = (h_{11} - h_{22}) - 2\sqrt{-1}h_{12}$ becomes holomorphic. The global quadratic differential $\Phi = \varphi(z)\,dz^2$ is thus a regular holomorphic quadratic differential on Σ. It is an elementary matter to show that the Riemann sphere S^2 cannot support a nonzero holomorphic quadratic differential. Thus if Σ is simply connected, then $\Phi \equiv 0$ and hence Σ is umbilic. Therefore Σ is a round sphere. Generally Φ is referred to as the Hopf differential of Σ.

The next chapter in the story is the result of A. D. Alexandrov [2] who proved that a closed embedded surface Σ of arbitrary genus with $H \equiv 1$ is a round sphere. In this case the method developed to prove this result has proved to be extremely useful and powerful. This is the Alexandrov method of plane reflection, a symmetry argument relying on the strong maximum principle.

From a variational point of view, the classical isoperimetric inequality implies that the round sphere is the unique least area surface bounding a given volume. That the round sphere is the only locally minimizing surface for the variational problem was shown by L. Barbosa and M. do Carmo [6].

We have now presented all of the evidence known before 1983 concerning the question of whether closed constant mean curvature surfaces are round. Since this evidence is all affirmative, it came as quite a surprise when H. Wente [65], [66] constructed a constant mean curvature torus. We first remark that the equation $H = 1$ can locally be reduced to either the sinh-Gordon equation $\Delta w + \sinh 2w = 0$ or to the harmonic map equation into the standard S^2. The reduction to the sinh-Gordon equation involves working in isothermal coordinates in which the Hopf differential is given by dz^2. This can be achieved globally for tori, and thus the problem can be reduced to studying periodic solutions. Wente's argument involves studying sufficiently interesting families of doubly periodic solutions and showing that some of these give rise to closed surfaces. After Wente's paper, the construction was made more explicit by U. Abresch [1] and R. Walter [63]. These ideas were extended, and a classification result proved by U. Pinkall and I. Sterling [46] for constant mean curvature tori. This makes contact with earlier work on the sinh-Gordon equation by N. Ercolani and G. Forest [18]. There has been further work done by Bobenko.

While there are useful general methods to deal with the construction of constant mean curvature tori, these use heavily the special structure of the torus and are difficult to generalize to higher genus. In 1987 N. Kapouleas [33], [34] introduced a new approach to the construction of complete and closed surfaces of constant mean curvature. To understand this approach one needs to understand slightly more complicated explicit surfaces of constant mean curvature. There is an important classical family of such surfaces introduced by Delaunay in 1841. These are the surfaces of revolution satisfying $H \equiv 1$. This family includes the cylinder as well as singly periodic surfaces which approximate in a singular limit infinite strings of tangent spheres. The idea is to use Delaunay surfaces as building blocks to construct more complicated surfaces. The proper way to think of the nearly singular Delaunay surfaces is that they are made up of positively curved pieces which are spherical, negatively curved which approximate catenoids with small necks, and flat regions which join the two. It turns out that the positively and negatively curved regions play more or less equivalent roles in the constructions. Kapouleas understood in a very precise way the kinds of configurations which lead to constant mean curvature surfaces and was able to construct closed surfaces of any genus greater than two. He also constructed larger classes of complete properly embedded surfaces of constant mean curvature. It turned out to be impossible using the Delaunay surfaces as building blocks to construct surfaces of genus 1 or 2. An analysis of Wente's tori suggests that they are made up of spherical positively curved regions, negatively curved regions which approximate scaled-down Enneper surfaces, and flat regions joining the two. Notice that the Enneper surface and the catenoid are closely related minimal surfaces, and are in fact the only two complete minimal surfaces whose Gauss map is injective. In addition certain of the Wente surfaces can be described rather explicitly, so Kapouleas has very recently succeeded in understanding Wente tori (or pieces of them) as building blocks along with the Delaunay surfaces. This has made it possible for him to construct closed constant mean curvature surfaces of arbitrary genus as well as many other previously unknown types of complete surfaces.

Since the work of Alexandrov gave us a method of controlling embedded constant mean curvature surfaces, it is reasonable to expect that complete noncompact embedded $H = 1$ surfaces may be more rigid than immersed surfaces. In fact a result of N. Korevaar, R. Kusner, and B. Solomon [36] shows that an annular end of such a surface is strongly asymptotic to a Delaunay surface. An earlier work of W. Meeks [39] should also be mentioned here. Korevaar and Kusner have also announced a compactness theorem which yields uniform control on an embedded constant mean curvature surface provided certain weak geometric quantities are controlled.

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