SURVEYS IN DIFFERENTIAL GEOMETRY
1 (1991) 243-310

TWO-DIMENSIONAL GRAVITY AND INTERSECTION THEORY ON MODULI SPACE

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Abstract

These are notes based on two lectures given at the Conference on Geometry and Topology (Harvard University, April 1990). The first was mainly devoted to explaining a conjecture according to which stable intersection theory on moduli space of Riemann surfaces is governed by the KdV hierarchy of integrable equations. The second lecture was primarily an introduction to the "hermitian matrix model" of two-dimensional gravity, which is a crucial part of the background for the conjecture. Analogous but more general theories also exist and are sketched in these notes. The generalization in the first lecture involves considering intersection theory on the moduli space of pairs consisting of a Riemann surface Σ and a holomorphic map of Σ to a fixed Kähler manifold K. The simplest analogous generalization in the second lecture involves a chain of hermitian matrices.

1. Introduction

At first sight, two-dimensional general relativity appears "trivial," at least as a physical theory, since for instance the Einstein-Hilbert action

$$I = \frac{1}{2\pi} \int \sqrt{g} R \tag{1.1}$$

is a topological invariant, so that the Einstein field equations are automatically obeyed.

Yet actually, on further investigation, two-dimensional quantum general relativity proves to be a strikingly rich theory. What is loosely called "critical" two-dimensional gravity is an essential ingredient in string theory. "Noncritical" two-dimensional gravity is a much more difficult subject which has been intensively studied with various motivations including possible applications to string theory and to the large N limit of quantum gauge theories with gauge group $SU(N)$.

In the earliest approach to the subject, introduced by Polyakov [54], noncritical two-dimensional gravity is related to a quantum field theory

Received June 14, 1990. Research supported in part by National Science Foundation Grant 86-20266 and National Science Foundation Waterman Grant 88-17521.

with Liouville action. This has been intensively studied [32], [11] by seeking to exploit at the quantum level the integrability of the classical Liouville equation $-\Box\phi+e^{\phi}=0$. A variant of this involves a different gauge choice that is claimed to lead to a sort of $SL(2, \mathbb{R})$ structure [42]. In a very different approach, two-dimensional gravity has been studied by counting triangulations of surfaces, which can be related to "matrix models" [14], [1], [41]. This approach has been developed with spectacular success, most recently with complete solutions in the "double scaling limit" by Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [13], [25], [35]. Another approach [52], [46] uses ideas of "topological quantum field theory" and can be reduced to a description in terms of the cohomology of the moduli space of Riemann surfaces. (Yet another approach was proposed at this meeting by I. Singer.)

A variety of arguments indicate that the theories constructed by these different approaches are equivalent. In addition to heuristic arguments, Liouville theory can be compared to the matrix models by comparison of critical exponents (which in Liouville theory can be computed by a scaling argument [21], [14]). Topological gravity is related to Liouville theory by an elegant argument due to Distler [21] that involves a variant [28] of the usual bose-fermi equivalence on Riemann surfaces. The topological field theory approach is related to the matrix models by explicit comparison in genus ≤ 3, by the "string equation" and another similar equation that can be derived in both frameworks, and by formal analogies.

Purely in mathematical terms, the proposal that topological gravity is equivalent to the one matrix model leads to a striking conjecture. Since topological gravity amounts to the study of stable intersection theory on the moduli space of Riemann surfaces, while the one matrix model is a soluble problem related to the KdV hierarchy, the conjecture that these are equivalent amounts to a conjecture that the KdV hierarchy governs the stable intersection theory on moduli space. §2 of this paper is devoted to a precise and self-contained formulation of this conjecture, and a description of the evidence for it. §3 is devoted to a generalization in which one considers a Riemann surface Σ together with a holomorphic map of Σ to a fixed complex manifold M. §4 is an introduction to the one matrix model and its relation to the KdV hiearchy. This section can be read independently of §§2 and 3. At the end of §4 we also briefly consider a matrix model analog of the generalization of the topological theory to include M.

§§2 and 3 are primarily an exposition of ideas that have appeared elsewhere [59], [19], with a few details added. §4 is an exposition of work of many authors, including the recent work of Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [13], [25], [35].

We are grateful to P. Deligne for help in understanding the compactification of moduli space and the string equation. Key points were also raised by M. F. Atiyah, H. Neuberger, D. Kazhdan, G. Segal, and I. M. Singer.

2. Stable intersection theory on moduli space

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n ordered punctures, and let $\overline{\mathcal{M}}_{g,n}$ be its compactification obtained by adjoining curves with double points [16]. This is the (compactified) "moduli space of stable curves," which arises naturally in string theory. $\overline{\mathcal{M}}_{g,n}$ is not a manifold but an orbifold (locally the quotient of a manifold by a finite group), so intersection theory is well defined on $\overline{\mathcal{M}}_{g,n}$, but intersection numbers are in general rational numbers rather than integers.

Such moduli spaces are endowed with natural cohomology classes, as described by Atiyah and Bott in the gauge theory case [4] and by Mumford, Morita, and Miller in the case we will be considering [50], [48], [45]. Let Σ be a stable curve with marked points $x_1, x_2, \cdots, x_n$. It is essential that, though Σ may have singularities (double points), the moduli space of stable curves is defined in such a way that the x_i never coincide with these singularities. Thus, each x_i has its complex cotangent space $T^*\Sigma|_{x_i}$, and these vary holomorphically with x_i, giving n holomorphic line bundles $\mathcal{L}_{(i)}$ over $\overline{\mathcal{M}}_{g,n}$. One can think of the x_i as sections of the universal curve $\mathcal{C}\overline{\mathcal{M}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$. If $K_{\mathcal{C}|\mathcal{M}}$ is the cotangent bundle to the fibers of $\mathcal{C}\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, then $\mathcal{L}_{(i)} = x_i^*(K_{\mathcal{C}|\mathcal{M}})$.

Let $d_1, d_2, \cdots, d_n$ be nonnegative integers such that

$$\sum_{i=1}^{n} d_i = 3g - 3 + n. \tag{2.1}$$

This is the dimensional condition under which the intersection number

$$\langle \bigwedge_{i=1}^{n} c_1(\mathcal{L}_{(i)})^{d_i}, \overline{\mathcal{M}}_{g,n} \rangle \tag{2.2}$$

may be nonzero. We will denote this number as

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle. \tag{2.3}$$

These quantities, which will be our main interest, are closely analogous to the intersection numbers on instanton moduli space that Donaldson introduced [22] in studying smooth four-manifolds.[1] The ordering of factors in (2.2) and (2.3) is of course immaterial, since the cohomology classes in question are even dimensional. If r_0 of the d_i are equal to 0, r_1 of them are equal to 1, r_2 of them are equal to 2, etc., then it is sometimes convenient to write

$$\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle = \langle\tau_0^{r_0}\tau_1^{r_1}\tau_2^{r_2}\cdots\rangle. \tag{2.4}$$

The notation in (2.3) reflects the fact that (like their analogs in Donaldson theory), these numbers have a quantum field theory interpretation [52], [46], [9]. Indeed, $\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle$ is the (unnormalized) expectation value of a product of "local operators" τ_{d_i} with respect to a certain Feynman path integral measure. Though we will not explain this path integral interpretation here, its existence is one of the things that makes plausible the conjecture that these objects are related to the hermitian one matrix model, which is also defined by a kind of path integral.

The $\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle$ are closely related to the intersection numbers of the stable cohomology classes on moduli space studied by Mumford, Morita, and Miller [50], [48], [45]. In that formulation, one considers the projection $\pi:\overline{\mathcal{M}}_{g,1}\to\overline{\mathcal{M}}_{g,0}=\overline{\mathcal{M}}_g$, and defines $2n$-dimensional cohomology classes κ_n on $\overline{\mathcal{M}}_{g,0}$ by

$$\kappa_n = \pi_*(c_1(\mathcal{L})^{n+1}). \tag{2.5}$$

($\mathcal{L}$ is again the line bundle whose fiber is the cotangent space to the one marked point of $\overline{\mathcal{M}}_{g,1}$. We define $\kappa_{-1}=0$.) It is known that the κ's obey no stable relations [45], and it is conjectured that the stable cohomology of moduli space (in a sense explained in [45], [36]) is a polynomial algebra generated by the κ's. It is natural to consider intersection numbers of the κ's, which we will denote as

$$\langle\kappa_{r_1}\kappa_{r_2}\cdots\kappa_{r_n}\rangle = \langle\kappa_{r_1}\wedge\kappa_{r_2}\wedge\cdots\wedge\kappa_{r_n}, \overline{\mathcal{M}}_g\rangle. \tag{2.6}$$

As a special case of the comparison between the τ's and the κ's, consider first the expectation value of a single τ,

$$\langle\tau_d\rangle = \int_{\overline{\mathcal{M}}_{g,1}} c_1(\mathcal{L})^d. \tag{2.7}$$

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[1] In [59], we worked with $\sigma_d = d!\tau_d$, in order to agree with the conventions of the literature on matrix models.

(Of course, this is nonzero only if $d = 3g-2$.) By performing first the integral over the fiber of $\pi: \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$, we get immediately that

$$\langle \tau_d \rangle = \int_{\overline{\mathcal{M}}_g} \kappa_{d-1} = \langle \kappa_{d-1} \rangle. \tag{2.8}$$

In general, we have by definition

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_{(1)})^{d_1} \wedge \cdots \wedge c_1(\mathcal{L}_{(n)})^{d_n}. \tag{2.9}$$

To get a similar expression for $\langle \kappa_{r_1} \cdots \kappa_{r_n} \rangle$, let $\mathcal{C}_n(\overline{\mathcal{M}}_g) = \mathcal{C}_{(1)}\overline{\mathcal{M}}_g \times_{\overline{\mathcal{M}}_g} \mathcal{C}_{(2)}\overline{\mathcal{M}}_g \times_{\overline{\mathcal{M}}_g} \cdots \times_{\overline{\mathcal{M}}_g} \mathcal{C}_{(n)}\overline{\mathcal{M}}_g$ be the n-fold fiber product of n copies $\mathcal{C}_{(1)}\overline{\mathcal{M}}_g, \cdots, \mathcal{C}_{(n)}\overline{\mathcal{M}}_g$ of the universal curve over $\overline{\mathcal{M}}_g$. Let π_i be the projection of $\mathcal{C}_n(\overline{\mathcal{M}}_g)$ onto the ith factor $\mathcal{C}_{(i)}\overline{\mathcal{M}}_g$, let $K_{\mathcal{C}_{(i)}/\mathcal{M}}$ be the cotangent bundle to the fibers of $\mathcal{C}_{(i)}\overline{\mathcal{M}}_g$, and let $\widehat{\mathcal{L}}_{(i)} = \pi_i^*(K_{\mathcal{C}_{(i)}/\mathcal{M}})$. Then by the definition of the κ's, we have

$$\langle \kappa_{d_1-1} \kappa_{d_2-1} \cdots \kappa_{d_n-1} \rangle = \int_{\mathcal{C}_n(\overline{\mathcal{M}}_g)} c_1(\widehat{\mathcal{L}}_{(1)})^{d_1} \wedge \cdots \wedge c_1(\widehat{\mathcal{L}}_{(n)})^{d_n}. \tag{2.10}$$

The key observation is now the following. A point in $\mathcal{C}_n(\overline{\mathcal{M}}_g)$ labels a stable curve Σ and n ordered marked points in Σ which are arbitrary so that in particular two or more of them are permitted to coincide. As long as we keep away from the locus in $\mathcal{C}_n(\overline{\mathcal{M}}_g)$ on which two or more of these points coincide, there is a natural 1-1 map $\mathcal{C}_n(\overline{\mathcal{M}}_g) \to \overline{\mathcal{M}}_{g,n}$. These varieties are thus birationally equivalent. The equivalence is only birational since $\overline{\mathcal{M}}_{g,n}$ parametrizes a family of genus g curves with n marked points which are never permitted to coincide. (Compactification is achieved by permitting Σ to degenerate to a curve with a larger number of components when naively two or more points are becoming coincident.) Thus, though $\mathcal{C}_n(\overline{\mathcal{M}}_g)$ and $\overline{\mathcal{M}}_{g,n}$ are birationally equivalent, they (and the curves they parametrize) differ on a certain divisor at infinity.

On the Zariski open set on which $\mathcal{C}_n(\overline{\mathcal{M}}_g)$ and $\overline{\mathcal{M}}_{g,n}$ (and the curves they parametrize) have a natural identification, the line bundles $\mathcal{L}_{(j)}$ and $\widehat{\mathcal{L}}_{(j)}$ also have a natural identification, as is immediate from their definitions. Thus, (2.9) and (2.10) differ only from the contribution of the divisor at infinity. The analysis of the effects of this divisor in comparing (2.9) and (2.10) is a universal local problem which naturally leads to additional terms involving the conjectured generators of the stable cohomology.

One finds

$$\begin{aligned} &\langle \tau_{d_1} \tau_{d_2} \rangle = \langle \kappa_{d_1 - 1} \kappa_{d_2 - 1} \rangle + \langle \kappa_{d_1 + d_2 - 2} \rangle , \\ (2.11) \quad &\langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \rangle = \langle \kappa_{d_1 - 1} \kappa_{d_2 - 1} \kappa_{d_3 - 1} \rangle + \langle \kappa_{d_1 + d_2 - 2} \kappa_{d_3 - 1} \rangle \\ &\qquad + \langle \kappa_{d_1 + d_3 - 2} \kappa_{d_2 - 1} \rangle + \langle \kappa_{d_2 + d_3 - 2} \kappa_{d_1 - 1} \rangle + 2 \langle \kappa_{d_1 + d_2 + d_3 - 3} \rangle , \end{aligned}$$

and so on, as we will sketch after explaining the "string equation." These relations are invertible (they are given by a triangular matrix with 1's on the diagonal) and show that the information contained in the intersection theory of the τ's is the same as the information contained in the intersection theory of the κ's.

Let us now heuristically explain the motivation for the way that we will organize the data. Given a quantum field theory Lagrangian $\mathscr{L}_0$ and operators τ_i, it is natural to consider a more general Lagrangian

$$\mathscr{L} = \mathscr{L}_0 - \sum_i t_i \int_\Sigma \tau_i , \tag{2.12}$$

where the t_i are known physically as "coupling constants." Thus the Feynman integral, in genus g, becomes

$$F_g(t_i) = \int (\text{FIELDS}) e^{-\mathscr{L}_0} e^{\sum_i t_i \int_\Sigma \tau_i} . \tag{2.13}$$

We can expand the exponential

$$e^{\sum_i t_i \int_\Sigma \tau_i} = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \left(\int_\Sigma \tau_i \right)^{n_i} , \tag{2.14}$$

with the sum running over all sequences $\{n_i\}$ of nonnegative integers, almost all of which are zero. So one sees that the path integral in (2.13) is the generating function of the intersection numbers $\langle \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle$. Summing over genus, as is natural in string theory, we would need to consider the "total free energy"

$$F(t_i) = \sum_{g=0}^{\infty} F_g(t_i). \tag{2.15}$$

With this motivation, the natural object that we wish to consider is the generating function of the stable intersection theory on moduli space, defined by

$$F(t_0, t_1, \cdots) = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n-2} \cdots \rangle . \tag{2.16}$$

Here it is understood that for every sequence $\{n_i\}$, $\langle\tau_0^{n_0}\tau_1^{n_1}\tau_2^{n_2}\cdots\rangle$ is to be computed in genus g, where

$$3g-3=\sum_i n_i(i-1). \tag{2.17}$$

If the g determined by this formula is not a nonnegative integer, one defines $\langle\tau_0^{n_0}\tau_1^{n_1}\tau_2^{n_2}\cdots\rangle$ to be zero. For $g=0, 1, 2, \cdots$, the genus g contribution $F_g(t_i)$ is defined by restricting the sum in (2.16) to sequences $\{n_i\}$ obeying (2.17). For our purposes, the sum in (2.16) can be regarded as a formal series, but the conjecture that will be stated presently would mean that (2.16) is an expansion of a natural function, defined in an open set in the space of the t_i.

There is actually a slight imprecision in the definition (2.16), since we have not given a meaning to the symbol $\langle 1\rangle$, which is the contribution from the zero sequence $n_0=n_1=\cdots=0$. This sequence corresponds according to (2.17) to curves of genus 1 with no marked points. This is a degenerate case, since the virtual dimension of $\overline{\mathcal{M}}_{1,0}$ (predicted from the Riemann-Roch formula for the moduli problem) is 0 but the actual dimension is 1. In the present paper, the object $\langle 1\rangle$ will play no role, and we could simply set it to zero, but the natural value is the Euler characteristic of $\overline{\mathcal{M}}_{1,0}$ as an orbifold, which is

$$\langle 1\rangle=-\tfrac{1}{12}. \tag{2.18}$$

We will now introduce a convenient notation for the derivatives of F. We define

$$\langle\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle\rangle=\frac{\partial}{\partial t_{d_1}}\frac{\partial}{\partial t_{d_2}}\cdots\frac{\partial}{\partial t_{d_n}}F(t_0, t_1, \cdots). \tag{2.19}$$

It is evident that $\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle$ reduces to $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$ if one sets $t_0=t_1=\cdots=0$. As a special case of (2.19), one occasionally uses the symbol $\langle\langle 1\rangle\rangle$ to represent the functional $F(t_0, t_1, \cdots)$. We also write

$$\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle_g=\frac{\partial}{\partial t_{d_1}}\cdots\frac{\partial}{\partial t_{d_n}}F_g. \tag{2.20}$$

2a. The conjecture. Our basic conjecture is that $F(t_0, t_1, \cdots)$ is determined by the following two conditions:

(1) The object $U=\partial^2F/\partial t_0^2$ obeys the KdV equations,

$$\frac{\partial U}{\partial t_n}=\frac{\partial}{\partial t_0}R_{n+1}(U, \dot U, \ddot U, \cdots), \tag{2.21}$$

where $\dot{U} = \partial U/\partial t_0$, $\ddot{U} = \partial^2 U/\partial t_0^2$, etc., are the derivatives of U with respect to t_0, and $R_{n+1}(U, \dot{U}, \ddot{U}, \cdots)$ are certain polynomials in U and its t_0 derivatives that are well known in the theory of the KdV equations (and can be defined by a recursion relation that is given below).

(2) In addition, F obeys the "string equation,"

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}. \tag{2.22}$$

The two statements can be summarized by saying that stable intersection theory on moduli space is equivalent to the "hermitian matrix model" of two-dimensional gravity. That formulation was the original context for the conjecture.

Consequences of the conjecture. We will first verify that the conjecture, if true, does uniquely determine F.

The first part of the conjecture obviously determines $U(t_0, t_1, t_2, t_3, \cdots)$ in terms of the "initial data" $U(t_0, 0, 0, 0, \cdots)$. The second part of the conjecture implies upon setting $t_i = 0$, for $i > 0$, that the initial data are

$$U(t_0, 0, 0, \cdots) = t_0. \tag{2.23}$$

Thus the conjecture suffices to determine U.

It is easy to see from the point of view of intersection theory on moduli space why the initial conditions for U must be those given in (2.23). In fact, the dimensional condition (2.17) implies right away that the numbers $\langle \tau_0^n \rangle$ for $n = 1, 2, 3, \cdots$ vanish unless $n = 3$, while $\langle \tau_0^3 \rangle$ receives a contribution only in genus zero. The moduli space of genus zero curves with three labeled marked points consists of a single point without symmetries (since the three points can be uniquely mapped to $0, 1, \infty$ by $SL(2, \mathbb{C})$), so $\langle \tau_0^n \rangle = \delta_{n,3}$. Hence (using (2.18) for the $n = 0$ contribution),

$$F(t_0, 0, 0, 0, \cdots) = \sum_{n=0}^{\infty} \frac{t_0^n}{n!} \langle \tau_0^n \rangle = \frac{t_0^3}{6} - \frac{1}{12}. \tag{2.24}$$

This indeed corresponds to the initial conditions $U = t_0$.

We now want to show that the conjecture suffices to determine F and not only $U = \ddot{F}$. Note first that the string equation is equivalent to an explicit relation among the intersection numbers

$$\left\langle \tau_0 \prod_{i=1}^{n} \tau_{d_i} \right\rangle = \sum_{j=1}^{n} \left\langle \prod_{i=1}^{n} \tau_{d_i - \delta_{ij}} \right\rangle + \delta_{n,2} \delta_{d_1,0} \delta_{d_2,0}. \tag{2.25}$$

(It is in this form that the string equation can naturally be deduced from algebraic geometry, as we will see soon.) This can be used to determine all intersection numbers, and thus F, once U is known. As a simple example, it follows from (2.25) that

$$\langle\tau_n\rangle = \langle\tau_{n+2}\tau_0\tau_0\rangle = \left.\frac{\partial U}{\partial t_{n+2}}\right|_{t_i=0}. \tag{2.26}$$

(In fact, one can easily show inductively that the KdV prediction for (2.26) is $\langle\tau_{3g-2}\rangle = 1/((24)^g \cdot g!)$.) This can be generalized in the following elementary but clumsy way to give an algorithm to compute an arbitrary genus g intersection number:

$$W = \langle\tau_{d_1} \cdots \tau_{d_k}\rangle. \tag{2.27}$$

We can suppose that none of the d_i are 0, since factors of τ_0 can be eliminated using (2.25). This being so, the dimensional equation $\Sigma_i(d_i - 1) = 3g - 3$ gives an upper bound $d_i \leq 3g - 2$ for (2.27) to be nonzero. Suppose, inductively, that for some integer r it is known that the KdV equations plus the string equation determine all intersection numbers in which all $d_i \geq 1$ and one of the d_i, say d_1, is $\geq r$. We can start the induction with $r = 3g - 1$. We want to improve the bound from r to $r - 1$. The quantity

$$W' = \langle\tau_{d_1+2}\tau_{d_2} \cdots \tau_{d_k}\tau_0\tau_0\rangle \tag{2.28}$$

can be determined form the KdV equations plus the string equation, since it is

$$W' = \left[\frac{\partial}{\partial t_{d_1+2}}\frac{\partial}{\partial t_{d_2}} \cdots \frac{\partial}{\partial t_{d_k}} U\right]_{t_i=0}, \tag{2.29}$$

and we know that the KdV equations plus the string equation suffice to determine U. Now, two uses of the string equation (2.25) to eliminate the two factors of τ_0 in (2.28) may leave us with an expression still containing τ_0's, since there may be factors of τ_1 or τ_2 in (2.28), and these may become τ_0's upon using the string equation. If so, use the string equation again until after finitely many steps all τ_0's are eliminated. After doing so, one obtains an expression exhibiting W' as a positive multiple of W plus genus g intersection numbers containing τ_{d_1+2} or τ_{d_1+1} that are already known from the induction hypothesis plus genus g correlation functions containing τ_{d_1} and a smaller number of τ_1's than are present in (2.27). Repetition of this procedure to eliminate all τ_1's eventually expresses W in terms of objects that are already known by the induction hypothesis. This completes the demonstration that the conjecture suffices to determine the generating function F.

Alternative ways of writing the equations. The polynomials R_n that appear in the first part of the conjecture can be defined inductively by the formulas

$$(2.30)\quad R_1 = U, \qquad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1}\left(\frac{\partial U}{\partial t_0}R_n + 2U\frac{\partial R_n}{\partial t_0} + \frac{1}{4}\frac{\partial^3}{\partial t_0^3}R_n\right).$$

(This recursion relation among the generalized KdV potentials R_n was obtained in [31] from the theory of the resolvent of the Schrödinger operator. The proof establishes the not obvious fact that the right-hand side of the second equation in (2.30) is indeed the t_0 derivative of a polynomial R_{n+1}. The Schrödinger operator enters via the inverse scattering method [30] which is the basis for the integrability of the KdV equation.) This recursion relation can be interpreted as stating that the KdV flows are Hamiltonian flows for two different symplectic structures. Since, in view of the definition of U, the left-hand side of (2.21) is the same as

$$(2.31)\qquad \frac{\partial}{\partial t_0}\langle\langle\tau_n\tau_0\rangle\rangle,$$

it is clear that upon integrating once in t_0 (a step that can be justified using the string equation), (2.21) amounts to the statement that

$$(2.32)\qquad \langle\langle\tau_n\tau_0\rangle\rangle = R_{n+1}(U, \dot{U}, \ddot{U}, \cdots).$$

With the aid of the recursion relation, we get the alternative version

$$(2.33)\qquad \begin{aligned}&\langle\langle\tau_n\tau_0\tau_0\rangle\rangle\\ &= \frac{1}{2n+1}(\langle\langle\tau_{n-1}\tau_0\rangle\rangle\langle\langle\tau_0^3\rangle\rangle + 2\langle\langle\tau_{n-1}\tau_0^2\rangle\rangle\langle\langle\tau_0^2\rangle\rangle + \frac{1}{4}\langle\langle\tau_{n-1}\tau_0^4\rangle\rangle),\end{aligned}$$

which captures the full content of the KdV equations. A still more explicit version is

$$(2.34)\qquad \begin{aligned}\langle\langle\tau_n\tau_0\tau_0\rangle\rangle_g = \frac{1}{2n+1}\Bigg(&\sum_{g'=0}^{g}\langle\langle\tau_{n-1}\tau_0\rangle\rangle_{g'}\langle\langle\tau_0^3\rangle\rangle_{g-g'}\\ &+2\sum_{g'=0}^{g}\langle\langle\tau_{n-1}\tau_0^2\rangle\rangle_{g'}\langle\langle\tau_0^2\rangle\rangle_{g-g'} + \frac{1}{4}\langle\langle\tau_{n-1}\tau_0^4\rangle\rangle_{g-1}\Bigg).\end{aligned}$$

2b. Evidence for the Conjecture. "Matrix models" and "topological gravity" were both proposed originally as candidates for simple approaches to two-dimensional gravity. They are based on similar apparently "trivial" Lagrangians, and this suggested the conjecture advanced in §2a that they might be equivalent. This thought was encouraged by formal analogies between the two that will be apparent in §4. It was also encouraged by the fact that the methods used [37], [51] to compute the Euler characteristic of moduli space (which is the partition function of an appropriate topological field theory) are close cousins of the methods used in the matrix models.

Apart from such heuristic considerations, the evidence for the conjecture consists mainly of the following:

(a) The "string equation" (2.30) can be verified directly, in the intersection theory.

(b) One can verify directly that the KdV equations (2.21) hold in the intersection theory for genus $g \leq 3$. (We will here only consider $g \leq 2$, and refer to work of Horne [37], comparing to results of Faber [26], for $g = 3$.)

(c) One can also, by direct methods, prove another equation analogous to the string equation which follows from the string equation together with the KdV equations.

This is the evidence for the conjecture that can be stated without any reference to physics. Physicists consider Distler's relation of topological gravity to Liouville theory [20] to be an important indication that the conjecture is true. Also, E. and H. Verlinde have proposed [57] physical arguments that may eventually lead to a proof of the conjecture.

The string equation and its cousins. To obtain the string equation, one considers the moduli space $\overline{\mathcal{M}}_{g,n+1}$ of stable curves Σ with $n+1$ marked points $x_0, \cdots, x_n$, which we regard as sections of the universal curve $\mathcal{C}\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n+1}$. Except for certain low values of g and n, which will have to be treated separately, there is a projection $\pi: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forgets the first point x_0. (This projection does not exist for $g = 0$, $n = 2$, and $g = 1, n = 0$, because then forgetting x_0 will render Σ unstable.) The line bundles of interest on $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{M}}_{g,n}$ are $\mathcal{L}_{(j)} = x_j^*(K_{\mathcal{C}|\mathcal{M}})$ and $\mathcal{L}'_{(j)} = x_j^*(K'_{\mathcal{C}|\mathcal{M}})$, respectively, where $K_{\mathcal{C}|\mathcal{M}}$ and $K'_{\mathcal{C}|\mathcal{M}}$ are the cotangent bundles along the fibers of $\mathcal{C}\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n+1}$ and $\mathcal{C}\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

There is a subtlety here that plays a crucial role in understanding the string equation and the analogous formulas (2.11) relating the τ_n's to the Mumford-Morita-Miller classes κ_{n-1}. A point in $\overline{\mathcal{M}}_{g,n+1}$ corresponds to a connected, stable curve Σ perhaps with more than one component. Stability means in particular that any genus zero component of Σ has at least three marked points, including possible double points. It may happen that "forgetting x_0" causes a particular genus zero component to contain only two marked points. If so, that component must be contracted to a point.

This occurs on the divisors D_j, indicated in Figure 1, in which a genus zero component contains x_0, x_j, and precisely one double point. (It also occurs if a genus zero component contains x_0 and two double points, but that happens in complex codimension two and will not affect comparisons of line bundles.) Because of this, though there is, analogous to the map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that "forgets x_0", a corresponding natural map of universal curves $\pi_{\mathscr{C}} : \mathscr{C}\overline{\mathcal{M}}_{g,n+1} \to \mathscr{C}\overline{\mathcal{M}}_{g,n}$, this map is not a fibering. $\pi_{\mathscr{C}}$ does not just "forget x_0"; it may be described by the instruction "forget x_0 and contract any genus zero components that become unstable as a result."

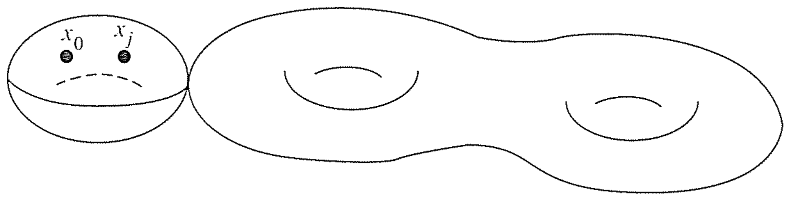


FIGURE 1. A CURVE THAT WILL BECOME UNSTABLE IF x_0 IS "FORGOTTEN."

Because of this, it is not the case, as one might have naively expected, that $\mathscr{L}_{(j)} = \pi^*(\mathscr{L}'_{(j)})$. To work out the discrepancy, note that a nonzero local section s of $K'_{\mathscr{C}/\mathcal{M}}$ determines a nonzero local section $x_j^*(s)$ of $\mathscr{L}'_{(j)}$ which intuitively is obtained by "evaluating s at x_j." Via the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, $x_j^*(s)$ pulls back to a section $\pi^* x_j^*(s)$ that vanishes precisely on the divisors D_j. In fact, if $\pi_{\mathscr{C}} : \Sigma' \to \Sigma$ is any map between curves, and s is any local holomorphic differential on Σ, then $\pi_{\mathscr{C}}^*(s)$ vanishes on any component of Σ' that is mapped to a point in Σ. Applying this to the fibers of $\pi_{\mathscr{C}} : \mathscr{C}\overline{\mathcal{M}}_{g,n+1} \to \mathscr{C}\overline{\mathcal{M}}_{g,n}$, we see that $\pi_{\mathscr{C}}^*(s) = 0$ on the locus $x_j(D_j) \subset \mathscr{C}\overline{\mathcal{M}}_{g,n+1}$. Hence, $\pi^* x_j^*(s) = x_j^* \pi_{\mathscr{C}}^*(s)$ vanishes on the divisor D_j, with a simple zero as one sees on more careful examination. The fact that a local nonzero section $x_j^*(s)$ of $\mathscr{L}'_{(j)}$ pulls back to a section $\pi^* x_j^*(s)$ with a simple zero on D_j corresponds to a formula

$$\mathscr{L}_{(j)} = \mathscr{L}'_{(j)} \bigotimes_{j=1}^{n} \mathscr{O}(D_j). \tag{2.35}$$

At the level of first Chern classes, this means that

$$c_1(\mathscr{L}_{(j)}) = \pi^*(c_1(\mathscr{L}'_{(j)})) + [D_j]. \tag{2.36}$$

We are now ready to consider the intersection number

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \int_{\overline{\mathscr{M}}_{g,n+1}} 1 \cdot \bigwedge_{j=1}^n c_1(\mathscr{L}_{(j)})^{d_j}, \tag{2.37}$$

temporarily avoiding the special cases $g = 0$, $n = 2$ and $g = 1$, $n = 0$. Note the factor of $1 = c_1(\mathscr{L}_0)^0$ in (2.37). Now, obviously,

$$0 = \int_{\overline{\mathscr{M}}_{g,n+1}} \pi^* \left(\bigwedge_{j=1}^n c_1(\mathscr{L}'_{(j)})^{d_j} \right), \tag{2.38}$$

since the pullback of a cohomology class from $\overline{\mathscr{M}}_{g,n}$ could not be a top-dimensional class on $\overline{\mathscr{M}}_{g,n+1}$. If it were so that $\mathscr{L}_{(j)} = \pi^*(\mathscr{L}'_{(j)})$, for $j = 1, \cdots, n$, then (2.37) would vanish. We must use instead the correct formula (2.35), which implies

$$c_1(\mathscr{L}_{(j)})^n = (\pi^* c_1(\mathscr{L}'_{(j)}))^n + [D_j] \cdot \sum_{m=0}^{n-1} c_1(\mathscr{L}_{(j)})^m \cdot (\pi^*(c_1(\mathscr{L}'_{(j)})))^{n-1-m}. \tag{2.39}$$

Now, the line bundle $\mathscr{L}_{(j)}$ is trivial when restricted to the divisor D_j, since on the universal curve over D_j, the point x_j is on a rigid object, a genus zero component with three marked points. So we can discard terms in (2.39) with $m > 0$. Since it follows directly from the definition that $D_i \cap D_j = 0$ for $i \neq j$, in evaluating (2.37), we can drop terms proportional to $[D_i] \cdot [D_j]$, so (2.37) becomes

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \sum_{j=1}^n \int_{\overline{\mathscr{M}}_{g,n+1}} [D_j] \cdot \bigwedge_{i=1}^n c_1(\mathscr{L}_{(i)})^{d_i - \delta_{ij}}. \tag{2.40}$$

(In case one of the exponents is negative, we set $c_1(\mathscr{L}_{(j)})^{-1} = 0$.) Integrating over the fibers of $\pi: \overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$, we get the final result

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \sum_{j=1}^n \left\langle \prod_{i=1}^n \tau_{d_i - \delta_{ij}} \right\rangle. \tag{2.41}$$

We still must consider the special cases $g = 0$, $n = 2$ and $g = 1$, $n = 0$. The only nonvanishing intersection number of this type is

$$\langle \tau_0 \tau_0 \tau_0 \rangle = 1 \tag{2.42}$$

for $g = 0$.

We will leave it to the reader to verify that (2.41) and (2.42) are precisely equivalent to the string equation (2.22). (2.41) alone, without the exceptional contribution for $g = 0$, $n = 2$, would give (2.22) without the $t_0^2/2$ term.

This completes the explanation of part (a) of the evidence for the conjecture. As for (c), another equation of a similar nature can be obtained by looking at

$$\left\langle \tau_1 \cdot \prod_{i=1}^{n} \tau_{d_i} \right\rangle = \int_{\overline{\mathscr{M}}_{g,n+1}} c_1(\mathscr{L}_{(0)}) \wedge_i (c_1(\mathscr{L}_{(i)}))^{d_i}. \tag{2.43}$$

Now, in evaluating (2.43) we may actually replace the $\mathscr{L}_{(i)}$ by $\pi^*(\mathscr{L}'_{(i)})$ for $i = 1, \cdots, n$. The second term in (2.35) does not contribute, since $c_1(\mathscr{L}_{(0)}) \cdot [D_j] = 0$. In evaluating (2.43) by integrating over the fiber of $\pi: \overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$, one has a natural identification $\alpha: \overline{\mathscr{M}}_{g,n+1} \cong \mathscr{C}\overline{\mathscr{M}}_{g,n}$. The relative canonical bundle $K'_{\mathscr{C}/\mathscr{M}}$ has degree $2g-2$ along the fibers of $\mathscr{C}\overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,n}$. It is not true, as one might think naively, that $\mathscr{L}_{(0)} \equiv \alpha^*(K'_{\mathscr{C}/\mathscr{M}})$. The correct relation, by reasoning just as that which led to (2.35), is

$$\mathscr{L}_{(0)} \cong \alpha^*(K'_{\mathscr{C}/\mathscr{M}}) \otimes_{j=1}^{n} \mathscr{O}(D_j). \tag{2.44}$$

(Intuitively, a differential form on a curve Σ with n marked points is permitted to have poles at the marked points.) Thus, $\mathscr{L}_{(0)}$ is a line bundle of degree $2g-2+n$ along the fibers of $\overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$. So integrating along the fibers in (2.43) we get

$$\left\langle \tau_1 \cdot \prod_{i=1}^{n} \tau_{d_i} \right\rangle = (2g-2+n) \left\langle \prod_{i=1}^{n} \tau_{d_i} \right\rangle, \tag{2.45}$$

with $2g-2+n$ being the degree of the canonical line bundle of a genus g curve with n marked points.

As in the discussion of the string equation, there is an exceptional case here, which arises for $g = 1$, $n = 0$, where there is no projection map $\overline{\mathscr{M}}_{1,1} \to \overline{\mathscr{M}}_{1,0}$. In this case, the exceptional contribution is

$$\langle \tau_1 \rangle = \tfrac{1}{24}. \tag{2.46}$$

This comes from a factor of $\frac{1}{12}$ which can be understood, for instance, for the existence of an elliptic modular form of weight 12 with a simple zero at the cusp, and a factor of $\frac{1}{2}$ because the generic elliptic curve has two symmetries. Now, (2.45) and (2.46) are equivalent to a differential equation

$$\frac{\partial F_g}{\partial t_1} = \left(2g - 2 + \sum_{i=0}^{\infty} t_i \frac{\partial}{\partial t_i}\right) F_g + \frac{1}{24}\delta_{g,1}. \tag{2.47}$$

This can be rewritten in a way that does not refer to g if one recalls that the right-hand side of (2.45) is nonzero only if $3g - 3 + n = \sum_i d_i$, which amounts to the statement that genus g correlation functions obey

$$\left(\sum_i (i-1) t_i \frac{\partial}{\partial t_i} - (3g-3)\right) F_g = 0. \tag{2.48}$$

The above equations combine to give

$$\frac{\partial F}{\partial t_1} = \frac{1}{3}\sum_{i=0}^{\infty}(2i+1) t_i \frac{\partial F}{\partial t_i} + \frac{1}{24}. \tag{2.49}$$

This result is part (c) of the evidence for our conjecture, because in fact it can alternatively be deduced if one assumes that $U = \ddot{F}$ obeys the KdV equations as well as the string equation.

Comparison of τ 's and κ 's. Let us now sketch how one similarly obtains the formulas (2.11) relating the τ_n 's to the κ_{n-1} 's. To explain the ideas, it should suffice to indicate the origin of the first equation in (2.11),

$$\langle \tau_{d_1} \tau_{d_2} \rangle = \langle \kappa_{d_1 - 1} \kappa_{d_2 - 1} \rangle + \langle \kappa_{d_1 + d_2 - 2} \rangle. \tag{2.50}$$

(We may assume that $d_1, d_2 > 0$, since otherwise, with $\kappa_{-1} = 0$, (2.50) is a consequence of the string equation.) To analyze this, we consider the moduli space $\overline{\mathcal{M}}_{g,2}$ of curves Σ with two marked points x_1 and x_2. It has two projections

$$\pi_i : \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}^{(i)}_{g,1}, \tag{2.51}$$

where π_1 "forgets" x_2 and π_2 "forgets" x_1. (The inverted naming of the π 's will make later formulas less painful.) Here $\overline{\mathcal{M}}^{(i)}_{g,1}$, $i = 1, 2$, are the two copies of $\overline{\mathcal{M}}_{g,1}$ obtained by "forgetting" one of the points. We also have the usual two line bundles over $\overline{\mathcal{M}}_{g,2}$ defined by $\mathcal{L}_{(i)} = x_i^*(K_{\mathcal{C}/\mathcal{M}})$; $K_{\mathcal{C}/\mathcal{M}}$ is the relative cotangent bundle of the universal curve. Similarly on $\pi_i(\overline{\mathcal{M}}_{g,2})$, we define $\mathcal{L}'_{(i)} = x_i^*(K'_{\mathcal{C}/\mathcal{M}})$, where $K'_{\mathcal{C}/\mathcal{M}}$ is the relative cotangent bundle to the universal curve over $\overline{\mathcal{M}}^{(i)}_{g,1}$.

According to (2.35),

$$c_1(\mathcal{L}_{(2)}) = \pi_2^*(c_1(\mathcal{L}'_{(2)})) + [D], \tag{2.52}$$

where D is the divisor in $\overline{\mathscr{M}}_{g,2}$ parametrizing curves which have a genus zero component containing x_1, x_2, and precisely one double point. Since $d_1 > 0$ and $\mathscr{L}_{(1)}$ is trivial when restricted to D, we can rewrite the definition of $\langle \tau_{d_1} \tau_{d_2} \rangle$ in the form

$$(2.53) \qquad \langle \tau_{d_1} \wedge \tau_{d_2} \rangle = \left(c_1(\mathscr{L}_{(1)})^{d_1} \wedge \pi_2^*(c_1(\mathscr{L}'_{(2)}))^{d_2}, \overline{\mathscr{M}}_{g,2} \right),$$

dropping the second term in (2.52). Similarly,

$$(2.54) \qquad c_1(\mathscr{L}_{(1)}) = \pi_1^*(c_1(\mathscr{L}'_{(1)})) + [D].$$

Writing out the analog of (2.39) and discarding the terms with $m \neq 0$ for the same reason as before, we get

$$(2.55) \qquad c_1(\mathscr{L}_{(1)})^{d_1} = \pi_1^*(c_1(\mathscr{L}'_{(1)}))^{d_1} + [D] \cdot \pi_1^*(c_1(\mathscr{L}'_{(1)}))^{d_1 - 1}.$$

So

$$(2.56) \qquad \begin{aligned} \langle \tau_{d_1} \tau_{d_2} \rangle = & \left(\pi_1^*(c_1(\mathscr{L}'_{(1)}))^{d_1} \wedge \pi_2^*(c_1(\mathscr{L}'_{(2)}))^{d_2}, \overline{\mathscr{M}}_{g,2} \right) \\ & + (\pi_1^*(c_1(\mathscr{L}'_{(1)}))^{d_1 - 1} \wedge \pi_2^*(c_1(\mathscr{L}'_{(2)}))^{d_2}, D). \end{aligned}$$

Now, D is a copy of $\overline{\mathscr{M}}_{g,1}$. When restricted to D, $\pi_1^*(\mathscr{L}'_{(1)}) \cong \pi_2^*(\mathscr{L}'_{(2)}) \cong K_{D/\mathscr{M}}$, where $K_{D/\mathscr{M}}$ is the canonical bundle to the fibers of $D \cong \overline{\mathscr{M}}_{g,1} \to \overline{\mathscr{M}}_g$. (Indeed, if one restricts to D and then forgets x_1 or x_2, then the genus zero component containing x_1 and x_2 in the curve parametrized by D "collapses" and x_2 or x_1 is identified with the one marked point of $D \cong \overline{\mathscr{M}}_{g,1}$.) Hence, by definition of the Mumford-Morita-Miller classes,

$$(2.57) \qquad \left(\pi_1^*(c_1(\mathscr{L}'_{(1)}))^{d_1 - 1} \wedge \pi_2^*(c_1(\mathscr{L}'_{(2)}))^{d_2}, D \right) = (K_{D/\mathscr{M}}^{d_1 + d_2 - 1}, D) = \langle \kappa_{d_1 + d_2 - 2} \rangle.$$

Now, practically from the definition in (2.10), we have

$$(2.58) \qquad \langle \kappa_{d_1 - 1} \kappa_{d_2 - 1} \rangle = \left(\pi_1^*(c_1(\mathscr{L}'_{(1)}))^{d_1} \wedge \pi_2^*(c_1(\mathscr{L}'_{(2)}))^{d_2}, \overline{\mathscr{M}}_{g,2} \right).$$

In fact, we can identify $\overline{\mathscr{M}}_{g,2}$ with $\mathscr{C}_2\overline{\mathscr{M}}_g = \mathscr{C}_{(1)}\overline{\mathscr{M}}_g \times_{\overline{\mathscr{M}}_g} \mathscr{C}_{(2)}\overline{\mathscr{M}}_g$.[2] The two factors of $\mathscr{C}\overline{\mathscr{M}}_g$ are $\overline{\mathscr{M}}_{g,1}^{(i)}$, and the two projections defined in (2.51) are the projections of the fiber product onto the factors. With this interpretation, the $\pi_i^*(\mathscr{L}'_{(i)})$ indeed coincide with $\widehat{\mathscr{L}}_{(i)}$ of (2.10), so (2.10) is equivalent to (2.58).

[2] That is, these two varieties are naturally isomorphic. The families of curves that they parametrize are not the same. Computing the effect of the difference is the point of the present computation.

Combining (2.56), (2.57), and (2.58), we have obtained (2.50). The analogous equations expressing $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ for $n > 2$ in terms of intersection products of κ's (and vice-versa) can be obtained similarly.

Analogy with homotopy theory. We will now make a brief digression. Equations (2.22) and (2.49) can be formulated as the statement that the functional $Z = e^F$, which physicists call the "partition function," is annihilated by the linear operators

$$L_{-1} = -\frac{\partial}{\partial t_0} + \frac{1}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}, \tag{2.59}$$
$$L_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}.$$

These operators generate a Lie algebra with $[L_0, L_{-1}] = L_{-1}$. This algebra is a subalgebra of the Virasoro algebra (the universal central extension of $\mathrm{Diff}(\mathbf{S}^1)$), and (2.59) has an obvious similarity to standard realizations of the Virasoro algebra. This fact has suggested a flight of fancy. The subject we are investigating in these notes has some notable analogies to the generalized K-theory investigated in [47]. The parameters t_i are analogous to the parameters $\tau_n = \tau(\mathbf{CP}^n)$ which in [47] determine a ring homomorphism $\tau: \mathscr{U}^* \to \mathbb{Z}$, with $\mathscr{U}^*$ being the complex cobordism ring. The critical hypersurfaces (corresponding to formal group laws of height n, for various n) of that theory have an analog in the present theory which will be apparent in §4. The invariants of almost complex manifolds that we will consider in §3, which depend on the parameters t_n, are somewhat similar to the complex cobordism invariant determined by τ which is essential in [47] (but the invariants considered in §3 are not cobordism invariants, so something is wrong with the analogy in its present form). Now, in K-theory, a sort of Virasoro algebra enters in the form of the Landweber-Novikov operations, and this motivated the guess (made in different forms by the author, G. Segal, and J. Morava) that (2.59) is in fact part of a Virasoro algebra that is relevant to the intersection theory problem. Recently, it has been shown to follow from the KdV equations that the partition function Z is indeed a highest weight vector for a Virasoro algebra of which (2.59) is part [18], [29].

Verification of the conjecture for low genus. In our sketch of the evidence for the conjecture, what remains is to explain statement (b)—that, at any rate, the genus ≤ 3 contributions to U obey the relations that follow from the KdV equations. We will first describe a shortcut for verifying this for $g \leq 2$, referring to [37] for a similar discussion in genus three, and then we will reconsider the genus zero and one cases in a more leisurely and perhaps more informative way.

In genus zero and one, the dimensional condition $3g - 3 = \sum_i (d_i - 1)$ makes it impossible to have all $d_i > 1$ in a nonzero intersection number $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle$. A nonzero result requires factors of τ_0 or τ_1. On the other hand, (2.22) and (2.49) can be used to eliminate factors of τ_0 and τ_1, reducing everything in genus zero to the special case $\langle \tau_0^3 \rangle = 1$ and reducing everything in genus one to the special case $\langle \tau_1 \rangle = \frac{1}{24}$. Since (2.22) and (2.49) as well as the values for these special cases follow either from algebraic geometry or from the KdV conjecture that we are testing, the conjecture is valid for arbitrary correlation functions in genus zero and one.

In genus two, (2.22) and (2.49) can be used to reduce everything to a knowledge of $\langle \tau_4 \rangle$, $\langle \tau_2 \tau_3 \rangle$, and $\langle \tau_2^3 \rangle$. Using the KdV equations plus the string equation, one determines these to be

$$\langle \tau_4 \rangle = \frac{1}{1152}, \qquad \langle \tau_2 \tau_3 \rangle = \frac{29}{5760}, \qquad \langle \tau_2^3 \rangle = \frac{7}{240}. \tag{2.60}$$

(An algorithm for computing these numbers was explained in (2.26) and the following discussion. An algorithm that is longer to prove but much quicker to use follows from the Virasoro equations of [18], [29].) Using (2.8) and (2.11), one has

$$\begin{aligned} \langle \tau_4 \rangle &= \langle \kappa_3 \rangle, \\ \langle \tau_2 \tau_3 \rangle &= \langle \kappa_1 \kappa_2 \rangle + \langle \kappa_3 \rangle, \\ \langle \tau_2^3 \rangle &= \langle \kappa_1^3 \rangle + 3\langle \kappa_1 \kappa_2 \rangle + 2\langle \kappa_3 \rangle. \end{aligned} \tag{2.61}$$

Mumford's formulas in genus two give

$$\langle \kappa_3 \rangle = \frac{1}{1152}, \quad \langle \kappa_1 \kappa_2 \rangle = \frac{1}{240}, \qquad \langle \kappa_1^3 \rangle = \frac{43}{2880}. \tag{2.62}$$

From these one can verify the genus two KdV formulas (2.60), completing the verification that the KdV and intersection theory results coincide in genus ≤ 2.

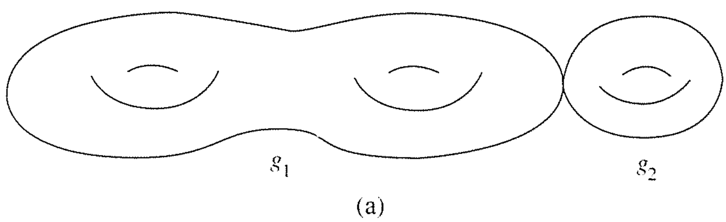
This completes the promised shortcut. To give a clearer picture of what is going on, we will now reconsider the genus zero and one situation in somewhat more detail.

2c. Leisurely approach to genus zero and one. Perhaps it is time to explain what is surprising about our conjecture and what is the difficulty in proving it. In genus zero, one, and two, the *uncompactified* moduli spaces $\mathcal{M}_{g,n}$ are affine varieties, and the cohomology classes $\kappa_n \in H^*(\overline{\mathcal{M}}_{g,n})$ vanish when restricted to $\mathcal{M}_{g,n}$. The κ_n's may thus be taken to have their support on the compactification divisor at infinity in $\overline{\mathcal{M}}_{g,n}$, and this makes computations relatively easy. In higher genus, $\mathcal{M}_{g,n}$ is far from being an affine variety, and [45] the κ_n's are definitely not zero when restricted to the "finite" part of moduli space.

Nevertheless, the KdV relations are surprisingly close to the sort of results that would hold if the cohomology classes of interest *were* trivial when restricted to $\mathcal{M}_{g,n}$. Let us reconsider (2.34):

$$\langle\langle\tau_n\tau_0\tau_0\rangle\rangle_g = \frac{1}{2n+1}\left(\sum_{g'=0}^{g}\langle\langle\tau_{n-1}\tau_0\rangle\rangle_{g'}\langle\langle\tau_0^3\rangle\rangle_{g-g'} + 2\sum_{g'=0}^{g}\langle\langle\tau_{n-1}\tau_0^2\rangle\rangle_{g'}\langle\langle\tau_0^2\rangle\rangle_{g-g'} + \frac{1}{4}\langle\langle\tau_{n-1}\tau_0^4\rangle\rangle_{g-1}\right). \tag{2.63}$$

The right-hand side of (2.63) looks very much like a sum over possible degenerations of a stable curve, to two branches of genus g' and $g-g'$, for $0 \le g' \le g$, or to a single branch of genus $g-1$. The possibilities are sketched in Figure 2. In each degeneration, a double point appears, which leads to two additional factors of τ_0 (one on each branch in the case of a separating degeneration).



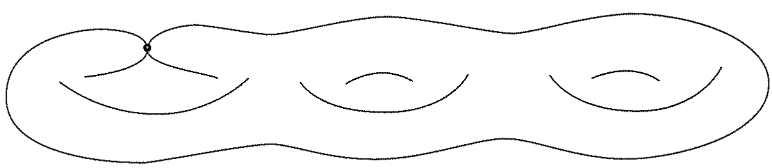


FIGURE 2. THE POSSIBLE DEGENERATIONS OF A STABLE CURVE OF GENUS g TO (A) TWO COMPONENTS WHOSE GENERA ADD TO g; OR (B) ONE COMPONENT OF GENUS $g-1$.

Now, as we will see, (2.63) is rather similar to the type of formula that would hold if the line bundles of interest were trivial on the finite part of moduli space for all g, n—yet strikingly different, because of the innocent looking factor of $1/(2n+1)$. To understand these assertions, we will consider genus zero (and one) in more detail. In genus zero, (2.63) reduces to

$$\langle\langle\tau_n\tau_0\tau_0\rangle\rangle_0 = \frac{1}{2n+1}(\langle\langle\tau_{n-1}\tau_0\rangle\rangle_0\langle\langle\tau_0^3\rangle\rangle_0 + 2\langle\langle\tau_{n-1}\tau_0^2\rangle\rangle_0\langle\langle\tau_0^2\rangle\rangle_0). \tag{2.64}$$

We will see how a similar but not identical formula arises from algebraic geometry.

Explicit treatment of genus zero. We will study the general n point function in genus zero:

$$\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle. \tag{2.65}$$

We recall that this is to be computed by intersection theory on $\overline{\mathscr{M}}_{0,n}$, which is the moduli space of stable genus zero curves Σ with n marked points $x_1, x_2, \cdots, x_n$. In particular, τ_{d_1} represents $c_1(\mathscr{L}_{(1)})^{d_1}$, where $\mathscr{L}_{(1)}$ is the cotangent bundle to Σ at x_1. Assuming $d_1 > 0$, we write schematically $\tau_{d_1} = c_1(\mathscr{L}_{(1)})\cdot\tau_{d_1-1}$, and we will evaluate $c_1(\mathscr{L}_{(1)})$ explicitly by computing the divisor of a rational section of $\mathscr{L}_{(1)}$. To write such a section explicitly, we use the fact that the finite part $\mathscr{M}_{0,n}$ of the moduli space consists of configurations of n distinct points on $\mathbf{CP}^1$ (which we represent as $\mathbf{C}\cup\infty$) modulo the action of $SL(2, \mathbb{C})$. A convenient section s of $\mathscr{L}_{(1)}$ on the finite part of moduli space can be described by the formula

$$s = dx_1\left(\frac{1}{x_1 - x_{n-1}} - \frac{1}{x_1 - x_n}\right), \tag{2.66}$$

which has the requisite $SL(2, \mathbb{C})$ invariance. This section obviously has neither zeros nor poles on the finite part $\mathscr{M}_{0,n}$ of moduli space. But we have to consider the possible degenerations.

The differential

$$\omega = dx\left(\frac{1}{x - x_{n-1}} - \frac{1}{x - x_n}\right) \tag{2.67}$$

on a smooth genus zero curve Σ may be characterized by saying that it has poles only at x_{n-1} and x_n, with residues 1 and -1, and no zeros. If Σ degenerates to a curve with two branches Σ_1 and Σ_2, one defines the sheaf of differentials on Σ as follows: a differential on Σ is a pair

(ω_1, ω_2), where ω_i for $i = 1, 2$ are differentials on Σ_i, and the ω_i are permitted to have simple poles, with equal and opposite residues, at the double point. With this definition, on a stable curve of genus zero, even a degenerate one, there is a unique differential ω with poles only at two given marked points x_{n-1} and x_n, of residues 1 and -1. This differential has no zeros on branches containing x_{n-1} or x_n, but, depending on the nature of the degeneration, may be identically zero on other branches. For instance, if, as in Figure 3, Σ has two branches Σ_1 and Σ_2 with x_{n-1} and x_n on the same branch, say Σ_2, then s is identically 0 on Σ_1 since otherwise its restriction to Σ_1 would be a differential with at most only one simple pole (at the node). Let D be the divisor that parametrizes such curves.

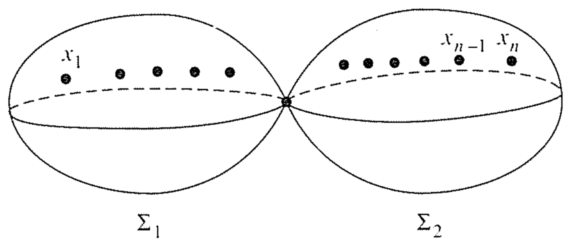


FIGURE 3. A GENUS ZERO CURVE WITH MARKED POINTS $x_1, \cdots, x_n$ DEGENERATING TO TWO COMPONENTS, ONE OF WHICH CONTAINS x_1 AND ONE OF WHICH CONTAINS x_{n-1}, x_n. THE OTHER POINTS MAY BE DISTRIBUTED IN AN ARBITRARY FASHION.

The section s in (2.66) is obtained by evaluating ω at $x = x_1$. Therefore, in view of the facts noted in the last paragraph, s has no poles, even at infinity in moduli space. But s vanishes on the divisor D. A closer examination shows that s has a simple zero on D. Let S denote the finite set $\{2, 3, \cdots, n-2\}$. The divisor D of zeros of s is a union of components $D_{X,Y}$, where $S = X \cup Y$ is a decomposition of S into disjoint subsets, and $D_{X,Y}$ is the divisor consisting of two component curves Σ, with one of the two components containing precisely x_1 and the x_j, $j \in X$, while the other component contains precisely x_{n-1}, x_n, and the x_j, $j \in Y$. We have schematically $\tau_{d_1} = [D] \cdot \tau_{d_1 - 1} = \sum_{S=X\cup Y} [D_{X,Y}] \cdot \tau_{d_1-1}$. This gives

$$\langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \cdots \tau_{d_n} \rangle = \sum_{S=X\cup Y} (\tau_{d_1-1} \wedge \tau_{d_2} \wedge \tau_{d_3} \wedge \cdots \wedge \tau_{d_n}, D_{X,Y}). \tag{2.68}$$

Part of the beauty of the compactified moduli spaces $\overline{\mathcal{M}}_{g,n}$ is, however, that the divisors at infinity are themselves moduli spaces of the same type. In this case, $D_{X,Y}$ can be identified with the product $\overline{\mathscr{M}}_{0,2+n_X}\times\overline{\mathscr{M}}_{0,3+n_Y}$, where n_X and n_Y are the cardinalities of the finite sets X and Y. So (2.68) is equivalent to the much more useful expression

$$\begin{aligned}(2.69)\qquad &\langle\tau_{d_1}\tau_{d_2}\tau_{d_3}\cdots\tau_{d_n}\rangle\\ &=\sum_{S=X\cup Y}\langle\tau_{d_1-1}\cdot\prod_{j\in X}\tau_{d_j}\cdot\tau_0\rangle\cdot\langle\tau_0\cdot\prod_{k\in Y}\tau_{d_k}\cdot\tau_{d_{n-1}}\tau_{d_n}\rangle.\end{aligned}$$

This is an inductive formula that determines the left-hand side in terms of a product of similar expressions with smaller values of n or of the d's. The factors of τ_0 that appear explicitly on the right-hand side of (2.69) represent the double point that appears on each branch in Figure 3.

Now, it is useful to write (2.69) in the following way. A special case of (2.69) is the case $n=3$, in which S is empty. One gets then

$$(2.70)\qquad \langle\tau_{d_1}\tau_{d_2}\tau_{d_3}\rangle=\langle\tau_{d_1-1}\tau_0\rangle\langle\tau_0\tau_{d_2}\tau_{d_3}\rangle.$$

Of course, we are working here at $t_i=0$, as is indicated by the use of the symbol $\langle\cdots\rangle$ (rather than $\langle\langle\cdots\rangle\rangle$). Let us differentiate (2.70) with respect to t_i for some i. The resulting equation

$$(2.71)\qquad \langle\tau_{d_1}\tau_i\tau_{d_2}\tau_{d_3}\rangle=\langle\tau_{d_1-1}\tau_i\tau_0\rangle\langle\tau_0\tau_{d_2}\tau_{d_3}\rangle+\langle\tau_{d_1-1}\tau_0\rangle\langle\tau_0\tau_i\tau_{d_2}\tau_{d_3}\rangle$$

is valid since it is simply the $n=4$ case of (2.69). In a similar way, one sees that the multiple derivatives of (2.70) with respect to the t_i all vanish; indeed the vanishing of the kth derivative of (2.70) is equivalent to the validity of (2.69) for $n=k+3$.

The fact that (2.70) vanishes *together with all of its derivatives* at $t_i=0$ is equivalent to the single statement

$$(2.72)\qquad \langle\langle\tau_{d_1}\tau_{d_2}\tau_{d_3}\rangle\rangle_0=\langle\langle\tau_{d_1-1}\tau_0\rangle\rangle_0\langle\langle\tau_0\tau_{d_2}\tau_{d_3}\rangle\rangle_0$$

(at least as a statement about formal power series, which is all we claim since here we are not considering analytical questions concerning the nature of these functions of the t_k). Indeed, this one equation is the generating function for the derivatives of (2.70) and thus for the totality of equations (2.69).

The special case $d_2=d_3=0$ of (2.72) is clearly very similar to (2.64). From this point of view, however, the factor of $1/(2n+1)$ in (2.64), which has no counterpart in (2.72), appears rather strange. Because of its dependence on n, it could not arise in a derivation on the above lines.

Actually, (2.64) and (2.72) are so similar without being identical that at first sight one is tempted to think that they could scarcely be consistent.

However, there are many ways to demonstrate their consistency. We will give one argument that will be useful background for §3.

To begin with, consider the objects $\langle\langle\tau_n\tau_m\rangle\rangle_0$. These are functions of $t_0, t_1, t_2, \cdots$. Let us, however, evaluate them at $t_1 = t_2 = \cdots = 0$ to get functions of t_0 only, defined by

$$G_{n,m}(t_0) = \langle\langle\tau_n\tau_m\rangle\rangle|_{t_1=t_2=\cdots=0}. \tag{2.73}$$

This process of restricting to functions of t_0 only will be important, so let us introduce some terminology. We will call the infinite-dimensional affine space $\mathbf{A}^\infty$ of the t_i the "phase space" (or "full phase space") of the theory, and we will call the line defined by $t_i = 0$, $i > 0$, the "small phase space." From (2.24), we have $t_0 = \langle\langle\tau_0\tau_0\rangle\rangle_0$ on the small phase space, so (2.73) is equivalent to the statement that on the small phase space,

$$\langle\langle\tau_n\tau_m\rangle\rangle_0 = G_{n,m}(\langle\langle\tau_0\tau_0\rangle\rangle_0). \tag{2.74}$$

We claim that (2.72) means that (2.74) is true, without modification, on the full phase space. We already know, of course, that (2.74), and therefore also its t_0 derivatives, vanish on the small phase space, so in particular

$$\langle\langle\tau_0\tau_n\tau_m\rangle\rangle_0 = G'_{n,m}(\langle\langle\tau_0\tau_0\rangle\rangle_0)\cdot\langle\langle\tau_0^3\rangle\rangle_0 \tag{2.75}$$

on the small phase space. We will use (2.72) to show that the t_k derivatives of (2.74) vanish on the small phase space also for $k > 0$.

The first derivative of the left-hand side of (2.74) with respect to t_k, on the small phase space, is

$$\begin{aligned}\langle\langle\tau_k\tau_n\tau_m\rangle\rangle_0 &= \langle\langle\tau_{k-1}\tau_0\rangle\rangle_0\langle\langle\tau_0\tau_n\tau_m\rangle\rangle_0\\ &= \langle\langle\tau_{k-1}\tau_0\rangle\rangle_0\cdot G'_{n,m}(\langle\langle\tau_0\tau_0\rangle\rangle_0)\cdot\langle\langle\tau_0^3\rangle\rangle_0,\end{aligned} \tag{2.76}$$

where (2.72) and (2.75) have been used. The first derivative of the right-hand side of (2.74) with respect to t_k is

$$G'_{n,m}(\langle\langle\tau_0\tau_0\rangle\rangle_0)\cdot\langle\langle\tau_k\tau_0\tau_0\rangle\rangle_0 = G'_{n,m}(\langle\langle\tau_0\tau_0\rangle\rangle_0)\cdot\langle\langle\tau_{k-1}\tau_0\rangle\rangle_0\cdot\langle\langle\tau_0^3\rangle\rangle_0. \tag{2.77}$$

Comparing these, one sees that the first derivative of (2.74) with respect to the t_k vanishes on the small phase space.

Inductively, if it is known that the r th derivatives of (2.74) with respect to the t_k all vanish on the small phase space, then precisely the same argument shows that the $(r+1)$ th derivatives of (2.74) vanish on the small phase space. Therefore, (2.74) is valid to all orders in an expansion in powers of the t_k and therefore (2.74) is valid on the full phase space.

To make this more concrete, let us now determine the functions $G_{n,m}$. We leave it to the reader to deduce from (2.69) that

$$\langle\tau_n\tau_m\tau_0^s\rangle = \delta_{s,n+m+1}, \tag{2.78}$$

from which it follows that on the small phase space

$$\langle\langle\tau_n\tau_m\rangle\rangle_0 = \frac{t_0^{n+m+1}}{(n+m+1)!}. \tag{2.79}$$

Thus,

$$G_{n,m}(t_0) = \frac{t_0^{n+m+1}}{(n+m+1)!}. \tag{2.80}$$

In particular, setting $m = 0$, we have

$$\langle\langle\tau_n\tau_0\rangle\rangle_0 = \frac{\langle\langle\tau_0\tau_0\rangle\rangle_0^{n+1}}{(n+1)!}. \tag{2.81}$$

Differentiating with respect to t_0, we then also have

$$\langle\langle\tau_n\tau_0^2\rangle\rangle_0 = \frac{\langle\langle\tau_0^3\rangle\rangle_0\langle\langle\tau_0^2\rangle\rangle_0^n}{n!}. \tag{2.82}$$

From these facts, the reader may straightforwardly deduce the genus zero KdV relation (2.64). Alternatively, we may say the following. In view of (2.32), (2.81) for $m = 0$ implies that the function which, according to our conjecture, is the generalized KdV potential $R_{n+1}(U, \dot{U}, \ddot{U}, \cdots)$ is

$$R_{n+1}(U, \dot{U}, \ddot{U}, \cdots) = \frac{U^{n+1}}{(n+1)!} + \cdots, \tag{2.83}$$

where '$\cdots$' denote terms involving derivatives of U which will arise as contributions from genus $g \geq 1$. It can indeed be seen (from the explicit form (2.34) of the KdV recursion relations) that if we consider the kth derivative $\partial^k U/\partial t_0^k$ to be of degree k, then the genus g contribution to R_{n+1} is homogeneous of degree $2g$. Of course, (2.83) agrees with the KdV theory for the contribution of degree 0.

In addition, (2.81) makes it possible to rewrite the genus zero approximation to the string equation in an interesting way. The t_0 derivative of the string equation (2.22) is

$$U = t_0 + \sum_{i=0}^{\infty} t_{i+1}\langle\langle\tau_i\tau_0\rangle\rangle, \tag{2.84}$$

and—in a genus zero approximation—we can now write this as

$$U = \sum_{i=0}^{\infty} t_i \frac{U^i}{i!}. \tag{2.85}$$

This looks like an equation for a fixed point of a general formal transformation $U \to \sum t_i U^i/i!$ of the affine line. This again suggests the analogy with homotopy theory that has been mentioned earlier.

Genus one. We will now, more briefly, indicate a similar treatment in genus one. We consider a genus one curve Σ with n marked points $x_1, \cdots, x_n$ and the general correlation function

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1. \tag{2.86}$$

Again the basic idea is to find a suitable section s of $\mathscr{L}_{(1)}$ and write $\tau_{d_1} = [s] \cdot \tau_{d_1 - 1}$. Actually, it is more convenient to take s to be a section of $\mathscr{L}_{(1)}^{\otimes 12}$, and then write

$$\tau_{d_1} = \frac{[s]}{12} \cdot \tau_{d-1}. \tag{2.87}$$

Indeed, if (on the Zariski open set in $\overline{\mathscr{M}}_{1,n}$ in which Σ has only one component) we regard Σ as an elliptic curve with x_1 as the origin, then the elliptic modular form Δ of weight 12 with a simple zero at the cusp can be interpreted as a section of $\mathscr{L}_{(1)}^{\otimes 12}$. It has no poles, and vanishes only when Σ degenerates. In the theory of modular forms, the only relevant degeneration is the degeneration of Σ to a rational curve with double point, where Δ and hence also s has a simple zero. Let us call this divisor D_0. In the present context there is an additional possibility: s vanishes when Σ degenerates to a union of two components, of genus zero and one, respectively, as sketched in Figure 4, provided that x_1 is on the genus zero component. (The reasoning showing that s vanishes in this situation, with a 12th order zero since it is a section of $\mathscr{L}_{(1)}^{\otimes 12}$, is similar to the reasoning that we used at an analogous point in the genus zero discussion. Heuristically, an elliptic curve, even if it degenerates to two components of genus zero and one, has a nonzero holomorphic differential, but this vanishes identically on the genus zero component as it would otherwise be a nonzero differential on that component with at most only a single pole at the node.) We will refer to this divisor as $D_{0,1}$. So we have

$$\frac{1}{12}[s] = \frac{[D_0]}{12} + [D_{0,1}]. \tag{2.88}$$

As in the genus zero discussion, it is essential that the divisors on which s vanishes are themselves moduli spaces of stable curves with marked points. Thus, the divisor D_0 on which Σ degenerates to a rational curve with double point is in the orbifold sense

$$D_0 = \tfrac{1}{2}\overline{\mathscr{M}}_{0,n+2}. \tag{2.89}$$

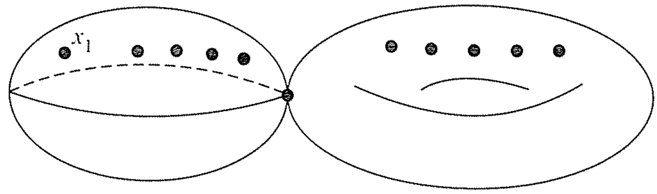


FIGURE 4. A GENUS ONE CURVE Σ WITH MARKED POINTS $x_1, x_2, \cdots, x_n$ DEGENERATING TO TWO COMPONENTS OF GENUS ZERO AND GENUS ONE RESPECTIVELY, WITH x_1 ON THE GENUS ZERO COMPONENT AND AN ARBITRARY DISTRIBUTION OF THE OTHER POINTS.

The $n+2$ marked points here are the n that were present originally and the two copies of the double point on the normalization of Σ; and the factor of $\frac{1}{2}$ arises because the two copies of the double point have no preferred ordering, but we have defined $\overline{\mathscr{M}}_{0,n}$ in terms of configurations of *ordered* points. The divisor $D_{0,1}$ is reducible, corresponding to the fact that the points $x_2, x_3, \cdots, x_n$ may be distributed on the two branches in an arbitrary fashion. For every decomposition $S = X \cup Y$ of the set $S = \{2, 3, \cdots, n\}$ as a union of disjoint subsets X and Y, $D_{0,1}$ has a component $D_{0,1;X,Y}$ in which precisely the x_j, $j \in Y$, are on the genus one branch. $D_{0,1;X,Y}$ is a copy of $\overline{\mathscr{M}}_{2+n_X} \times \overline{\mathscr{M}}_{1+n_Y}$, where n_X and n_Y are the cardinalities of X and Y, respectively. Writing $\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle = (\tau_{d_1-1}\tau_{d_2}\cdots\tau_{d_n}, [s])/12$ and using (2.88) and the facts just indicated, we get

$$\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_1 = \sum_{S=X\cup Y} \langle \tau_{d_1-1}\prod_{j\in X}\tau_{d_j}\cdot\tau_0\rangle_0 \cdot \langle \tau_0 \cdot \prod_{j\in Y}\tau_{d_j}\rangle_1 + \frac{1}{24}\langle \tau_{d_1-1}\tau_{d_2}\cdots\tau_{d_n}\cdot\tau_0^2\rangle_0. \tag{2.90}$$

By reasoning exactly analogous to that which led from (2.69) to (2.72), the totality of equations (2.90) is equivalent to a single statement about generating functionals,

$$\langle\langle \tau_n \rangle\rangle_1 = \langle\langle \tau_{n-1}\tau_0\rangle\rangle_0 \langle\langle \tau_0\rangle\rangle_1 + \tfrac{1}{24}\langle\langle \tau_{n-1}\tau_0\tau_0\rangle\rangle_0. \tag{2.91}$$

One can also deduce a sort of genus one analog of (2.74), namely

$$F_1 = \tfrac{1}{24}(\ln(\langle\langle \tau_0^3\rangle\rangle_0) - 2). \tag{2.92}$$

Indeed, this equation holds on the small phase space (where $\langle \tau_0^3\rangle_0 = 1$; the -2 in (2.92) comes from (2.18)). Its derivatives with respect to the t_n

can, inductively, be shown to vanish, using (2.91) and the genus zero recursion relations; this establishes that (2.92) holds on the full phase space. By differentiating (2.92) with respect to the t_n's, one can get formulas expressing $\langle\langle\tau_n\rangle\rangle_1$, $\langle\langle\tau_n\tau_m\rangle\rangle_1$, etc., in terms of genus zero quantities (which, using (2.74), can all be expressed in terms of the $\langle\langle\tau_0^n\rangle\rangle_0$.

In §3, we will generalize this story by introducing a rather general "target space" M, and see that a considerable amount of the structure that we have described generalizes.

3. Coupling to sigma models

The analogy of the problem that we have been studying to ordinary string theory is greatly strengthened if one couples the purely "gravitational" problem to "topological sigma models." (These correspond to a mathematical problem studied by Floer, Gromov, and others [27], [34]. The relation of this problem to quantum field theory was suggested in [2]. A Lagrangian realization was found in [58], and was developed further in [8].)

Fix a compact Kähler manifold M and a Riemann surface Σ with a fixed complex structure, to begin with. (The Kähler condition on M can be weakened, as noted below, but this is not central for our purposes.) Let $\mathcal{S}_\lambda$ be the moduli space of holomorphic maps of Σ to M, of a fixed homotopy type λ, and let $\mathcal{S} = \bigcup_\lambda \mathcal{S}_\lambda$.

Let us begin with a few comments on different methods and goals in the mathematical and physical work in this area. In the mathematical literature, the goal has been to relax the Kähler condition. Intersection theory on the $\mathcal{S}_\lambda$ is usually regarded as the definition of the problem, and analytical problems involving these spaces are the crux of the matter. From this point of view, the theory has been developed for general compact symplectic manifolds; a symplectic structure determines an almost complex structure uniquely up to homotopy by requiring that the symplectic form is positive and of type $(1, 1)$, and a notion of almost holomorphic maps from Riemann surfaces exists for general almost complex manifolds. The main success of the mathematical theory has been to obtain exotic invariants of symplectic manifolds and to use them to prove theorems about such manifolds.

Physically, the starting point is not intersection theory on moduli space but the existence of an appropriate topological sigma model Lagrangian, which one attempts to quantize. The Lagrangian exists for arbitrary almost complex manifolds M, and one aims to develop the theory in this generality. From this point of view, a priori one is studying Feynman path integrals, not intersection theory on moduli space. However, there are formal arguments [58] which, when the $\mathcal{S}_\lambda$ are smooth and compact, give a reduction of the Feynman path integral to classical intersection theory. Those arguments break down when the behavior of the moduli spaces $\mathcal{S}_\lambda$ is bad, but one would expect the Feynman integral itself to provide a more general definition of the desired "intersection numbers" even in such a case. The goal of the physical discussion is to construct quantum field theories, to explore a possible "unbroken phase" of string theory, and, in the present context, to explore how much of the discussion of the last section has an analog.

Letting K denote the canonical line bundle of M, the content of the theory that we will be discussing depends very much on the sign of $c_1(K)$. If $c_1(K) > 0$ and the dimension of M is not very small, the theory will be rather dull since the formal dimensions of the moduli spaces will almost all be negative. A much more interesting situation arises when $c_1(K) < 0$ (a condition that singles out a much smaller class of Kähler manifolds, including $\mathbb{C}P^n$), since then the moduli spaces generally have positive formal dimension.

For $c_1(K) < 0$, one has nice finiteness conditions; any correlation function (as introduced in the next subsection) receives contributions only from finitely many components of moduli space, and it seems likely that all of the unknown functions that will be introduced later can be determined from computation of finitely many special cases. Also, for $c_1(K) < 0$, the topological sigma model if studied by conventional physical methods is "asymptotically free," and has much better properties.

The discussion in this section will be particularly informal. We will not attempt to determine the appropriate class of target spaces M, and we will assume that the moduli spaces $\mathcal{S}_\lambda$ behave favorably. We should also note that, although we will concentrate on models derived from Kahler manifolds, there are other classes of models that obey the same general conditions (such as the models derived from matrix chains, which we will consider in §4d, and much less well understood models associated with the two-dimensional analogs of Donaldson theory.)

We will limit ourselves to the case $\pi_1(M) = 0$ to avoid a number of questions that have not yet been elucidated.

3a. Correlation functions. Let $\mathcal{S}$ be the moduli space of holomorphic maps of $\varphi: \Sigma \to M$, and let $\Phi: \mathcal{S} \times \Sigma \to M$ be the corresponding "universal instanton." For x a point in Σ, let Φ_x be the restriction of Φ to $\mathcal{S} \times x$. Every cohomology class $\alpha \in H^*(M)$ determines a cohomology class $W_\alpha = \Phi_x^*(\alpha) \in H^*(\mathcal{S})$, which is obviously independent of x since any two points on Σ are cohomologous. (One can consider integral cohomology at this point, but we will eventually be thinking in terms of real cohomology.) If α is the Poincaré dual to a submanifold $H \subset M$, then W_α is Poincaré dual to $W_H = \{\varphi | \varphi(x) \in H\}$. One denotes the intersection numbers of the W_α as

$$\langle \mathcal{O}_{\alpha_1}(x_1)\mathcal{O}_{\alpha_2}(x_2)\cdots\mathcal{O}_{\alpha_n}(x_n)\rangle = (W_{\alpha_1} \wedge W_{\alpha_2} \wedge \cdots \wedge W_{\alpha_n}, \mathcal{S}). \tag{3.1}$$

As in (2.3), the motivation for the notation is that these intersection numbers can be represented as the expectation value of a product of local operators $\mathcal{O}_\alpha$, with respect to some Feynman path integral measure. In (3.1), the x_i are arbitrary distinct points in Σ, and $W_{\alpha_i} = \Phi_{x_i}^*(\alpha_i)$. To avoid cluttering this section with minus signs, we will assume that M has only even-dimensional cohomology so that the ordering of terms in (3.1) is immaterial.

Of course, (3.1) vanishes except for contributions from homotopy classes λ such that

$$\sum_i \dim \alpha_i = \dim \mathcal{S}_\lambda. \tag{3.2}$$

According to the Riemann-Roch formula, the virtual dimension of $\mathcal{S}_\lambda$ is

$$\dim_{\mathbb{C}} \mathcal{S}_\lambda = (1-g)\cdot \dim_{\mathbb{C}} M - (\varphi^*(c_1(K)), \Sigma), \tag{3.3}$$

where φ is any map of the homotopy type λ. Thus, for $c_1(K) < 0$, only finitely many $\mathcal{S}_\lambda$'s have the right dimension to contribute to (3.1), and there are no problems of infinite sums. In general, we would have to modify the discussion to keep track of the homotopy type. (The right modification is actually part of what we will do anyway later in forming the generating functional F, but we do not wish to introduce it now in an ad hoc fashion.)

Actually (for an appropriate class of target spaces M), the intersection numbers (3.1) are independent of the complex structure on Σ. Mathematically, this is a corollary of the existence of an appropriate moduli space $\mathcal{N}_{g,n}$ of pairs of objects (Σ, φ), where Σ is a stable curve of genus g with n marked points $x_1, \cdots, x_n$, and $\varphi: \Sigma \to M$ is a holomorphic map. Physically, one would deduce the independence of complex structure by using the BRST invariance of the Lagrangian.

3b. Topological field theory. In fact, these intersection numbers obey the axioms of a "topological quantum field theory," in a sense spelled out in detail in [3], adapted from Segal's axiomatization of conformal field theory [55]. These axioms require one to associate to every circle C a vector space V_C, and to every Riemann surface Σ bounding a collection of "incoming" circles C_i, $i \in X$, and "outgoing" circles C_j, $j \in Y$, a linear transformation $\Phi_\Sigma : \bigotimes_{i\in X} V_{C_i} \to \bigotimes_{j\in Y} V_{C_j}$. The main requirement (sketched in Figure 5) is that if Σ is obtained by joining the outgoing boundary of Σ_1 to the incoming boundary of Σ_2, then one wants

$$\Phi_\Sigma = \Phi_{\Sigma_2} \circ \Phi_{\Sigma_1}. \tag{3.4}$$

Physicists describe this by saying that "one can calculate the transition amplitude by summing over physical intermediate states." In addition to (3.4), one imposes a similar condition relating Φ_Σ and Φ'_Σ, where Σ' is obtained from Σ by cutting on a nonseparating cycle.

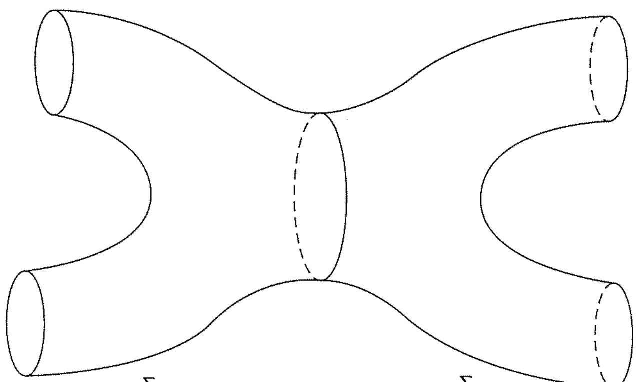


FIGURE 5. A KEY TOPOLOGICAL FIELD THEORY AXIOM IS A COMPOSITION LAW $\Phi_\Sigma = \Phi_{\Sigma_2} \circ \Phi_{\Sigma_1}$ THAT MUST HOLD IN THIS SITUATION.

To realize these axioms in the case at hand, one takes V_C for every circle C to be a copy of $H^*(M, \mathbb{R})$. This vector space has a natural metric given by Poincaré duality, so one need to distinguish incoming and outgoing circles. The metric will play an important role in what follows. If H_σ, $\sigma \in L$, is a basis for the real cohomology of M, then the metric is $\eta_{\sigma\tau} = (H_\sigma \wedge H_\tau, M)$. This is an invertible matrix whose inverse will be denoted by $\eta^{\sigma\tau}$.

If Σ is a surface bounded by circles $C_1, \cdots, C_n$, then the linear transformation Φ_Σ which is part of the topological field theory data is simply a vector in $\bigotimes_{i=1}^n V_{C_i} = \bigotimes_{i=1}^n H^*(M, \mathbb{R})$. This vector is given by the correlation function $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n} \rangle$. The only property that needs to be verified is the composition law (3.4) (and its analog for nonseparating cuts). We will investigate this in the next paragraph. To simplify notation, we will specialize the following discussion to the case of genus 0 with $n = 4$, which proves to play a special role. Other cases (and the rest of the topological field theory axioms) can be discussed similarly.

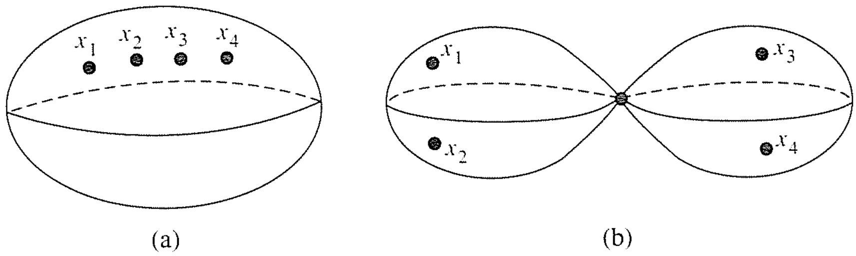


FIGURE 6. A CONFIGURATION OF FOUR POINTS $x_1, \cdots, x_4$ IN GENUS ZERO, AND ONE OF ITS THREE DEGENERATIONS.

The four point function $\langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\gamma(x_3)\mathcal{O}_\delta(x_4)\rangle$ is computed, as in Figure 6, by fixing a configuration of four points $x_1, \cdots, x_4$ on a curve Σ of genus zero, and evaluating an appropriate intersection on $\mathcal{S}_{0,4}$. Suppose that one degenerates the configuration of four points to the boundary of moduli space, where Σ decomposes into two components Σ_1 and Σ_2 which share a double point P. A holomorphic map $\varphi : \Sigma \to M$ is by definition a pair (φ_1, φ_2), where for $i = 1, 2$, φ_i is a holomorphic map $\Sigma_i \to M$, obeying $\varphi_1(P) = \varphi_2(P)$. If $\mathcal{S}, \mathcal{S}_1$, and $\mathcal{S}_2$ are the moduli spaces of holomorphic maps of Σ, Σ_1, and Σ_2 to M, then the condition $\varphi_1(P) = \varphi_2(P)$ defines a cycle X in $\mathcal{S}_1 \times \mathcal{S}_2$, and we can identify $\mathcal{S}$ with X. Since $X \subset \mathcal{S}_1 \times \mathcal{S}_2$, we can compute the four point function by counting intersections on $\mathcal{S}_1 \times \mathcal{S}_2$. Moreover, if x_1, x_2 lie on Σ_1 and x_3, x_4 lie on Σ_2, then we can think of $\mathcal{O}_\alpha, \mathcal{O}_\beta$ as representing classes $W_\alpha^{(1)}, W_\beta^{(1)}$ in $H^*(\mathcal{S}_1)$, and $\mathcal{O}_\gamma, \mathcal{O}_\delta$ as representing classes $W_\gamma^{(2)}, W_\delta^{(2)}$ in $H^*(\mathcal{S}_2)$, which are then pulled back to $H^*(\mathcal{S}_1 \times \mathcal{S}_2)$. We then have

$$(3.5) \qquad \langle \mathcal{O}_\alpha\mathcal{O}_\beta\mathcal{O}_\gamma\mathcal{O}_\delta\rangle = (W_\alpha^{(1)} \wedge W_\beta^{(1)} \wedge W_\gamma^{(2)} \wedge W_\delta^{(2)} \wedge [X], \mathcal{S}_1 \times \mathcal{S}_2).$$

In the last step we are thinking of $[X]$, the Poincaré dual to X, as a class in $H^*(\mathcal{S}_1 \times \mathcal{S}_2)$. To put this in a more useful form, we reexpress $[X]$ by using the Kunneth decomposition of the diagonal $\Delta \subset M \times M$, which in real cohomology reads

$$(3.6) \qquad [\Delta] = \sum_{\sigma, \tau \in L} \eta^{\sigma\tau} H_\sigma \times H_\tau.$$

Here $[\Delta]$ is the Poincaré dual of the diagonal in $M \times M$. Correspondingly, one has $[X] = \sum_{\sigma,\tau} \eta^{\sigma\tau} W_\sigma^{(1)} \times W_\tau^{(2)}$, so (3.5) becomes

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \mathcal{O}_\delta \rangle = \sum_{\sigma\tau} \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\sigma \rangle \eta^{\sigma\tau} \langle \mathcal{O}_\tau \mathcal{O}_\gamma \mathcal{O}_\delta \rangle. \tag{3.7}$$

This is precisely the topological field theory axiom which we had aimed to explain, and though we have focused on the four point function in genus zero, the general case is no different.

The four point function in genus zero has, however, a special significance. Since the left-hand side is symmetric in α, β, γ, and δ, but the right-hand side does not obviously possess this symmetry, we deduce that

$$\sum_{\sigma,\tau} \eta^{\sigma\tau} \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\sigma \rangle \langle \mathcal{O}_\tau \mathcal{O}_\gamma \mathcal{O}_\delta \rangle = \sum_{\sigma,\tau} \eta^{\sigma\tau} \langle \mathcal{O}_\alpha \mathcal{O}_\delta \mathcal{O}_\sigma \rangle \langle \mathcal{O}_\tau \mathcal{O}_\gamma \mathcal{O}_\beta \rangle. \tag{3.8}$$

This equation means that if we define

$$f_{\alpha\beta\gamma} = \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle \tag{3.9}$$

and

$$f_{\alpha\beta}{}^{\delta} = \eta^{\gamma\delta} f_{\alpha\beta\gamma}, \tag{3.10}$$

then the formula

$$\mathcal{O}_\alpha \mathcal{O}_\beta = \sum_\gamma f_{\alpha\beta}{}^{\gamma} \mathcal{O}_\gamma \tag{3.11}$$

defines a commutative, associative multiplication law on $H^*(M, \mathbb{R})$ (which in addition is compatible with the metric η in the sense that $\eta(A, BC) = \eta(BA, C)$ for any A, B, C; this amounts to the statement, clear from the definition, that $f_{\alpha\beta\gamma}$ is completely symmetric). This ring has an identity, namely $\mathcal{O}_1$, where $1 \in H^0(M, \mathbb{R})$ is the identity in the ring $H^*(M, \mathbb{R})$. (Conversely, it can be shown in an elementary fashion [17], [59] that from such a ring structure one can reconstruct a two-dimensional topological field theory. In higher dimensions, topological field theories are not classified so easily.)

By our definitions, (3.9) is to be computed by summing over all homotopy classes of holomorphic maps $\Sigma \to M$, with Σ a curve of genus zero. A particularly simple role is played by the null-homotopic maps. This component of $\mathcal{S}$ can be identified with M itself, since a holomorphic map that is also homotopic to zero is constant. Unwinding the definitions, one finds that if one considers only the null-homotopic maps in computing $f_{\alpha\beta\gamma}$, then one recovers the classical ring structure on $H^*(M, \mathbb{R})$. This is a graded ring in which all elements of positive degree are nilpotent. The higher homotopy classes of holomorphic maps contribute in such a way as to deform $H^*(M, \mathbb{R})$ to a structure which tends to be less degenerate, especially if $c_1(K) > 0$, and in general is no longer graded. For example, for $M = \mathbf{CP}^n$, the classical cohomology ring is $\mathbb{R}[x]/(x^{n+1})$, where x is a generator in degree 2, but it is straightforward to compute that the "quantum corrections," that is, the contributions of the higher homotopy classes, deform this to a "quantum cohomology ring" which is isomorphic to $\mathbb{R}[x]/(x^{n+1}-1)$. If $c_1(M)$, as an integral cohomology class, is divisible by r, then the quantum cohomology ring is graded by $\mathbb{Z}/2r\mathbb{Z}$, as one can see from the Riemann-Roch formula for the dimensions of the $\mathcal{N}_\alpha$'s.

We will later generalize this in the following perhaps surprising way. We will find a function F on the vector space $H^*(M, \mathbb{R})$, with the property that at any point in $H^*(M, \mathbb{R})$, the third derivatives of F are the structure constants of a commutative associative algebra. Thus, if y^σ, $\sigma \in L$, are affine coordinates for $H^*(M, \mathbb{R})$, and

$$f_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial y^\alpha \partial y^\beta \partial y^\gamma} \tag{3.12}$$

(F will not be a cubic function, in general, so the f's are not constant) and $f_{\alpha\beta}{}^\delta = \eta^{\gamma\delta} f_{\alpha\beta\gamma}$, then at *any* point in $H^*(M, \mathbb{R})$, the formula

$$\mathcal{O}_\alpha \mathcal{O}_\beta = f_{\alpha\beta}{}^\gamma \mathcal{O}_\gamma \tag{3.13}$$

defines a commutative, associative (and of course metric compatible) algebra. This is equivalent to saying that F obeys the overdetermined system of equations

$$\eta^{\sigma\tau} \frac{\partial^3 F}{\partial y^\alpha \partial y^\beta \partial y^\sigma} \cdot \frac{\partial^3 F}{\partial y^\tau \partial y^\gamma \partial y^\delta} = \eta^{\sigma\tau} \frac{\partial^3 F}{\partial y^\alpha \partial y^\gamma \partial y^\sigma} \cdot \frac{\partial^3 F}{\partial y^\tau \partial y^\beta \partial y^\delta}, \tag{3.14}$$

for which we will find for each M a canonical solution.

3c. Coupling to gravity. This structure becomes considerably more interesting if we let the complex structure on Σ vary. Thus, the basic object of study will henceforth be the (compactified) moduli space $\mathcal{N}_{g,n}$ of stable pairs (Σ, φ), where Σ is a curve of genus g with n distinct, ordered marked points $x_1, \cdots, x_n$ and $\varphi: \Sigma \to M$ is a holomorphic map. It is now possible to combine the two constructions that we have considered in this paper. On the one hand, each marked point x_i has a complex cotangent bundle $T^*\Sigma|_{x_i}$, and these vary over $\mathcal{N}_{g,n}$ to give n line bundles $\mathcal{L}_{(i)}$. On the other hand, let $\mathcal{C}\mathcal{N}_{g,n}$ be the "universal curve" over $\mathcal{N}_{g,n}$. We can regard the marked point $x_i: \mathcal{N}_{g,n} \to \mathcal{C}\mathcal{N}_{g,n}$ as a section of the universal curve. If $\Phi: \mathscr{C}\mathscr{N}_{g,n} \to M$ is the "universal instanton," then for $i = 1, \cdots, n$ one has maps $\Phi \circ x_i: \mathscr{N}_{g,n} \to M$, and this gives for each i a natural map from $\alpha \in H^*(M)$ to $W_{i,\alpha} = (\Phi \circ x_i)^*(\alpha) \in H^*(\mathscr{N}_{g,n})$. Corresponding to each of the marked points x_i, we can thus define natural cohomology classes in $H^*(\mathscr{N}_{g,n})$ of the form

$$c_1(\mathscr{L}_{(i)})^d \cdot W_{i,\alpha}. \tag{3.15}$$

Here d is a nonnegative integer, and we may as well consider α to run over a finite set corresponding to a basis L of $H^*(M, \mathbb{R})$. Symbolically, as in §2, we represent these classes by "quantum field theory operators" $\tau_d(\mathscr{O}_\alpha)$,[3] or, for brevity, simply $\tau_{d,\alpha}$, and we write

$$\begin{aligned}&\langle \tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\cdots\tau_{d_n,\alpha_n}\rangle \\ &\quad = (c_1(\mathscr{L}_{(1)})^{d_1} \wedge W_{1,\alpha_1} \cdot \cdots \cdot c_1(\mathscr{L}_{(n)})^{d_n} \wedge W_{n,\alpha_n}, \mathscr{N}_{g,n}).\end{aligned} \tag{3.16}$$

Again as in §2, we introduce formal variables ("coupling constants") t_r^α, $r = 0, 1, 2, \cdots$, $\alpha \in L$, and we define the "generating functional"

$$F(t_r^\alpha) = \langle e^{\Sigma_{r,\alpha} t_r^\alpha \tau_{r,\alpha}} \rangle. \tag{3.17}$$

More concretely, this is to be

$$F(t_r^\alpha) = \sum_{\{n_{r,\alpha}\}} \prod_{r,\alpha} \frac{(t_r^\alpha)^{n_{r,\alpha}}}{n_{r,\alpha}!} \cdot \left\langle \prod_{r,\alpha} \tau_{r,\alpha}^{n_{r,\alpha}} \right\rangle, \tag{3.18}$$

where the $n_{r,\alpha}$ are arbitrary collections of nonnegative integers, almost all zero, labeled by r, α. The correlation function on the right-hand side is to be summed over all values of the genus g of Σ, and all homotopy classes of holomorphic maps $\varphi: \Sigma \to M$. However, a nonzero contribution arises only if the genus and the homotopy class obey an appropriate dimensional condition. If $d(\alpha)$ is the dimension of the cohomology class $\alpha \in H^*(M, \mathbb{R})$, then the dimensional condition is now, from the Riemann-Roch formula,

$$6g - 6 + (2g-2)\dim_{\mathbb{C}} M - 2\int_\Sigma \varphi^*(c_1(K)) = \Sigma_{r,\alpha} n_{r,\alpha}(2r - 2 + d(\alpha)). \tag{3.19}$$

For $c_1(K) < 0$, this condition ensures that the coefficient of a given monomial $\prod (t_r^\alpha)^{n_{r,\alpha}}$ receives a contribution only from finitely many homotopy classes of maps; in this case F can be regarded as a formal power series,

[3] By analogy with standard terminology in conformal field theory, one sometimes refers to the $\mathscr{O}_\alpha = \tau_0(\mathscr{O}_\alpha)$ as "primaries" and the $\tau_d(\mathscr{O}_\alpha)$, $d > 0$, as "descendants." If $m = \dim H^*(M, \mathbb{R})$, the model we are discussing then "has m primaries."

as in §2. (Otherwise, F must be expanded as a series in $e^{\sum_i n_i y_i}$, where the n_i are integers and $y_i = t_0^{\alpha_i}$; here the α_i run over a basis of the *two-dimensional* cohomology of M.)

As in §2, it is convenient to write

$$\langle\langle \tau_{d_1}(\alpha_1)\cdots\tau_{d_n}(\alpha_n)\rangle\rangle = \frac{\partial}{\partial t_{d_1}^{\alpha_1}}\cdot\cdots\cdot\frac{\partial}{\partial t_{d_n}^{\alpha_n}}\cdot F(t_r^\alpha). \tag{3.20}$$

Thus, if one sets all $t_r^\alpha = 0$, the symbol $\langle\langle\ \rangle\rangle$ reduces to $\langle\ \rangle$. As a special case of (3.20), we write $\langle\langle 1\rangle\rangle = F$.

Our goal in the rest of this section is to show that at least part of the discussion of §2 generalizes in this more elaborate situation.

3d. Analogs of the string and KdV equations. First of all, one important part of the discussion of §2, namely the string equation (2.22), generalizes straightforwardly. We recall that the origin of that equation was that one of the operators (namely τ_0) correspond to the identity, a zero-dimensional cohomology class. The analogous object in the present context is $\tau_0(\mathscr{O}_1)$ (or simply $\tau_{0,1}$), where $\mathscr{O}_1$ corresponds to $1 \in H^0(M, \mathbb{R})$. We consider a general genus g correlation function

$$\left\langle \tau_{0,1}\cdot\prod_{i=1}^n \tau_{d_i,\alpha_i}\right\rangle \tag{3.21}$$

of this operator with n other operators. As in §2, this is to be evaluated by integrating a certain cohomology class over $\mathscr{N}_{g,n+1}$; if the cohomology class in question were a pullback from $\mathscr{N}_{g,n}$, then (3.21) would vanish. A nonzero result comes, again, only from the second term in (2.35). Treating this in a similar way, we arrive at the generalized string equation:

$$\frac{\partial F}{\partial t_0^1} = \frac{1}{2}\eta_{\sigma\tau}t_0^\sigma t_0^\tau + \sum_{i=0}^\infty\sum_\alpha t_{i+1}^\alpha\frac{\partial}{\partial t_i^\alpha}F. \tag{3.22}$$

It is also possible to find a general analog of (2.49), but we will not enter into this here.

In particular, it follows from the string equation that if all $t_r^\alpha = 0$ for $r > 0$, then

$$\langle\langle \tau_{0,1}\tau_{0,\alpha}\rangle\rangle = \eta_{\alpha\beta}t_0^\beta, \tag{3.23}$$

which will be useful later.

In the rest of this section, we will see that at least in genus zero and genus one, one can find analogs of the KdV flows. To begin with, we consider a genus zero curve Σ with marked points $x_1, \cdots, x_n$, and a general correlation function

$$\langle\tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\cdots\tau_{d_{n-1},\alpha_{n-1}}\tau_{d_n,\alpha_n}\rangle. \tag{3.24}$$

This can be treated precisely as we treated (2.65) in §2. Symbolically, $\tau_{d_1,\alpha_1} = [s]\cdot\tau_{d_1-1,\alpha_1}$, where $[s]$ is the divisor of a section s of $\mathscr{L}_{(1)}$. For s we pick the same section that we used in §2, namely

$$s = dx_1\cdot\left(\frac{1}{x_1-x_{n-1}}-\frac{1}{x_1-x_n}\right). \tag{3.25}$$

Its divisor again consists of certain degenerate configurations in which Σ has two branches Σ_1 and Σ_2. As a holomorphic map $\varphi:\Sigma\to M$ is a pair $\varphi_i:\Sigma_i\to M$ obeying a condition $\varphi_1(P)=\varphi_2(P)$ at the node P, we must again, as in the derivation of (3.7), carry out the Kunneth decomposition of the diagonal in $M\times M$ to express this condition in terms of the cohomology classes we are using. Upon thus modifying the derivation of (2.69), we arrive at the generalization of that equation, namely

$$\begin{aligned}\langle\tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}&\cdots\tau_{d_{n-1},\alpha_{n-1}}\tau_{d_n,\alpha_n}\rangle\\ &=\sum_{\sigma,\tau}\sum_{S=X\cup Y}\left\langle\tau_{d_1-1,\alpha_1}\prod_{j\in X}\tau_{d_j,\alpha_j}\cdot\tau_{0,\sigma}\right\rangle\\ &\quad\cdot\eta^{\sigma\tau}\cdot\left\langle\tau_{0,\tau}\cdot\prod_{j\in Y}\tau_{d_j,\alpha_j}\cdot\tau_{d_{n-1},\alpha_{n-1}}\tau_{d_n,\alpha_n}\right\rangle.\end{aligned} \tag{3.26}$$

Just as in the derivation of (2.72), we may now assert that the totality of equations (3.26) for $n\ge 3$ is actually equivalent to a single relation among the genus zero generating functions, namely

(3.27)
$$\langle\langle\tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\tau_{d_3,\alpha_3}\rangle\rangle_0=\sum_{\sigma,\tau}\langle\langle\tau_{d_1-1,\alpha_1}\tau_{0,\sigma}\rangle\rangle_0\cdot\eta^{\sigma\tau}\langle\langle\tau_{0,\tau}\tau_{d_2,\alpha_2}\tau_{d_3,\alpha_3}\rangle\rangle_0.$$

Now, (3.27) has the following consequence. The derivative of (3.27) with respect to $t_{d_4}^{\alpha_4}$ is the equation

$$\begin{aligned}\langle\langle\tau_{d_1,\alpha_1}&\tau_{d_2,\alpha_2}\tau_{d_3,\alpha_3}\tau_{d_4,\alpha_4}\rangle\rangle_0\\ &=\sum_{\sigma\tau}\langle\langle\tau_{d_1-1,\alpha_1}\tau_{d_4,\alpha_4}\tau_{0,\sigma}\rangle\rangle_0\cdot\eta^{\sigma\tau}\langle\langle\tau_{0,\tau}\tau_{d_2,\alpha_2}\tau_{d_3,\alpha_3}\rangle\rangle_0\\ &\quad+\sum_{\sigma\tau}\langle\langle\tau_{d_1-1,\alpha_1}\tau_{0,\sigma}\rangle\rangle_0\cdot\eta^{\sigma\tau}\langle\langle\tau_{0,\tau}\tau_{d_2,\alpha_2}\tau_{d_3,\alpha_3}\tau_{d_4,\alpha_4}\rangle\rangle_0.\end{aligned} \tag{3.28}$$

The left-hand side of (3.28) is symmetric under permutations of (d_2, α_2), (d_3, α_3), and (d_4, α_4), but the right-hand side is not. Therefore we can infer that

$$\sum_{\sigma,\tau}\langle\langle\tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\tau_{0,\sigma}\rangle\rangle_0\eta^{\sigma\tau}\langle\langle\tau_{0,\tau}\tau_{d_3,\alpha_3}\tau_{d_4,\alpha_4}\rangle\rangle_0 = \sum_{\sigma,\tau}\langle\langle\tau_{d_1,\alpha_1}\tau_{d_4,\alpha_4}\tau_{0,\sigma}\rangle\rangle_0\cdot\eta^{\sigma\tau}\langle\langle\tau_{0,\tau}\tau_{d_2,\alpha_2}\tau_{d_3,\alpha_3}\rangle\rangle_0. \tag{3.29}$$

This amounts to the statement that

$$\eta^{\sigma\tau}\frac{\partial^3 F}{\partial t_{d_1}^{\alpha_1}\partial t_{d_2}^{\alpha_2}\partial t_0^{\sigma}}\frac{\partial^3 F}{\partial t_0^{\tau}\partial t_{d_3}^{\alpha_3}\partial t_{d_4}^{\alpha_4}} = \eta^{\sigma\tau}\frac{\partial^3 F}{\partial t_{d_1}^{\alpha_1}\partial t_{d_3}^{\alpha_3}\partial t_0^{\sigma}}\frac{\partial^3 F}{\partial t_0^{\tau}\partial t_{d_2}^{\alpha_2}\partial t_{d_4}^{\alpha_4}}. \tag{3.30}$$

If all d_i are set to zero, this reduces to (3.14), so the free energy F is the promised function whose third derivatives with respect to the $t_{0,\alpha}$ at any point define a commutative, associative algebra.

Actually, (3.30) is more than was promised. Thus, (3.14) was formulated as an equation for a function defined on the finite-dimensional vector space $H^*(M, \mathbb{R})$, but in (3.30) we have a function F of infinitely many variables t_r^α. To reduce to (3.14), in addition to setting $d_i = 0$, we restrict F to a finite-dimensional subspace characterized to $t_r^\alpha = c_r^\alpha$, $r \geq 1$ (where the c_r^α are arbitrary constants). Thus, F is really a family of solutions of (3.14) depending on the c_r^α as parameters.

To give a very simple concrete example, let $M = \mathbf{CP}^1$. $H^*(\mathbf{CP}^1, \mathbb{R})$ is two dimensional, generated by a zero-form 1 and a two-form ω. If $x = t_0^1$ and $y = t_0^\omega$, then, from §2.3 of [19], the function F on the small phase space is $F(x, y) = x^2y + e^y$.

Equation (3.27) can be given an interpretation analogous to (2.74). We introduce the infinite-dimensional affine "phase space" $\mathbf{A}^\infty$ of the t_r^α, and the "small phase space" characterized by $t_r^\alpha = 0$, for $r \geq 1$. Thus, the small phase space is a copy of $H^*(M, \mathbb{R})$, and has coordinates t_0^α. On the small phase space, the genus zero two point functions $\langle\langle\tau_{n,\alpha}\tau_{m,\beta}\rangle\rangle_0$ are functions of the t_0^γ:

$$\langle\langle\tau_{n,\alpha}\tau_{m,\beta}\rangle\rangle_0 = G_{n,\alpha;m,\beta}(t_0^\gamma). \tag{3.31}$$

According to (3.23), if we define

$$U_\alpha = \langle\langle\tau_{0,1}\tau_{0,\alpha}\rangle\rangle_0, \qquad U^\alpha = \eta^{\alpha\beta}U_\beta, \tag{3.32}$$

then we can rewrite this as

$$\langle\langle\tau_{n,\alpha}\tau_{m,\beta}\rangle\rangle_0 = G_{n,\alpha;m,\beta}(U^\gamma). \tag{3.33}$$

Precisely the argument that led to (2.74) can now be repeated to show that (3.33) is valid, without modification, on the full phase space, not just the small phase space.

As in §2, it is possible at this stage to reinterpret the string equation (3.22). Differentiating (3.22) with respect to $t_{0,\alpha}$, we get

$$U_\alpha = \eta_{\alpha\beta} t_0^\beta + \sum_{i=0}^{\infty} \sum_\beta t_{i+1}^\beta \langle\langle \tau_{i,\beta} \tau_{0,\alpha} \rangle\rangle \tag{3.34}$$

for all α. Using (3.33), we can rewrite this as

$$U_\alpha = \eta_{\alpha\beta} t_0^\beta + \sum_{i=0}^{\infty} \sum_\beta t_{i+1}^\beta G_{i,\beta;0,\alpha}(U^\gamma). \tag{3.35}$$

If $H^*(M, \mathbb{R})$ is m dimensional, this is a system of m equations for the m unknowns U_α; these equations (insofar as their solution is unique, which is actually true in an open set in phase space) determine the U_α's as functions of the parameters t_i^β.

It is interesting to note that the equations (3.35) can be given an interpretation as the equations for a critical point (with respect to the U's) of a certain generalized potential $W(U_\sigma; t_i^\beta)$.[4] To see this, note that on the small phase space there are some functions $G_{i,\alpha}(t_0^\gamma)$ such that

$$\langle \tau_{i,\alpha} \rangle = G_{i,\alpha}(t_0^\gamma). \tag{3.36}$$

(As far as we know, (3.36) does not extend in any nice way on the full phase space.) Comparing (3.36) and (3.33), we see that

$$G_{i,\alpha;0,\beta}(t_0^\gamma) = \frac{\partial G_{i,\alpha}(t_0^\gamma)}{\partial t_0^\beta}. \tag{3.37}$$

This identity of course remains valid if the arguments of the functions on the left and right are U^γ instead of t_0^γ. So (3.35) can be written

$$U_\alpha = \eta_{\alpha\beta} t_0^\beta + \sum_{i=0}^{\infty} \sum_\beta t_{i+1}^\beta \frac{\partial}{\partial U^\alpha} G_{i,\beta}(U^\sigma). \tag{3.38}$$

This is tantamount to the critical point equation

$$\frac{\partial W}{\partial U^\sigma} = 0, \tag{3.39}$$

[4] In the context of matrix chains, of which we will give a very brief sketch in §4d, Ginsparg, Goulian, Plesser, and Zinn-Justin [33] and Jevicki and Yoneya [38] have shown that the string equations are the variational equations of an appropriate Lagrangian. The argument that we are about to give shows that, at least in genus zero, this is true for arbitrary topological field theories coupled to topological gravity.

where

$$W = -\frac{1}{2}U^\alpha U_\alpha + t_0^\beta U_\beta + \sum_{i=0}^{\infty}\sum_\beta t_{i+1}^\beta G_{i,\beta}. \tag{3.40}$$

Conjectured generalization. What will become of the structure that we have found here for genus zero when one considers contributions of higher genus? The analogy with the results of matrix models and the structure of the generalized KdV equations suggests the following. Use the symbols $\dot{U}_\sigma$, $\ddot{U}_\sigma$, etc., to denote derivatives of U_σ with respect to t_0^1, so

$$\dot{U}_\sigma = \frac{\partial U}{\partial t_0^1} = \langle\langle \tau_{0,1}^2 \tau_{0,\sigma}\rangle\rangle, \qquad \ddot{U}_\sigma = \frac{\partial^2 U_\sigma}{\partial (t_0^1)^2} = \langle\langle \tau_{0,1}^3 \tau_{0,\sigma}\rangle\rangle, \tag{3.41}$$

and so on. Let us consider $U, \dot{U}, \ddot{U}, \dddot{U}$, etc., to be of degree 0, $1, 2, 3, \cdots$. By a differential function of degree k we mean a function $G(U, \dot{U}, \ddot{U}, \dddot{U}, \cdots)$ which is of degree k in that sense. (Thus, in particular, such a function has only a polynomial dependence on $\dot{U}, \ddot{U}, \dddot{U}, \cdots$, but its dependence on U need not be polynomial.)

Let us recall now that the free energy F_g has an expansion $F = \sum_{g=0}^\infty F_g$, where F_g is the genus g contribution. Similarly, all other generating functionals that we have considered, such as $U_\sigma = \langle\langle \tau_{0,1}\tau_{0,\sigma}\rangle\rangle$, or $\langle\langle \tau_{n,\alpha}\tau_{m,\beta}\rangle\rangle$, etc., are derivatives of F and in particular have similar expansions. More generally, we may be interested in products of generating functionals. Such a product of course also has a genus expansion, which explicitly is

$$(\langle\langle A\rangle\rangle\langle\langle B\rangle\rangle)_g = \sum_{g'=0}^{g} \langle\langle A\rangle\rangle_{g'}\langle\langle B\rangle\rangle_{g-g'}. \tag{3.42}$$

Then the following conjecture is a tempting generalization of (3.33):

Conjecture. For every $g \geq 0$, there are differential functions $G_{m,\alpha;n,\beta}(U_\sigma, \dot{U}_\sigma, \ddot{U}_\sigma, \cdots)$ of degree $2g$ such that

$$\langle\langle \tau_{n,\alpha}\tau_{m,\beta}\rangle\rangle = G_{m,\alpha;n,\beta}(U_\sigma, \dot{U}_\sigma, \ddot{U}_\sigma, \cdots) \tag{3.43}$$

up to and including terms of genus g.

To explain the rationale for the conjecture, let me point out that for $M =$ a point, it is a consequence of the main conjecture of §2, since the KdV hierarchy has the stated property. Indeed, the KdV hierarchy has a stronger property— $G_{m,\alpha;n,\beta}$ is a differential function of degree at most $2(m+n)$. This means that for any given correlation function of fixed m and n, the $G_{m,\alpha;n,\beta}$ are differential functions of finite degree even for

$g \to \infty$. To put it more forcefully, this means that in the KdV case, there are differential functions $G_{m,\alpha;n,\beta}$ of finite degree (depending on m and n) such that (3.43) is true *exactly*, not just up to some genus g. But the conjecture stated above permits the possibility that for general M, in going to higher and higher genus, one will have to add to the $G_{m,\alpha;n,\beta}$ terms of higher and higher degree.

The conjecture is an attempt to interpret (3.33), which hold for a very large class of target manifolds M, as the genus zero approximation to a systematic picture that would hold in arbitrary genus g, without proposing that there is an integrable hierarchy of differential equations associated with every compact Kähler manifold M or even every such manifold in a large class. Apart from the case $M =$ a point, $g \leq 3$, the only situation in which we know the conjecture is true is the following. If the dimension of $H^*(M, \mathbb{R})$ is 2, then the conjecture can be verified in genus one (in a tedious and unilluminating way, which we will not present here) using the formulas of the next subsection. (In practice, $\dim H^*(M, \mathbb{R}) = 2$ only for $M = \mathrm{CP}^1$, but the reasoning applies also to an arbitrary model that obeys the general properties assumed here and has "two primaries" in a sense described in a previous footnote.)

To make the conjecture sound a little more plausible, let me point out the following reinterpretation of the above genus zero equations. We have

$$
\begin{aligned}
\frac{\partial}{\partial t_n^\alpha} U_\sigma &= \langle\langle \tau_{0,1} \tau_{n,\alpha} \tau_{0,\sigma} \rangle\rangle = \frac{\partial}{\partial t_0^1} \langle\langle \tau_{n,\alpha} \tau_{0,\sigma} \rangle\rangle \\
&= \frac{\partial}{\partial t_0^1} G_{n,\alpha;0,\sigma}(U_\gamma) = \frac{\partial}{\partial t_0^1} \frac{\partial}{\partial U^\sigma} G_{n,\alpha}(U_\gamma).
\end{aligned} \tag{3.44}
$$

In the last two steps, (3.33) and (3.37) have been used.

Now, (3.44) has the following interpretation. Think of the U_σ as functions of $x = t_0^1$, and introduce Poisson brackets, with

$$
\{U_\sigma(x), U_\tau(x')\} = \eta_{\sigma\tau} \frac{d}{dx} \delta(x - x'). \tag{3.45}
$$

These Poisson brackets correspond to one of the two symplectic structures of the KdV equations. Introduce the "Hamiltonians"

$$
H_{n,\alpha} = \int dx\, G_{n,\alpha}(U_\gamma). \tag{3.46}
$$

Then (3.44) can be regarded as the Hamiltonian equation of motion:

$$
\frac{\partial U_\sigma}{\partial t_n^\alpha} = \{U_\sigma, H_{n,\alpha}\}. \tag{3.47}
$$

Thus, the genus zero correlation functions for any M (of an appropriate type to justify the above considerations) can be described by a family of commuting Hamiltonian flows! In genus zero, the Hamiltonian densities $G_{n,\alpha}$ are simply functions of the U's. A somewhat sharpened version of the above conjecture would assert that in a genus g approximation, the correlation functions are generated by a system of Hamiltonian flows with the Hamiltonian densities being differential functions of degree $2g$ (which Poisson commute up to terms of degree $2g+2$). If the conjecture is true, then one would expect, upon taking the limit as $g \to \infty$ in a suitable sense, to obtain commuting Hamiltonians that would no longer be differential functions of finite order, so that the commuting Hamiltonian flows would be governed by integral equations rather than differential equations.

The KdV flows have the much stronger property of being bi-Hamiltonian; that is, they preserve two different symplectic structures. This is closely related to the fact that the differential functions in equation (3.43) have a degree that is bounded by $2(m+n)$, independent of g, and thus, one really gets commuting differential operators of finite order. (The other symplectic structure is also related to a kind of Virasoro algebra.) We do not know of any evidence for a second symplectic structure playing a role for general M.

Genus one structure. We will now much more briefly discuss how the genus one equations of §2 generalize in the present situation. The generalization of (2.91) to include a target space M can be obtained by reasoning that should by now be familiar, giving

(3.48)
$$\begin{aligned}\langle\langle\tau_{n,\alpha}\rangle\rangle_1 \\ = \sum_{\sigma,\tau}\langle\langle\tau_{n-1,\alpha}\tau_{0,\sigma}\rangle\rangle_0\eta^{\sigma\tau}\langle\langle\tau_{0,\tau}\rangle\rangle_1 + \frac{\eta^{\sigma\tau}}{24}\langle\langle\tau_{n-1,\alpha}\tau_{0,\sigma}\tau_{0,\tau}\rangle\rangle_0.\end{aligned}$$

It is also possible to obtain an analog of (2.92). To this aim, introduce the matrix

(3.49)
$$M_{\sigma\tau} = \langle\langle\tau_{0,1}\tau_{0,\sigma}\tau_{0,\tau}\rangle\rangle_0.$$

Define a function $E(t_0^\alpha)$ by requiring that the genus one part of the free energy, on the small phase space, is

(3.50)
$$F_1 = \tfrac{1}{24}\ln\det M + E(t_0^\alpha).$$

Then we claim that the genus one free energy, on the full phase space, is

(3.51)
$$F_1 = \tfrac{1}{24}\ln\det M + E(U^\alpha).$$

Since (3.51) is valid on the small phase space (by definition of E), it suffices to prove that the repeated derivatives of (3.51) with respect to the $t_{n,\alpha}$ all vanish. This can be proved inductively using (3.48).[5]

As we have already mentioned, with the above equations and some patience, one can verify the conjecture of the last subsection for genus one in the case $\dim H^*(M) = 2$. We do not know if this restriction is necessary.

4. Introduction to matrix models

Our goal in this section is to give a relatively self-contained but far from complete introduction to the matrix model approach to two-dimensional gravity and some of the remarkable results obtained recently by Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [3], [25], [35]. In §4a, we explain the physical problem and the strategy for discretizing it; §4b is an explanation of how the discretized problem can be interpreted in terms of matrix integrals, and in §4c, the matrix integrals are described in terms of (discrete analogs of) the KdV flows. The reader who is willing to take it on faith that the problem of interest is to compute matrix integrals $\int (dM)\exp(-\mathrm{tr}(V(M)))$ can read §4c without understanding all the previous details.

4a. The physical problem. Let Σ be a smooth two-dimensional surface of genus g (no complex structure given), and let h be a metric on Σ. The curvature scalar of this metric will be denoted as R. The space MET_g of metrics is itself an infinite-dimensional Riemannian manifold. Indeed, let h_t be a one-parameter family of metrics. Then $\delta h = (dh_t/dt)_{t=0}$ is a tangent vector to MET_g at $h = h_{t=0}$, and one defines its norm to be

$$|\delta h|^2 = \int_\Sigma \sqrt{h}(\delta h\,,\,\delta h) = \int_\Sigma \sqrt{h}h^{\alpha\gamma}h^{\beta\delta}\delta h_{\alpha\beta}\delta h_{\gamma\beta}. \tag{4.1}$$

This determines a metric on MET_g and thus, formally, a Riemannian measure, which we will denote as (Dh). The physical problem is to learn how to integrate over MET_g. Naively speaking, one would like to compute

[5] In models based on matrix chains, there is strong evidence that all general relations that hold for arbitrary M are valid [19], and therefore, by the reasoning just indicated, one expects that (3.51) holds with some E. However, in the matrix chains, the "primary fields" (the $\tau_{0,\sigma}$) have negative dimension, and the virtual dimension of moduli space in genus one is zero. These facts force $E = \text{constant}$. This explains results that were noted in [19].

the integral

$$F(g) = \int_{\mathrm{MET}_g} (Dh) \exp\left(-\lambda_1 \int_\Sigma \sqrt{h} - \lambda_2 \int_\Sigma \sqrt{h} \frac{R}{2\pi}\right), \tag{4.2}$$

with arbitrary real numbers λ_1 and λ_2. Of course, the term multiplying λ_2 is a topological invariant, the Euler characteristic $\chi(\Sigma) = 2 - 2g$, and plays a trivial role as long as g is fixed; but we will be interested in the dependence on g.

Let $\mathrm{MET}_{A,g}$ be the space of metrics of total area A on a genus g surface. It too has an induced Riemannian structure, and therefore it should have a volume $\mathrm{Vol}(g, A)$. Computing (4.2) is equivalent to knowing $\mathrm{Vol}(g, A)$, since from

$$\begin{aligned} F(g, A) &= \int_{\mathrm{Met}_{A,g}} (Dh) \exp\left(-\lambda_1 A - \lambda_2 \int_\Sigma \sqrt{h} \frac{R}{2\pi}\right) \\ &= \mathrm{Vol}(g, A) \cdot \exp(-\lambda_1 A - \lambda_2 \chi(\Sigma)) \end{aligned} \tag{4.3}$$

we can recover (4.2) by integrating over A.

Of course, a priori one does not quite know what integration theory on these infinite-dimensional spaces is supposed to mean. Usually, in quantum field theory one introduces some sort of "cut-off," which one might imagine to be an approximation to MET_g of some finite-dimension Λ, such that the desired integrals become well defined. Then one tries to "remove the cutoff," that is, one considers a sequence of better and better approximations to MET_g with increasing Λ, and one tries to determine the limit of the integrals for $\Lambda \to \infty$. It then will typically occur even in good cases that such a limit does not exist unless one adjusts ("renormalizes") the "coupling constants" λ_1 and λ_2 in a suitable fashion. So we come to the basic problem of renormalization theory:

Problem. Adjust λ_1 and λ_2 as $\Lambda \to \infty$ so that $F(g, A)$ and $F(g)$ converge to well-defined functions of g and A.

Now, notice that if this problem has a solution, the solution cannot quite be unique. For one could add to λ_2 a finite constant, that is, a constant c independent of the cutoff, and the F's would change by

$$F(g, A) \to F(g, A) \cdot e^{c\chi(\Sigma)}. \tag{4.4}$$

Similarly, there is a potential ambiguity from the ability to add a constant to λ_1, but it turns out that this ambiguity can be canonically removed by requiring that $F(g, A)$ varies only as a power of A (adding a constant to λ_1 would introduce an exponential factor).

The problem of renormalization posed above has analogs in many other quantum field theories; and, usually, it is very difficult to get full control over this problem (including possible strong coupling fixed points)—or to explain the partial results in an introductory lecture. But the particular problem we are discussing here can be treated in a remarkably effective way using a discrete cutoff that was proposed in [14], [41], [1]. In this approach, one considers not metrics on Σ, but triangulations of Σ, or certain generalizations that we will consider later. Every triangulation of Σ determines a metric; for instance, one can consider the triangles to be equilateral triangles of area ε. Of course, the metrics determined this way, except in a flat case where all the coordination numbers are six, cannot be smooth; the curvature consists of delta functions with coefficients that are integral multiples of $2\pi/6$. Nevertheless, one can hope that if a surface is covered with a very large number of triangles, and one averages over the local irregularities, then on a large scale one can effectively see a general metric.

Let $V(g, n)$ be the number of isomorphism classes of triangulations of a genus g surface with n triangles.[6] Now, for small n, $V(g, n)$ is determined by "accidents." But for large n, we can hope that the metrics determined by triangulations approximate any given point in MET_g with equal probability, and that counting triangulations becomes an approximation to computing integrals on MET_g. In fact, one proves (a rather precise account is given in [10]) that the large n behavior of V is

$$V(g, n) \sim e^{cn} \cdot n^{\gamma(2-2g)-1} \cdot b_g \cdot (1 + O(1/n)). \tag{4.5}$$

If we consider every triangle to have area ε, then the total area is $A = n\varepsilon$, so

$$n = A/\varepsilon. \tag{4.6}$$

We regard ε as a cutoff, and we regard $\sum_n V(g, n)$, which is the sum over metrics of arbitrary area, as an approximation to $\int dA\,\mathrm{Vol}(g, A)$. With $\sum_n \sim \int dA/\varepsilon$, we interpret $V(g, n)/\varepsilon$ as an approximation $\mathrm{Vol}_\varepsilon(g, A)$ to the volume of $\mathrm{MET}_{A,g}$. We want to take $\varepsilon \to 0$ while keeping A fixed. (4.5) means that

$$\mathrm{Vol}_\varepsilon(g, A) \sim \frac{1}{\varepsilon} \cdot e^{cA/\varepsilon} \cdot \left(\frac{A}{\varepsilon}\right)^{\gamma(2-2g)-1} \cdot b_g. \tag{4.7}$$

[6] To get a precise relation to the matrix model formulation later, one should define $V = \sum_T 1/q(T)$, where the T's are isomorphism classes of triangulations, and $q(T)$ is the order of the automorphism group of the triangulation T. We may ignore this at present, since $q(T)$ is 1 for almost all T, and the factors of $q(T)$ do not affect the large n behavior.

So from (4.3), the corresponding cutoff version of $F(g, A)$ is

$$F_\varepsilon(g, A) = \mathrm{Vol}_\varepsilon(g, A)\exp(-\lambda_1 A - \lambda_2(2-2g)) \tag{4.8}$$
$$= \frac{1}{\varepsilon}\cdot e^{cA/\varepsilon}\cdot\left(\frac{A}{\varepsilon}\right)^{\gamma(2-2g)-1}\cdot b_g\cdot\exp(-\lambda_1 A - \lambda_2(2-2g)).$$

Now we must "renormalize," that is, take $\varepsilon \to 0$ while adjusting λ_1 and λ_2 as functions of ε so that $F_\varepsilon(g, A)$ converges to a well-defined function $F(g, A)$. Obviously, what we need is

$$\lambda_1 = c/\varepsilon, \qquad \lambda_2 = \gamma\ln(A_0/\varepsilon), \tag{4.9}$$

where A_0 is a constant. The limiting or "renormalized" function $F(g, A)$ is then

$$F(g, A) = \frac{1}{A}\cdot\left(\frac{A}{A_0}\right)^{\gamma(2-2g)}\cdot b_g. \tag{4.10}$$

A_0 is the arbitrary constant discussed already in (4.4). In the above derivation, one might feel that it is natural to set $A_0 = 1$. Intuitively, this is in fact unnatural, because A, ε, and A_0 all have dimensions of area, and there is no natural unit of area. In including the arbitrary A_0 in (4.10), we are simply bringing out into the open the need to choose such an arbitrary unit. (Shortly we will replace triangles with squares. If one sets $A_0 = 1$ using triangles, one gets a different answer from what one would get if one sets $A_0 = 1$ using squares, so neither choice is truly natural.)

Now, apart from the fundamental fact that renormalization works and the theory exists, the moral of the above discussion is that the dominant looking term e^{cn} in (4.5) did not matter and disappeared after renormalization. On the other hand, the subleading power $n^{\gamma(2-2g)-1}$ does matter, as does b_g. However, the b_g are well defined only up to

$$b_g \to b_g\cdot t^{1-g}, \tag{4.11}$$

for constant t.

The above computation is interesting, but it is not so fundamental if we are just exploring quirks of triangles. We want to see that we would obtain the same theory if we make different arbitrary choices of regularization. For instance, we could construct the theory with squares instead of triangles. Let $W(g, n)$ be the number of ways to cover a genus g surface with n squares. Then[7] one finds that the large n behavior of $W(g, n)$ is just like the asymptotic formula in (4.5), but with a different value of c, the

[7] Apart from a delicate factor of two that will be pointed out at the end of §4c.

same value of γ, and the b_g's differing only by a transformation of the type (4.11). Thus, up to the one inevitable ambiguity, the same theory is obtained if one uses squares instead of triangles. Similarly, one obtains the same results if one uses the number of ways to cover a surface with pentagons, hexagons, etc., as the way to regularize $\mathrm{MET}_{A,g}$.

There is a fascinating variant of this, originally proposed in [40]. Instead of covering a surface with, say, only squares or only hexagons, we could permit both squares and hexagons. Let $W(g; n_4, n_6)$ be the number of ways to cover a genus g surface with n_4 squares and n_6 hexagons. We pick a real number x and let

$$W_x(g, n) = \sum_{n_4+n_6=n} W_x(g; n_4, n_6)x^{n_6}. \tag{4.12}$$

For generic x, the large n behavior of $W_x(g, n)$ is the same as (4.5), except for the usual modifications—an x dependent value of c, and an x dependent transformation of the sort in (4.4). At a critical value of x, however, one finds a new theory, with a different value of γ and very different b_g's.

This in turn can be generalized. We can consider coverings of a surface with s-gons of various s. It turns out that nothing essential is lost if one considers the s-gons of even s only. So we let $W(g; n_2, n_4, n_6, \cdots)$ be the number of ways to cover a genus g surface with n_2 2-gons, n_4 4-gons, n_6 6-gons, and so on. Picking real numbers $x_2, x_4, x_6, \cdots$, we let

$$W_{\{x\}}(g, n) = \sum_{n_2+n_4+\cdots=n} W(g; n_2, n_4, \cdots)x_2^{n_2} \cdot x_4^{n_4} \cdot x_6^{n_6} \cdots. \tag{4.13}$$

To avoid analytical questions, we can restrict this to a theory with s_0 parameters, for arbitrary s_0, by supposing that the x_{2s} are zero for $s > s_0$. Then, the generic large n behavior of $W_{\{x\}}$, for fixed $\{x\}$, is that of (4.5) (up to the usual irrelevant modifications). On a codimension one subvariety, one finds the exceptional behavior that we already mentioned in the theory with only squares and hexagons. Generically, on this subvariety, the large n behavior of $W_{\{x\}}(g, n)$ is independent of the x's. But on the codimension two subvariety, one finds again a new theory, with a new value of γ and essentially new b_g's. This process continues indefinitely; in every codimension there is a new critical subvariety. The kth theory arises on a codimension $k-1$ subvariety, for $k = 1, 2, 3, \cdots$. This nested hierarchy of critical subvarieties is, as we have already noted in §2, reminiscent of the situation considered in [47].

Now, let us return to the theory in which Σ is covered by squares only, but let us enrich the theory by permitting a few impurities. We consider as impurities u_2 2-gons, u_4 4-gons, u_6 6-gons, etc. (Since we are covering Σ by 4-gons anyway, it is necessary to specify that by a 4-gon impurity we mean a marked 4-gon, in a sea of 4-gons which are generically unmarked.) Let $W(g, n; u_2, u_4, u_6, \cdots)$ be the number of ways to cover a surface of genus g with n unmarked 4-gons and u_{2r} $2r$-gon impurities, for $r = 1, 2, \cdots$. We consider the $2r$-gon impurities of each r to be ordered (otherwise, one must simply divide by $\prod_i u_{2i}!$).[8] Then the methods that give (4.5) yield

(4.14)

$$W(g, n; u_0, u_1, \cdots) \sim e^{cn} n^{\gamma(2-2g)-1+\sum_{r=1}^{\infty} u_{2r}\gamma_r} \cdot f_g(u_2, u_4, u_6, \cdots),$$

with certain constants γ_r and $f_g(u_2, u_4, u_6, \cdots)$. (The γ_r are "universal," that is, they are unchanged if one considers impurities in a sea of hexagons, etc., instead of squares. The $f_g(u_2, u_4, \cdots)$ similarly are universal up to a transformation, analogous to (4.11), that can be absorbed in a rescaling of the variables t_i that we will introduce in a moment.) The general methods for computing the $f_g(u_2, u_4, u_6, \cdots)$ can be found in [10] (and an introduction is given below).

The dramatic development of the last year is that it has been found [12], [23], [34], [7] that the generating function defined as

(4.15)

$$f(t_0, t_1, t_2, \cdots) = \sum_g \sum_{\{u_{2i}\}} \prod_{i=1}^{\infty} \frac{t_{i-1}^{u_{2i}}}{u_{2i}!} f_g(u_2, u_4, u_6, \cdots) \cdot \text{trivial constants}$$

obeys the KdV equations as well as the string equation, described in §2.[9] The main conjecture of §2 is equivalent to the statement that the function $F(t_0, t_i, \cdots)$ defined there coincides, after some slight shifts in the variables, with the function f. That conjecture indeed was an attempt to propose for $f(t_0, t_1, \cdots)$ a geometrical interpretation more direct than the one by which it is defined.

4b. Random matrices. A powerful tool for obtaining the results just sketched comes from the interpretation, given long ago by 't Hooft [56], of Feynman diagrams with matrix-valued fields in terms of triangulations of

[8] To make the ideas clear, we will in this and the next paragraph overlook a few important details including the need to take linear combinations of the u's corresponding to "scaling variables."

[9] The "trivial constants" arise because of the integral over are needed to go from (4.3) to (4.2). The f_g's were defined in terms of the behavior with a fixed large number of squares, corresponding to fixed area; but the generating function f that one really wants should be defined with an integral over the area. This gives some trivial constants, as we will discuss later.

Riemann surfaces. 'T Hooft's motivation was to understand the behavior of quantum gauge theories with gauge group $SU(N)$ in the large N limit. This problem, which is outstandingly interesting from a physical point of view, has so far been intractable except in two space time dimensions. A few years after the original suggestion, it was realized [12] that drastically simplified models of this program could be understood by methods of random matrix theory, which had been developed in the 1950s and 1960s by Wigner, Dyson, Mehta, and others with the aim of understanding the statistics of nuclear energy levels. A classic reference is Mehta's book [43]. The paper [10] gives a highly readable account, not assuming any prior familiarity with Feynman diagrams, of the application of random matrix methods to count triangulations of surfaces. The reader not acquainted with these matters is strongly urged to consult §§2 and 3 of that paper, as we will only offer a few indications here.

Suppose that we wish to compute $W(g, n)$, the number of ways to cover a genus g surface with n squares. The dual to a covering by squares is a four-valent graph, as indicated in Figure 7. So we can interpret $W(g, n)$ as the number of connected four-valent graphs that can be drawn on a surface of genus g.

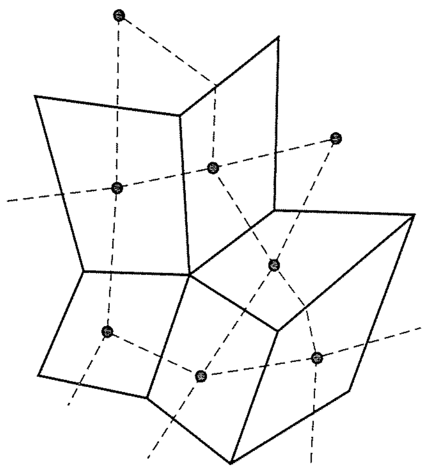


FIGURE 7. A PORTION OF A COVERING OF A SURFACE BY SQUARES AND ITS DUAL FOUR-VALENT GRAPH.

Consider first the slightly easier problem of counting abstract four-valent graphs with n vertices (without reference to any Riemann surface). Let $u(n)$ be the number of graphs which are *connected* and let $y(n)$ be the number of such graphs that are not necessarily connected.[10] The corresponding generating functions are

$$U(-\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n u(n), \qquad Y(-\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n y(n). \tag{4.16}$$

It is easy to see that these are simply related,

$$Y = e^U. \tag{4.17}$$

Now, Y has a convenient integral representation

$$Y(-\lambda) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2} - \lambda\frac{\phi^4}{4!}\right). \tag{4.18}$$

(In other words, the function of λ that is well defined for $\mathrm{Re}\,\lambda > 0$ by (4.18) has an asymptotic, not convergent, expansion near $\lambda = 0$ with coefficients $y(n)$.) We will explain the origin of (4.18) momentarily, but first let us note that (4.18) leads to a quick determination of the large n behavior of $y(n)$ and $u(n)$. By taking the nth derivative of (4.18) we have

$$\begin{aligned} y(n) &= \frac{1}{4!^n n!} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} (\phi^4)^n \\ &= \frac{1}{4!^n n!} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2} + 4n \ln \phi\right). \end{aligned} \tag{4.19}$$

The integral in (4.19) can be estimated for large n by noting that the main contribution comes from the neighborhood of the maxima of the integrand at $\phi = \pm\sqrt{4n}$, and this gives $y(n) \sim (4n)^{2n} e^{-2n}/(4!^n n!)$. It is easy to see that the growth with n of $y(n)$ is so fast that for large n almost every four-valent graph with n vertices is connected, and thus asymptotically $u(n) \sim y(n)$. In particular, $u(n)$ and $y(n)$ grow faster than exponentially with n, so that the series in (4.16) has zero radius of convergence.

To understand (4.18), we first note the elementary integral

$$\int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} = 1, \tag{4.20}$$

and as a result

$$\int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2 + J\phi} = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-(\phi - J)^2/2 + J^2/2} = e^{J^2/2}. \tag{4.21}$$

---

[10] We will consider the vertices to be unordered; otherwise, the numbers $u(n)$ and $y(n)$ are larger by a factor of $n!$, and a factor of $1/n!$ would be included in the following definition. The minus signs in the definitions of the generating functions are for later convenience.

So

$$\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-\phi^2/2}\phi^{2k}=\left[\frac{d^{2k}}{dJ^{2k}}\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-\phi^2/2+J\phi}\right]_{J=0} = \left[\frac{d^{2k}}{dJ^{2k}}e^{J^2/2}\right]_{J=0}. \tag{4.22}$$

Now since

$$\frac{d}{dJ}e^{J^2/2}=Je^{J^2/2}, \tag{4.23}$$

a derivative d/dJ, when acting on $e^{J^2/2}$, "creates" a factor of J. More generally, when we compute a repeated derivative

$$\frac{d}{dJ}\frac{d}{dJ}\cdots\frac{d}{dJ}e^{J^2/2}, \tag{4.24}$$

each derivative either "creates" a factor of J when it acts on the exponential, or "annihilates" a factor of J that has been created by a derivative further to the right. Since in (4.22), we are to set $J=0$ at the end, every factor of J that is "created" by one derivative must be "annihilated" by another. So, finally, (4.22) is equal to the number of ways to group $2k$ objects in pairs.

What we actually want is

$$y(n)=\frac{1}{n!}\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-\phi^2/2}\left(\frac{\phi^4}{4!}\right)^n. \tag{4.25}$$

The factors of ϕ were "born" in groups of four by expanding the exponential in (4.18). So we are counting the possible ways of pairing $4n$ objects which come in groups of four. As shown in Figure 8, it is natural to represent a group of four by a vertex from which four lines emerge, and a pairing of two objects as a connection between the corresponding lines. (The factors of $1/4!$ in (4.25) mean that the four objects in each group are unordered, and the factor of $1/n!$ means that the vertices in the graph are unordered.) In such a way we obtain a four valent graph with n vertices, and the argument shows that $y(n)$ is indeed the number of such graphs. The graphs obtained by such perturbative expansions of integrals are known in quantum field theory as Feynman graphs or Feynman diagrams.

The faster than exponential growth of $y(n)$ and $u(n)$ should be compared with the prediction (4.5) that the number $V(g, n)$ or $W(g, n)$ of three-valent or four-valent graphs that can be drawn on a surface of genus

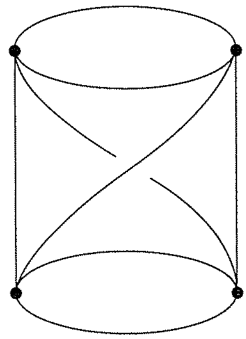


FIGURE 8. A FOUR-VALENT "FEYNMAN GRAPH" OBTAINED BY TAKING CLUSTERS OF FOUR OBJECTS (SUCH A CLUSTER IS DEPICTED AS A VERTEX FROM WHICH FOUR LINES EMERGE) AND PAIRING THEM (BY CONNECTING THE LINES IN PAIRS). IN THIS INSTANCE, THERE ARE FOUR VERTICES.

g grows only exponentially with the number of vertices. How can one modify the above graph counting to construct the generating function of the number of graphs that can be drawn on a surface of fixed genus? The simple modification that is required goes back to [56]. One simply replaces ϕ by an $N \times N$ hermitian matrix M. The space of such matrices is a Euclidean space $\mathbb{R}^{N^2}$, on which one introduces a translationally invariant measure (dM) normalized so that

$$\int (dM) \exp -\operatorname{Tr} M^2 = 1. \tag{4.26}$$

Then, the claim is that the integral

$$Z(N, -\lambda) = \int (dM) \exp -\operatorname{Tr}\left(\frac{M^2}{2} + \lambda \frac{M^4}{4N}\right) \tag{4.27}$$

is essentially the generating function that we need. Indeed, $F(N, -\lambda) = \ln Z$ has the expansion

$$F(N, -\lambda) = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} (-\lambda)^n W(g, n), \tag{4.28}$$

where $W(g, n)$ is the number of ways to cover a surface of genus g with n squares.

Derivation of (4.28). Equation (4.28), which is essentially due to 't Hooft [56], is explained in [10] and in [51]. Here is a very brief account. To begin with, by completing the square, one proves that

$$\int (dM) \exp(-\mathrm{Tr}(\tfrac{1}{2}M^2 + MJ)) = \exp(\mathrm{Tr}(J^2)/2). \tag{4.29}$$

Hence

$$\begin{aligned}
&\int (dM) \exp\left(-\mathrm{Tr}\left(\frac{M^2}{2}\right)\right) M^{i_1}_{j_1} M^{i_2}_{j_2} \cdots M^{i_n}_{j_n} \\
&= \left[\frac{\partial}{\partial J^{j_1}_{i_1}} \cdots \frac{\partial}{\partial J^{j_n}_{i_n}} \int (dM) \exp\left(-\mathrm{Tr}\left(\frac{M^2}{2} + MJ\right)\right)\right]_{J=0} \\
&= \left[\frac{\partial}{\partial J^{j_1}_{i_1}} \cdots \frac{\partial}{\partial J^{j_n}_{i_n}} e^{\mathrm{Tr}\, J^2/2}\right]_{J=0}.
\end{aligned} \tag{4.30}$$

Now, as in the previous case, each derivative $\partial/\partial J^j_i$ either "creates" or "annihilates" a factor of J^i_j. Since one is to set $J = 0$ at the end of the computation, every factor of J^i_j that is created must be annihilated, so that the evaluation of (4.30) involves a sum over pairings. Again, it is natural to represent such a pairing by a line connecting two vertices. The difference is now that there are N^2 distinct "objects" J^i_j that may be propagating in such a line. Following 't Hooft, we denote this by a "double line notation" in which each line is thickened slightly to a band, and the edges are labeled by i or j, as in Figure 9(a). The two edges of the band correspond to the two indices of the matrix J^i_j, and the N possible labels of each edge correspond to the N possible values of the corresponding index.

Now, expanding (4.27) in powers of λ, the coefficient of $(-\lambda)^n$ is

$$\left(\frac{1}{4N}\right)^n \frac{1}{n!} \int (dM) \exp\left(-\frac{1}{2}\mathrm{Tr}(M^2)\right) (\mathrm{Tr}(M^4))^n. \tag{4.31}$$

Again, we must integrate a polynomial of order $4n$ in the matrix elements of M; again, this can be done using (4.30), and will lead to a sum over four-valent graphs of an appropriate type. However, we must pay attention to just what kind of $4n$th order polynomial in matrix elements of M we have in (4.31). If we bear in mind that $\mathrm{Tr}\, M^4 = M^i_j M^j_k M^k_l M^l_i$, then in the double line notation, the four-valent vertices have the structure indicated in Figure 9(b), and the diagrams with "double lines" connecting such vertices are as in Figure 9(c). The key point is now that, though an abstract graph does not naturally determine a Riemann surface on which it can be drawn, the "double line" structure has had the effect of thickening

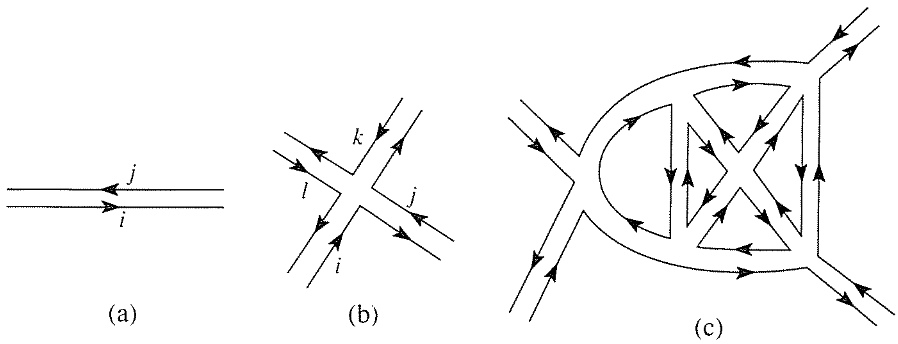


FIGURE 9. A CONVENIENT NOTATION FOR FEYNMAN DIAGRAMS OBTAINED BY PERTURBATIVE EXPANSION OF MATRIX INTEGRALS. EVERY LINE IS THICKENED AS IN (A) TO A "BAND," WHOSE TWO EDGES CORRESPOND TO THE TWO INDICES OF AN $N \times N$ MATRIX M^i_j; EACH EDGE CARRIES A LABEL THAT MAY RUN FROM $1, \cdots, N$, CORRESPONDING TO THE POSSIBLE VALUES OF THE CORRESPONDING INDEX. IN THIS NOTATION, THE FOUR-VALENT VERTEX CORRESPONDING TO A FACTOR OF $\mathrm{Tr}(M^4)$ IN AN INTEGRAL IS DEPICTED AS IN (B). COMBINING THE THICKENED LINES OF (A) WITH THE THICKENED VERTICES OF (B) ONE OBTAINS GRAPHS (C) IN WHICH THE EDGES FIT TOGETHER SMOOTHLY INTO "INDEX LOOPS." FILLING IN THE INDEX LOOPS WITH DISCS, ONE CANONICALLY CONSTRUCTS A TWO-DIMENSIONAL SURFACE.

the lines slightly, in a way which is compatible with the structure of the vertices, and this gives the extra information that is needed in order to reconstruct a Riemann surface. Indeed, with the vertices drawn as indicated, the edges of the double lines join together into circles, and upon filling in these circles with discs, we obtain a surface Σ together with a simplicial decomposition.

Let n_0, n_1, and n_2 be the number of 0, 1, and 2 simplices in this decomposition. Then n_0 is the same as the number n in equation (4.31), and it is a fact of life for graphs drawn with four-valent vertices that

$$n_1 = 2n_0. \tag{4.32}$$

On the other hand, n_2 is the same as the number of circles that were filled in to reconstruct Σ.

These circles are usually called "index loops". The terminology reflects the fact that each edge of one of the thickened lines has a labeling or "index" that takes an arbitrary value in the range $1, \cdots, N$; because of the structure of the vertices, the labelings are constant in running around the circles or index loops, and there is no correlation between the labeling of different loops.

Now, the evaluation of (4.31) proceeds by drawing all the possible four-valent thickened graphs, and then assigning to each graph a numerical factor which comes from factors explicitly present in (4.31) and from summing over the various types of "object" that can be propagating in each double line, that is, by summing over the labelings of the edges. The sum over labelings gives a factor of N for each index loop, or altogether a factor of N^{n_2}. In addition, a factor of N^{-n_0} is explicit in (4.31). The N dependence is thus

$$N^{-n_0+n_2} = N^{n_0-n_1+n_2} = N^{2(1-g)}, \tag{4.33}$$

where in the first step we use (4.32), and in the second step we use the fact that $n_0 - n_1 + n_2$ is the Euler characteristic $2-2g$. The power of N is the main result that is claimed in (4.28). The other numerical factors that arise are the trivial factors that appear explicitly in (4.31). The factor of $1/n!$ means simply that the vertices are unordered, and the factor of $(1/4)^n$ means that the four objects emanating from a vertex carry only a cyclic order.

The double scaling limit. Granting (4.28), what must we do to understand two-dimensional quantum gravity? The problem is that (4.27) generates, via (4.28), all the numbers $W(g, n)$, but this is far more than we want. According to the discussion in §4a, we are only interested in the large n behavior of $W(g, n)$, where one sees an approximation to a random metric on a surface of genus g. Therefore, we want to take a limit of (4.28) in which the extraneous information will be eliminated. This occurs in the limit in which λ approaches a critical value at which the infinite sum in (4.28) is ceasing to converge—and exhibits a singularity that is determined by the asymptotic behavior of the series. The issue has been analyzed (nonrigorously) as follows in the literature. According to (4.5), for large n, $W(g, n) \sim e^{cn} \cdot n^{\gamma(2-2g)-1} b_g$. The genus g contribution to $F(N, -\lambda)$,

$$F_g(-\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n W(g, n) \tag{4.34}$$

thus has a singularity at $\lambda = \lambda_c = -e^{-c}$. The leading singular behavior of

F_g is

$$\sum_{n=0}^{\infty}\left|\frac{\lambda}{\lambda_c}\right|^n n^{\gamma(2-2g)-1} b_g \sim \left|\frac{\lambda-\lambda_c}{\lambda_c}\right|^{-\gamma(2-2g)} \cdot b_g \cdot \Gamma(\gamma(2-2g)). \tag{4.35}$$

Therefore, with

$$y = N^2\left|\frac{\lambda-\lambda_c}{\lambda_c}\right|^{-2\gamma}, \tag{4.36}$$

the sum over g of the leading singular contributions to the F_g is

$$F_{\text{sing}} = \sum_{g=0}^{\infty} y^{1-g} \cdot b_g \cdot \Gamma(\gamma(2-2g)). \tag{4.37}$$

(The sum over n in (4.35) corresponds to the integral over area to go from (4.3) to (4.2), and the resulting Γ function is the "trivial constant" in (4.15). It is the generating functional F_{sing} with these factors included that (a) corresponds to an ensemble with a random metric of any area on a surface of any genus; (b) can be represented as a matrix integral.) Thus, the prescription that has been followed in the recent literature is to extract the leading singularity of F in the limit $N \to \infty$, $\lambda \to \lambda_c$, with y fixed. It is in this limit, which is known as the double scaling limit, that the matrix integral (4.27) is governed by the interesting numbers b_g and γ.

More generally, if, as in (4.13), we wish to consider arbitrary mixtures of 2-gons, 4-gons, 6-gons, etc., one must consider a generalization of (4.27), namely

$$Z(\lambda_i) = \int (DM) \exp\left(-\operatorname{Tr}\left((1-\lambda_2)\frac{\phi^2}{2} - \lambda_4\frac{\phi^4}{4N} - \lambda_6\frac{\phi^6}{6N^2} - \cdots\right)\right). \tag{4.38}$$

In effect, in order to study coverings of genus g surfaces by n-gons of various n, we must discuss a general integral

$$Z = \int (dM) \exp(-\operatorname{Tr} V(M)), \tag{4.39}$$

for general V. It is in this form that we will discuss the problem in the next subsection.

4c. Orthogonal polynomials and discrete KdV flows. In this subsection, we will, finally, explain the origin of some of the key recent results [13], [25], [35], [7] that are important physically and motivated the conjecture about intersection theory on moduli space presented in §2. Following some preliminaries that can be found in [10], we will take a point of view that for the most part follows the exposition by Douglas [23] (with some modifications suggested in part by G. Segal).

The first step in analyzing integrals of the form (4.39) is to diagonalize the matrix M, so $M = U\Lambda U^{-1}$, where U is a unitary matrix and $\Lambda = \mathrm{diag}(s_1, s_2, \cdots, s_n)$. Then as computed in appendix (2) of [10], the measure can be written

$$(dM) = \text{constant} \cdot (dU) \cdot ds_1 ds_2 \cdots ds_n \cdot \prod_{i<j} (s_i - s_j)^2 , \tag{4.40}$$

where (dU) denotes Haar measure on the unitary group, and the constant factor is not important because it does not affect the singularity in the double scaling limit. The double zero of the measure at $s_i = s_j$ reflects the fact that in the space of hermitian matrices, the matrices with two equal eigenvalues are of codimension three rather than codimension one since the stabilizer of a hermitian matrix with two equal eigenvalues has dimension two more than the stabilizer of a generic hermitian matrix.

The integral (4.39) can therefore be replaced by

$$Z = \int_{-\infty}^{\infty} ds_1 ds_2 \cdots ds_N \prod_{i<j} (s_i - s_j)^2 \cdot \prod_i e^{-V(s_i)} . \tag{4.41}$$

Let $d\mu$ be the measure $ds\, e^{-V(s)}$ on the real line. Introduce the monic orthogonal polynomials $P_r(s)$, $r = 0, 1, 2, \cdots$, for this measure, defined by

$$P_r(s) = s^r + \text{lower order terms} \tag{4.42}$$

and

$$\int d\mu\, P_k(s) P_r(s) = h_r \delta_{k,r} . \tag{4.43}$$

Let Q be the $N \times N$ matrix whose i, j matrix element is s_i^{j-1}. Then $\det Q$ is a polynomial of order $N(N-1)/2$ which vanishes whenever $s_i = s_j$ for any i, j since in that case Q has two equal rows. These facts fix the relation

$$\det Q = (-1)^{N(N-1)/2} \prod_{i<j} (s_i - s_j) \tag{4.44}$$

up to a numerical factor which can be verified by, for instance, working out the coefficient of $\prod_{i=1}^N s_i^{i-1}$. On the other hand, consider instead of Q the matrix $\widetilde{Q}$ whose ij element is $P_i(s_j)$. $\det \widetilde{Q} = \det Q$, since, in view of (4.42), $\tilde{Q}$ differs from Q by column rearrangements of a triangular kind. Hence, we may rewrite (4.41) in the form

$$Z = \int d\mu(s_1) \cdots d\mu(s_N) (\det \tilde{Q})^2. \tag{4.45}$$

If now we explicitly write

$$\det \tilde{Q} = \sum_{\pi} (-1)^{\pi} \prod_i P_{\pi(i)}(s_i), \tag{4.46}$$

where π ranges over the permutations of N objects, then the integral becomes

$$Z = \int d\mu(s_1) \cdots d\mu(s_N) \sum_{\pi, \pi'} (-1)^{\pi+\pi'} \prod_i P_{\pi(i)}(s_i) P_{\pi'(i)}(s_i). \tag{4.47}$$

By using the orthogonality relation (4.43), this gives

$$Z = N! \cdot h_0 \cdot h_1 \cdot \cdots \cdot h_{N-1}. \tag{4.48}$$

Thus, to solve the problem, it suffices to know the normalization constants h_i of the monic orthogonal polynomials. The constant $N!$ in (4.48) is, again, irrelevant in the double scaling limit.

At this point, it is convenient to switch to *orthonormal* polynomials,

$$\hat{P}_r = \frac{P_r}{\sqrt{h_r}}. \tag{4.49}$$

Let $\mathscr{V}$ be the vector space consisting of polynomial functions $\sum_{i=0}^{n} b_i s^i$, of arbitrary degree. It is a fixed vector space, given once and for all. Since we do not want to introduce any Hilbert space structure on $\mathscr{V}$ (the only natural L^2 structure in the problem is determined by the measure $d\mu$, which depends on the potential V, but we want to consider objects that are independent of V), by a *basis* of $\mathscr{V}$ we will mean a vector space basis, that is, a set of vectors $q_r \in \mathscr{V}$ such that every element of $\mathscr{V}$ can be uniquely written as a *finite* linear combination of the q_r. In particular, every choice of a potential V determines a canonical basis, namely the basis consisting of the polynomials $\hat{P}_i$, which may be characterized completely as the orthogonal polynomials for the measure $d\mu$ with positive leading term.

On $\mathscr{V}$, there are certain natural operators, such as the operation $\mathscr{S}$ of "multiplication by s", which maps the polynomial $P(s)$ to the polynomial $sP(s)$, and the operation $\mathscr{T}$ of "differentiation with respect to s," which maps $P(s)$ to dP/ds. If one is given a particular basis for $\mathscr{V}$, such as the basis of orthogonormal polynomials, then $\mathcal{S}$ or $\mathcal{T}$ can be written out as a concrete $(\infty \times \infty)$ matrix,

$$\mathcal{S}\widehat{P}_r = \sum_k S_{k,r}\widehat{P}_k. \tag{4.50}$$

It is obvious that $S_{k,r} = 0$ for $k - r > 1$, and noting that

$$\int d\mu(s\widehat{P}_r)\widehat{P}_k = \int d\mu\widehat{P}_r(s\widehat{P}_k), \tag{4.51}$$

we see that $S_{k,r} = S_{r,k}$, so $S_{k,r}$ vanishes for $r - k > 1$. We have learned that, in the basis of orthonormal polynomials, $\mathcal{S}$ is a "Jacobi matrix," that is, a symmetric matrix whose matrix elements $S_{k,l}$ vanish for $|k - l| > 1$. For $|k - l| = 1$, the $S_{k,l}$ are determined by the leading coefficients of the orthonormal polynomials, so concretely

$$\mathcal{S} \cdot \widehat{P}_r = \sqrt{\frac{h_{r+1}}{h_r}}\widehat{P}_{r+1} + S_r\widehat{P}_r + \sqrt{\frac{h_r}{h_{r-1}}}\widehat{P}_{r-1}, \tag{4.52}$$

with some constants S_r.

Now, for *every* choice of potential V, we get a canonical basis of orthonormal polynomials in which S can be written out as a Jacobi matrix. Considering explicitly as arbitrary polynomial V,

$$V(s) = \sum w_i s^i \tag{4.53}$$

(with all but finitely many w_i vanishing), we get a family of Jacobi matrices $S(w_1, w_2, w_3, \cdots)$. However, since the matrix $S(w_1, w_2, \cdots)$ is obtained by writing out a *fixed* operator $\mathcal{S}$ on a fixed vector space $\mathcal{V}$ in a basis that depends on the w's all that happens to it when the w_i are changed is that it is written out in terms of a new basis. If the derivative of the basis with respect to the w_i is

$$\frac{\partial \widehat{P}_k}{\partial w_i} = \sum_l (O_{(i)})_{kl}\widehat{P}_l, \tag{4.54}$$

then the derivative of the Jacobi matrix S with respect to the w_i is

$$\frac{\partial S}{\partial w_i} = [O_{(i)}, S]. \tag{4.55}$$

Since we know a priori that S is well defined as a function of the w_i, the $O_{(i)}$, which we have not yet determined, must be such that the flows defined by (4.55) in the space of Jacobi matrices are *commuting*.

Now, under appropriate conditions, a symmetric Jacobi matrix can behave as a discrete approximation to a second-order differential operator in one dimension of the form $Q = d^2/dx^2 + U(x)$. The usual KdV flows are commuting flows in the space of such operators, of the form

$$\frac{\partial Q}{\partial t_n} = [M_n , Q], \tag{4.56}$$

where the M_n are certain differential operators of order n. Now, as explained by Moser [49], the space of Jacobi matrices, like the space of second-order operators of the indicated type, has a natural symplectic structure, and moreover Moser proposed commuting flows of the type (4.55) as discrete analogs of the KdV flows. (See also P. van Moerbeke, Inv. Math. **37** (1976) 45, and references therein.)

Now actually, in the continuum case, that is, the case of differential operators, the M_n in (4.56) are almost uniquely determined by requiring that $[M_n , Q]$ is a zeroth order differential operator (which can be interpreted as $\partial U/\partial t_n$). Indeed, according to Gelfand and Dikii [31], a differential operator with this property is a linear combination of the operators

$$M_n = (Q^{n/2})_+ , \tag{4.57}$$

where $Q^{n/2}$ is the $n/2$ power of Q as a pseudodifferential operator, and $(Q^{n/2})_+$ is the unique differential operator such that $Q^{n/2} - (Q^{n/2})_+$ is of negative order. The KdV flows are precisely the flows (4.56) with these M_n. Half of the flows are trivial, since if n is even, $n/2$ is an integer, and $(Q^{n/2})_+ = Q^{n/2}$ is a differential operator that commutes with Q.

Let us now consider the discrete analogs of these flows. We will call a matrix W local, of order p, if the matrix elements $W_{k,l}$ vanish for $|k - l| > p$. A local matrix is a natural candidate for approximating a differential operator. Given a Jacobi matrix S, in looking for local matrices W such that $[W , S]$ is a local matrix, the interesting W's are the antisymmetric ones. For given any W, if we write $W = W_+ + W_-$, where W_+ is symmetric and W_- is antisymmetric, the condition that $[W , S]$ is a Jacobi matrix means (since Jacobi matrices are symmetric by definition) that $[W_- , S]$ is a Jacobi matrix and $[W_+ , S] = 0$. For a generic Jacobi matrix S with distinct eigenvalues, the condition on W_+ has only the trivial solutions that $W_+ = \sum_i a_i S^i$, analogous to the trivial KdV flows. So we may as well take W antisymmetric. An antisymmetric local matrix W such that $[W , S]$ is a Jacobi matrix is (as shown in the concluding pages of [49]) a linear combination of certain matrices B_p, $p = 1 , 2 , 3 , \cdots$, with B_p being local of degree p. (If $S^p = A_+ + A_-$, where A_+ is upper triangular and A_- is the transpose of A_+, then one can take $B_p = A_+ - A_-$.) The flows

$$\frac{dS}{dt_n} = [B_n, S] \tag{4.58}$$

are discrete analogs of the KdV flows.

Let us now verify that the matrices $O_{(i)}$ defined in (4.54) are a linear combination of the B_p. By differentiating the orthonormality relation, we get

$$0 = \frac{\partial}{\partial w_i}\int d\mu \widehat{P}_k \widehat{P}_l = \int d\mu \left(\frac{d\widehat{P}_k}{dw_i}\widehat{P}_l + \widehat{P}_k\frac{d\widehat{P}_l}{dw_i} - s^i \widehat{P}_k \widehat{P}_l\right). \tag{4.59}$$

The s^i term comes from differentiating the measure $d\mu = dse^{-V}$ with respect to w_i. This gives

$$(O_{(i)})_{k,l} + (O_{(i)})_{l,k} = \int d\mu s^i \widehat{P}_k \widehat{P}_l. \tag{4.60}$$

Since the $\widehat{P}_k$ are polynomials of order k, it follows immediately from the definition in (4.54) that $(O_{(i)})_{k,l} = 0$ for $l > k$. As multiplication by s is a Jacobi matrix, it follows from (4.60) that $(O_{(i)})_{k,l}$ also vanishes for $k - l > i$, and thus $O_{(i)}$ is local, of degree i, as we wished to show. Notice that (4.60) can be written in the form

$$O_{(i)} - \frac{S^i}{2} = -\left(O_{(i)} - \frac{S^i}{2}\right)^T. \tag{4.61}$$

Thus, $\widetilde{O}_{(i)} = O_{(i)} - S^i/2$, which obviously generates the same flow as $O_{(i)}$, is antisymmetric and indeed coincides with B_i as defined above.

So far, we have determined that $S(w_1, w_2, \cdots)$ is an orbit of the discrete KdV flows. It remains to determine which orbit arises, that is, to determine the initial conditions. To this aim, we will appeal to an elegant argument by Douglas [24]. In addition to $\mathcal{S} =$ multiplication by s being local, it is also true that $\mathcal{T} = d/ds$ is local in the basis of orthonormal polynomials, provided the potential V is a polynomial (provided almost all of the w's vanish). It is indeed obvious that if we write

$$\mathcal{T}\widehat{P}_k = \sum_{r=1}^{k} T_{k,r}\widehat{P}_r, \tag{4.62}$$

then $T_{k,r} \neq 0$ only for $r < k$. By considering

$$0 = \int ds \frac{d}{ds}(e^{-V}\widehat{P}_k \widehat{P}_r) = \int d\mu \left(-\frac{dV}{ds}\widehat{P}_k \widehat{P}_r + \frac{d\widehat{P}_k}{ds}\widehat{P}_r + P_k \frac{d\widehat{P}_r}{ds}\right), \tag{4.63}$$

one sees that $T_{k,r} = 0$ unless $k - r \leq n - 1$, where n is the degree of V. Thus $\mathcal{T}$ is represented in the basis of orthonormal polynomials by a matrix T that is a local of degree $n - 1$, and so has an expansion

$$T = \sum_{j=1}^{n-1} (v_j O_{(j)} + p_j S^j), \tag{4.64}$$

for some real numbers v_j and p_j. In fact, more incisively, it follows from (4.63) that

$$\mathcal{T}' = \mathcal{T} - \tfrac{1}{2} V'(\mathcal{S}) \tag{4.65}$$

is antisymmetric, and this expression for an antisymmetric matrix as the sum of a triangular matrix and a polynomial in S determines $\mathcal{T}'$ as a linear combination of the B_p's.

The underlying relation

$$[\mathcal{T}, \mathcal{S}] = 1 = [\mathcal{T}', \mathcal{S}] \tag{4.66}$$

may then be written out in the form

$$\sum_{j=1}^{n-1} v_j [O_{(j)}, S] = 1. \tag{4.67}$$

The requirement that there exist constants v_j such that (4.67) is obeyed determines a particular orbit for the discretized KdV flows on the space of Jacobi matrices. In fact, (4.67) is a discrete analog of (2.22) which served in §2 to determine the initial conditions for the solution of the KdV equations. To see this, write (2.22) in the form

$$\langle\langle \tau_0 \rangle\rangle - \sum_{i=0}^{\infty} t_{i+1} \langle\langle \tau_i \rangle\rangle = \frac{t_0^2}{2}. \tag{4.68}$$

Differentiating twice with respect to t_0, this becomes

$$\sum_{i=0}^{\infty} y_i \langle\langle \tau_i \tau_0 \tau_0 \rangle\rangle = 1, \tag{4.69}$$

where $y_i = \delta_{i,0} - t_{i+1}$. Alternatively, this can be written

$$\sum_{i=0}^{\infty} y_i \frac{\partial}{\partial t_i} U = 1. \tag{4.70}$$

According to the main conjecture of §2, $\partial U / \partial t_i$ is the ith KdV flow.

Interpreting S as a discrete approximation to a differential operator $Q = d^2/dt_0^2 + U$, and $[O_{(i)}, S]$ as a discrete approximation to the ith KdV flow, we see that we can indeed identify the initial conditions (4.67) in the matrix model formulation as a discrete approximation to the initial conditions defined by the string equation.

We have carried out all of this discussion *without* considering the double scaling limit, discussed at the end of §4b, in which it is expected that two-dimensional gravity can be extracted from the matrix model. It is argued in the literature that in the double scaling limit, S converges to a differential operator, and the discrete KdV flows in the space of Jacobi matrices converge to the ordinary continuum KdV flows in the space of differential operators. We refer to the original papers [13], [25], [35] for these arguments.

The role of the odd polynomials. To conclude, we will attempt to explain a detail that has been left unclear in the previous literature, though somewhat similar points are raised in [24].[11] This detail is important in a careful comparison of intersection theory to matrix models.

On the space $\mathscr{V}$ of polynomials in s, let U be the operator that maps $P(s)$ to $P(-s)$. Let $\widehat{S} = -USU$. It is easy to see that like S, $\widehat{S}$ is a Jacobi matrix, or more precisely, a family of Jacobi matrices parametrized by $w_1, w_2, \cdots$. Moreover,

$$\frac{\partial \widehat{S}}{\partial w_r} = [\widehat{O}_{(r)}, \widehat{S}], \tag{4.71}$$

where $\widehat{O}_{(r)} = UO_{(r)}U$ is local of the same degree as $O_{(r)}$. So like S, $\widehat{S}$ evolves by the discrete analog of the KdV flows.

Therefore, the question arises of whether $\widehat{S}$, like S, might in the double scaling limit converge to a second order differential operator. Actually, it is really necessary to specify more precisely that the statement "S (or $\widehat{S}$) converges to a differential operator in a certain limit" will mean that in acting on vectors $\sum_i a_i\widehat{P}_i$, where the a_i are *slowly varying with* i, S (or $\widehat{S}$) approximates a differential operator. It is evident that $\widehat{S}$ converges to a differential operator in this sense if and only if in the same limit S approximates a differential operator when acting on vectors of the type $\sum_i(-1)^i b_i\widehat{P}_i$, with slowly varying b_i. Thus, the consideration of convergence of $\widehat{S}$ to a differential operator is equivalent to consideration

[11] After writing these notes, I received a paper developing the same point more extensively [6].

of a generalized possibility for how S might converge to a differential operator.[12]

Now, in general, in the double scaling limit, upon appropriate adjustment of the couplings w_i, it is possible for both S and $\widehat{S}$ to converge to differential operators, of the form $Q = d^2/dx^2 + V$ and $\widehat{Q} = d^2/dx^2 + \widehat{V}$, and there is absolutely no general relation between the two potentials V and $\widetilde{V}$[13]. Both Q and $\widehat{Q}$ evolve separately according to the KdV flows, so in fact the hermitian matrix model leads to two entirely independent, commuting copies of the KdV hierarchy! The initial conditions are of the same structure for each, since the argument that led to (4.67) could just as well be made for $\widehat{Q}$.

It is dull to study two decoupled copies of the same structure, and what is usually done in the matrix model literature is to eliminate half the variables. The usual way to do this is to take the potential V to be even, $V(s) = V(-s)$. It is evident that in this case, $\widehat{S} = S$, so one is seeing, in effect, the diagonal combination of the two theories. The free energy computed this way, which is the result usually reported in the matrix model literature, receives half its contribution from S and half its contribution from $\widetilde{S}$, and is precisely twice that of the basic system. In a generic double scaling limit with a noneven potential, S or $\widehat{S}$ would converge to a differential operator, but not both, and the free energy would be precisely half of the result for an even potential. By careful comparison of intersection theory on moduli space, discussed in §2, to the matrix model results, one can see that (at least in genus ≤ 3, where all of the conjectures of §2 have been verified) the free energy defined by intersection theory is equal to that of the matrix models for a generic, noneven potential, and is half of the matrix model result as usually quoted.

4d. Matrix chains. In §3, we generalized intersection theory on moduli space of Riemann surfaces to include maps to a Kähler manifold M. One may wonder whether the hermitian matrix model has an analogous generalization. In fact, it has a very beautiful generalization, which we will now indicate very briefly.

First of all, the physical problem is to study two-dimensional quantum gravity coupled to quantum fields. Once one agrees to describe

[12] There are yet more elaborate possibilities for how S might converge to a differential operator, but they do not arise for generic even potentials, and thus are not relevant to elucidating the existing literature, which is our goal in the present discussion.

[13] To explicitly achieve this, take a matrix model potential V which in the naive large N limit is even. Add to it odd terms, suppressed by just the right powers of N so as to give contributions of order 1 in the double scaling limit. In this way, one gets an explicit double scaling solution with Q and $\widehat{Q}$ completely independent.

quantum gravity by a sum over triangulations of a surface, it is natural to describe the quantum fields by lattice statistical mechanics on the triangulated surface. This, again, can be accommodated in the framework of matrix models—provided that one introduces more than one matrix. The most general type of example that has so far been tractable is the "matrix chain", in which one considers n hermitian matrices M_i, $i = 1, \cdots, n$, and an integral of the form

(4.72)
$$Z = \int (dM_1) \cdots (dM_n) \exp\left(-\mathrm{Tr}\left(\sum_{i=1}^{n} V_i(M_i) + \sum_{i=1}^{n-1} c_i M_i M_{i+1}\right)\right).$$

Such an integral has an interpretation as the generating function for coverings of a Riemann surface (of variable genus) by graphs with certain additional information. The additional information arises because a "vertex" in the graph may come from expanding the factor of $\exp(-\mathrm{Tr}(V_i(M_i)))$ for any value of $i = 1, \cdots, n$. In addition to summing over all isomorphism classes of graphs in evaluating (4.72), one sums over all maps of the set of vertices in the graph to the finite set $\{1, 2, 3, \cdots, n\}$. The possible maps (from a given graph) are not weighted equally; they are weighted by local factors, which one finds by further study of (4.72), and which are similar to the characteristic Boltzmann weights of statistical mechanics.

The integral in (4.72) can again be analyzed very effectively using orthogonal polynomials. One requires certain additional tricks originally introduced by Mehta and collaborators [44]. (Mehta's crucial formula for integrating over angular variables has been explained as an application of the Duistermaat-Heckman stationary phase formula [53].) The main difference in the result that eventually emerges is that the matrix analogous to S is still a local matrix but has degree > 1. As a result, S does not converge to a second-order differential operator, but in general to a differential operator of higher degree.

Let $D = d/dx$, and let S be an $(N+1)$th order differential operator of the form

$$S = D^{N+1} + \sum_{\alpha=0}^{N-1} v_\alpha D^\alpha. \tag{4.73}$$

For $n = 1, 2, 3, \cdots$, let $K_n = (S^{n/(N+1)})_+$ be the differential operator part of the pseudodifferential operator $S^{n/(N+1)}$. The flows

$$\frac{\partial S}{\partial y_n} = [K_n, S] \tag{4.74}$$

on the space of S's are the commuting flows of the Nth generalized KdV hierarchy. Arguments of Douglas [23] indicate that this hierarchy governs the double scaling limit of the N matrix chain.

In §3, we generalized intersection theory on moduli space to include a target space M, and we described general properties of the resulting models that hold for a large class of M's. It turns out that the generalized KdV hierarchies obey all of the same general properties! I refer to the second half of [19] for an explanation of this, and merely note that the Nth KdV hierarchy has a behavior similar to that of the models studied in §3 with a target space M such that the dimension of $H^*(M, \mathbb{R})$ is N (and the signature of M is 1 or 0 for odd or even N). In this correspondence, $v_{N-\alpha}$, $\alpha = 1, \cdots, N$ (or more precisely a certain differential polynomial of the form $v_{N-\alpha}+$ higher order terms), corresponds to $\langle\langle\tau_{0,1}\tau_{0,\alpha}\rangle\rangle$, where the $\tau_{0,\alpha}$ are the "primary fields" associated to a basis of $H^*(M, \mathbb{R})$. The variables $t_{n,\alpha}$ correspond to $y_{n(N+1)+\alpha}$. With this translation, the string equation of the N matrix chain has precisely the structure (3.22), and the other key conclusions of §3, such as the equations (3.27) and (3.48) that determine the genus zero and genus one correlation functions, may be deduced from standard properties of the generalized KdV hierarchies!

We do not actually believe that there is a mysterious Kähler manifold with N-dimensional cohomology that underlies the N matrix chain and on which the holomorphic curves are governed by the Nth generalized KdV hierarchy. It seems likely, though, that the model based on the N matrix chain has a geometrical interpretation in terms of an appropriate kind of intersection theory on some suitable moduli space.

Note added in proof: Recently K. Li (*Topological strings with minimal matter*, Caltech preprint CALT-68-1662) has answered the question raised in the last paragraph by showing which topological field theory coupled to topological gravity is equivalent to the N matrix model. This has been further clarified in R. Dijkgraaf and E. and H. Verlinde (*Topological strings in* $D < 1$, Institute for Advanced Study preprint, October, 1990). The interpretation of the N matrix model in algebraic geometry turns out to involve intersection theory on a cover of moduli space obtained by taking certain fractional roots of the canonical line bundle of a surface, as will be explained elsewhere (E. Witten, to appear).

Improved derivations of some of the foundational questions related to §4.3 have been given by H. Neuberger (*Regularized string and flow equations*, Rutgers preprint RU-90-50).

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