

DIRAC CHARGE QUANTIZATION AND GENERALIZED DIFFERENTIAL COHOMOLOGY

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*To the Gang Of Four*¹

The *classical* Maxwell equations, which describe electricity and magnetism in four-dimensional spacetime, may be generalized in many directions. For example, nonabelian generalizations play an important role in both geometry and physics. The equations also admit abelian generalizations in which differential forms of degree greater than two come into play. Such forms enter into high dimensional supergravity theories, so also into string theory and M-theory. There are analogs of electric and magnetic currents for these higher degree forms. In the classical theory these are also differential forms, and their de Rham cohomology classes in real cohomology (with support conditions) are the corresponding electric and magnetic charges.

In *quantum* theories *Dirac charge quantization* asserts that these charges are constrained to lie in a lattice in real cohomology. In many examples this lattice is the suitably normalized image of integer cohomology, but recently it was discovered that Ramond-Ramond charges² in Type II superstring theory lie in the suitably normalized image of complex K -theory instead. (See [39] and the references contained therein.) Furthermore, physical arguments suggest that there is a refined Ramond-Ramond charge in K -theory whose image in real cohomology is the cohomology class of the Ramond-Ramond current. Inspired by this example,

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²These are often called “D-brane charges,” but that is a misnomer. After all, in ordinary electromagnetism the notion of charge is attached to the abelian gauge field, not to the point particles, monopoles, etc. which are charged with respect to it. Similarly, Ramond-Ramond fields have associated charges. D-branes are Ramond-Ramond charged, just as point particles are electrically charged.

we argue in §2 that the group of charges associated to *any* abelian gauge field is a generalized cohomology group. The rationale is that the group of charges attached to a manifold X should depend locally on X , and generalized cohomology groups are more or less characterized as being topological invariants which satisfy locality (in the form of the Mayer-Vietoris property). The choice of generalized cohomology theory and its embedding into real cohomology affects both the lattice of charges measured by the gauge field and also the possible torsion charges. Both integral cohomology and K -theory (in many of its variations) occur in examples; I do not know an argument to rule out more exotic cohomology theories. Particular physical properties—decay processes, anomalies, etc.—are used to determine which generalized cohomology theory applies to a particular gauge field. We do not review such arguments in this paper.

Our concern instead is a more formal question: How do we implement generalized Dirac charge quantization in a functional integral formulation of the quantum theory? The quantization of charge means that the currents have the local degrees of freedom of a differential form yet carry a global characteristic class in a generalized cohomology theory. Furthermore, the fact that currents and gauge fields couple means that gauge fields are the same species of geometric object. We answer this query using *generalized differential cohomology theories*. The marriage of integral cohomology and differential forms, which we term ordinary differential cohomology, appears in the mathematics literature in two guises: as Cheeger-Simons differential characters [13] and as smooth Deligne cohomology [16]. For field theory we must go beyond differential cohomology groups and use cochains and cocycles. Again this is due to locality—gauge fields have automorphisms (gauge transformations) and we cannot cut and paste equivalence classes. For more subtle reasons electric and magnetic currents must also be refined to cocycles. Many aspects of a cocycle theory for ordinary differential cohomology are developed in [28], and for generalized cohomology theories it is an ongoing project of the author, M. Hopkins, and I. M. Singer. That theory is the mathematical foundation for the discussion in this paper; we give a provisional summary in §1. The application to abelian gauge theory is one motivation for the development of generalized differential cohomology theory, and indeed the presentation here will help shape the theory. There are other mathematical motivations for generalized differential cohomology as well.

The heart of the paper is §2, where we write the action for an abelian gauge field in the language of generalized differential cohomology. Both

electric and magnetic currents are cocycles for a differential cohomology class. The gauge field is a cochain which trivializes the magnetic current; this is a geometric version of the Maxwell equation $dF = j_B$. The electric current appears in the action; in classical electromagnetism the other Maxwell equation is the Euler-Lagrange equation. That term in the action is anomalous if there is both electric and magnetic current, and the anomaly has a natural expression (2.30) in the language of differential cohomology. It is a bilinear form in the electric and magnetic currents j_E and j_B . The various ingredients which enter the discussion are collected in Summary 2.32. We also describe how twistings of differential cohomology enter; they are closely related to orientation issues.

Our illustrations in §2 are mostly for 0-form and 1-form gauge fields. In §3 we turn to theories of more current interest, where higher degree gauge fields occur. After a brief comment on Chern-Simons theory, we focus on superstring theories in 10 dimensions. There is a new theoretical ingredient: *self-dual* gauge fields. In Definition 3.11 we specify the additional data we need to define a self-duality constraint. The main ingredient is a quadratic form, whose use in defining the partition function of a self-dual field was elucidated in [38]. Here we also observe that the same quadratic form is used to divide the usual electric coupling term by two; see (3.26) for the action of a self-dual gauge field. Therefore, the quadratic form enters into the formula for the anomaly as well. Note that for self-dual fields the electric and magnetic currents are (essentially) equal. In this paper we do not explain how the data which define the self-duality constraint are used in the quantum theory. These ideas were applied to D-branes in Type II superstring theory in [21], where the focus is anomaly cancellation. We review that argument briefly here. For the theory with nonzero Neveu-Schwarz B -field twistings of generalized cohomology play a crucial role. Indeed, the Ramond-Ramond fields are cochains in B -twisted differential K -theory. In this language a certain restriction on D-branes (equation (3.34)) appears naturally. We also explain a puzzle [4] about the formula for Ramond-Ramond charge with nonzero B -field.

At the end of §3 we treat the Green-Schwarz anomaly cancellation in the low energy limit of Type I superstring theory, including global anomalies. (As far as we know these global anomalies have not previously been discussed.) Since the charges in Type I have been shown to live in KO -theory, the 2-form gauge field is naturally interpreted in differential KO -theory. As with Type II the formulation is self-dual. But here there are background electric and magnetic currents which are present even in the absence of D-branes. Their presence is most naturally

explained in our framework by the observation that the KO quadratic form which defines the self-duality constraint is not symmetric about the origin. Rather, the center is a differential KO class which determines the background charges. The theory of this center is discussed in Appendix B, written jointly with M. Hopkins. The gauge field in Type I is a trivialization of the background magnetic current, and this leads to a constraint (3.46) in KO -theory which generalizes the usual cohomological constraint.³ For spacetimes of the form Minkowski spacetime cross a compact r -dimensional manifold the KO constraint is no new information if $r \leq 7$. The computational aspects of the anomaly cancellation in our treatment are not different than the original, though novel computations are required to relate our self-dual KO -formulation with the standard formulation in terms of a 2-form field. We also verify the local and global anomaly cancellation for D1- and D5-branes. The case of Type I theories “without vector structure” also fits naturally into our approach—it involves a twisted version of KO —but we do not develop the underlying mathematical ideas. These global anomaly cancellations are further evidence that KO -theory is the correct generalized cohomology theory for the gauge field in Type I.

The Atiyah-Singer index theorem, in a geometric form, computes the pfaffian of a Dirac operator as an integral in differential KO -, KSp -, or K -theory, depending on the dimension. In quantum field theories it appears as the anomaly of the fermionic functional integral. Sometimes these fermion anomalies cancel among each other. In the Green-Schwarz mechanism these fermion anomalies cancel against a boson anomaly: the anomaly in the electric coupling of an abelian gauge field in the presence of nonzero magnetic current. The gauge field is quantized by some flavor of K -theory, and so the anomaly in the electric coupling is also an integral in a version of differential K -theory. This idea was first presented in [21]. It indicates that gauge fields involved in this type of anomaly cancellation will always be quantized by some variation of K -theory.

Each factor in an exponentiated (effective) action is a section of a complex line bundle with metric and connection. That geometric line bundle is called the “anomaly,” and to say the anomaly cancels between two factors is to say that the tensor product of the corresponding geometric line bundles is isomorphic to the trivial bundle. To define the product of those factors as a function—so to define the partition function—one needs a choice of isomorphism. In this paper we do not address the

³Let E be the rank 32 real bundle over spacetime X^{10} . The usual constraint asserts that both X and E are spin, and that $\lambda(E) = \lambda(X)$, where $2\lambda = p_1$.

construction of such isomorphisms. It is undoubtedly true that the geometric form of the index theorem gives a canonical isomorphism between the pfaffian line bundle of a family of Dirac operators and an integral in some differential K -theory. The definition of the partition function in cases where the Green-Schwarz mechanism operates depends on this.⁴

There are two appendices. The first is a heuristic discussion of Wick rotation. We include it since some elementary points, especially in the context of self-dual gauge fields, cause confusion. As mentioned above, the second (with M. Hopkins) contains mathematical arguments needed for the anomaly cancellation in Type I.

It is a pleasure to dedicate this paper to Michael Atiyah, Raoul Bott, Fritz Hirzebruch, and Is Singer. I hope they enjoy seeing the full-blown K -theory form of the index theorem for families of Dirac operators appear in physics. Discussions with many mathematicians and physicists over a long period of time contributed to the presentation here. I particularly thank Jacques Distler and Willy Fischler for clarifying many aspects and Mike Hopkins for his collaboration on a variety of topological issues in the main text and in Appendix B.

1. Generalized differential cohomology

In differential geometry we encounter the *real* cohomology of a manifold via representative closed differential forms. In this section we describe differential geometric objects which represent *integral* generalized cohomology classes. For example, a principal circle bundle with connection is a differential geometric representative of a degree two integral cohomology class. A detailed development of the ideas outlined here is the subject of ongoing work with M. Hopkins and I. M. Singer. The treatment here is only a sketch, offered as background for the discussion of abelian gauge fields in §2.

Let Γ be a multiplicative generalized cohomology theory.⁵ We give examples shortly, but in brief Γ obeys the axioms of ordinary cohomology H except that the ring⁶ $\Gamma^*(pt)$ may differ from $H^*(pt) \cong \mathbb{Z}$. We

⁴I thank Ed Witten for emphasizing this point.

⁵For simplicity of exposition we assume that the cohomology theory Γ is multiplicative—all of our examples are—but much of what we say does not require this hypothesis.

⁶Throughout, A^\bullet denotes a \mathbb{Z} -graded abelian group $A^\bullet = \bigoplus_{q \in \mathbb{Z}} A^q$. Often it has a graded ring structure as well. If A^\bullet, B^\bullet are graded groups, then $A^\bullet \otimes B^\bullet$ is double graded. We denote the associated simply graded group as $(A \otimes B)^\bullet$.

introduce the notation

$$\pi_{-n}\Gamma = \Gamma^0(S^{-n}) = \Gamma^n(pt), \quad n \in \mathbb{Z},$$

for the cohomology of a point. (Another typical notation for this graded ring is ‘ Γ^\bullet ’.) The most important property of a generalized cohomology theory is the Mayer-Vietoris exact sequence, which we view as asserting the locality of the assignment $X \mapsto \Gamma^\bullet(X)$, where X ranges over a suitable category of finite dimensional spaces. Now after tensoring with the reals, Γ is isomorphic to ordinary real cohomology. More precisely, there is for each X a *canonical* map

$$(1.1) \quad \begin{array}{ccc} \Gamma^\bullet(X) & \longrightarrow & (H(X; \mathbb{R}) \otimes \Gamma(pt))^\bullet \\ \lambda & \longmapsto & \lambda_{\mathbb{R}} \end{array}$$

It is natural to introduce the notation

$$\pi_{-n}\Gamma_{\mathbb{R}} = \Gamma_{\mathbb{R}}^n(pt) = \Gamma^n(pt) \otimes \mathbb{R}.$$

Then the codomain of (1.1) is the (hyper)cohomology of X with coefficients in the graded ring $\pi_{-\bullet}\Gamma_{\mathbb{R}}$. The image of (1.1) is a full lattice $\bar{\Gamma}^\bullet(X) \subset H(X; \pi\Gamma_{\mathbb{R}})^\bullet$; the kernel is the torsion subgroup of $\Gamma^\bullet(X)$.

Example 1.2 (integral cohomology). There are many cochain models for integral cohomology: singular, Čech, Alexander-Spanier, etc. Such cochains have integral coefficients, and on the cochain level the map (1.1) is the standard inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. A class in the image of (1.1) is represented by a closed differential form ω on X such that $\int_Z \omega$ is an integer for all cycles Z in X .

Example 1.3 (K -theory). Historically, this is the first example of a generalized cohomology theory [8], [3]. For X compact we can represent an element of $K^0(X)$ by a $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle $E = E^0 \oplus E^1$, thought of as the formal difference $E^0 - E^1$. The cohomology ring of a point is $\pi_{-\bullet}K \cong \mathbb{Z}[[u, u^{-1}]]$, where $\deg u = 2$. The element $u^{-1} \in K^{-2}(pt) \cong K^0(S^2)$ is called the *Bott element*; multiplication by u^{-1} is the Bott periodicity map. The element u^{-1} is represented by the hyperplane (Hopf) complex line bundle over $\mathbb{C}\mathbb{P}^1 \cong S^2$. The map (1.1) is the Chern character

$$\text{ch } K^\bullet(X) \longrightarrow H(X; \mathbb{R}[[u, u^{-1}]])^\bullet.$$

Example 1.4 (KO - and KSp -theory). These are the variations of K -theory for real and quaternionic bundles, respectively. Whereas

$KO^\bullet(X)$ is a ring—the tensor product of real bundles is real— $KSp^\bullet(X)$ is not. In fact, $KSp^\bullet(X)$ is a module over $KO^\bullet(X)$ and there is also a tensor product $KSp^\bullet(X) \otimes KSp^\bullet(X) \rightarrow KO^\bullet(X)$. So it is natural to consider the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z})$ -graded theory $KOSp^\bullet = KO^\bullet \times KSp^\bullet$, which does have a multiplicative structure. Notice that the ring $\pi_{-\bullet}KOSp$ has torsion in this case. Over the reals there is an isomorphism $\pi_{-\bullet}KOSp_{\mathbb{R}} \cong \mathbb{R}[[u^2, u^{-2}]]$. The element $u^{-2} \in KSp^{-4}(pt) \cong KSp^0(S^4)$ is represented by the hyperplane (Hopf) quaternionic line bundle over $\mathbb{H}\mathbb{P}^1 \cong S^4$. Odd powers of u^{-2} are quaternionic; even powers are real. Note also that twice a quaternionic bundle (e.g. $2u^{-2}$) is real.

Differential Γ -theory, which we denote $\check{\Gamma}$, combines Γ with closed differential forms Ω_{cl} . It is defined on the category of smooth manifolds. Loosely speaking, it is the pullback in the diagram

$$(1.5) \quad \begin{array}{ccc} \check{\Gamma}^\bullet(\cdot) & \longrightarrow & \Omega_{cl}(\cdot; \pi\Gamma_{\mathbb{R}})^\bullet \\ \downarrow & & \downarrow \\ \Gamma^\bullet(\cdot) & \longrightarrow & H(\cdot; \pi\Gamma_{\mathbb{R}})^\bullet \end{array}$$

The northeast corner is the set of closed differential forms with coefficients in $\pi_{-\bullet}\Gamma_{\mathbb{R}}$. As a first approximation to $\check{\Gamma}$, for a manifold X define the group $A_{\check{\Gamma}}^\bullet(X)$ by the pullback diagram

$$\begin{array}{ccc} A_{\check{\Gamma}}^\bullet(X) & \longrightarrow & \Omega_{cl}(X; \pi\Gamma_{\mathbb{R}})^\bullet \\ \downarrow & & \downarrow \\ \Gamma^\bullet(X) & \longrightarrow & H(X; \pi\Gamma_{\mathbb{R}})^\bullet \end{array}$$

In other words, for each $q \in \mathbb{Z}$

$$(1.6) \quad A_{\check{\Gamma}}^q(X) = \left\{ (\lambda, \omega) \in \Gamma^q(X) \times \Omega_{cl}(X; \pi\Gamma_{\mathbb{R}})^q : \lambda_{\mathbb{R}} = [\omega]_{dR} \right\}.$$

Here $[\omega]_{dR}$ is the de Rham cohomology class of the form ω . But $\check{\Gamma}$ —the pullback in (1.5)—is a pullback *as a cohomology theory*.⁷ So a class

⁷There is a subtlety which we avoid in the main text. Namely, the cohomology theory whose q^{th} cohomology is Ω_{cl}^q depends on q . Precisely, on a manifold X we use the cochain complex

$$\Omega(X; \pi\Gamma_{\mathbb{R}})^q \xrightarrow{d} \Omega(X; \pi\Gamma_{\mathbb{R}})^{q+1} \xrightarrow{d} \Omega(X; \pi\Gamma_{\mathbb{R}})^{q+2} \rightarrow \dots$$

to define the theory in the northeast corner of (1.5). So for each $q \in \mathbb{Z}$ we have a pullback diagram (1.5). This leads to a *bigraded* cohomology theory in the northwest corner: the p^{th} cohomology in the q^{th} theory is denoted $\check{\Gamma}(q)^p$. The groups we call $\check{\Gamma}^q$ are the diagonal groups $\check{\Gamma}(q)^q$ in the bigraded theory.

in $\check{\Gamma}^q(X)$ is a pair (λ, ω) with $\lambda_{\mathbb{R}} = [\omega]_{\text{dR}}$ as in (1.6), together with an “isomorphism” of $\lambda_{\mathbb{R}}$ and $[\omega]_{\text{dR}}$ in $H(X; \pi\Gamma_{\mathbb{R}})^q$. If we understand cohomology classes on X to be homotopy classes of maps from X into some universal space B , then an “isomorphism” is an explicit choice of homotopy (up to homotopies of the homotopy). Even when $\lambda = \omega = 0$ there may be nontrivial isomorphisms, and this is the sense in which $\check{\Gamma}$ carries topological information beyond Γ . Equivalence classes of nontrivial isomorphisms appear as the kernel torus in the exact sequence

$$(1.7) \quad 0 \longrightarrow \frac{H(X; \pi\Gamma_{\mathbb{R}})^{q-1}}{\Gamma^{q-1}(X)} \longrightarrow \check{\Gamma}^q(X) \xrightarrow{c} A_{\Gamma}^q(X) \longrightarrow 0.$$

In some situations the kernel torus sometimes captures topological information not detected by the topological group $\Gamma^q(X)$. If $\check{\lambda} \in \check{\Gamma}^q(X)$ with $c(\check{\lambda}) = (\lambda, \omega)$ it is natural to call λ the *characteristic class* of $\check{\lambda}$ and ω the *curvature* of $\check{\lambda}$. We can rewrite this exact sequence as

$$(1.8) \quad 0 \longrightarrow \frac{\Omega(X; \pi\Gamma_{\mathbb{R}})^{q-1}}{\Omega_{\text{cl}}(X; \pi\Gamma_{\mathbb{R}})_{\Gamma}^{q-1}} \longrightarrow \check{\Gamma}^q(X) \longrightarrow \Gamma^q(X) \longrightarrow 0,$$

where $\Omega_{\text{cl}}(X; \pi\Gamma_{\mathbb{R}})_{\Gamma}^{\bullet}$ is the set of closed differential forms whose cohomology class lies in $\check{\Gamma}^{\bullet}(X)$. The second map is the characteristic class. The curvature of a differential cohomology class defined by a global $(q-1)$ -form B is the exact q -form dB . A third way to present (1.7) and (1.8) is the exact sequence

$$(1.9) \quad 0 \longrightarrow \Gamma^{q-1}(X; \mathbb{R}/\mathbb{Z}) \longrightarrow \check{\Gamma}^q(X) \longrightarrow \Omega_{\text{cl}}(X; \pi\Gamma_{\mathbb{R}})_{\Gamma}^q \longrightarrow 0.$$

The second map is the curvature. The kernel is the set of “flat” differential cohomology classes, an abelian group whose identity component is the kernel torus in (1.7) and whose group of components is the torsion subgroup of $\Gamma^q(X)$.

As with topological cohomology theories there are many possible ways to represent classes in differential cohomology theories. In computations we are free to use whichever model is most convenient. We use the usual notations $\check{C}_{\Gamma}^{\bullet}(X)$, $\check{Z}_{\Gamma}^{\bullet}(X)$, $\check{B}_{\Gamma}^{\bullet}(X)$ for cochains, cocycles, and coboundaries in a given model for $\check{\Gamma}$. (This is schematic, as models do not necessarily involve cochain complexes.) In any model we construct a category⁸ whose set of equivalence classes is $\check{\Gamma}^{\bullet}(X)$. The homotopy

⁸We work in the bigraded theory. Then the objects in the category form a set, the set of cocycles $\check{Z}_{\Gamma}(q)^q(X)$. If $\check{a}', \check{a} \in \check{Z}_{\Gamma}(q)^q(X)$, then a morphism $\check{b} \check{a}' \rightarrow \check{a}$ is a cochain $\check{b} \in \check{C}_{\Gamma}(q)^{q-1}(X)$ such that $\check{a} = \check{a}' + d\check{b}$ in $\check{Z}_{\Gamma}(q)^q(X)$, but we take such

theory neatly encodes the categorical (and multi-categorical) structure in cochain complexes, or better in spaces of maps. We need the notion of a “trivialization” of a cocycle $\check{a} \in \check{Z}_\Gamma^q(X)$. For our purposes⁹ we take it to be a cochain $\check{b} \in \check{C}_\Gamma^{q-1}(X)$ such that $d\check{b} = \check{a}$. The meaning of ‘ d ’ in this equation depends on the model. Associated to \check{b} is a differential form $\eta \in \Omega^{q-1}(X; \pi\Gamma_\mathbb{R})$ —the *covariant derivative* of \check{b} —such that $d\eta = \omega$, where $\omega \in \Omega_{\text{cl}}^q(X; \pi\Gamma_\mathbb{R})$ is the curvature of \check{a} .

We next give some explicit models for $\Gamma = H$ (integral cohomology) and $\Gamma = K$ (complex K -theory). The differential cohomology groups $\check{H}^\bullet(X)$ are also known as the groups of *Cheeger-Simons differential characters* [13] or as the *smooth Deligne cohomology* groups [16].

Example 1.10 (differential cohomology [28]). We represent an element of $\check{H}^q(X)$ by a triple

$$(c, h, \omega) \in C^q(X; \mathbb{Z}) \times C^{q-1}(X; \mathbb{R}) \times \Omega^q(X)$$

of differentiable singular cochains [36, §5.31] and differential forms which satisfy

$$\begin{aligned} \delta c &= 0, \\ d\omega &= 0, \\ \delta h &= \omega - c_\mathbb{R}. \end{aligned}$$

In the last equation we view the differential form ω as a singular cochain by integration over (smooth) chains. This last equation very directly expresses the pullback diagram (1.5); h is the isomorphism of the images of c, ω in a set of cochains representing $H^\bullet(X; \mathbb{R})$. We also have maps of such triples:

$$(s, t)(c, h, \omega) \longrightarrow (c', h', \omega'),$$

cochains up to equivalence. An equivalence $\check{c} \check{b}' \rightarrow \check{b}$ is a cochain $\check{c} \in \check{C}_\Gamma(q)^{q-2}(X)$ with $\check{b} = \check{b}' + d\check{c}$ in $\check{Z}_\Gamma(q)^{q-1}(X)$. The group of automorphisms of any cocycle is $\check{\Gamma}(q)^{q-1}(X) \cong \Gamma^{q-1}(X) \otimes \mathbb{R}/\mathbb{Z}$. The construction of a category from a cochain complex is standard. It may be continued to construct higher categories as well.

⁹There are different notions of trivialization, and they appear naturally in the bigraded theory. The most useful notion makes precise the one mentioned in the text: A trivialization of a cocycle $\check{a} \in \check{Z}_\Gamma(q)^q(X)$ is a cochain $\check{b} \in \check{C}_\Gamma(q-1)^{q-1}(X)$ such that $d\check{b} = \check{a}$ in $\check{C}_\Gamma(q-1)^q(X)$. A map $\check{c} \check{b}' \rightarrow \check{b}$ of trivializations is then a cochain $\check{c} \in \check{C}_\Gamma(q-1)^{q-2}(X)$ with $\check{b} = \check{b}' + d\check{c}$ in $\check{C}_\Gamma(q-1)^{q-1}(X)$. One can go on to discuss equivalence classes of such maps and make a category of trivializations of \check{a} , analogous to the discussion in the previous footnote.

where $s \in C^{q-1}(X; \mathbb{Z})$, $t \in C^{q-2}(X; \mathbb{R})$, and

$$\begin{aligned} c' &= c + \delta s, \\ \omega' &= \omega, \\ h' &= h - s_{\mathbb{R}} - \delta t. \end{aligned}$$

In the category representing $\check{H}^q(X)$ we equate maps (s, t) and $(s + \delta e, t - e_{\mathbb{R}} - \delta f)$ for $e \in C^{q-2}(X; \mathbb{Z})$, $f \in C^{q-3}(X; \mathbb{R})$.

This may be more neatly formulated as a cochain complex whose q^{th} cohomology is $\check{H}^q(X)$.

Example 1.11 (differential cohomology). Whereas the last model was based on singular cochains, this model is based on Čech theory. Fix $q \geq 0$. Let $\{U_i\}_{i \in I}$ be an open cover of X with ordered index set I , and set

$$\check{C}^{r,s}(X) = \begin{cases} 0, & s < -1 \text{ or } s > q - 1; \\ \prod_{i_0 < \dots < i_p} C^0(U_{i_0} \cap \dots \cap U_{i_r} \rightarrow \mathbb{Z}), & s = -1; \\ \prod_{i_0 < \dots < i_p} \Omega^s(U_{i_0} \cap \dots \cap U_{i_r}), & 0 \leq s \leq q - 1. \end{cases}$$

Then $\check{C}^{\bullet, \bullet}$ is a double complex with the Čech differential δ of degree $(1, 0)$ and a differential \check{d} of degree $(0, 1)$ defined by

$$\check{d} = \begin{cases} \text{inclusion} & \text{on } \check{C}^{r,-1}; \\ d & \text{on } \check{C}^{r,s}, 0 \leq s \leq q - 1. \end{cases}$$

The degree q cohomology of the total complex is $\check{H}^q(X)$. This is the model which is described, for example, in [23, §6].

Example 1.12 (differential K -theory). Here we only discuss models for \check{K}^0 . Our first model requires fixing an infinite dimensional manifold B whose homotopy type is the classifying space $\mathbb{Z} \times BU$ of K^0 . There are many possibilities, for example the space of Fredholm operators on a separable complex Hilbert space. For complex K -theory we have $\pi_{-\bullet} K_{\mathbb{R}} = K_{\mathbb{R}}^{\bullet}(pt) \cong \mathbb{R}[[u, u^{-1}]]$. Fix a closed differential form $\omega_B \in \Omega_{\text{cl}}(B; \pi K_{\mathbb{R}})^0$ which represents the Chern character of the universal K -theory class on B . Then

$$\omega_B = \omega_B^0 + \omega_B^1 u^{-1} + \omega_B^2 u^{-2} + \dots,$$

where $\omega_B^i \in \Omega^{2i}(B)$ represents the i^{th} universal Chern character class. A representative of an element in $\check{K}^0(X)$ is a triple

$$(f, \eta, \omega) \in \text{Map}(X, B) \times \Omega(X; \pi K_{\mathbb{R}})^{-1} \times \Omega(X; \pi K_{\mathbb{R}})^0$$

with

$$\begin{aligned} d\omega &= 0 \\ d\eta &= \omega - f^*\omega_B. \end{aligned}$$

Let $\pi [0, 1] \times X \rightarrow X$ be projection. Then a map of triples is

$$(F, \sigma) (f, \eta, \omega) \longrightarrow (f', \eta', \omega'),$$

where $F [0, 1] \times X \rightarrow B$ and $\sigma \in \Omega(X; \pi K_{\mathbb{R}})^{-2}$ satisfy

$$\begin{aligned} F_0 &= f \\ F_1 &= f' \\ \omega' &= \omega \\ \eta' &= \eta + \pi_* F^* \omega_B + d\sigma. \end{aligned}$$

There is an equivalence relation on maps (F, σ) : the maps (F_0, σ_0) and (F_1, σ_1) are equivalent if there exists a homotopy $\tilde{F} [0, 1] \times [0, 1] \times X \rightarrow B$ from F_0 to F_1 and a form $\phi \in \Omega(X; \pi K_{\mathbb{R}})^{-3}$ such that $\sigma_1 = \sigma_0 + \Pi_* \tilde{F}^* \omega_B + d\phi$, where $\Pi [0, 1] \times [0, 1] \times X \rightarrow X$ is projection.

Example 1.13 (differential K -theory). Next we give a more geometric picture of elements in $\check{K}^\bullet(X)$, though we do not give a complete “cochain model” which computes differential K -theory. In other words, we do not specify maps between representatives.

Simply stated: A vector bundle $E \rightarrow X$ with connection ∇ represents an element of $\check{K}^0(X)$. Certainly (E, ∇) determines a pair $(\lambda, \omega) \in A_K^0(X)$. Namely, $\lambda \in K^0(X)$ is the equivalence class of E and $\omega = \text{ch}(\nabla)$ is the Chern-Weil representative of the Chern character using the connection ∇ . To make contact with our previous model, assume $\omega_B = \text{ch}(\nabla_B)$ is the Chern character form of a universal vector bundle with connection (E_B, ∇_B) on B . Choose a classifying map $\tilde{f} E \rightarrow E_B$ and let $f X \rightarrow B$ be the induced map. Then $\tilde{f}^* \nabla_B$ is connection on E , and there is a secondary (Chern-Simons) form $\eta = \eta(\tilde{f}^* \nabla_B, \nabla)$ with $d\eta = f^* \omega_B - \omega$. This gives a triple (f, ω, η) as in our previous model.

We can also use Quillen’s superconnections [33] to represent elements of $\check{K}^0(X)$. A superconnection on $E = E^0 \oplus E^1$ has a 0-form piece which is a pair of maps $E^0 \rightrightarrows E^1$. We can allow E^0, E^1 to be infinite dimensional if we restrict these maps to be Fredholm. Such infinite dimensional superconnections play a prominent role in Bismut’s treatment of index theory [6]. The expression of that work and of other geometric developments in index theory in terms of differential K -theory is part of ongoing

research. We learned recently that some versions of differential K -theory and close relatives, together with applications to index theory, appear in the literature. See [30] and the references therein.

There are other, more geometric, models for differential cohomology in low degree. First, we have

$$\begin{aligned}\check{H}^0(X) &\cong H^0(X; \mathbb{Z}), \\ \check{H}^1(X) &\cong \text{Map}(X, \mathbb{R}/\mathbb{Z}).\end{aligned}$$

A circle bundle with connection represents an element of $\check{H}^2(X)$. There are various concrete models for elements of $\check{H}^3(X)$, often called “circle gerbes with connection”. See [10], [27], [11], [24] for example.

Multiplication and pushforward on $\check{\Gamma}$ are induced from the corresponding operations on Γ and Ω_{cl} . Explicit formulas for these operations depend on the particular cochain model. Multiplication in $\check{\Gamma}$ combines multiplication in Γ and Ω_{cl} . Thus the map c in (1.7) is a ring homomorphism. In particular, if $\check{\lambda}, \check{\lambda}'$ are differential cohomology classes with curvatures ω, ω' , and we have a (locally defined) form α with $d\alpha = \omega$, then the curvature of the product $\check{\lambda} \cdot \check{\lambda}'$ is (locally) the differential of $\alpha \wedge \omega'$. Pushforward is defined for suitably oriented maps. In this paper we encounter fiber bundles $X \rightarrow T$ with compact fibers and inclusions $i: W \hookrightarrow X$ of submanifolds. For a fiber bundle $X \rightarrow T$ we need at least to orient the tangent bundle along the fibers in topological Γ -cohomology. For ordinary cohomology this suffices. For the various forms of K -theory we also need a Riemannian structure on the family, i.e., a Riemannian metric on the relative tangent bundle $T(X/T)$ and a distribution of horizontal planes on X . This data determines a Levi-Civita connection on $T(X/T)$. (See [20, §1].) We call $X \rightarrow T$ a “Riemannian fiber bundle.” Pushforward is integration along the fibers

$$(1.14) \quad \int_{X/T} \check{\Gamma}^\bullet(X) \longrightarrow \check{\Gamma}^{\bullet-n}(T),$$

where the relative dimension is n . This map refines to a map on cochain representatives, and suitable versions of Stokes’ theorem hold for this extension. For example, if the fibers are closed and a cochain \check{b} is a trivialization of a cocycle \check{a} , then $\int_{X/T} \check{b}$ is a trivialization of $\int_{X/T} \check{a}$. For an inclusion $i: W \hookrightarrow X$ we must orient the normal bundle to W in X in Γ -cohomology and also choose a smooth closed differential form Poincaré dual to W . Then pushforward

$$i_* \check{\Gamma}^\bullet(W) \longrightarrow \check{\Gamma}^{\bullet+r}(X)$$

is defined, where r is the codimension of W in X . Curvature does not commute with pushforward. For example, if $X \rightarrow T$ is a $\check{\Gamma}$ -oriented fiber bundle, then there is a closed differential form $\hat{A}_\Gamma(X/T)$ on X so that if $\check{\lambda} \in \check{\Gamma}^\bullet(X)$ has curvature ω , then the curvature of $\int_{X/T} \check{\lambda}$ is

$$(1.15) \quad \int_{X/T} \hat{A}_\Gamma(X/T) \wedge \omega.$$

For integral cohomology this form is the constant $\hat{A}_H(X/T) \equiv 1$; for K -theory it is $\hat{A}(X/T) \wedge e^{\eta/2}$, where \hat{A} is the usual \hat{A} -genus of the curvature and $-2\pi i \eta$ the curvature of a spin^c connection on X . For KO - and KSp -theory it is¹⁰ $\hat{A}(X/T)$.

Generalized cohomology theories admit twistings, and so too do generalized differential cohomology theories. For example, if $F \rightarrow X$ is a flat real vector bundle, then $H^\bullet(X; F)$ is a twisted version of real cohomology. It may be computed by an extension of the de Rham complex to F -valued forms. Quite generally, for any cohomology theory E (which could be a topological theory $E = \Gamma$ or a differential theory $E = \check{\Gamma}$) a real vector bundle $V \rightarrow X$ determines a one-dimensional twisting $\zeta(V) = \zeta_E(V)$ of $E^\bullet(X)$. We denote the $\zeta(V)$ -twisted E -cohomology as $E^{\bullet+\zeta(V)}(X)$. Then there is a Thom homomorphism

$$(1.16) \quad E^{\bullet+\zeta(V)}(X) \longrightarrow E_{cv}^{\bullet+r}(V),$$

where $\text{rank } V = r$. In the codomain we use cohomology with compact vertical support. In topological theories (1.16) is an isomorphism, but in differential theories it only has a left inverse. For a manifold X we use the notation $\zeta(X) = \zeta(TX)$ for the twisting derived from its tangent bundle. An E -orientation of V is a trivialization of the twisting $\zeta(V)$, which then induces an isomorphism $E^{\bullet+\zeta(V)}(X) \cong E^\bullet(X)$. For example, in ordinary

¹⁰Write the curvature of an element $\check{\lambda} \in (KO)^0(X)$ as $\omega = \omega_0 + \omega_4 u^{-2} + \omega_8 u^{-4} + \dots$, where ω_i are the Chern character forms of the complexification $\check{\lambda}_\mathbb{C}$. If, for example, $\dim X/T = 4$, then since $\hat{A}(X/T) = 1 - p_1(X/T)/24 + \dots$ we have

$$\int_{X/T} \hat{A}(X/T) \wedge \omega = \int_{X/T} u^{-2} (\omega_4 - \omega_0 p_1(X/T)/24) + \dots,$$

and the curvature of $\int_{X/T} \check{\lambda}$ is

$$(2u^{-2}) \int_{X/T} \frac{1}{2} (\omega_4 - \omega_0 p_1(X/T)/24) + \dots$$

Since $2u^{-2}$ is the generator of $KO^{-4}(pt) \cong \mathbb{Z}$, the coefficient (with the factor 1/2) computes the KO index.

cohomology $\zeta(V) = w_1(V) \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ is the first Stiefel-Whitney class, the characteristic class of the real line bundle $\text{Det } V$. In topological K -theory $\zeta(V) = (w_1(V), W_3(V)) \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \times H^3(X; \mathbb{Z})$, so a K -theory orientation of V is an orientation in the usual sense (trivialization of $w_1(V)$) together with a spin^c structure (trivialization of $W_3(V)$). In differential K -theory the twisting class is

$$(1.17) \quad \check{\zeta}(V) = (w_1(V), \check{w}_2(V)) \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \times \check{H}^3(X),$$

where we use the map $H^2(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(X; \mathbb{R})/H^2(X; \mathbb{Z}) \rightarrow \check{H}^3(X)$ (cf. (1.7)) to regard the second Stiefel-Whitney class as a differential cohomology class (“flat gerbe”) of order two. A twisting in a generalized differential cohomology theory induces a twisting of differential forms. If $\check{\zeta}(V)$ is the twisting of a real vector bundle V , the induced twisting on forms is by the real line bundle $\text{Det } V \rightarrow X$. When V is the tangent bundle to X , then $\text{Det } V$ is the orientation bundle. A twisted n -form is simply a density. (See [18, §2.2] for more about twisted forms and densities.)

Pushforward is defined using the Thom isomorphism, so without choice of topological orientation makes sense in twisted cohomology. For example, for suitable¹¹ fiber bundles $X \rightarrow T$ we have

$$(1.18) \quad \int_{X/T} \check{\Gamma}^{\bullet-\check{\zeta}(X/T)}(X) \longrightarrow \check{\Gamma}^{\bullet-n}(T),$$

where $\check{\zeta}(X/T) = \check{\zeta}(T(X/T))$ is the twisting class of the tangent bundle along the fibers.

2. Gauge fields and quantization

In a classical nonrelativistic formulation Maxwell’s equations concern a time-varying electric field $E \in \Omega^1(\mathbb{R}^3)$, a time varying magnetic field $B \in \Omega^2(\mathbb{R}^3)$, a time-varying electric current $J_E \in \Omega^2(\mathbb{R}^3)$, and a time varying electric charge density¹² $\rho_E \in \Omega^3(\mathbb{R}^3)$. The relativistic invariance is manifest if we work instead on Minkowski space-time $M^4 = \mathbb{R}_t \times \mathbb{R}^3$, where t is the time coordinate and the speed of light

¹¹For the various forms of K -theory we need a Riemannian fiber bundle; for ordinary cohomology (1.18) is true for any fiber bundle.

¹²We write ρ_E, J_E as forms, rather than densities, using the canonical orientation of \mathbb{R}^3 . We discuss the role of orientation at the end of this section. Using the standard metric and volume form on \mathbb{R}^3 as well, we can write E, B, J_E as vector fields and ρ_E as a function.

has been set to unity. Introduce

$$\begin{aligned} F &:= B - dt \wedge E && \in \Omega^2(M^4), \\ j_E &:= \rho_E - dt \wedge J_E && \in \Omega^3(M^4). \end{aligned}$$

Then Maxwell's equations assert

$$(2.1) \quad \begin{aligned} dF &= 0, \\ d * F &= j_E. \end{aligned}$$

With an eye towards generalizations we introduce a magnetic current $j_B \in \Omega^3(M^4)$ and allow dF to be nonzero:

$$(2.2) \quad \begin{aligned} dF &= j_B, \\ d * F &= j_E. \end{aligned}$$

This version of Maxwell's equations is our starting point.

The form F is called the *field strength*, j_E the *electric current*, and j_B the *magnetic current*. We assume that on any spacelike slice j_E, j_B have compact support (or more generally satisfy some integrability condition). The integral of j_E (j_B) over a spacelike slice $N \cong \mathbb{R}^3$ is the total *electric* (*magnetic*) *charge*. Maxwell's equations (2.2) imply that the currents j_E, j_B are closed, and as a consequence the total charge is constant in time. Letting j denote either of the closed forms j_E or j_B we have a de Rham cohomology interpretation of the charges \overline{Q}_E and \overline{Q}_B :

$$(2.3) \quad \overline{Q} = [j|_N]_{dR} \in H_c^3(N; \mathbb{R}).$$

The subscript 'c' indicates that the cohomology is taken with compact support. Notice that the field strength F need not have compact support, so equation (2.2) does not imply that the charge vanishes. However, it does imply

$$(2.4) \quad \overline{Q} \in \ker(H_c^3(N; \mathbb{R}) \longrightarrow H^3(N; \mathbb{R})).$$

When $N \cong \mathbb{R}^3$ this refinement is vacuous, but for more general manifolds N (of higher dimension, for example) it may be nontrivial. For example, if N is compact (2.4) implies that $\overline{Q} = 0$. For a similar discussion of charge, see [32, §2].

So far we have presented the equations of *classical* electromagnetism. A new feature enters in the *quantum* theory: charge is quantized. *Dirac charge quantization* asserts that in appropriate units the total charge

is an integer. Equivalently, the cohomology class \bar{Q} in (2.3), (2.4) lies in the image of the map $H_c^3(N; \mathbb{Z}) \rightarrow H_c^3(N; \mathbb{R})$. This is the correct quantization condition for Maxwell theory. For general abelian gauge fields integral cohomology may be replaced by a generalized cohomology theory, as we now explain.

We work in arbitrary dimensions and allow space N to be any oriented Riemannian manifold of dimension $n - 1$. Let F be an (abelian) field strength and j_E, j_B currents. These are real differential forms on $\mathbb{R} \times N$. We allow F to have arbitrary degree; it may or may not be homogeneous. (Shortly we will consider F as a form with coefficients, as in §1.) The degrees of the currents are then determined from (2.2). The currents are assumed to have compact support on spacelike slices, but we do not make the support condition explicit in the notation. Quantization of the charge associated to F means that the integral of a current j over a closed cycle in N is not an arbitrary real number, but rather takes discrete values. Therefore, the charge $\bar{Q} = [j]$ is restricted to lie in a lattice $\bar{\Gamma}^\bullet(N) \subset H^\bullet(N; \mathbb{R})$. It is natural from a mathematical point of view—there are physical arguments which motivate this—to postulate an abelian group $\Gamma^\bullet(N)$ and a map $\Gamma^\bullet(N) \rightarrow H^\bullet(N; \mathbb{R})$ with image $\bar{\Gamma}^\bullet(N)$ such that the charges $\bar{Q} \in \bar{\Gamma}^\bullet(N)$ are refined to charges $Q \in \Gamma^\bullet(N)$. Furthermore, the locality of quantum field theory implies that the group of possible charges $\Gamma^\bullet(N)$ should depend locally on N . As stated at the beginning of §1, locality is a characteristic feature of generalized cohomology theories whose expression is the Mayer-Vietoris exact sequence. We are led, then, to postulate that the group of charges $\Gamma^\bullet(N)$ assigned to a space N is a generalized cohomology group. We will not discuss the physical motivation behind the choice of Γ and the choice of map to real cohomology. There are detailed discussions of particular cases in the physics literature (most recently concerning Ramond-Ramond fields in Type II superstring theory and its close cousins). Notice that different choices of Γ and the map to real cohomology lead to different lattices $\bar{\Gamma}^\bullet(N)$, so to different quantization conditions on charges measured around cycles. Also, different choices of Γ lead to different torsion phenomena for charges.

Now we Wick rotate to Euclidean field theory and formulate the theory on oriented¹³ Riemannian manifolds X of dimension n . (Appendix A reviews Wick rotation in general terms, so provides the setting for our discussion here.) We will not specify explicit support conditions on currents, though the reader should keep in mind the compact spatial

¹³At the end of this section we relax the orientation assumption.

support condition above on manifolds of the form $X = \mathbb{R} \times N$. Correlation functions in the Euclidean theory are defined (formally) by a functional integral over Euclidean fields using a Euclidean action. Our task is to describe the Euclidean fields and Euclidean action precisely. We implement our conclusion in the previous paragraph in the Euclidean setting by choosing: (i) a generalized cohomology theory Γ , and (ii) a map

$$(2.5) \quad \Gamma^\bullet(X) \rightarrow H(X; \pi\Gamma_{\mathbb{R}})^\bullet$$

to real cohomology. This is precisely the data we need to define differential Γ -theory $\check{\Gamma}$. Now given a generalized cohomology theory Γ there is a canonical map (1.1), and any other map (2.5) is obtained by multiplying by an invertible element in $H(X; \pi\Gamma_{\mathbb{R}})^0$. In gauge theory there is an invertible closed differential form

$$(2.6) \quad \omega_X \in \Omega_{\text{cl}}(X; \pi\Gamma_{\mathbb{R}})^0$$

which represents this class; it depends locally on X in a suitable sense. Then as we will see shortly, it is then natural to lift the currents j to differential cohomology classes $\check{j} \in \check{\Gamma}^\bullet(X)$ whose image in $A_\Gamma^\bullet(X)$ is¹⁴

$$(2.7) \quad c(\check{j}) = (Q, \frac{j}{\omega_X}).$$

(Recall the definition of A_Γ^\bullet from (1.6).) Note that \overline{Q} is $[\omega_X]_{dR}$ times the image of Q under (2.5). A particularly good choice for the normalizing factor (2.6) is

$$(2.8) \quad \omega_X = 2\pi\sqrt{\hat{A}_\Gamma(X)},$$

where $\hat{A}_\Gamma(X)$ is defined in (1.15). (The 2π is convention; the $\sqrt{\hat{A}_\Gamma(X)}$ is to make bilinear pairings in Γ compatible with integration of curvatures.) To make sense of the first Maxwell equation $dF = j_B$ we must refine the field strength F to differential Γ -theory as well. We will see that in fact we must lift F and j to *cocycles* representing generalized differential cohomology classes. (If $j_B \neq 0$ then F is lifted to a cochain rather than a cocycle.) In this setting the differential forms F and j have coefficients in $\pi_{-\bullet}\Gamma_{\mathbb{R}}$. We proceed with the construction of the Euclidean theory after describing two motivating examples.

¹⁴We continue to use “charges” Q, \overline{Q} in the Euclidean setting. The physical interpretation of these quantities as charges is in the Hamiltonian situation $X = \mathbb{R} \times N$ after rotating back to real time.

Example 2.9 (typical p -form gauge field). Suppose F is a homogeneous form of degree $p + 1$, so that $\deg j_B = p + 2$ and $\deg j_E = n - p$. Typically the quantization law asserts that charges lie in integral cohomology $\Gamma = H$ with $\omega_X = 2\pi$. In other words, the pair (Q_E, Q_B) lives in $H^{p+2}(X; \mathbb{Z}) \oplus H^{n-p}(X; \mathbb{Z})$.

Example 2.10 (Ramond-Ramond fields). These occur in the low energy field theory description of the Type II superstring. Here X^{10} is a spin Riemannian manifold. If the B -field vanishes, then the Ramond-Ramond charges naturally live in K -theory. (For a physical discussion of the choice of K -theory, see [39] and the references therein.) The charge is a homogeneous class in $K^\bullet(X)$, and by Bott periodicity only the parity of the degree matters. The parity is odd in Type IIA and even in Type IIB. For definiteness we suppose the charge lives in $K^1(X)$ in Type IIA and $K^0(X)$ in Type IIB. (For Type IIB—and probably for any theory—the charge is in $\epsilon^{-1}(0)$ for $\epsilon: K^0(X) \rightarrow H^0(X)$ the augmentation.) Thus the Ramond-Ramond field strengths and currents are refined to differential K -theory classes of degree 0 and -1 , respectively. In this case (2.8) is $\omega_X = 2\pi\sqrt{\hat{A}(X)}$. Also, there is a self-duality condition which enters in the construction of the functional integral. We discuss self-dual fields in §3.

The B -field, which heretofore was assumed zero, is locally a 2-form. Its field strength, usually denoted H , is a closed 3-form on X which obeys integrality constraints corresponding to integral cohomology. In other words, we postulate a class $\zeta \in H^3(X)$ with $\zeta_{\mathbb{R}} = [H]_{dR}$, and suppose that globally the B -field is a cocycle $\check{\zeta}$ for a class in the differential cohomology group $\check{H}^3(X)$. Then the Ramond-Ramond charges live in twisted K -theory $K^{\bullet+\zeta}(X)$, and currents are lifted to twisted differential K -theory $\check{K}^{\bullet+\zeta}(X)$.

There is a lagrangian formulation of the classical Maxwell equations (2.1) (with no magnetic current) in the classical Lorentzian theory. The field variable is a gauge field. The first Maxwell equation is the Bianchi identity and holds off-shell. The second Maxwell equation is the variational equation of a classical action for the gauge field. As stated above, our task is to incorporate Dirac charge quantization into the Wick rotated Euclidean theory. (We summarize our answer in Summary 2.32.) As we have seen charge quantization means choosing a generalized cohomology theory Γ and an embedding (2.5), refined to a differential form (2.6). We begin with the case where the currents j_E

and j_B both vanish. Fix a degree¹⁵ \mathbf{d} . Then we: (i) refine the characteristic class $[F/\omega_X]_{dR}$ of the normalized field strength $F \in \Omega_{cl}(X; \pi\Gamma_{\mathbb{R}})^{\mathbf{d}}$ to a class $\lambda \in \Gamma^{\mathbf{d}}(X)$, and (ii) refine the field strength itself to a cohomology class $\check{F} \in \check{\Gamma}^{\mathbf{d}}(X)$ such that

$$c(\check{F}) = \left(\lambda, \frac{F}{\omega_X} \right).$$

Here we use a multi-normalization $\omega_X = ((\omega_X)_1, \dots, (\omega_X)_k)$ corresponding to the components of $F = (F_1, \dots, F_k)$. The differential cohomology group $\check{\Gamma}^{\mathbf{d}}(X)$ is the space of abelian gauge fields (or gauge potentials) *up to gauge transformations*. It is the space over which one integrates in the Euclidean functional integral. Cocycles in $\check{Z}_{\Gamma}^{\mathbf{d}}(X)$ representing a class $\check{F} \in \check{\Gamma}^{\mathbf{d}}(X)$ are particular *gauge fields*. If $\check{A} \in \check{Z}_{\Gamma}^{\mathbf{d}}(X)$ is such a cocycle, we denote its cohomology class in $\check{\Gamma}^{\mathbf{d}}(X)$ as \check{F}_A . Cocycles are the proper variables for field theory—they are local. We are led, then, to a Euclidean theory in which the space of fields is the category $\check{Z}_{\Gamma}^{\mathbf{d}}(X)$ of cocycles (of particular degrees). The Wick rotated version of the classical Lorentzian action makes sense for our refined fields: If $\check{A} \in \check{Z}_{\Gamma}^{\mathbf{d}}(X)$ is a field with curvature F_A/ω_X , then the Euclidean action is

$$(2.11) \quad S(\check{A}) = \frac{1}{2e^2} \int_X F_A \wedge *F_A.$$

Here $e = (e_1, \dots, e_k)$ is a set of coupling constants, and the notation implies a sum over components:

$$S(\check{A}) = \sum_{i=1}^k \frac{1}{2e_i^2} \int_X (F_A)_i \wedge *(F_A)_i.$$

Since the curvature depends only on the cohomology class \check{F}_A of \check{A} in $\check{\Gamma}^{\mathbf{d}}(X)$, the action is gauge-invariant. Of course, in writing (2.11) we are implicitly assuming either that X is compact or some support condition on the fields.

Example 2.12 (1-form gauge field). Let X^n be an oriented n -manifold and suppose $F \in \Omega_{cl}^2(X)$ obeys quantization using integral cohomology (with $\omega_X = 2\pi$). As a model for $\check{Z}_{\check{H}}^2(X)$ we take the category of connections on principal circle bundles over X . A gauge field \check{A}

¹⁵We use a multi-index notation

$$\mathbf{d} = (d_1, \dots, d_k).$$

Then if A^\bullet is a graded group, an element $a \in A^{\mathbf{d}}$ is a k -tuple (a_1, \dots, a_k) with $a_i \in A^{d_i}$. Arithmetic is done componentwise. For example, $\mathbf{d} + 1 = (d_1 + 1, \dots, d_k + 1)$.

is such a connection and \check{F}_A its equivalence class under isomorphisms of circle bundles with connection. The field strength F_A is $\sqrt{-1}$ times the curvature of \check{A} ; the characteristic class is the first Chern class. If the characteristic class vanishes, then the gauge field may be represented by a global 1-form, uniquely up to differentials of circle-valued functions (see (1.8)).

Example 2.13 (periodic scalar field). Again X^n is an oriented n -manifold. Suppose $F \in \Omega_{\text{cl}}^1(X)$, with quantization specified by integer cohomology. Now $\check{Z}_H^1(X)$ may simply be taken to be the set $\text{Map}(X, \mathbb{R}/\mathbb{Z})$ of periodic real scalar fields on X . Taking $\omega_X = 2\pi$ a gauge field is a map $\phi: X \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ and its field strength is $d\phi$. Then (2.11) is the usual action $\frac{1}{2e^2} \int_X d\phi \wedge *d\phi$. It is convenient in this case to write formulas in terms of the exponentiated circle-valued scalar field $e^{i\phi}: X \rightarrow \mathbb{T}$.

Next, we allow the currents to be nonzero. First, consider $j_E \neq 0$. In the classical Lorentzian theory there is an additional term in the action whose variation gives the right-hand side of the second Maxwell equation¹⁶ in (2.1). To write its Wick rotation in our framework we need to postulate maps¹⁷

$$\begin{aligned}
 (2.14) \quad & \Gamma^1 \longrightarrow H^1, \\
 & \Gamma^2 \longrightarrow H^2, \\
 & \pi_{-\bullet} \Gamma_{\mathbb{R}} \longrightarrow \pi_{-\bullet} H_{\mathbb{R}} \cong \mathbb{R}.
 \end{aligned}$$

Using the pullback square (1.5) there are induced maps

$$\begin{aligned}
 (2.15) \quad & \check{\Gamma}^1 \longrightarrow \check{H}^1 \\
 & \check{\Gamma}^2 \longrightarrow \check{H}^2.
 \end{aligned}$$

We must also assume that X is $\check{\Gamma}$ -oriented so that integration over X in $\check{\Gamma}$ -theory is defined. Recall that the closed differential form j_E is refined to a differential cohomology class $\check{j}_E \in \check{\Gamma}^{n-d+1}(X)$. (See (2.7).) The additional term in the Euclidean action is the purely imaginary expression

$$(2.16) \quad 2\pi i \int_X \check{j}_E \cdot \check{F}_A.$$

¹⁶The coupling constant is $e^2 = 2\pi$ in (2.1).

¹⁷For complex K -theory there are natural “determinant” maps $K^1 \rightarrow H^1$ and $K^2 \rightarrow H^2$. The map $\pi_{-\bullet} K_{\mathbb{R}} \cong \mathbb{R}[[u, u^{-1}]] \rightarrow \pi_{-\bullet} H_{\mathbb{R}} \cong \mathbb{R}$ sends u to zero. We will also make use of the natural “pfaffian” map $KSp^2 \rightarrow H^2$. (See [20, §3]; note $KSp^2 \cong KO^{-2}$ by Bott periodicity.) The map in (2.14) in degree 1 is used in (2.17); that in degree 2 is used later in (2.30). For general Γ maps to low degree cohomology with coefficients exist canonically, using the Postnikov tower, and with some choice one produces (2.14).

The product takes place in $\check{\Gamma}^\bullet(X)$, and the degrees are such that the integrand is an element of¹⁸ $\check{\Gamma}^{n+1}(X)$. Hence the integral lands in $\check{\Gamma}^1(pt)$, and using (2.15) we map to $\check{H}^1(pt) \cong \mathbb{R}/\mathbb{Z}$. Therefore, the exponentiated action

$$(2.17) \quad e^{-S}(\check{A}) = \exp\left(-\frac{1}{2e^2} \int_X F_A \wedge *F_A\right) \exp\left(-2\pi i \int_X \check{j}_E \cdot \check{F}_A\right)$$

is a well-defined complex number.

Several comments are in order. First, it is more illuminating to work with a family of gauge fields \check{A} parametrized by a manifold T . Then the exponentiated action is a function $e^{-S} T \rightarrow \mathbb{C}$ and we can use Stokes' theorem to differentiate the second term of the action. Also, the fact that (2.17) depends on \check{A} only through F_A, \check{F}_A means that the exponentiated action e^{-S} is gauge-invariant. Finally, for gauge fields quantized by integer cohomology, as in Example 2.12 and Example 2.13, the electric coupling term (2.16) is usually written as " $\frac{i}{2\pi} \int_X j_E \wedge A$ ". Indeed, (2.16) reduces to this if $F_A = dA$ for some form A ; otherwise this is only valid locally. The correct global expression is (2.16).

Example 2.18 (1-form gauge field). Continuing Example 2.12, a typical electric current j_E is induced from point charges. For manifolds of the form $\mathbb{R} \times N$ the point charges are described by a finite set P of points in N with integers attached. If the particles are static, then $\mathbb{R} \times P$ is the set of their "worldlines"; if they move the worldline is the graph of a function $\mathbb{R} \rightarrow N$. More generally, let $W \subset X$ be a 1-dimensional oriented submanifold and

$$(2.19) \quad q_E W \longrightarrow \mathbb{Z}$$

a (locally constant) function.¹⁹ Let $iW \hookrightarrow X$ denote the inclusion. Then the electric charge is the pushforward of q_E in cohomology:

$$(2.20) \quad Q_E = i_* q_E \in H^{n-1}(X; \mathbb{Z}).$$

We can regard q_E as a class in $\check{H}^0(X)$ (or better as a cocycle in $\check{Z}_H^0(W)$). Then the refined electric current is the pushforward of q_E in differential cohomology:

$$(2.21) \quad \check{j}_E = i_* q_E \in \check{H}^{n-1}(X).$$

¹⁸If $\mathbf{d} = (d_1, \dots, d_k)$ is a multi-degree, then (2.16) is the sum of k terms, one for each component of \check{j}_E and \check{F}_A . For the signs to work out properly, we should regard \check{j}_E as a form twisted by the orientation bundle, so of degree $-(d-1)$. (A density—twisted n -form—has degree 0.) This is formalized in (2.33) below.

¹⁹For manifolds of the form $\mathbb{R} \times N$ we require $W \cap (\{\tau\} \times N)$ to be compact for all $\tau \in \mathbb{R}$. For general X we do not specify support conditions, and in any case omit them from the notation as usual.

Recall from §1 that this depends on choosing a smooth closed differential form Poincaré dual to W . Without making any choices one could define a distributional electric current—a current in the sense of de Rham—supported on W . But we prefer to remain in the smooth category.²⁰ The second factor in the exponentiated action (2.17) may be rewritten as

$$(2.22) \quad \exp(-2\pi i \int_W \check{F}_A)^{q_E}.$$

This is the product over components of W of the q_E^{th} power of the holonomy of the connection A .

Example 2.23 (periodic scalar field). There is an analogous story for the periodic scalar $e^{i\phi} X \rightarrow \mathbb{T}$, continuing Example 2.13. In this case W is 0-dimensional—a “(-1)-brane”—and (2.19)–(2.21) hold with $n-1$ replaced by n . Expression (2.22) becomes a product over the points of W :

$$(2.24) \quad \prod_{w \in W} e^{-iq_E(w)\phi(w)}$$

This factor is often viewed as a local operator inserted into the functional integral, but in our context it is the electric coupling in the exponentiated action.

Next, we consider nonzero magnetic current $j_B \neq 0$. As before, we suppose j_B is refined to a differential cohomology class $\check{j}_B \in \check{\Gamma}^{\mathbf{d}+1}(X)$, and assume that we have a fixed cocycle representative, also denoted \check{j}_B . Recall the first Maxwell equation in (2.2). If the magnetic current is nonzero, then the field strength is not closed and it makes no sense to refine it to a differential cohomology class. Rather, we postulate that the gauge field \check{A} is a trivialization of the refined magnetic current \check{j}_B , in the sense explained before Example 1.10. This means that $\check{A} \in \check{C}_{\Gamma}^{\mathbf{d}}(X)$ with $d\check{A} = \check{j}_B$. The equivalence class of \check{A} (under the equivalence of trivializations of \check{j}_B discussed in a footnote preceding Example 1.10) is denoted \check{F}_A as before. The covariant derivative F_A/ω_X of \check{A} is a global differential form, and $dF_A = j_B$. Notice that the existence of a global trivialization of \check{j}_B implies that the magnetic charge Q_B vanishes in the global cohomology group (no support condition), which is consistent with (2.4). Also, if $\check{j}_B = 0$ we recover the previous definitions of $\check{A}, \check{F}_A, F_A$.

Example 2.25 (periodic scalar field). We continue Example 2.23. In this case $\deg j_B = 2$ and in our model for \check{H}^2 the refined magnetic

²⁰Smoothness plays a greater role when we come to magnetic currents. For example, it was a key idea in [22]. The point is to avoid illegal products of distributions.

current $\check{j}_B \in \check{H}^2(X)$ is a principal circle bundle with connection on X whose curvature is $\sqrt{-1}$ times j_B . The exponentiated gauge field $e^{i\phi}$ is now a global section of this bundle, and therefore the bundle is topologically trivial. The field strength $F_A = d\phi$ is the “covariant derivative” of this section, that is, the pullback to X of the connection form on the total space of the circle bundle. Notice that in case $\check{j}_B = 0$ —the cocycle \check{j}_B is the trivial circle bundle with product connection—we recover our previous description of the gauge field as a map $X \rightarrow \mathbb{T}$.

Specialize now to $n = \dim X = 2$. Then we can consider a magnetically charged (-1) -brane. Thus suppose $i: W \hookrightarrow X$ is the inclusion of an oriented 0-manifold and

$$(2.26) \quad q_B W \longrightarrow \mathbb{Z},$$

which we regard as a class in $\check{H}^0(W)$. Then we set

$$(2.27) \quad \check{j}_B = i_* q_B \in \check{H}^2(X).$$

Recall that the pushforward depends on a choice of Poincaré dual form, which in this case is a closed bump 2-form localized near points $w \in W$ and whose integral near w is $q_B(w)$. A cocycle representative of the refined magnetic current is a circle bundle with connection whose curvature is this Poincaré dual form. This construction should be compared to the construction in complex geometry of a holomorphic line bundle from a divisor.

Example 2.28 (1-form gauge field). In this case $\check{j}_B \in \check{Z}_H^3(X)$ is intuitively a “gerbe with connection” and \check{A} is a trivialization—a “translated version” of a circle bundle with connection. The theory outlined in §1 is a natural home for these notions and their generalizations to higher degrees.

If the electric current j_E vanishes, then the action (2.11) is well-defined and gauge-invariant. In case both j_E and j_B are nonzero we must re-examine the second factor in the exponentiated action (2.17), whose form does not change, but whose geometric nature does. As before it is gauge-invariant, since it only depends on \check{A} through \check{F}_A . In a family of gauge fields parametrized by T , we compute the action as a function on T . Suppose now that the refined electric current \check{j}_E has been lifted to a cocycle. Since \check{F}_A is a trivialization of \check{j}_B , by Stokes’ theorem (see

the text following (1.14))

$$(2.29) \quad \exp\left(-2\pi i \int_{X/T} \check{j}_E \cdot \check{F}_A\right)$$

is a trivialization of $\exp\left(-2\pi i \int_{X/T} \check{j}_E \cdot \check{j}_B\right)$.

The degrees work out so that

$$(2.30) \quad \exp\left(-2\pi i \int_{X/T} \check{j}_E \cdot \check{j}_B\right)$$

lives in²¹ $\check{Z}_H^2(T)$, i.e., is a circle bundle with connection over T . Equivalently, the first expression in (2.29) is a section of a hermitian line bundle with connection, as then is the entire exponentiated action (2.17). Actions which are sections of hermitian line bundles with connection are potentially anomalous; the anomaly is the obstruction to trivializing the line bundle with connection. Here (2.30) is the formula for the anomaly, where we interpret the integral as a differential cohomology class in $\check{H}_H^2(T)$.

Example 2.31 (periodic scalar field). We continue Example 2.25 in any dimension n . Recall that $i: W \hookrightarrow X$ is the inclusion of a 0-manifold—which we assume to be compact, i.e., a finite set of points—and $q_E: W \rightarrow \mathbb{Z}$ encodes the electric charges of the points of W . The refined magnetic current $\check{j}_B = i_* q_B$ is a circle bundle with connection over X and the gauge field $e^{i\phi}$ is a section of \check{j}_B . Let $L \rightarrow X$ denote the associated hermitian line bundle with connection. The electric coupling in the exponentiated action is (2.24), and the anomaly formula reduces to the obvious assertion that it is an element of the hermitian line

$$\bigotimes_{w \in W} (L_w)^{\otimes (-q_E(w))}.$$

As we vary over a family T of connections (and also embeddings $W \hookrightarrow X$) these lines assemble into a smooth hermitian line bundle with connection over T .

The anomaly (2.30) is nonzero in this example since W is both electrically and magnetically charged. The example generalizes to higher dimensional submanifolds which are both electrically and magnetically

²¹Note we use the degree 2 map in (2.14) to define (2.30) as an element in $\check{Z}_H^2(T)$, though we do not make it explicit in the notation.

charged. In these higher dimensional cases the Euler class $\chi_{\check{\Gamma}}(\nu)$ in $\check{\Gamma}$ -theory of the normal bundle ν to W in X enters.

For convenience we collate the various parts of the discussion.

Summary 2.32. The data needed to define an abelian gauge field is:

- (i) a generalized cohomology theory Γ ;
- (ii) maps (2.14) to ordinary cohomology;
- (iii) a multidegree $\mathbf{d} = (d_1, \dots, d_k)$;
- (iv) normalizing differential forms $\omega_X = ((\omega_X)_1, \dots, (\omega_X)_k)$ which depend functorially and locally on X ; and
- (v) coupling constants $e = (e_1, \dots, e_k)$.

Then the gauge field, magnetic current, and electric current live in:

$$\begin{aligned} \check{A} &\in \check{C}_{\Gamma}^{\mathbf{d}}(X), \\ \check{j}_B &\in \check{Z}_{\Gamma}^{\mathbf{d}+1}(X), \\ \check{j}_E &\in \check{Z}_{\Gamma}^{n-\mathbf{d}+1}(X). \end{aligned}$$

The gauge field \check{A} is a nonflat trivialization of the magnetic current \check{j}_B . The exponentiated action is (2.17), and the electric coupling has an anomaly given by (2.30).

Finally, we relax the orientability assumption on X . For this we use the discussion of twistings and orientation at the end of §1. Thus let X be a Riemannian manifold which is not oriented and possibly not orientable. The only change²² from Summary 2.32 is that the refined electric current \check{j}_E lives in $-\check{\zeta}(X)$ -twisted $\check{\Gamma}$ -theory:

$$(2.33) \quad \check{j}_E \in \check{Z}_{\Gamma}^{n-\mathbf{d}+1-\check{\zeta}(X)}(X).$$

The twisting refers to differential $\check{\Gamma}$ -theory. Then the integral in (2.16) is well-defined (cf. (1.18)). The refined magnetic current \check{j}_B still lives in the untwisted theory. Suppose these currents are induced from submanifolds $i: W \hookrightarrow X$ and cocycles \check{q}_E, \check{q}_B on W , as in (2.19) and (2.27), but now we

²²There is another possible scenario in which the gauge field is twisted, hence the magnetic current is twisted, and the electric current is untwisted. This occurs in M-theory, for example.

allow twisted cocycles. In other words, postulate twistings $\check{\tau}_E, \check{\tau}_B$ on W such that

$$\begin{aligned}\check{q}_E &\in \check{Z}_\Gamma^{\bullet+\check{\tau}_E}(W), \\ \check{q}_B &\in \check{Z}_\Gamma^{\bullet+\check{\tau}_B}(W).\end{aligned}$$

Then if $\check{j}_E = i_*\check{q}_E$ and $\check{j}_B = i_*\check{q}_B$ the twistings must satisfy

$$(2.34) \quad \begin{aligned}\check{\tau}_E &= \check{\zeta}(\nu) - i^*\check{\zeta}(X), \\ \check{\tau}_B &= \check{\zeta}(\nu),\end{aligned}$$

where $\nu \rightarrow W$ is the normal bundle to W in X .

In some theories the gauge fields and currents live in twisted versions of differential cohomology. Precisely, we have for some twisting $\check{\zeta}$:

$$(2.35) \quad \begin{aligned}\check{A} &\in \check{C}_\Gamma^{\mathbf{d}+\check{\zeta}}(X), \\ \check{j}_B &\in \check{Z}_\Gamma^{\mathbf{d}+1+\check{\zeta}}(X), \\ \check{j}_E &\in \check{Z}_\Gamma^{n-\mathbf{d}+1-\check{\zeta}-\check{\zeta}(X)}(X).\end{aligned}$$

Note the sign change in the twisting for the electric current. This makes the electric coupling (2.16) well-defined. Equations (2.34) are now

$$(2.36) \quad \begin{aligned}\check{\tau}_E &= \check{\zeta}(\nu) + i^*\check{\zeta} - i^*\check{\zeta}(X), \\ \check{\tau}_B &= \check{\zeta}(\nu) + i^*\check{\zeta}.\end{aligned}$$

3. Applications

Chern-Simons class

Recall that the origin of differential cohomology lies in the work of Cheeger-Simons [13]. Their primary motivation is the application to secondary characteristic classes. We focus on 4-dimensional characteristic classes.

Fix a compact Lie group G , and suppose BG is a *smooth* classifying space. The odd real cohomology of BG vanishes, so from (1.7) we conclude

$$(3.1) \quad \begin{aligned}\check{H}^4(BG) &\cong A_H^4(BG) \\ &= \{(\lambda, \omega) \in H^4(BG) \times \Omega_{\text{cl}}^4(BG) : \lambda_{\mathbb{R}} = [\omega]_{dR}\}.\end{aligned}$$

Fix a connection A_{univ} on the universal bundle $EG \rightarrow BG$ and suppose $\lambda \in H^4(BG)$ is a characteristic class. Let $\omega_{\text{univ}} \in \Omega_{\text{cl}}^4(BG)$ be its Chern-Weil representative. Then $(\lambda, \omega_{\text{univ}}) \in A_H^4(BG)$, and by (3.1) this data determines a universal *Chern-Simons class* in $\check{H}^4(BG)$. Fix a cocycle representative $\check{\alpha}_{\text{univ}} \in \check{Z}_H^4(BG)$.

Now suppose $P \rightarrow X$ is a principal G -bundle over a smooth manifold M . Let A be a connection on P . A *classifying map* for A is a G -equivariant map $f: P \rightarrow EG$ such that $f^*A_{\text{univ}} = A$. It is well-known that classifying maps exist. Let $\bar{f}: M \rightarrow BG$ be the map induced from a classifying map f . Then

$$(3.2) \quad \check{\alpha}(A) = \bar{f}^* \check{\alpha}_{\text{univ}} \in \check{Z}_H^4(M)$$

is the Chern-Simons cocycle of A . Note that the curvature of $\check{\alpha}(A)$ is the Chern-Weil 4-form $\omega(A)$ of A . As stated here $\check{\alpha}(A)$ depends on the classifying map f . Any two classifying maps are homotopic through classifying maps, so the cohomology class of $\check{\alpha}(A)$ in $\check{H}^4(M)$ is well-defined. In fact, there is a more refined context in which we can work so that $\check{\alpha}(A)$ is canonically defined as a cocycle.

In 3-dimensional Chern-Simons theory [19] one considers a family of connections A on a compact oriented manifold X parametrized by T , where $n = \dim X \leq 3$. Then the associated classical Chern-Simons invariant is

$$\int_{(X \times T)/X} \check{\alpha}(A) \in \check{C}_H^{4-n}(T).$$

If X is closed the result is a cocycle, so represents a differential cohomology class. For example, if $\dim X = 3$ this cocycle is a map $T \rightarrow \mathbb{R}/\mathbb{Z}$, the usual classical Chern-Simons action. For $\dim X < 3$ we obtain other geometric invariants.

If a field theory (in arbitrary dimensions) contains a nonabelian gauge field A , we can couple a “2-form field” B to it in a nontrivial way using the Chern-Simons cocycle (3.2). Let the underlying manifold be X . Then we interpret the field B globally as an element $B \in \check{C}_H^3(X)$ which trivializes $\check{\alpha}(A)$, as explained before Example 1.10. Schematically, we write

$$(3.3) \quad d\check{B} = \check{\alpha}(A).$$

There are theories of A, B alone which use this coupling [5], and it appears in more complicated theories as well, as we explain next.

Type I B -field: differential cohomology

The coupling (3.3) between a nonabelian gauge field A and an abelian 2-form field B occurs in type I supergravity in 10 dimensions [12]. The B -field occurs in *pure* supergravity, where it may be interpreted as a cocycle in $\check{Z}_H^3(X)$. But in classical Type I supergravity coupled to super Yang-Mills there is also a nonabelian gauge field A . Suppose $\check{\alpha}(A) \in \check{Z}_H^4(X)$ is its Chern-Simons cocycle, as in (3.2). Then the B -field is a cochain $\check{B} \in \check{C}_H^3(X)$ such that \check{B} trivializes $\check{\alpha}(A)$:

$$(3.4) \quad d\check{B} = \check{\alpha}(A).$$

The *Green-Schwarz anomaly cancellation mechanism* in the low energy description of Type I superstring theory [25] is a modification to the global geometric nature of B and an additional term in the action. Here we interpret it in differential cohomology, along the lines of the standard story. Namely, (3.4) is replaced by

$$(3.5) \quad d\check{B} = \check{\alpha}(A) - \check{\alpha}(g),$$

where $\check{\alpha}(g)$ is the Chern-Simons cocycle of the Levi-Civita connection. The additional term in the action has the form

$$(3.6) \quad 2\pi i \int_X \check{\gamma}(g, A) \cdot \check{B},$$

where the curvature of the differential cocycle $\check{\gamma}(g, A) \in \check{Z}_H^8(X)$ is an 8-form $P_8(g, A)$ which occurs in the anomaly computation from the fermionic functional integrals. (The ability to write the fermion anomaly in this form restricts the gauge algebra to a few possibilities. The precise formula for $P_8(g, A)$ may be found in (3.38).) The exponential of (3.6) is a section of a hermitian line bundle whose curvature is

$$2\pi i \int_X P_8(g, A) \wedge [\omega(A) - \omega(g)].$$

(Recall that ω denotes the Chern-Weil 4-form.) This cancels the curvature from the fermion Pfaffian line bundles. Note that the existence of the trivialization \check{B} in (3.5) implies the topological constraint

$$(3.7) \quad \lambda(A) = \lambda(g),$$

where λ is the *integral* characteristic class used to define the Chern-Simons cocycle. Equation (3.7) plays an important role in heterotic

string theory, for example in the cancellation of worldsheet anomalies. It also appears in the Type I superstring.

In the scenario presented here the local anomaly (curvature) cancels, but there remains a global anomaly. Later we revise this discussion for the Type I superstring (and one of the heterotic strings). We replace integral cohomology by KO -theory; then the global anomaly cancels as well. Equation (3.7) is refined to equation (3.46) in KO -theory.

Self-dual gauge fields²³

Often these are termed *chiral gauge fields* or *gauge fields with self-dual field strength*; for simplicity we call them *self-dual gauge fields*. In Lorentzian field theory on $\mathbb{R} \times N$ the self-duality condition makes sense classically, and it states $F_A = *F_A$. This is not an equation of motion from an action principle, but rather is an auxiliary condition imposed by hand. The set of self-dual solutions (up to equivalence) to the classical equations of motion is a symplectic submanifold of the set of all solutions, and so gives a well-defined classical system. Note that electric and magnetic currents and charges are equal for a self-dual gauge field: $j_E = j_B$. In this section we outline the additional data needed to define self-dual gauge fields—including charge quantization—in Euclidean quantum field theory. We assume throughout that the Riemannian manifold X is compact; if not, one should add convergence conditions on the integrals over X .

There are three main examples we have in mind.

Example 3.8 (doubling). Here $n = \dim X$ is arbitrary. Let $\check{A} \in \check{Z}_\Gamma^{\mathbf{d}}(X)$ be any gauge field, and consider now $\check{\mathbf{d}} = (\mathbf{d}, n - \mathbf{d})$ and $\check{A} = (\check{A}, \check{A}') \in \check{Z}_\Gamma^{\check{\mathbf{d}}}(X) = \check{Z}_\Gamma^{\mathbf{d}}(X) \times \check{Z}_\Gamma^{n-\mathbf{d}}(X)$. The self-duality condition asserts that \check{A}' is the electromagnetic dual of \check{A} , and it allows us to recover the theory of \check{A} from the theory of the pair $\check{A} = (\check{A}, \check{A}')$.

Example 3.9 (integer cohomology). Here $n = \dim X = 4\ell + 2$ for some integer ℓ . Then on a middle-dimensional gauge field $\check{A} \in \check{Z}_H^{2\ell+1}(X)$ we can impose the self-duality constraint. Returning momentarily to the Lorentzian framework on Minkowski spacetime M^n , free theories

²³We take this opportunity to point out a conceptual mistake in [21]. It occurs in the paragraph following equation (6), and also in the footnote which follows. In fact, there is no change in the quantization law of the gauge field, but rather it is the quadratic form introduced in (3.13) below which is needed to make sense of the electric coupling. See Example 3.27 for an analogous case. Also, there is no constraint on the Ramond-Ramond gauge field as proposed before equation (15) in [21]; the factor of $1/2$ is implemented by the quadratic form (3.13).

correspond to representations of the Poincaré group. In this case we obtain an irreducible massless representation induced from the action of the little group $\text{Spin}_{n-2} = \text{Spin}_{4\ell}$ on self-dual 2ℓ -forms.

Example 3.10 (*K*-theory). We continue Example 2.10. Here $n = \dim X = 10$ and X is spin. Then for vanishing B -field the self-duality is imposed on $\check{A} \in \check{Z}_K^d(X)$. This occurs in the low energy description of the Type II superstring— \check{A} is the *Ramond-Ramond gauge field*.

We list the data and constraints necessary to define a self-dual gauge field.

Definition 3.11. Fix a dimension n , a cohomology theory Γ , a multi-degree $\mathbf{d} = (d_1, \dots, d_k)$, a multi-coupling constant $e^2 = (e_1^2, \dots, e_k^2)$, and maps (2.14) from Γ to integer cohomology H in low degrees. A *self-duality constraint* on the corresponding gauge field \check{A} is the additional data:

(i) an automorphism $\theta \check{\Gamma}^\bullet \rightarrow \check{\Gamma}^{n+2-\bullet}$ of a product of k copies of generalized differential cohomology so that for any $\check{\Gamma}$ -oriented fiber bundle $X \rightarrow T$ with closed fibers of dimension n , the bilinear form

$$(3.12) \quad \begin{aligned} B_{X/T} \check{Z}_\Gamma^{\mathbf{d}+1}(X) \times \check{Z}_\Gamma^{\mathbf{d}+1}(X) &\longrightarrow \check{Z}_\Gamma^2(T) \\ \check{a}_1 \times \check{a}_2 &\longmapsto \int_{X/T} \theta(\check{a}_1) \cdot \check{a}_2 \end{aligned}$$

is symmetric;

(ii) for each fiber bundle $X \rightarrow T$ as above a quadratic map

$$(3.13) \quad q_{X/T} \check{Z}_\Gamma^{\mathbf{d}+1}(X) \longrightarrow \check{Z}_\Gamma^2(T)$$

which refines the bilinear map (3.12) in the sense that there is a natural isomorphism

$$(3.14) \quad \begin{aligned} q_{X/T}(\check{a}_1 + \check{a}_2) - q_{X/T}(\check{a}_1) - q_{X/T}(\check{a}_2) + q_{X/T}(0) \\ \cong B_{X/T}(\check{a}_1, \check{a}_2), \quad \check{a}_1, \check{a}_2 \in \check{Z}_\Gamma^{\mathbf{d}+1}(X). \end{aligned}$$

If $\check{b} \in \check{C}_\Gamma^{\mathbf{d}}(X)$ is a trivialization of $\check{a} \in \check{Z}_\Gamma^{\mathbf{d}+1}(X)$, then there is a canonically induced trivialization $q_{X/T}(\check{b})$ of $q_{X/T}(\check{a})$.

In addition, we take the specific normalizing form $\omega_X = ((\omega_X)_1, \dots, (\omega_X)_k)$ (see (2.6)) defined by

$$(3.15) \quad (\omega_X)_i = \sqrt{\pi e_i^2 \hat{A}_\Gamma(X)},$$

where $\hat{A}_\Gamma(X)$ is the form in (1.15). Finally, the electric and magnetic currents are constrained to satisfy

$$(3.16) \quad \check{j}_E = \theta(\check{j}_B).$$

The definition requires several comments. The quadratic form q is modeled after [28], who treat Example 3.9 in great detail. They specify a more precise set of axioms for q which we do not state explicitly here but implicitly require.²⁴ One statement we will use is: The quadratic form (3.13) is a map of categories. (Recall the discussion before Example 1.10.) The quadratic form q is determined by an analogous quadratic refinement in Γ -cohomology, so is really a topological choice. Typically, q has no constant term— $q_{X/T}(0) = 0$ —and often q has no linear term— $q_{X/T}(-\check{a}) = q_{X/T}(\check{a})$. However, we will encounter one example (involving KO -theory) where the symmetry is of the form $q_{X/T}(\check{\lambda} - \check{a}) = q_{X/T}(\check{a})$ for some class $\check{\lambda}$. If the gauge field and magnetic current live in $\check{\zeta}$ -twisted differential cohomology, in which case the electric current lives in $(-\check{\zeta})$ -twisted differential cohomology (see (2.35)), then $\theta \check{\Gamma}_{\check{\zeta}}^\bullet \rightarrow \check{\Gamma}_{-\check{\zeta}}^{n+2-\bullet}$ and the domains of $B_{X/T}$ and $q_{X/T}$ are suitably twisted. In some cases (e.g. Example 3.22 below) the codomain of $q_{X/T}$ does not involve $\check{\Gamma}$ but rather a different differential theory which maps into $\check{\Gamma}$. In fact, we really only use the quadratic form obtained by composing $q_{X/T}$ with the map $\check{Z}_\Gamma^2(T) \rightarrow \check{Z}_H^2(T)$ obtained from (2.14). The constraint (3.16) on the currents means that there is only one current and one charge in the theory. This is the basic meaning of self-duality. Finally, we note that the quadratic refinement q of B is used twice in the self-dual theory: it determines the partition function (see [38]) and is used in the electric coupling term in the action (see (3.25) below).

Example 3.17 (doubling). The map θ is

$$\theta(\check{a}, \check{a}') = ((-1)^{(\mathbf{d}+1)(n-\mathbf{d}+1)} \check{a}', \check{a}), \quad \check{a}, \check{a}' \in \check{\Gamma}^\bullet.$$

Let $\tilde{a}_1 = (\check{a}_1, \check{a}'_1)$ and $\tilde{a}_2 = (\check{a}_2, \check{a}'_2)$ be elements of $\check{\Gamma}^{\mathbf{d}+1} \times \check{\Gamma}^{n-\mathbf{d}+1}$. Then the bilinear form

$$(3.18) \quad \tilde{a}_1, \tilde{a}_2 \mapsto \theta(\tilde{a}_1) \cdot \tilde{a}_2 = \check{a}_1 \check{a}'_2 + \check{a}_2 \check{a}'_1$$

²⁴They extend to maps $q_{X/T} \check{Z}_\Gamma^{\mathbf{d}+1}(Y) \rightarrow \check{Z}_\Gamma^i(T)$ over fiber bundles $Y \rightarrow T$ with relative dimension $n+2-i$ for $i = 0, 1, 2$. The additional axioms involve functoriality under base change and composition of fiber bundles.

has a canonical quadratic refinement

$$(3.19) \quad \tilde{a} = (\check{a}, \check{a}') \longmapsto \check{a} \cdot \check{a}'.$$

The required $q_{X/T}$ is obtained from this by integration.

Example 3.20 (integer cohomology). Set $n = 4\ell + 2$ and $d = 2\ell + 1$. The map θ is the identity. Hopkins and Singer [28], following Browder [9], explain that on the category of compact oriented manifolds with “Wu structure” there is a functorial quadratic refinement defined. In the familiar case $n = 2$ of a self-dual scalar field, a Wu structure is simply a spin structure. The normalization (3.15) corresponds to the “free fermion radius”. Namely, with $\omega_X = 2\pi$ so that the gauge field ϕ has periodicity 2π , the kinetic action is

$$(3.21) \quad S_{\text{kin}}(\phi) = \frac{1}{8\pi} \int_X d\phi \wedge *d\phi.$$

Example 3.22 (K -theory). Recall $n = 10$ and X is spin. Then

$$(3.23) \quad \begin{aligned} \theta \check{K}^q(X) &\longrightarrow \check{K}^{12-q}(X) \\ \check{a} &\longmapsto u^{6-q} \bar{\check{a}} \end{aligned}$$

where $u \in K^2(pt)$ is the inverse Bott element. If $\check{a} \in \check{K}^0(X)$ is represented by a complex vector bundle $E \rightarrow X$ with connection, then $\bar{\check{a}}$ is represented by the complex conjugate vector bundle $\bar{E} \rightarrow X$ with the conjugate connection. The quadratic refinement is defined by Witten in²⁵ [40], and it uses the fact that we integrate over Riemannian *spin* fiber bundles $X \rightarrow T$. Namely, for $\check{a} \in \check{Z}_K^\bullet(X)$ the element $\theta(\check{a}) \cdot \check{a}$ has a canonical lift to $\check{Z}_{KSp}^{12}(X)$, and topologically $q_{X/T}(a)$ is the pushforward of this lift in KSp -theory.

In the quantum theory the Euclidean partition function and correlation functions of self-dual fields are not defined by the usual functional integral, but rather a special procedure is needed which accounts for the self-duality constraints. The reader should keep in mind that the interpretation of this action is not that of the usual functional integral.

We work in the situation of Definition 3.11. Let $\check{j} \in \check{Z}_\Gamma^{\mathbf{d}+1}(X)$ and set $\check{j}_B = \check{j}$, $\check{j}_E = \theta(\check{j})$. The gauge field $\check{A} \in \check{C}_\Gamma^{\mathbf{d}}(X)$ is a trivialization of the refined magnetic current \check{j} . The kinetic term in the action is unchanged

²⁵Witten actually defines a related quadratic form, but the basic idea is the same.

from (2.11), but we do use the normalization (3.15). For a gauge field quantized by integer cohomology with $\omega_X = 2\pi$ we find

$$S_{\text{kin}}(A) = \frac{1}{8\pi} \int_X F_A \wedge *F_A,$$

generalizing (3.21). In the presence of the self-duality constraint the electric coupling term is half²⁶ the usual term (2.16). Recall first the anomaly (2.30); in the self-dual case it is the exponential of $2\pi i$ times

$$\text{“} -\frac{1}{2} \int_{X/T} \theta(\check{j}) \cdot \check{j} = -\frac{1}{2} B_{X/T}(\check{j}, \check{j})\text{”}$$

where we use the degree 2 map in (2.14) to land in $\check{H}^2(T)$. Taking half the integral is precisely what the quadratic form $q_{X/T}$ does, so the self-dual anomaly is

$$(3.24) \quad \exp\left(-2\pi i q_{X/T}(\check{j})\right).$$

The electric coupling term is formally the exponential of $2\pi i$ times

$$\text{“} -\frac{1}{2} \int_{X/T} \theta(\check{j}) \cdot \check{F}_A = -\frac{1}{2} B_{X/T}(\check{j}, \check{F}_A)\text{”},$$

which is

$$(3.25) \quad \exp\left(-2\pi i q_{X/T}(\check{F}_A)\right).$$

This is a trivialization of the circle bundle (3.24). The entire exponentiated action is

$$(3.26) \quad e^{-S}(\check{A}) = \exp\left(-\frac{1}{8\pi} \int_X F_A \wedge *F_A\right) \exp\left(-2\pi i q_{X/T}(\check{F}_A)\right).$$

Example 3.27 (doubling). It is instructive to work out the quadratic map (3.13) in the case of a pair of \mathbb{T} -valued scalar fields, a special case of Example 3.8 and Example 3.17. Thus $\Gamma = H$ is integer cohomology and the degree is $\mathbf{d} = (1, 1)$. Let the dimension be $n = 2$. Assume $X \rightarrow T$ has fibers which are closed oriented surfaces. We use formulas (3.18) and (3.19) for the bilinear form and its quadratic refinement. Consider first $\check{j} = 0$ so that the gauge field $\check{A} = (e^{i\phi}, e^{i\phi'})$ is a pair of maps $X \rightarrow \mathbb{T}$. Then the anomaly (3.24) vanishes so the electric coupling (3.25) is an

²⁶Some justification for the factor 1/2 comes by considering the classical Lorentzian field theory; see [21, (6)].

ordinary function $T \rightarrow \mathbb{T}$. The square of this function is identically 1 as it is the exponential of $2\pi i$ times $-B_{X/T}(\check{j}, \check{F}_A)$. Thus (3.25) is a locally constant function with value ± 1 . In fact, it is the exponential of $2\pi i$ times

$$(3.28) \quad -\frac{1}{2} \int_{X/T} \frac{d\phi}{2\pi} \wedge \frac{d\phi'}{2\pi}.$$

Note that $\frac{d\phi}{2\pi} \wedge \frac{d\phi'}{2\pi}$ represents an integral cohomology class on X , the characteristic class of the product $e^{i\phi} \cdot e^{i\phi'}$ in $\check{H}^2(X)$.

Now consider $\check{j} = (0, \check{k}')$ where $\check{k}' \in \check{H}^2(X)$ is represented by a circle bundle $P' \rightarrow X$ with connection. Then the gauge field is $\check{A} = (e^{i\phi}, f')$, where $e^{i\phi} X \rightarrow \mathbb{T}$ and f' is a section of P' . The exponential of $2\pi i$ times $B_{X/T}(\check{j}, \check{F}_A) = \int_{X/T} e^{i\phi} \cdot \check{k}'$ is again an ordinary function $T \rightarrow \mathbb{T}$, the anomaly $\exp(-2\pi i q_{X/T}(\check{j}))$ vanishes, and $\exp(-2\pi i q_{X/T}(\check{F}_A))$ is an ordinary function—a square root of $\exp(-2\pi i \int_{X/T} e^{i\phi} \cdot \check{k}')$. Let $\theta \in \Omega^1(X)$ be the covariant derivative of f' ; then $d\theta$ is the curvature of P' . A computation in differential cohomology yields

$$(3.29) \quad \exp(-2\pi i q_{X/T}(\check{F}_A)) = \exp\left(-\frac{2\pi i}{2} \int_{X/T} \frac{d\phi}{2\pi} \wedge \frac{\theta}{2\pi}\right),$$

a generalization of (3.28). As a special singular case let P' be flat outside a “divisor” of degree 0 on each component of X . For example, let P' have curvature $\delta_p - \delta_q$ where p and q are sections of $X \rightarrow T$ whose images lie in the same component and δ_p and δ_q are distributional 2-forms supported on these images. Then (compare (2.24))

$$\exp(-2\pi i B_{X/T}(\check{F}_A, \check{j})) = \frac{e^{i\phi(q)}}{e^{i\phi(p)}}.$$

(3.29) is a square root of this function, but as f' is discontinuous at p, q it is not easy to describe geometrically.

Type II Ramond-Ramond fields ($B = 0$)

As described in Example 2.10, in the low energy description of the Type II superstring X is a spin 10-manifold. If the B -field vanishes, then the Ramond-Ramond gauge field \check{A} lives in $\check{C}_K^\bullet(X)$ with $\bullet = 0$ for Type IIA and $\bullet = -1$ for Type IIB. Furthermore, the gauge field is self-dual and the extra data (see Definition 3.11) needed to describe it is specified in Example 3.22. Our goal here is to make everything a bit

more explicit for the current \check{j} induced from a submanifold $iW \hookrightarrow X$, the worldvolume of a *D-brane*.

Suppose W has codimension r in X ; r is odd in Type IIA and even in Type IIB. We assume given $\check{q} \in \check{Z}_{K,\check{\tau}}^0(W)$, analogous to (2.19) and (2.26), but where $\check{\tau}$ is a twisting of differential K -theory. (Twistings are discussed at the end of §1 and in this context at the end of §2.) Define the magnetic current as

$$(3.30) \quad \check{j} = u^{-[\frac{e}{2}]} i_* \check{q}.$$

Concretely (see Example 1.13) \check{q} is usually described as a complex vector bundle with connection $Q \rightarrow W$, the “Chan-Paton vector bundle”. For the pushforward to be well-defined we have equation (2.34) for the twistings. Recall from (1.17) that the twisting class in \check{K} -theory of a real vector bundle V is the pair $(w_1(V), \check{w}_2(V))$. Now since X is assumed spin, its twisting class vanishes. Hence (2.34) asserts that \check{q} is a twisted cocycle on W , the twisting being $(w_1(\nu), \check{w}_2(\nu))$, where $\nu \rightarrow W$ is the normal bundle to W in X . Usually one assumes W is oriented, in which case the twisting is by $\check{w}_2(\nu)$. For example, a (locally) rank one element \check{q} is not represented by a complex line bundle over W with connection, but rather by a spin^c connection on ν . This was derived from perturbative string theory in [23].

Next we work out the electric coupling term (3.25) and the associated anomaly. First, the electric current is

$$\theta(\check{j}) = u^{e-[\frac{e}{2}]} i_* \bar{\check{q}},$$

where $e = 5$ in Type IIA and $e = 6$ in Type IIB. The electric coupling term is a section of a hermitian line bundle with connection over a parameter space T ; the line bundle represents the anomaly. We compute the Chern class of this line bundle from (3.24) as

$$-q_{X/T}(\check{j}) = - \int_{X/T} u^{6-p} i_* \bar{\check{q}} \cdot i_* \check{q},$$

where the integral is in KSp . We can write this as an integral over W/T

$$\begin{aligned} -q_{X/T}(\check{j}) &= - \int_{W/T} u^{6-p} \bar{\check{q}} \cdot i^* i_* \check{q} \\ &= - \int_{W/T} u^{6-p} \bar{\check{q}} \cdot \check{q} \cdot \text{Euler}_{\check{K}}(\nu), \end{aligned}$$

but the integrals can no longer be interpreted in KSp . The last section in [21] explains how to interpret this computation depending on the

dimension of W , and also gives a formula for the \check{K} -theory Euler class. (The subtle point is while for r odd the Euler class vanishes in topological K -theory, it is an element of order two in differential \check{K} -theory.) As explained there, this anomaly cancels the anomaly from the fermions on W . The electric coupling (3.25) may be written formally as

$$(3.31) \quad 2\pi i q_{X/T}(\check{F}_A) \text{ " = " } \frac{2\pi i}{2} \int_{X/T} i_* \bar{q} \cdot \check{F}_A = \frac{2\pi i}{2} \int_{W/T} \bar{q} \cdot i^* \check{F}_A.$$

The integrals are in \check{K} -theory and the factor of $1/2$ is because of the quadratic form. The electric coupling appears in this form in [21, (15)]. To convert to a formula with differential forms, we assume that \bar{q} is defined by a complex vector bundle $Q \rightarrow W$ with connection. Suppose also that W is spin^c and the curvature of the spin^c connection is $-2\pi i \eta \in \Omega^2(W)$. Finally, suppose the Ramond-Ramond field is determined by a differential form $A/(2\pi\sqrt{\hat{A}(X/T)})$ with $dA = F_A$, at least over W (see (1.8)). Then (3.31) reduces to

$$(3.32) \quad 2\pi i q_{X/T}(\check{F}_A) = \pi i \int_{W/T} \hat{A}(W/T) \wedge e^{\eta/2} \wedge \text{ch}(\bar{Q}) \wedge i^* \left(\frac{A}{2\pi\sqrt{\hat{A}(X/T)}} \right).$$

The formula appears in roughly this form in the physics literature (e.g. [31], [14]). Notice that we ignore the magnetic current in writing this expression. We have already given a precise definition of the electric coupling; (3.32) is included to make contact with the literature.

Type II Ramond-Ramond fields ($B \neq 0$)

The Type II B -field is a cocycle $\check{B} \in \check{Z}_H^3(X)$. In other words, it is a "usual" 2-form gauge field quantized by integer cohomology. As explained at the end of §1 it determines a twisted version $\check{K}^{\bullet+\check{B}}(X)$ of differential K -theory, and the Ramond-Ramond fields are cochains in this twisted theory. The Ramond-Ramond charges take values in the twisted K -group $K^{\bullet+\zeta}(X)$, where $\zeta \in H^3(X)$ is the characteristic class of the B -field \check{B} .

The previous discussion may be reconsidered with this twist. The automorphism θ has the same formula (3.23) as before, but it reverses the twisting:

$$\theta \check{K}^{q+\check{B}}(X) \longrightarrow \check{K}^{12-q-\check{B}}(X).$$

By (2.36) the twisting $\check{\tau}$ of the Chan-Paton bundle $\check{q} \in \check{Z}_K^{0+\check{\tau}}(W)$ satisfies

$$(3.33) \quad \check{\tau} = \check{w}_2(\nu) + i^* \check{B}.$$

This equation was deduced from perturbative open string theory in [23]; it is one of many pieces of evidence that Ramond-Ramond charge lives in K -theory. Equation (3.33) is a nontrivial constraint on D-branes. Traditionally one thinks of the Chan-Paton vector bundle on a “single” D-brane as having rank one. The concept of rank does not make sense in every twisted K -theory, and the only reasonable interpretation is that a rank one element is a cochain in $\check{C}_H^2(W)$ which trivializes $\check{\tau} \in \check{Z}_H^3(W)$. Such trivializations exist if and only if the topological class of the twisting vanishes:

$$(3.34) \quad W_3(\nu) + i^*\lambda(\check{B}) = 0.$$

This constraint on a single D-brane was derived from different points of view in [41], [39], and [23]. If the class $W_3(\nu) + i^*\lambda(\check{B})$ is torsion of order N , then again it makes sense to talk about twisted K -theory elements of finite rank, but the rank is constrained to be a multiple of N . Elements of virtual rank zero exist for any twisting; one might instead formally consider them to have infinite rank [37].

Explicit formulas for twisted \check{K} -theory are difficult to write in general, but can be written if we assume \check{B} to be defined by a global real 2-form $B \in \Omega^2(X)$ (cf. (1.8)). Note that the characteristic class ζ of this \check{B} vanishes and the curvature is dB . The form B induces a map

$$(3.35) \quad \psi \check{Z}_H^2(X) \longrightarrow \check{C}_H^2(X)$$

whose image consists of trivializations of \check{B} . In the model of Example 1.10 the triple $(0, B, dB)$ represents \check{B} and the map (3.35) is

$$\psi(c, h, \omega) \longmapsto (c, h, \omega + B).$$

In our current notation, if $\check{A} \in \check{Z}_H^2(X)$ has field strength F_A , then the field strength (covariant derivative) of $\psi(A)$ is $F_A + B$. Note that trivializations of \check{B} are “rank one” cocycles for $\check{K}^{\bullet+\check{B}}(X)$.

We can apply these remarks to construct $\check{q} \in \check{Z}_K^{0+i^*(\check{B})}(W)$ in case W is spin. Thus we suppose $\check{A} \in \check{Z}_H^2(W)$ is an ordinary 1-form gauge field on W and set $\check{q} = \psi(\check{A})$. In explicit formulas like (3.32) the field strength F_A is replaced by $F_A + B$, for example in the factor $\text{ch}(\overline{Q})$. Also, these remarks explain a puzzle [4] about Ramond-Ramond charge which was resolved in [35], [1]. Namely, since the characteristic class of \check{B} vanishes the Ramond-Ramond charges take values in ordinary K -theory: \check{B} -twisted topological K -theory is not twisted. The remarks above tell that the \check{B} -twisted K -theory class of $\check{j} = u^{-[\frac{2}{2}]} i_* \check{q} \in \check{Z}_K^{\bullet+\check{B}}(X)$

(see (3.30)) is the ordinary K -theory class of $i_*\check{A}$. So in explicit formulas for the Ramond-Ramond charge—as in the papers just cited—one finds the field strength F_A , not $F_A + B$.

Type I B -field: differential KO -theory

We have already discussed the B -field in Type I superstring theory and the Green-Schwarz local anomaly cancellation from the point of view of integral cohomology. But according to [39] the charges in Type I superstring theory lie in KO -theory. Hence we expect the B -field to be interpreted in differential KO -theory. In fact, this 2-form field is related to the Ramond-Ramond field in Type IIB, and this also leads us to expect that the corresponding charge is quantized in terms of some form of K -theory. Since the Ramond-Ramond fields are self-dual, we expect that in the differential KO formulation the Type I B -field is also self-dual. Finally, the Atiyah-Singer index theorem computes the fermion anomaly as an integral in differential KO -theory. For this anomaly to cancel against local and global anomalies involving bosonic gauge fields, we expect the gauge fields to be cochains in differential KO -theory. We develop these ideas in this section. Proofs of some mathematical assertions made in this discussion are deferred to Appendix B, written jointly with M. Hopkins.

Let $X \rightarrow T$ be a Riemannian spin fiber bundle with fibers closed 10-manifolds. Recall this means that there is a Riemannian metric on the relative tangent bundle $T(X/T)$ and a distribution of horizontal planes on X , as well as a spin structure on $T(X/T)$. In Type I superstring theory there is a real rank 32 vector bundle $E \rightarrow X$ with connection A . The fermion anomaly has three contributions: a chiral spinor field with values in $\bigwedge^2 E$, the adjoint bundle to E ; a chiral Rarita-Schwinger field, which is a spinor field coupled to $T(X/T) - 1$; and a chiral spinor field of the opposite chirality. (The trivial bundle 1 is subtracted from the relative tangent bundle to obtain the pure spin-3/2 field.) The fermion anomaly is a complex line bundle $\mathcal{L} \rightarrow T$ with connection, and it is computed by a geometric form of the index theorem [7]. We express²⁷ the answer in differential KO -theory:

$$(3.36) \quad \mathcal{L} = \text{pfaff} \int_{X/T} \bigwedge^2 \check{E} + \check{T}(X/T) - 2.$$

Here $\check{E} \in \check{Z}_{KO}^0(X)$ is the cocycle corresponding to the real vector bundle

²⁷As mentioned at the end of Example 1.13, the rigorous derivation of formulas like (3.36) is part of an ongoing project with M. Hopkins and I. Singer.

with connection E ; similarly, $\check{T}(X/T) \in \check{Z}_{KO}^0(X)$; the integral is a map $\int_{X/T} \check{Z}_{KO}^0(X) \rightarrow \check{Z}_{KO}^{-10}(T)$; and $\text{pfaff } \check{Z}_{KO}^{-10}(T) \rightarrow \check{Z}_{KO}^2(T)$ is the pfaffian line bundle. The standard formula in the physics literature ([26, §13.5], for example) is for the curvature of \mathcal{L} , which we write as

$$(3.37) \quad \text{curv } \mathcal{L} = 2\pi i \int_{X/T} \frac{1}{2} P_8(g, A) \wedge [p_1(g) - \text{ch}_2(A)],$$

where

$$(3.38) \quad P_8(g, A) = -\text{ch}_4(A) + \frac{1}{48} p_1(g) \text{ch}_2(A) - \frac{1}{64} p_1(g)^2 + \frac{1}{48} p_2(g).$$

The integrand in (3.37) is a rational combination of Chern-Weil differential forms for the Pontrjagin classes of $T(X/T)$ and the Chern character classes of the complexification of E . The extra factor of $1/2$ is due to the fact that \mathcal{L} is the *pfaffian* line bundle, a square root of the determinant line bundle. The factorization of the integrand in (3.37) is a crucial ingredient in the standard story. Usually $P_8(g, A)$ is expressed in terms of characteristic forms of $\wedge^2 E$ rather than E ; in that case there is a term $[\text{ch}_2(\wedge^2 A)]^2$. The fact that (3.38) is affine linear in A is important to our argument. Set

$$\check{\mathcal{T}} = \check{T}(X/T) + 22.$$

When $\check{E} = \check{\mathcal{T}}$ the curvature of \mathcal{L} vanishes, as the first factor in the integrand does. We claim, and provide a proof in Proposition B.1, that \mathcal{L} itself is trivial for $\check{E} = \check{\mathcal{T}}$, and so in general we can rewrite the formula (3.36) for the fermion anomaly \mathcal{L} as

$$(3.39) \quad \mathcal{L} \cong \text{pfaff} \int_{X/T} \wedge^2 \check{E} - \wedge^2 \check{\mathcal{T}}.$$

We turn now to the B -field, a local 2-form field whose global description we now make precise. The charges associated to this gauge field lie in²⁸ $KO^0(X)$, so the gauge field \check{B} is at first glance a cocycle for $(KO)^{-1}(X)$. In fact, there is a background magnetic current, which we have already seen in a different scenario in (3.5). We now give the self-duality data of Definition 3.11. The cohomology theory underlying this example is the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z})$ -graded theory $KOSp$ which was described in Example 1.4. The automorphism θ in the degree we need is

$$\begin{aligned} \theta(KO)^0(X) &\longrightarrow (KSp)^{12}(X) \\ \check{a} &\longmapsto u^6 \check{a}. \end{aligned}$$

²⁸As in Type IIB, charges lie in the *augmentation ideal* $\epsilon^{-1}(0)$, where $\epsilon KO^0(X) \rightarrow H^0(X)$ is the augmentation.

(Recall that u^6 is quaternionic.) For the quadratic refinement (3.13) of the bilinear form (3.12) we set

$$(3.40) \quad \begin{aligned} q_{X/T} \check{Z}_{KO}^0(X) &\longrightarrow \check{Z}_{KSp}^2(T) \\ \check{a} &\longmapsto \int_{X/T} u^6 \lambda^2(\check{a}), \end{aligned}$$

where $\lambda^2 \check{Z}_{KO}^0(X) \rightarrow \check{Z}_{KO}^0(X)$ is the second exterior power operation and the integral is a map $\int_{X/T} \check{Z}_{KSp}^{12}(X) \rightarrow \check{Z}_{KSp}^2(T)$. If \check{a} is the KO -cocycle of a real vector bundle with connection E , then $\lambda^2(\check{a})$ is the KO -cocycle of $\wedge^2 E$ with the induced connection. The normalizing form is

$$\omega_X = 2\pi \sqrt{\hat{A}(X)},$$

and the coupling constant is $e^2 = 4\pi$.

The quadratic form $q_{X/T}$ satisfies $q_{X/T}(0) = 0$, but it is not symmetric about the origin. In Appendix B we define a $\mu(X) \in KO^0(X)$ canonically associated to spin 10-manifolds X . The class $2\mu(X) \in KO^0(X)$ is a KO -theoretic analog of the Wu class in cohomology. If X has a Riemannian structure then there is a canonical lift of $2\mu(X)$ to differential KO theory (and a canonical cocycle representative), but there may be many lifts of $\mu(X)$ to a differential KO class $\check{\mu}(X)$. (An analogy: Square roots of the canonical bundle of a Riemann surface exist, but none is canonically picked out.) We term a choice of $\check{\mu}(X)$ a $\check{\mu}$ -structure. Furthermore, we only consider families $X \rightarrow T$ in which an appropriate class $\check{\mu}(X/T)$ is defined. These ideas are developed in Appendix B, where the following facts are part definition, part proposition:

- The quadratic form pfaff $q_{X/T}$ has a symmetry:

$$(3.41) \quad \text{pfaff } q_{X/T}(2\check{\mu} - \check{a}) \cong \text{pfaff } q_{X/T}(\check{a}), \quad \check{a} \in \check{Z}_{KO}^0(X).$$

In other words, $\check{\mu}$ is a center for pfaff $q_{X/T}$.

- Set

$$\check{\epsilon} = \check{\mu} - \check{T}.$$

Then $\check{\epsilon}$ restricts to zero on the 7-skeleton of X . For example, the Chern character of $\check{\epsilon}$ contains only forms of degree ≥ 8 . There is an explicit formula for $\check{\epsilon}$ after inverting 2:

$$(3.42) \quad \check{\epsilon} = \frac{1}{64} (7\lambda^2(\check{T}) - 5\text{Sym}^2(\check{T}) + 4\check{T} - 80) + \dots,$$

where $\check{T} = \check{T}(X/T)$ is the relative tangent bundle.

- We have

$$(3.43) \quad \text{pfaff } q_{X/T}(\check{\mu}) \cong \text{pfaff } q_{X/T}(\check{T}) \cong \text{pfaff } \int_{X/T} \wedge^2 \check{T}.$$

From the formula (3.42) for $\check{\mu}$ we compute the Chern character

$$\text{ch}(\check{\mu}) = 32 + p_1(g) u^{-2} + \left[\frac{1}{192} p_1(g)^2 + \frac{1}{48} p_2(g) \right] u^{-4} + \dots$$

Also, from (3.14) and (3.41) we deduce

$$(3.44) \quad \text{pfaff } 2q_{X/T}(\check{a}) \cong \text{pfaff } B_{X/T}(\check{a}, \check{a} - 2\check{\mu}), \quad \check{a} \in \check{Z}_{KO}^0(X).$$

If $X^{10} \cong \mathbb{R}^3 \times Y^7$ then according to the second point above there is a canonical $\check{\mu}$ -structure $\check{\mu}(X) = \check{T}(X)$. Furthermore, in this case the characteristic class $[\check{T}] \in KO^0(X)$ is determined by: $\text{rank}(\check{T}) = 32$; $w_1(\check{T}) = w_2(\check{T}) = 0$; and $\lambda(\check{T}) \in H^4(X)$, where λ is the canonical $\frac{1}{2}p_1$ for spin bundles.

Our postulate for the 2-form field in Type I on a Riemannian spin 10-manifold X is:

$$(3.45) \quad \check{B} \text{ is a nonflat isomorphism } \check{\mu}(X) \longrightarrow \check{E}.$$

The notion of nonflat trivialization was explained in the paragraph following (1.9); it is the same notion which underlies equation (3.5). A nonflat isomorphism is similar. One version is this: $\check{\mu}(X)$ and \check{E} are elements of a category and \check{B} is a morphism between $\check{\mu}(X)$ and $\check{E} - H$ for some global differential form H (see (1.8)). Another version uses the bigraded theory referred to in the footnotes of §1. The form H has components in degrees 3 and 7 on a 10-manifold. Assertion (3.45) means that there is a background magnetic current equal to $\check{E} - \check{\mu}(X)$. A necessary and sufficient condition for \check{B} to exist is that

$$(3.46) \quad [E] = [\mu(X)] \quad \text{in } KO^0(X).$$

From the remark in the previous paragraph, we see that (3.46) is equivalent to the standard condition (3.7) if $X = \mathbb{R}^3 \times Y^7$; in many cases it is a stronger condition. There is also a background electric current, manifested through an electric coupling term of the form (3.25). We write it for a family $X \rightarrow T$ of Riemannian spin 10-manifolds for which $\check{\mu} = \check{\mu}(X/T)$ exists. Recall that in (3.25) the map $(KSp)^2 \rightarrow \check{H}^2$ is omitted from the notation and the factor $2\pi i$ is part of the identification of \check{H}^2

with connections on circle bundles. Now since \check{B} is a nonflat isomorphism $\check{\mu} \rightarrow \check{E}$, by applying the quadratic form, which is a map of categories, we obtain a nonflat isomorphism $q_{X/T}(\check{B}) q_{X/T}(\check{\mu}) \rightarrow q_{X/T}(\check{E})$. Thus the background electric coupling term—the *inverse*²⁹ pfaffian applied to $q_{X/T}(\check{B})$ —is a nonflat trivialization of

$$(3.47) \quad \text{pfaff} - \{q_{X/T}(\check{E}) - q_{X/T}(\check{\mu})\}.$$

In other words, the anomaly in the background electric coupling term is (3.47). Using (3.40), (3.43), and Bott periodicity in KO -theory we see that (3.47) precisely cancels the anomaly (3.39) from the fermions.

There is also a version of Type I theories in which E is a projective bundle with a nontrivial cocycle $w \in H^2(X; \mathbb{Z}/2\mathbb{Z})$.³⁰ This means that \check{E} lies in a twisted version of KO , namely $\check{E} \in \check{Z}_{KO}^{0+w}(X)$. Note that w is a torsion version of the 2-form field (called ‘ B ’) in Type II. It seems that arguments parallel to those in Appendix B yield a “twisted” KO class $\mu_w(X) \in KO^{0+w}(X)$ associated to a spin 10-manifold, and so a parallel discussion with $\check{\mu}$ replaced by $\check{\mu}_w$, but we do not discuss such twisted classes in this paper.

Our motivation for (3.45) is not simply the anomaly cancellation. After all, because of (3.43) we could substitute \check{T} for $\check{\mu}(X)$ in (3.45) and still cancel the anomaly. An additional motivation for the choice of $\check{\mu}(X)$ is that it is the choice which renders the magnetic current equal to the electric current, which we require for a self-dual field. Another motivation is the presumed twisted analog of $\check{\mu}$ just described; there is no such twisted analog of \check{T} , for example.

To make contact with the usual presentation of the *local* anomaly cancellation, we now relate the electric coupling to the standard formula for the Green-Schwarz term. We write \check{B} as a differential form $(B_2 u^{-2} + B_6 u^{-4}) / (2\pi \sqrt{\hat{A}(X/T)})$ relative to a fixed trivialization of the background magnetic current $\check{E} - \check{\mu}$. The differential of the covariant derivative $(H_3 u^{-2} + H_7 u^{-4}) / (2\pi \sqrt{\hat{A}(X/T)})$ is the Chern character of

²⁹due to the minus sign in (3.25).

³⁰It is usually asserted that the gauge group of Type I is $\text{Spin}_{32}/(\mathbb{Z}/2\mathbb{Z})$; the cocycle w is the obstruction to lifting the associated $\text{Spin}_{32}/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ bundle to an SO_{32} bundle.

the background magnetic current, and so

$$\begin{aligned}
 d\left(\frac{H_3 u^{-2} + H_7 u^{-4}}{2\pi}\right) &= \sqrt{\hat{A}(X/T)} \operatorname{ch}(\check{E} - \check{\mu}) \\
 &= \left[\operatorname{ch}_2(A) - p_1(g)\right] u^{-2} \\
 (3.48) \quad &+ \left[\operatorname{ch}_4(A) - \frac{1}{48} p_1(g) \operatorname{ch}_2(A) \right. \\
 &\quad \left. + \frac{1}{64} p_1(g)^2 - \frac{1}{48} p_2(g)\right] u^{-4}.
 \end{aligned}$$

The fixed global trivialization of the background magnetic current gives a fixed trivialization of the anomaly (3.47), relative to which the electric coupling term may be written as an integral of differential forms. Set $\check{a} = \check{E} - \check{\mu}$. Then

$$\begin{aligned}
 q_{X/T}(\check{E}) - q_{X/T}(\check{\mu}) &= q_{X/T}(\check{\mu} + \check{a}) - q_{X/T}(\check{\mu}) \\
 &= q_{X/T}(\check{a}) + B_{X/T}(\check{\mu}, \check{a}),
 \end{aligned}$$

so combining with (3.44) we find

$$(3.49) \quad \operatorname{pfaff} 2[q_{X/T}(\check{E}) - q_{X/T}(\check{\mu})] = \operatorname{pfaff} B_{X/T}(\check{a}, \check{a}).$$

Thus the electric coupling in the unexponentiated action is

$$\begin{aligned}
 (2\pi i) \frac{1}{2} u^6 \int_{X/T} &\left[\hat{A}(X/T) \wedge \operatorname{ch}(\check{a}) \wedge \left(\frac{B_2 u^{-2} + B_6 u^{-4}}{2\pi \sqrt{\hat{A}(X/T)}} \right) \right]_{(10)} \\
 (3.50) \quad &= (2\pi i) \frac{1}{2} u^6 \int_{X/T} \left[\sqrt{\hat{A}(X/T)} \wedge [\operatorname{ch}(\check{E} - \check{\mu})] \wedge \left(\frac{B_2 u^{-2} + B_6 u^{-4}}{2\pi} \right) \right]_{(10)}.
 \end{aligned}$$

The factor of $1/2$ in (3.50) is due to the factor of 2 in (3.49); the u^6 is in (3.40). We expand $\sqrt{\hat{A}(X/T)} \wedge \operatorname{ch}(\check{E} - \check{\mu})$ using (3.48).

Now (B_2, B_6) is a self-dual pair of gauge fields (with no Dirac quantization condition). The self-dual Euclidean action has the form

$$\frac{1}{2} \operatorname{kinetic}(B_2) + \frac{1}{2} \operatorname{kinetic}(B_6) + \frac{2\pi i}{2} \int \frac{j_E}{2\pi} \wedge \frac{B_2}{2\pi} + \frac{2\pi i}{2} \int \frac{j_B}{2\pi} \wedge \frac{B_6}{2\pi}$$

and the magnetic currents are determined by

$$\begin{aligned}
 dH_3 &= j_B, \\
 dH_7 &= j_E.
 \end{aligned}$$

The “ $\frac{1}{2}$ kinetic” terms are one-half the value for non-self-dual gauge fields. If we eliminate B_6 and write the same system in terms of a single gauge field \tilde{B}_2 , then the corresponding Euclidean action is

$$(3.51) \quad \text{kinetic}(\tilde{B}_2) + 2\pi i \int \frac{j_E}{2\pi} \wedge \frac{\tilde{B}_2}{2\pi}$$

with magnetic current determined by

$$(3.52) \quad d\tilde{H}_3 = j_B.$$

From (3.48) and (3.50) we read off

$$j_B = 2\pi \left[\text{ch}_2(A) - p_1(g) \right],$$

$$j_E = 2\pi \left[\text{ch}_4(A) - \frac{1}{48} p_1(g) \text{ch}_2(A) + \frac{1}{64} p_1(g)^2 - \frac{1}{48} p_2(g) \right],$$

Thus (3.52) yields

$$d \left(\frac{\tilde{H}_3}{2\pi} \right) = \text{ch}_2(A) - p_1(g),$$

in agreement with (3.5). The electric coupling in (3.51) is the Green-Schwarz term

$$-2\pi i \int_{X/T} P_8(g, A) \wedge \frac{\tilde{B}_2}{2\pi},$$

where $P_8(g, A)$ is given in (3.38). Therefore, our electric coupling term is indeed a refinement of the standard Green-Schwarz term.

Finally, we consider the anomaly for D1- and D5-branes in Type I; the discussion is similar to the treatment [21] of D-brane anomalies in Type II. A D1-brane is a compact spin submanifold $iW^2 \hookrightarrow X^{10}$. It is endowed with a real vector bundle Q with connection, which as usual we write as a cocycle $\tilde{q} \in \check{Z}_{KO}^0(W)$. The corresponding contribution to the magnetic current is

$$\check{j} = u^{-4} i_* \tilde{q}.$$

As usual magnetic current modifies the geometric nature of the gauge field, so when added to the background magnetic current the appropriate modification of (3.45) is:

$$\check{B} \text{ is a nonflat isomorphism } \check{\mu} \longrightarrow \check{E} + \check{j}.$$

The electric coupling $\text{pfaff}(-q_{X/T}(\check{B}))$ is now the inverse pfaffian of a nonflat isomorphism

$$q_{X/T}(\check{\mu}) \longrightarrow q_{X/T}(\check{E} + \check{j}) \cong q_{X/T}(\check{E}) + q_{X/T}(\check{j}) + B_{X/T}(\check{E}, \check{j}).$$

Recall that $q_{X/T}$ is defined in (3.40) and $B_{X/T}(\check{a}, \check{a}') = \int_{X/T} u^6 \check{a} \cdot \check{a}'$ for $\check{a}, \check{a}' \in \check{Z}_{KO}^0(X)$. Thus the new contribution to the anomaly (beyond (3.47)) is

$$(3.53) \quad \text{pfaff} -\{q_{X/T}(\check{j}) + B_{X/T}(\check{E}, \check{j})\}.$$

Next, we rewrite the expression in braces as an integral over W . The second term is immediate using the push-pull formula:

$$(3.54) \quad B_{X/T}(\check{E}, \check{j}) = u^2 \int_W i^* E \cdot \check{q}.$$

For the first term we claim, and prove in Proposition B.40, that

$$(3.55) \quad q_{X/T}(\check{j}) = u^2 \int_W \Delta^+(\nu) \cdot \lambda^2(\check{q}) - \Delta^-(\nu) \cdot \text{Sym}^2(\check{q}),$$

where $\nu \rightarrow W$ is the normal bundle and Δ^\pm are the half-spin bundles.

The low energy theory on the D1-brane W has fermions³¹ whose anomaly exactly cancels (3.53). Namely, the D1-D9 strings give massless positive chirality spinor fields with coefficients in the bundle $\text{Hom}(Q, i^* E)$, and the D1-D1 strings give positive chirality spinor fields with coefficients in $\Delta^+(\nu) \otimes \Lambda^2(Q)$ as well as negative chirality spinor fields with coefficients in $\Delta^-(\nu) \otimes \text{Sym}^2(Q)$. As before, a geometric form of the Atiyah-Singer index theorem computes the anomaly to be the pfaffian of (3.54) times the pfaffian of (3.55), and this cancels the anomaly (3.53) from the electric coupling.

The story for the D5-brane is parallel, except that Q is quaternionic.

Appendix: Wick rotation

We include these well-known remarks since some of these issues caused the author confusion from which we hope to spare others. The exposition has no pretense to rigor.

On Minkowski spacetime there are both classical and quantum versions of field theory. A classical theory consists of a symplectic manifold of fields (often there are classical field equations which define it) and a symplectic action of the Poincaré group. A quantum theory consists of a Hilbert space and operators (in particular for the Lie algebra of the Poincaré group), and from this data one defines correlation functions of operators. Such classical and quantum theories exist as well on

³¹I warmly thank Jacques Distler for computing the low energy theory on Type I D-branes.

Lorentzian spacetimes of the form $\mathbb{R}_t \times N$, where (N, g_N) is Riemannian and \mathbb{R}_t represents time; the Lorentz metric is $dt^2 - g_N$. Wick rotation occurs in the quantum theory as follows. Correlation functions depend on the positions $(t, n) \in \mathbb{R}_t \times N$ of local operators. Assuming³² they extend to holomorphic functions of t , one restricts to purely imaginary values of $t = \sqrt{-1} \tau$ to obtain correlation functions on $\mathbb{R}_\tau \times N$, which is a Riemannian manifold with metric $d\tau^2 + g_N$. In good cases one can functorially define correlation functions on all Riemannian manifolds X (of fixed dimension) not necessarily of the form $\mathbb{R}_\tau \times N$. This, then, is the substance of Euclidean field theory: quantum correlation functions. There is not a Hilbert space interpretation, nor does one try to make physical sense of classical field theory.

In the Euclidean context one often introduces “classical fields” and an action functional and then writes correlation functions as a functional integral over fields. We review that process briefly, but issue the warning that despite the language of classical fields one is *not* doing classical field theory. In particular, the classical field equations derived mathematically from the Euclidean action have no physical meaning in Euclidean field theory.³³

The Euclidean functional integral on $\mathbb{R}_\tau \times N$ is derived formally from a corresponding (formal) functional integral in the Lorentzian theory on the spacetime $\mathbb{R}_t \times N$ in case the quantum theory on $\mathbb{R}_t \times N$ is obtained by quantizing a classical theory which has a lagrangian description. The derivation of the Lorentzian functional integral is given in standard texts.³⁴ The main result has the schematic form

$$(A.1) \quad \langle \mathcal{O} \rangle = \iint D\phi D\psi e^{iS(\phi, \psi)} \mathcal{O}(\phi, \psi),$$

³²In axiomatic formulations of quantum field theory on Minkowski spacetime (see [29, §2], for example) this is a consequence of more basic axioms.

³³For example, the Euler-Lagrange equation derived from the action (2.17) is

$$d * F_A = -\sqrt{-1} C_X j_E,$$

for some real form C_X . As both F_A and j_E are real (see below), this equation clearly has no solutions with nonzero j_E . Of course, variational equations do have a distinguished place in Riemannian geometry. We also remark that Euclidean functional integrals have other applications in physics. But here our concern is strictly Euclidean quantum field theory.

³⁴One subtlety, which already occurs in quantum mechanics ($N = pt$), is that the action should be at most quadratic in the first time derivative of the fields (see [34, p.163], for example).

where

$$(A.2) \quad S(\phi, \psi) = \int_{\mathbb{R}_t \times N} L(\phi, \psi).$$

In these formulas ϕ stands for a collection of Bose fields and ψ for a collection of Fermi fields. The fields are defined on $\mathbb{R}_t \times N$, and in (A.1) the integral is over fields with finite action. It is an important principle of unitary quantum field theory that the fields and action are *real*. (Complex or quaternionic notation may be used, but the point remains.) In the quantum theory this leads to the fact that operators corresponding to real observables are symmetric. The integrand which defines the action S is the lagrangian density L . The symbol \mathcal{O} in (A.1) denotes a functional (or finite product of functionals) of the fields. The left-hand side $\langle \mathcal{O} \rangle$ is the quantum correlation function. Now a typical operator

$$(A.3) \quad \mathcal{O}_{(t,n)}\phi = \phi(t, n)$$

evaluates the field at a point. It is the t -dependence of the left-hand side of (A.1) which one (formally) analytically continues to complex values of t . We now discuss the corresponding Wick rotation of the right-hand side. For this one simultaneously rotates both the finite dimensional integral in (A.2) and the functional integral in (A.1). Bose fields and Fermi fields in (A.1) are treated differently. We emphasize that our presentation is formal and algebraic. In any case the Lorentzian functional integral (A.1) is usually oscillatory and badly behaved, and one should view this heuristic argument as motivation for the definition of correlation functions by Euclidean functional integrals.³⁵

First, we discuss Wick rotation in the space of fields. Partially complexify the fields to spaces of complex-valued functions which depend holomorphically on a complex variable t . Real fields on $\mathbb{R}_t \times N$ satisfy the reality condition

$$(A.4) \quad \phi(\bar{t}, n) = \overline{\phi(t, n)}.$$

Operators such as (A.3) extend to complex operators defined on complexified fields; they also depend holomorphically on t . The restriction of complexified fields to purely imaginary values of $t = \sqrt{-1}\tau$, i.e., to $\mathbb{R}_\tau \times N$, we term *Euclidean fields*. Here is where the treatment of Bose and Fermi fields differs. There is no reality constraint imposed on Euclidean Fermi fields ψ_E : the fermionic functional integral is algebraic,

³⁵As Coleman [15, p.148] says, the Lorentzian functional integral is “ill-defined, even by our sloppy standards.”

and extending the coefficients from \mathbb{R} to \mathbb{C} does not affect the answer. For example, the pfaffian of a real operator equals the pfaffian of its complexification. For Euclidean Bose fields ϕ_E , on the other hand, (A.4) is “rotated” to the reality constraint

$$(A.5) \quad \phi_E(-\bar{t}, n) = \overline{\phi_E(t, n)}.$$

In particular, Euclidean Bose fields are real-valued on $\mathbb{R}_\tau \times N$. This rotation in the complexified function space is the change of integration domain from the Lorentzian functional integral to the Euclidean functional integral.

In the finite dimensional integral (A.2) rotate the domain of integration from $\mathbb{R}_t \times N$ to $\mathbb{R}_\tau \times N$. The Euclidean lagrangian density L_E on $\mathbb{R}_\tau \times N$ is defined by analytic continuation from L as a function of Euclidean fields:

$$L_E(\phi, \psi) = \frac{1}{\sqrt{-1}} L(\phi, \psi).$$

The reality constraint (A.5) is not used in defining L_E ; one essentially substitutes $t = \sqrt{-1} \tau$ into L and divides by $\sqrt{-1}$. This has the usual effect of changing the sign of potential energy terms, etc.³⁶ Notice that imposing (A.5) does *not* render L_E real in general. The integral of L_E over $\mathbb{R}_\tau \times N$ is the Euclidean action S_E . By a similar change of variables, operators \mathcal{O} are re-expressed as Euclidean operators \mathcal{O}_E .

With this understood the Euclidean correlation functions are

$$\langle \mathcal{O}_E \rangle_E = \iint D\phi_E D\psi_E e^{-S_E(\phi_E, \psi_E)} \mathcal{O}_E(\phi_E, \psi_E).$$

The functional integral is over Euclidean fields of finite Euclidean action.

Usually the Euclidean theory may be formulated on Riemannian manifolds X not necessarily of the form $\mathbb{R}_\tau \times N$. The reality condition on the classical fields and action in the Lorentzian case is replaced by (i) the reality condition on Euclidean Bose fields, and (ii) the condition that the Euclidean action undergo complex conjugation when the orientation is reversed.³⁷ The corresponding reality condition on the quantum Euclidean correlation functions is called *reflection positivity*.

³⁶See [18, §7] for typical examples. However, be warned that there is a crucial notational mistake in the second paragraph: the analytic continuation of a real scalar field on Minkowski spacetime is not real when restricted to Euclidean space. This confusion—which extends to more complicated fields—was one of our motivations to include this appendix.

³⁷Orientation reversal should be construed locally so that the condition makes sense on unorientable manifolds.

(See [17, p.690] for a formal derivation of reflection positivity from the reality condition (ii) on the Euclidean action.)

Appendix: KO and anomalies in type I Daniel S. Freed & Michael J. Hopkins

In this appendix we provide proofs of several assertions needed in the anomaly cancellation arguments for Type I given in §3. We begin in Proposition B.1 with a special computation of a pfaffian line bundle, stated before (3.39). Then we turn to the theory of the quadratic form (B.31) in differential KO -theory which is relevant to the self-dual field in Type I. It is not symmetric about the origin; rather, the symmetry involves a (differential) KO -theoretic analog of the Wu class. We restrict to situations where this center exists. In Proposition B.40 we carry out a computation needed for D-brane anomaly cancellation in Type I. At the end of the appendix we prove that the Adams operation ψ_2 deloops once, a fact used earlier in this appendix. We do not resolve all questions about the quadratic form—for example, existence and uniqueness questions concerning the center—so view this account as provisional.

Proposition B.1. *Let $X \rightarrow T$ be a Riemannian spin fiber bundle with fibers smooth closed spin 10-manifolds. Let \mathcal{L} be the pfaffian line bundle of the family of Dirac operators on the fibers of X coupled to*

$$(B.2) \quad \wedge^2(T(X/T) + 22) + T(X/T) - 2.$$

Then \mathcal{L} , together with its natural metric and connection, is trivial.

The natural connection is defined, and the curvature and holonomy computed, in [7]. We do not claim here to give a *canonical* trivialization, so as explained at the end of the introduction while this proposition can be used to prove the cancellation of anomalies, it is not strong enough to construct correlation functions.

Proof. The proof is similar to the proof of [23, Theorem 4.7].

The curvature of \mathcal{L} was computed in (3.37), and was seen to vanish. To compute the holonomy we pull back over a loop in T , so consider a family $X \rightarrow S^1$. Let the circle have its bounding spin structure; then X is a closed spin 11-manifold. The holonomy of \mathcal{L} around S^1 is³⁸ $\exp(-2\pi i \xi_X/2)$, where $\xi_X = \frac{1}{2}(\eta_X + \dim \ker D_X)$ is the Atiyah-Patodi-Singer invariant for the Dirac operator coupled to (B.2). Now

³⁸The extra factor of 2 is due to the pfaffian (as opposed to a determinant); the absence of an adiabatic limit is due to the vanishing of the curvature.

any closed spin 11-manifold is the boundary $X = \partial M$ of a compact spin 12-manifold, and we take M to have a Riemannian metric which is a product near the boundary. We compute $\xi_X/2 \pmod{1}$ via the Atiyah-Patodi-Singer theorem, but for that we need to extend (B.2) to M , which is not necessarily fibered over S^1 . A short computation shows that

$$\mathcal{E} = \wedge^2(TM + 20) + TM - 4$$

restricts on ∂M to (B.2). Then

$$\xi_X/2 \equiv \int_M \left[\frac{1}{2} \hat{A}(M) \text{ch}(\mathcal{E}) \right]_{(12)} \pmod{1},$$

and a straightforward computation shows that the integrand vanishes. Hence the holonomy of \mathcal{L} is trivial. q.e.d.

As a preliminary to the rest of this appendix we recall some facts about KO -theory. Let X be any manifold. First, there is a canonical filtration, the *Atiyah-Hirzebruch filtration*. A class $a \in KO(X)$ has filtration q if q is the largest integer such that the pullback of a via any smooth map $\Sigma^k \rightarrow X$ of a k -dimensional manifold Σ into X vanishes for all $k < q$. The product of classes of filtration q and filtration q' has filtration $\geq q + q'$. Second, suppose $\nu \rightarrow X$ is a real spin bundle of even rank $2r$. The *K-theory Thom class* U is an element in $K_{cv}^{2r}(\nu)$, where ‘cv’ denotes ‘compact vertical support.’ Let $iX \hookrightarrow \nu$ be the zero section. By the splitting principle we formally write

$$(B.3) \quad \nu \otimes \mathbb{C} = \bigoplus_{i=1}^r (\ell_i + \ell_i^{-1}).$$

Then

$$(B.4) \quad i^*U = u^r [\Delta^+(\nu) - \Delta^-(\nu)] = u^r \prod_{i=1}^r (\ell_i^{1/2} - \ell_i^{-1/2}),$$

where Δ^\pm are the half-spin representations. If $r \equiv 0 \pmod{4}$, then U is real; if $r \equiv 2 \pmod{4}$, then r is quaternionic. When $r \equiv 0 \pmod{4}$ there is a KO -theory Thom class (also called ‘ U ’) whose complexification is the K -theory Thom class. Next, we state the form of Poincaré duality in KO -theory which we need. Let X be a spin manifold of dimension n . Then

$$(B.5) \quad \text{Hom}(KO_c^q(X; \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong KO^{n+4-q}(X),$$

where ‘c’ denotes ‘compact support.’ The analogous statement in ordinary cohomology does not have the shift by 4. Ordinarily, a generalized cohomology theory does not contain such a duality statement. Finally, we make use of the Adams operation ψ_2 . It is a natural *ring endomorphism* of $KO^0(X)$ for any manifold X , and is related to the exterior square λ^2 (which is not a ring homomorphism) by

$$(B.6) \quad 2\lambda^2(a) = a^2 - \psi_2(a).$$

The operation ψ_2 is defined on line bundles ℓ by the formula $\psi_2(\ell) = \ell^2$; it extends to arbitrary elements of KO using the splitting principle and the fact that ψ_2 is a ring homomorphism. The same definitions for ψ_2 work on complex K -theory, and then ψ_2 extends to K^{-q} for $q \geq 0$. Its action on the Bott element u^{-1} is³⁹

$$(B.7) \quad \psi_2(u^{-1}) = 2u^{-1}.$$

If we invert 2, then ψ_2 extends to K^q for all q . Furthermore, after inverting 2 there is an inverse operation $\psi_{1/2}$. If ℓ is a line bundle, and $x = 1 - \ell$, then

$$\psi_{1/2}(\ell) = (1 - x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

Note that x^q has filtration $\geq q$, so on a finite dimensional manifold the infinite series terminates. On an infinite dimensional manifold we must work in a certain completion \hat{K} of K -theory. We also need a single delooping of ψ_2 .

Proposition B.8. *There exists an operation $\psi_2 KO^1(X) \rightarrow KO^1(X)$ which is compatible with the standard ψ_2 under suspension, i.e., the diagram*

$$(B.9) \quad \begin{array}{ccc} KO^0(X) & \xrightarrow{\psi_2} & KO^0(X) \\ \downarrow & & \downarrow \\ KO^1(\Sigma X) & \xrightarrow{\psi_2} & KO^1(\Sigma X) \end{array}$$

commutes. Furthermore, ψ_2 extends to an operation on the differential group $(KO)^1(X)$ and so restricts to an operation on $KO^0(X; \mathbb{R}/\mathbb{Z})$.

³⁹Since $u^{-1} = H - 1$ for H the hyperplane bundle on $S^2 \cong \mathbb{C}P^1$, we compute $\psi_2(u^{-1}) = H^2 - 1 = 2(H - 1)$ in the reduced K -theory of S^2 , which is isomorphic to $K^{-2}(pt)$.

The proof, which involves homotopy-theoretic techniques, is deferred to the end of the appendix. More elementary is the extension of the operations ψ_2 and λ^2 to the differential group $(KO)^0$. For that we use the fact that the topological operations are defined on the level of cochains, not just cohomology, and there are compatible operations on differential forms. Then such operations on KO are defined from the basic pullback square (1.5).

Concerning the Atiyah-Hirzebruch filtration, we have the following easy statement.

Lemma B.10. *Suppose $F \rightarrow X$ is a real vector bundle with $w_1(F) = w_2(F) = 0$. Then the class of $(F - \text{rank } F)$ in $KO(X)$ has filtration ≥ 4 .*

Proof. The classifying map $X \rightarrow \mathbb{Z} \times BO$ of $(F - \text{rank } F)$ lifts to $B\text{Spin}$, and the 3-skeleton of $B\text{Spin}$ is trivial. \square e.d.

We introduce a characteristic class $\hat{\rho}(F) \in \widehat{KO}[\frac{1}{2}](X)$ associated to a real vector bundle $F \rightarrow X$. Let $i: X \rightarrow F$ be the zero section. Suppose first that F is spin of even rank $2r$. Let $\ell_i, i = 1, \dots, r$ as in (B.3) and set

$$(B.11) \quad \hat{\rho}(F) = i^* \left(\frac{U}{\psi_{1/2}(U)} \right) = \prod_{i=1}^r \frac{\ell_i^{1/4} + \ell_i^{-1/4}}{2},$$

where we use (B.4) and (B.7). Here $\psi_{1/2}(U) \in \widehat{KO}[\frac{1}{2}]_{\text{cv}}^{2r}(F)$ and under the Thom isomorphism it corresponds to a class $i^*(\psi_{1/2}(U) / U) \in \widehat{KO}[\frac{1}{2}]^0(X)$ which has the form $1 + z$ for z of filtration ≥ 1 . The characteristic class $\hat{\rho}(F)$ is its inverse. From the last expression in (B.11) we see that $\hat{\rho}(F)$ is defined for any real vector bundle F . To compute a formula for $\hat{\rho}(F)$, write $\ell_i = 1 - x_i$ and $\ell_i^{-1} = 1 - y_i$. Expand (B.11) using the binomial theorem, take the log, and write the result in terms of $s_i = x_i + y_i = x_i y_i$ using the Newton polynomials for $x_i^n + y_i^n$:

$$\log \hat{\rho}(F) = \sum_{i=1}^r \left(-\frac{1}{32} s_i - \frac{3}{1024} s_i^2 - \frac{5}{12288} s_i^3 + \dots \right).$$

Note $\sum s_i$ is the reduced bundle $2r - F$, so has filtration ≥ 1 . We then express power sums in s_i in terms of the elementary symmetric polynomials p_1, p_2, \dots in s_i and exponentiate:

$$(B.12) \quad \begin{aligned} \hat{\rho}(F) = & 1 - \left(\frac{1}{32} p_1 \right) + \left(\frac{3}{512} p_2 - \frac{5}{2048} p_1^2 \right) \\ & + \left(-\frac{5}{4096} p_3 + \frac{17}{16384} p_1 p_2 - \frac{21}{65536} p_1^3 \right) + \dots \end{aligned}$$

Finally, we compute

$$\begin{aligned} p_1 &= 2r - F \\ p_2 &= (2r^2 - 3r) - (2r - 2)F + \lambda^2 F \\ p_3 &= \left(\frac{4r^3 - 18r^2 + 20r}{3} \right) - (2r^2 + 7r - 5)F \\ &\quad + 2(r - 2)\lambda^2 F - \lambda^3 F, \end{aligned}$$

so find

$$\begin{aligned} \hat{\rho}(F) &= 1 + \frac{1}{2^5}[F - 2r] \\ &\quad + \frac{1}{2^{11}} \left[7\lambda^2(F) - 5 \text{Sym}^2(F) + (24 - 4r)F + (4r^2 - 36r) \right] \\ &\quad + \frac{1}{2^{16}} \left[33\lambda^3(F) - 26R(F) + 21 \text{Sym}^3(F) \right. \\ (B.13) \quad &\quad \left. - (14r - 184)\lambda^2 F + (10r - 136) \text{Sym}^2(F) \right. \\ &\quad \left. + (4r^2 + 1036r - 400)F \right. \\ &\quad \left. - \left(\frac{8r^3 - 216r^2 + 1600r}{3} \right) \right] + \dots \end{aligned}$$

Here $R(F)$ is the associated bundle to F which satisfies

$$F^{\otimes 3} \cong \text{Sym}^3 F \oplus 2R(F) \oplus \lambda^3 F, \quad F \otimes \lambda^2 F \cong \lambda^3 F \oplus R(F).$$

Note from Lemma B.10 and (B.12) that if $w_1(F) = w_2(F) = 0$, then the second term in (B.13) has filtration ≥ 4 , the third term has filtration ≥ 8 , etc.

For a finite dimensional manifold Y define $\hat{\rho}(Y) = \hat{\rho}(TY) \in KO(Y)[\frac{1}{2}]$.

Proposition B.14. *Let Y^{8n+4} be a spin manifold. Then $2^{4n+2}\hat{\rho}(Y)$ is the image in $KO[\frac{1}{2}](Y)$ of a canonical class $\lambda_n(Y) \in KO(Y)$.*

The class $\lambda_n(Y)$ is a KO -theoretic analog of a Wu class. It is defined for any spin manifold Y^d of dimension $\leq 8n + 4$; apply Proposition B.14 to $Y^d \times \mathbb{R}^{8n+4-d}$.

Proof. By Poincaré duality (B.5) the functional

$$\begin{aligned} (B.15) \quad KO_c^0(Y; \mathbb{R}/\mathbb{Z}) &\longrightarrow KO^{-(8n+4)}(pt; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \\ a &\longmapsto \int_Y \psi_2(a) \end{aligned}$$

is represented by a class $u^{4n+4}\lambda_n(Y) \in KO^{8n+8}(Y)$:

$$(B.16) \quad \int_Y \psi_2(a) = \int_Y \lambda_n(Y) \cdot a, \quad a \in KO_c^0(Y; \mathbb{R}/\mathbb{Z}).$$

Note that the existence of the functional (B.15) relies on Proposition B.8. Now integration in KO -theory is defined using an embedding $iY \hookrightarrow \mathbb{R}^N$ with $N = (8n + 4) + 8k$ for some k ; then

$$i_* KO^0(Y; \mathbb{R}/\mathbb{Z}) \longrightarrow KO_c^{8k}(\mathbb{R}^N; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$$

is the integral. Let U be the KO Thom class of the normal bundle $\nu \rightarrow Y$ of Y in \mathbb{R}^N . Then we compute

$$(B.17) \quad \begin{aligned} i_*\psi_2(a) &= \psi_2(a) \cdot U \\ &= \psi_2(a \cdot \psi_{1/2}(U)) \\ &= 2^{-4k} a \cdot \psi_{1/2}(U) \\ &= 2^{-4k} \left(a \cdot \frac{\psi_{1/2}(U)}{U} \right) \cdot U. \end{aligned}$$

In the first equation we pull $\psi_2(a)$ back to ν and extend $\psi_2(a)U$ to \mathbb{R}^N using the fact that U has compact vertical support. In the second equation we regard U in KO with 2 inverted. In the third equation we use the fact that $KO_c^{8k}(\mathbb{R}^N)$ is generated by u^{-4k} . Thus from (B.7) we know that ψ_2 acts on $KO_c^{8k}(\mathbb{R}^N)$ and $KO_c^{8k}(\mathbb{R}^N; \mathbb{R})$ as multiplication by 2^{-4k} ; it now follows from (B.9) that ψ_2 also acts on $KO_c^{8k}(\mathbb{R}^N; \mathbb{R}/\mathbb{Z})$ as multiplication by 2^{-4k} . Let V be the KO Thom class of $TY \rightarrow Y$. Since $TY \oplus \nu$ is trivial of rank N , we deduce

$$\frac{\psi_{1/2}(U)}{U} \frac{\psi_{1/2}(V)}{V} = \frac{\psi_{1/2}(u^N)}{u^N} = 2^{N/2}.$$

This is an equation in $KO(Y)$; the factors on the left-hand side are implicitly restricted to Y . Substituting into (B.17) we find

$$(B.18) \quad i_*\psi_2(a) = a \cdot \left(2^{4n+2} \frac{V}{\psi_{1/2}(V)} \right) \cdot U = i_*(a \cdot 2^{4n+2} \hat{\rho}(V)).$$

Thus

$$\int_Y \psi_2(a) = \int_Y 2^{4n+2} \hat{\rho}(Y) \cdot a, \quad a \in KO_c^0(Y; \mathbb{R}/\mathbb{Z}).$$

Comparing with (B.16), and using the Poincaré duality isomorphism (B.5), we deduce that the image of $\lambda_n(Y)$ in $KO[\frac{1}{2}](Y)$ is $2^{4n+2} \hat{\rho}(Y)$, as desired. q.e.d.

On a spin manifold of dimension $\leq 8n + 3$, the class λ_n is canonically divisible by 2. (Compare with a similar assertion about Wu classes in [28].)

Proposition B.19. *Let X^{8n+3} be a spin manifold. Then there is a canonically associated class $\mu_n(X) \in KO(X)$ with $2\mu_n(X) = \lambda_n(X)$.*

The proposition applies to manifolds of dimension $< 8n + 3$ by taking the product with a vector space as before.

Proof. The operation λ^2 loops to an operator $\Omega\lambda^2$ on $KO^{-1}(X)$. It is linear since products of suspended classes vanish. Similarly, there is a linear operator $\Omega\lambda^2$ on $KO^{-1}(X; \mathbb{R}/\mathbb{Z})$ compatible with $\Omega\lambda^2$ on $KO^{-1}(X; \mathbb{R})$ and λ^2 on $KO^0(X)$ in the long exact sequence. From (B.6) we have

$$(B.20) \quad 2\Omega\lambda^2 = -\psi_2.$$

Now Poincaré duality (B.5) implies that the linear functional

$$\begin{aligned} KO_c^{-1}(X; \mathbb{R}/\mathbb{Z}) &\longrightarrow KO^{-(8n+4)}(pt; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \\ a &\longmapsto \int_X \Omega\lambda^2(a) \end{aligned}$$

is represented by a class $-u^{4n+4}\mu_n(X) \in KO^{8n+8}(X)$:

$$(B.21) \quad \int_X \Omega\lambda^2(a) = \int_X -\mu_n(X) \cdot a, \quad a \in KO_c^{-1}(X; \mathbb{R}/\mathbb{Z}).$$

From (B.20) and (B.16) we have

$$(B.22) \quad \int_X 2\Omega\lambda^2(a) = \int_X -\psi_2(a) = \int_X -\lambda_n(X) \cdot a.$$

Comparing (B.22) and (B.21) we deduce $2\mu_n(X) = \lambda_n(X)$. q.e.d.

Turning to *differential KO*-theory we have the following.

Proposition B.23. *Let Y^{8n+4} be a Riemannian spin manifold. Then there is a canonical lift $\check{\lambda}_n(Y) \in (KO)_c^0(Y)$ of $\lambda_n(Y)$ such that*

$$\int_Y \psi_2(\check{a}) = \int_Y \check{\lambda}_n(Y) \cdot \check{a} \in \mathbb{R}/\mathbb{Z}$$

for all $\check{a} \in (KO)_c^1(Y)$.

The proof is parallel to the proof of Proposition B.14. It relies on Poincaré duality for KO , which on an n -dimensional Riemannian spin manifold Y states that there is an “almost perfect” pairing

$$\begin{aligned} (KO)_c^{q+1}(Y) \otimes (KO)^{n+4-q}(Y) &\longrightarrow \mathbb{R}/\mathbb{Z} \\ \check{a} \otimes \check{b} &\longmapsto \int_Y \check{a} \cdot \check{b}. \end{aligned}$$

Note that the integral lands in $(KO)^5(pt) \cong KO^4(pt; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. This duality combines the topological duality (B.5) with a duality on differential forms—see (1.8) and (B.18). The “almost perfect” refers to the fact that the dual of a differential form is a de Rham current. Thus the application of KO -theory Poincaré duality to the functional $\check{a} \mapsto \int_Y \psi_2(\check{a})$, $\check{a} \in (KO)_c^1(Y)$ only gives a distributional class $\check{\lambda}_n(Y)$. Then the computation of its image in $(KO[\frac{1}{2}])^1(Y)$, parallel to the computation in the proof of Proposition B.14, shows that its curvature is in fact smooth.

There is no *canonical* lift of $\mu_n(X)$ to a differential class on a Riemannian spin $(8n + 3)$ -manifold, though lifts do exist. We define suitable lifts below.

Specialize to $n = 1$, so to the classes $\lambda(Y) = \lambda_1(Y) \in KO^0(Y)$ and $\check{\lambda}(Y) = \check{\lambda}_1(Y) \in (KO)^0(Y)$ canonically associated to a Riemannian spin 12-manifold Y . We also have the topological class $\mu(X) = \mu_1(X) \in KO^0(X)$ canonically associated to a spin 11- or 10-manifold X . Note by (B.13) that

$$\lambda(Y) = 2(TY + 22) + (\text{classes of filtration } \geq 8) \in KO[\frac{1}{2}]^0(Y);$$

there are similar equations for $\check{\lambda}(Y)$ and $\mu(X)$ after inverting 2.

Lemma B.24. *Let Y be a Riemannian spin 12-manifold and X a Riemannian spin 11- or 10-manifold. Then (without inverting 2)*

$$(B.25) \quad \lambda(Y) = 2(TY + 22) + \epsilon_1$$

$$(B.26) \quad \check{\lambda}(Y) = 2(\check{T}Y + 22) + \check{\epsilon}_1$$

$$(B.27) \quad \mu(X) = (TX + 22) + \epsilon_2$$

where $\epsilon_1, \check{\epsilon}_1, \epsilon_2$ have filtration ≥ 8 .

Here $\check{T}Y \in (KO)^0(Y)$ is the class of the tangent bundle of Y with its Levi-Civita connection. Also, we induce a filtration on $KO(X)$ from the Atiyah-Hirzebruch filtration on $KO(X)$ via the characteristic class $KO(X) \rightarrow KO(X)$.

Proof. The first assertion (B.25) is equivalent to

$$(B.28) \quad \begin{aligned} f(Y, a) &:= \int_Y \psi_2(a) - 2(TY + 22)a = 0, \\ a &\in KO^0(Y; \mathbb{R}/\mathbb{Z}), \quad \text{filtration } a \geq 8. \end{aligned}$$

(Note that an element of $KO^0(Y; \mathbb{R}/\mathbb{Z})$ of filtration ≥ 5 has filtration ≥ 8 .) There is a similar rewriting of the other two assertions. As a first step we argue that it suffices to assume that Y is compact. Namely, using a proper Morse function we can find a compact manifold with boundary Y' contained in Y such that the support of a lies in the interior of Y' ; then replace Y by the double of Y' . This does not change the value of (B.28). Then our proof of (B.28) relies on the fact that $f(Y, a)$ depends only on the bordism class of (Y, a) : If $Y^{12} = \partial Z^{13}$ for a compact spin manifold Z , and a extends to a class on Z , then (B.28) vanishes. Since $f(Y, 0) = 0$ it suffices to consider the bordism class of $(Y, a) - (Y, 0)$. The spectrum which classifies such reduced pairs is $M\text{Spin} \wedge BO\langle 8 \rangle$; an element of $\pi_n(M\text{Spin} \wedge BO\langle 8 \rangle)$ represents a spin n -manifold which bounds together with a class in KO^0 of filtration ≥ 8 . One computes

$$(B.29) \quad \begin{aligned} \pi_{10}(M\text{Spin} \wedge BO\langle 8 \rangle) &\cong \mathbb{Z}/2\mathbb{Z}, \\ \pi_{11}(M\text{Spin} \wedge BO\langle 8 \rangle) &= 0, \\ \pi_{12}(M\text{Spin} \wedge BO\langle 8 \rangle) &\cong \mathbb{Z} \times \mathbb{Z}, \\ \pi_{13}(M\text{Spin} \wedge BO\langle 8 \rangle) &= 0. \end{aligned}$$

From these facts one deduces that the reduced bordism group of pairs (Y, a) with $a \in KO^0(Y; \mathbb{R}/\mathbb{Z})\langle 8 \rangle$ is isomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. In particular, it is arbitrarily divisible. Since $f(Y, 32b) = 0$ for all b (see (B.13)), and any $(Y, a) - (Y, 0)$ is bordant to $(Y, 32b) - (Y, 0)$ for some b by the divisibility, we obtain the desired result $f(Y, a) = 0$.

The proof of (B.27) is similar; the relevant bordism group of reduced pairs $(X, a) - (X, 0)$, $a \in KO^{-1}(X; \mathbb{R}/\mathbb{Z})\langle 8 \rangle$ is again isomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.

For (B.26) we must show

$$g(Y, \check{a}) := \int_Y \psi_2(\check{a}) - 2(\check{T}Y + 11) \cdot \check{a}$$

vanishes for all $\check{a} \in (KO)^1(Y)$ of filtration ≥ 8 . From the exact sequence (1.7), the topological result (B.25), and the fact that differential forms are divisible we conclude that $g(Y, \check{a})$ depends only on the characteristic class $[a] \in KO^1(Y)$ of \check{a} . But the assertion about π_{11} in (B.29)

implies that $(Y, [a]) - (Y, 0)$ vanishes in the appropriate reduced bordism group, whence g vanishes.

We remark that Lemma B.24 follows formally from the stronger Lemma B.36 below using more bordism theory. q.e.d.

The following definition is analogous to the definition of a square root of the canonical bundle of a Riemann surface.

Definition B.30. Let X be a Riemannian spin 11- or 10-manifold. Then a $\check{\mu}$ -structure on X is a class $\check{\mu}(X) \in (KO)^0(X)$ such that

- (i) $2\check{\mu}(X) = \check{\lambda}(X)$;
- (ii) The cohomology class of $\check{\mu}(X)$ is $\mu(X)$;
- (iii) $\check{\mu}(X)$ differs from $\check{T}X + 22$ by an element of filtration ≥ 8 .

The preceding shows that $\check{\mu}$ -structures exist; differences of $\check{\mu}$ -structures are certain points of order 2 on the torus $KO^{-1}(X; \mathbb{R})/KO^{-1}(X)$.

Let $X \rightarrow T$ be a Riemannian spin fiber bundle with fibers closed manifolds of dimension 10. Recall from (3.40) the quadratic form

$$(B.31) \quad \begin{aligned} q = q_{X/T} \check{Z}_{KO}^0(X) &\longrightarrow \check{Z}_{KSp}^2(T) \\ \check{a} &\longmapsto \int_{X/T} u^6 \lambda^2(\check{a}) \end{aligned}$$

which refines the bilinear form

$$\begin{aligned} B = B_{X/T} \check{Z}_{KO}^0(X) \times \check{Z}_{KO}^0(X) &\longrightarrow \check{Z}_{KSp}^2(T) \\ \check{a} \times \check{a}' &\longmapsto \int_{X/T} u^6 \check{a} \cdot \check{a}'. \end{aligned}$$

Note that $q(0) = 0$. The quadratic form q does not necessarily have a symmetry; we restrict to fiber bundles for which a symmetry exists for the pfaffian.

Definition B.32. A $\check{\lambda}$ -structure on a Riemannian spin fiber bundle $X \rightarrow T$ of closed 10-manifolds is a cocycle $\check{\lambda} = \check{\lambda}(X/T) \in \check{Z}_{KO}^0(X)$ and isomorphisms

$$(B.33) \quad \text{pfaff } q(\check{a}) \cong \text{pfaff } q(\check{\lambda} - \check{a})$$

natural in $\check{a} \in \check{Z}_{KO}^0(X)$.

An easy computation shows that (B.33) is equivalent to natural isomorphisms

$$(B.34) \quad \text{pfaff} \int_{X/T} \psi_2(\check{a}) \cong \text{pfaff} \int_{X/T} \check{\lambda} \cdot \check{a}$$

together with an isomorphism $\text{pfaff} q(\check{\lambda}) \cong 0$. (Note (B.34) implies $\text{pfaff} 2q(\check{\lambda}) \cong 0$.) Also, Proposition B.23 implies that the equivalence class of the restriction of $\check{\lambda}$ to the fiber is canonically determined. A computation parallel to that in the proof of Proposition B.14, now for fiber bundles and in differential KO , computes the image of $\check{\lambda}$ in $(KO[\frac{1}{2}])^0(X)$ as

$$(B.35) \quad \check{\lambda} = 2\check{T} + \frac{1}{32}(7\lambda^2(\check{T}) - 5\text{Sym}^2(\check{T}) + 4\check{T} - 80) + \dots,$$

where $\check{T} = \check{T}(X/T)$ and $\check{T} = \check{T} + 22$. More precisely, analogous to Lemma B.24 we have the following.

Lemma B.36. $\check{\lambda} \cong 2\check{T}$ modulo cocycles of filtration ≥ 8 .

Proof. As in the proof of Lemma B.24 we must show

$$\text{pfaff} \int_{X/T} \psi_2(\check{a}) \cong \text{pfaff} \int_{X/T} 2(\check{T} + 22) \cdot \check{a}$$

for all \check{a} of filtration ≥ 8 . It follows from (B.29) that the universal family of spin 10-manifolds (up to bordism) together with a class in KO of filtration ≥ 8 is simply-connected. Thus to prove (B.39)—an isomorphism of circle bundles with connection—it suffices to prove that some powers are isomorphic over the universal parameter space, since there are no flat circle bundles there (cf. (1.7) for \check{H}^2). But this follows from (B.35). q.e.d.

As for a single manifold, there is no canonical division of $\check{\lambda}$ by 2, so no canonical center for q . We restrict to fiber bundles for which a center exists.

Definition B.37. A $\check{\mu}$ -structure on a Riemannian spin fiber bundle $X \rightarrow T$ with $\check{\lambda}$ -structure is a cocycle $\check{\mu} = \check{\mu}(X/T) \in \check{Z}_{KO}^0(X)$ and an isomorphism $2\check{\mu} \cong \check{\lambda}$ such that $\check{\mu} = \check{T}$ modulo terms of filtration ≥ 8 .

These are the fiber bundles used in §3. We leave to the future an investigation of existence and uniqueness questions for $\check{\lambda}$ - and $\check{\mu}$ -structures.

Next, we prove a fact used in (3.39).

Proposition B.38. *Let $X \rightarrow T$ be a fiber bundle with a $\check{\mu}$ -structure, as in Definition B.37. Then*

$$\text{pfaff } q(\check{\mu}) = \text{pfaff } q(\check{T}) = \text{pfaff } \int_{X/T} \wedge^2 \check{T}.$$

Proof. Set $\check{\epsilon} = \check{\mu} - \check{T}$. Then $\check{\epsilon}$ has filtration ≥ 8 , whence $\text{pfaff } B(\check{\epsilon}, \check{\epsilon}) = 0$. Thus

$$q(\check{T}) = q(\check{\mu} - \check{\epsilon}) = q(\check{\mu}) - q(\check{\epsilon}) + B(\check{\epsilon}, \check{\epsilon} - \check{\mu}),$$

so it suffices to prove

$$(B.39) \quad \text{pfaff } q(\check{\epsilon}) \cong \text{pfaff } B(\check{\epsilon}, -\check{\mu}).$$

In fact, (B.39) holds for *any* class $\check{\epsilon}$ of filtration ≥ 8 . As in the proof of Lemma B.36, we must only prove some power of (B.39). Now

$$\text{pfaff } B(\check{\epsilon}, -2\check{\mu}) \cong \text{pfaff } B(\check{\epsilon}, \check{\epsilon} - 2\check{\mu}) \cong \text{pfaff } B(\check{\epsilon}, \check{\epsilon} - \check{\lambda}),$$

and from (B.33) or (B.34) it follows that this is isomorphic to $\text{pfaff } 2q(\check{\epsilon})$, which gives the square of (B.39). q.e.d.

We now prove (3.55), which we restate as follows.

Proposition B.40. *Let $X \rightarrow T$ be a fiber bundle of 10-manifolds with Riemannian, spin, and $\check{\mu}$ -structures, and $W \rightarrow T$ a fiber bundle of 2-dimensional spin submanifolds. Denote the inclusion map as $i : W \hookrightarrow X$. Then for $\check{q} \in \check{Z}_{KO}^0(W)$,*

$$(B.41) \quad q(u^{-4} i_* \check{q}) = u^2 \int_W \Delta^+(\nu) \cdot \lambda^2(\check{q}) - \Delta^-(\nu) \cdot \text{Sym}^2(\check{q}),$$

where $\nu \rightarrow W$ is the normal bundle and Δ^\pm are the half-spin bundles.

Proof. Quite generally, for any manifold W let $\pi \nu \rightarrow W$ be a rank 8 real spin bundle⁴⁰ and $Q \rightarrow W$ a real vector bundle of rank r . Denote the zero section of ν as $iW \hookrightarrow \nu$. We first compute the element $x \in KO^0(W)$ defined by

$$(B.42) \quad x := u^4 \pi_* \lambda^2(u^{-4} i_* Q).$$

We claim that

$$(B.43) \quad x = \Delta^+(\nu) \cdot \lambda^2(Q) - \Delta^-(\nu) \cdot \text{Sym}^2(Q).$$

⁴⁰The computation holds for any even rank over the complexes. For rank 8 the half-spin bundles associated to ν are real; for rank 4 they are quaternionic.

Let $U \in KO_{cv}^8(\nu)$ be the Thom class. Then (B.42) implies

$$U \cdot \pi^*x = u^4 \lambda^2(U \cdot \pi^*(u^{-4}Q)).$$

Apply i^* to conclude

$$(B.44) \quad i^*U \cdot x = u^4 \lambda^2(i^*U \cdot u^{-4}Q).$$

This equation, and its solution (B.43), may be viewed as equations in the representation ring $RSpin_8 \times RSO_r$; the corresponding relations in $KO^0(W)$ then follow by passing to the principal bundles underlying ν, Q and the vector bundles associated to the representations. Note (B.44) is an equation of real representations, but we prove it by working in the complex representation ring. To compute the right-hand side of (B.44) we use the Adams operation ψ_2 in the representation ring. Use the splitting principle—i.e., restrict to the maximal torus of $Spin_8$ —to write $\nu \otimes \mathbb{C} = \bigoplus_{i=1}^4 (\ell_i \oplus \ell_i^{-1})$. From (B.4) we compute (the factors of u cancel)

$$\begin{aligned} \psi_2(i^*U \cdot u^{-4}Q) &= \prod_{i=1}^4 (\ell_i - \ell_i^{-1}) \cdot \psi_2(Q) \\ &= \prod_{i=1}^4 (\ell_i^{1/2} - \ell_i^{-1/2}) \cdot (\ell_i^{1/2} + \ell_i^{-1/2}) \cdot \psi_2(Q) \\ &= \frac{i^*U}{u^4} \cdot [\Delta^+(\nu) + \Delta^-(\nu)] \cdot \psi_2(Q). \end{aligned}$$

Hence from (B.6)

$$\begin{aligned} (B.45) \quad 2u^4 \lambda^2(i^*U \cdot u^{-4}Q) &= i^*U \cdot \left\{ [\Delta^+(\nu) - \Delta^-(\nu)] \cdot Q^2 \right. \\ &\quad \left. - [\Delta^+(\nu) + \Delta^-(\nu)] \cdot \psi_2(Q) \right\} \\ &= 2i^*U \cdot \left\{ \Delta^+(\nu) \cdot \lambda^2(Q) - \Delta^-(\nu) \cdot \text{Sym}^2(Q) \right\}, \end{aligned}$$

where we use $Q^2 = \lambda^2(Q) + \text{Sym}^2(Q)$. We deduce the desired result (B.43) from (B.44) and (B.45) using the fact that the ring $RSpin_8 \times RSO_r$ has no zero divisors. Finally, this universal relation in the representation ring applies to bundles with connection, so to differential KO -theory, whence (B.41) holds. q.e.d.

Finally, we provide the proof of the delooping of ψ_2 .

Proof of Proposition B.8. The construction is based on Atiyah's construction of Adams operations [2]. Start with $x \in KO^0(X)$ and square it, remembering the $\mathbb{Z}/2$ -action, to get $P(x) \in KO_{\mathbb{Z}/2}^0(X)$. Since the group $\mathbb{Z}/2$ is not acting on X , there is an isomorphism

$$KO_{\mathbb{Z}/2}^0(X) \approx RO(\mathbb{Z}/2) \otimes KO^0(X),$$

where

$$RO(\mathbb{Z}/2) = \mathbb{Z}[t]/(t^2 - 1)$$

is the real representation ring of $\mathbb{Z}/2$. The Adams operation is the image of $P(x)$ in

$$\mathbb{Z} \xrightarrow{RO(\mathbb{Z}/2)} \otimes KO_{\mathbb{Z}/2}^0(X) = KO^0(X),$$

where the ring homomorphism $RO(\mathbb{Z}/2) \cong \mathbb{Z}[t]/(t^2 - 1) \rightarrow \mathbb{Z}$ sends t to -1 .

This whole discussion would make sense for X a *spectrum*, provided we had an equivariant map $X \rightarrow X \wedge X$ to play the role of the diagonal. We'll define the operation ψ_2 on $KO^1(X)$ by defining it on $KO^0(S^{-1} \wedge X)$. So start with $x \in KO^0(S^{-1} \wedge X)$, form the equivariant external square, and restrict along the diagonal $X \rightarrow X \wedge X$ to define

$$P(x) \in KO_{\mathbb{Z}/2}^0(S^{-1} \wedge S^{-1} \wedge X).$$

For $a, b \geq 0$, let S^{a+b} be the 1-point compactification of the representation of $\mathbb{Z}/2$ on $\mathbb{R}_{(+1)}^a \times \mathbb{R}_{(-1)}^b$. (The subscript indicates the eigenvalue of action of the non-trivial element of $\mathbb{Z}/2$.) By forcing the exponents to add under smash product, we define equivariant spectra S^{a+b} for all $a, b \in \mathbb{Z}$. The shearing isomorphism implies the sphere $S^{-1} \wedge S^{-1}$ with the flip action is isomorphic to S^{-1-t} , so we can regard

$$P(x) \in KO_{\mathbb{Z}/2}^0(S^{-t} \wedge S^{-1} \wedge X).$$

We'll produce below, for any spectrum Y with trivial $\mathbb{Z}/2$ action, a canonical isomorphism

$$(B.46) \quad KO_{\mathbb{Z}/2}^0(S^{-t} \wedge Y) \approx KO^0(Y).$$

In particular, this gives an isomorphism

$$(B.47) \quad KO_{\mathbb{Z}/2}^0(S^{-1} \wedge S^{-1} \wedge X) \approx KO^0(S^{-1} \wedge X).$$

We then define

$$\psi_2(x) \in KO^0(S^{-1} \wedge X)$$

to be the image of $P(x)$ under (B.47).

To construct (B.46) consider the cofibration

$$S^0 \rightarrow S^t \rightarrow S^t \wedge \mathbb{Z}/2_+$$

in which the first map is map of suspension spectra gotten by suspending the inclusion of the fixed points. Smash this with S^{-t} to get

$$S^{-t} \rightarrow S^0 \rightarrow S^0 \wedge \mathbb{Z}/2_+.$$

Passing to equivariant KO -groups leads to a sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{1+t} RO(\mathbb{Z}/2) \rightarrow KO_{\mathbb{Z}/2}^0(S^{-t}) \rightarrow 0,$$

from which it follows that

$$\begin{aligned} KO_{\mathbb{Z}/2}^0(S^{-t}) &\cong RO(\mathbb{Z}/2)/(1+t) \cong \mathbb{Z} \\ KO_{\mathbb{Z}/2}^1(S^{-t}) &= 0. \end{aligned}$$

Smashing this sequence with Y then leads to a short exact sequence

$$0 \rightarrow KO^0(Y) \xrightarrow{1+t} RO(\mathbb{Z}/2) \otimes KO^0(Y) \rightarrow KO_{\mathbb{Z}/2}^0(S^{-t} \wedge Y) \rightarrow 0,$$

which gives the desired result (B.46).

It is useful to note that the map $S^{-t} \rightarrow S^0$ is also the one derived from the diagonal map $S^1 \rightarrow S^1 \wedge S^1$ in

$$(B.48) \quad (S^{-1} \wedge_{\text{flip}} S^{-1}) \wedge S^1 \longrightarrow (S^{-1} \wedge S^1) \wedge_{\text{flip}} (S^{-1} \wedge S^1) = S^0$$

with the $\mathbb{Z}/2$ action as indicated.

Now suppose that $X = S^1 \wedge Y$. We need to show that the diagram

$$\begin{array}{ccc} KO^0(Y) & \xrightarrow{\psi_2} & KO^0(Y) \\ \downarrow & & \downarrow \\ KO^0(S^{-1} \wedge (S^1 \wedge Y)) & \xrightarrow[\psi_2]{} & KO^0(S^{-1} \wedge (S^1 \wedge Y)) \end{array}$$

commutes. The main thing to check is that the map

$$(S^{-1} \wedge_{\text{flip}} S^{-1}) \wedge S^1 \longrightarrow (S^{-1} \wedge S^1) \wedge_{\text{flip}} (S^{-1} \wedge S^1) = S^0,$$

derived from the diagonal map of S^1 , leads to a factorization

$$\begin{array}{ccc} RO(\mathbb{Z}/2) & \longrightarrow & KO_{\mathbb{Z}/2}^0(S^{-1} \wedge S^{-1} \wedge S^1) \\ t \mapsto -1 \downarrow & & \downarrow \approx \\ \mathbb{Z} & \longleftarrow & KO^0(S^{-1} \wedge S^1) \end{array}$$

in which the isomorphism labeled “ \approx ” is the one of (B.47) with $X = S^1$. But this follows immediately from the previous discussion, especially (B.48). q.e.d.

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