Berman’s inequality under random scaling

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Berman’s inequality is the key for establishing asymptotic properties of maxima of Gaussian random sequences and supremum of Gaussian random fields. This contribution shows that, asymptotically an extended version of Berman’s inequality can be established for randomly scaled Gaussian random vectors. Two applications presented in this paper demonstrate the use of Berman’s inequality under random scaling.

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1. INTRODUCTION

In the analysis of extreme values of Gaussian processes and Gaussian random fields, Berman’s inequality plays a crucial role. Essentially, for given two Gaussian distribution functions in $\mathbb{R}^d$ it bounds their difference by comparing the covariances. The key result that motivated this comparison method is Plackett’s partial differential equation given in [27]. As explained in [20], it was then developed by Slepian [28], Berman [1, 2], Cramér [4], Piterbarg [25, 26] and then by Li and Shao [22]. Specifically, the developed results are summarised by Berman’s inequality which we formulate below in the most general form derived in [22]. Let therefore $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two Gaussian random vectors with $N(0, 1)$ components and covariance matrices $A_1 = (\lambda_1^{(1)})$ and $A_2 = (\lambda_2^{(2)})$, respectively. For arbitrary constants $u_i, i \leq n$, [22] obtained

$$\mathbb{P}(X_i \leq u_i, 1 \leq i \leq n) - \mathbb{P}(Y_i \leq u_i, 1 \leq i \leq n) \leq \frac{1}{2\pi} \sum_{1 \leq i < j \leq n} A_{ij} \exp \left( \frac{-u_i^2 + u_j^2}{2(1 + \rho_{ij})} \right) ,$$

where

$$\rho_{ij} := \max(|\lambda_{ij}^{(1)}|, |\lambda_{ij}^{(2)}|), \quad A_{ij} = |\arcsin(\lambda_{ij}^{(1)}) - \arcsin(\lambda_{ij}^{(2)})|.

(1)

Clearly, for arbitrary constants $v_i, u_i, i \leq n$, set $w := \min_{1 \leq i \leq n} \min(|u_i|, |v_i|)$

$$\mathbb{P}(-v_i < X_i \leq u_i, 1 \leq i \leq n) - \mathbb{P}(-v_i < Y_i \leq u_i, 1 \leq i \leq n) \leq \frac{2}{\pi} \sum_{1 \leq i < j \leq n} A_{ij} \exp \left( -\frac{w^2}{1 + \rho_{ij}} \right) .$$

(2)

for a detailed proof see [21]. Berman’s inequality can be applied also to non-Gaussian random vectors. For instance, consider two random vectors $X = (S_1X_1, \ldots, S_nX_n)$ and $Y = (S_1Y_1, \ldots, S_nY_n)$ with $S, S_i, i \leq n$ some positive independent random variables with common distribution function $G$ being further independent from $X$ and $Y$. In the special case $G$ is the uniform distribution on $(0, 1)$, the upper bound in (2) implies

$$\Delta_S(u, v) := \mathbb{P}(-v_i < S_iX_i \leq u_i, 1 \leq i \leq n) - \mathbb{P}(-v_i < S_iY_i \leq u_i, 1 \leq i \leq n) \leq \frac{2}{\pi} \sum_{1 \leq i < j \leq n} A_{ij} \exp \left( -\frac{w^2}{1 + \rho_{ij}} \right) .$$

(3)

Another tractable case is when $G(x) = 1 - \exp(-x), x > 0$ is the exponential distribution. Indeed, by (2) for all $0 < a, b < 1$ we have

$$\Delta_S(u, v) \leq \frac{2}{\pi} \sum_{1 \leq i < j \leq n} A_{ij} \int_0^\infty \int_0^\infty \exp\left( \frac{-(w/s_i)^2 + (w/s_j)^2}{2(1 + \rho_{ij})} - s_i - s_j \right) ds_i ds_j$$

$$= \frac{2}{\pi} \sum_{1 \leq i < j \leq n} A_{ij} \int_0^\infty \int_0^\infty \exp\left( \frac{-(w/s_i)^2 + (w/s_j)^2}{2(1 + \rho_{ij})} - as_i - bs_j \right)$$

$$\times \exp\left( -(1 - a)s_i - (1 - b)s_j \right) ds_i ds_j .$$

(4)

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we shall deal with two large classes of random scaling: a) varying, and b) in particular the case that its survival function is regularly at the right endpoint of its distribution function, including a bounded random variable with a tractable tail behaviour.

Section 3 we display two applications, while the proofs are Berman’s inequality for scaled Gaussian random vectors. In some tractable tail assumptions on the random vector \( \mathbf{X} \) which has right endpoint equal to 1. Intuitively, large values of the product say do not influence significantly the \( \mathbf{X} \)’s and \( \mathbf{Y} \)’s sufficiently large. We shall consider two particular cases for the random vector \( \mathbf{S} = (S_1, \ldots, S_n) \). Our results can be in fact extended for certain tractable dependence models. We shall focus on simplicity only with these two cases.

Since random scaling is a natural phenomena related to the time-value of money in finance, measurement errors in these two cases.

Of course, Berman’s inequality alone is not enough for extending [17] to randomly scaled Gaussian triangular arrays; some additional results (see [15, 16]) which show that for some tractable tail assumptions on \( \mathbf{S} \) the scaled random vector \( \tilde{\mathbf{X}} \) behaves similarly to \( \mathbf{X} \) are also important. Specifically, we shall deal with two large classes of random scaling: a) \( \mathbf{S} \) is a bounded random variable with a tractable tail behaviour at the right endpoint of its distribution function, including in particular the case that its survival function is regularly varying, and b) \( \mathbf{S} \) is a Weibull-type random variable.

In view of our findings, several known results for Gaussian random sequences and processes can be extended to the scaled Gaussian framework; we shall demonstrate this with two representative applications.

Organisation of the rest of the paper: Section 2 presents Berman’s inequality for scaled Gaussian random vectors. In Section 3 we display two applications, while the proofs are relegated to Section 4.

2. MAIN RESULTS

We consider first the case that \( \mathbf{S} \) is non-negative with distribution function \( G \) which has right endpoint equal to 1. Intuitively, large values of \( \mathbf{S} \) do not influence significantly large values of the product say \( 3X \) if \( X \) is a Gaussian random variable being independent of \( \mathbf{S} \). It turns out that the following asymptotic upper bound valid for all \( u \) large and some \( c_A > 0, \tau > 0 \) is sufficient for the derivation of a useful upper bound for \( \Delta_S(u, v) \) defined above.

A canonical example of such \( \mathbf{S} \) is the beta random variable, which is a special case of a power-tail random variable \( \mathbf{S} \), namely

\[
\mathbb{P}(S > 1 - 1/u) \leq c_A u^{-\tau}
\]

Corollary 2.1. Let \( \mathbf{X}, \tilde{\mathbf{X}}, \mathbf{Y}, \tilde{\mathbf{Y}}, \mathbf{S}, \mathbf{S}', \mathbf{S}_1, i \leq n \) be as above. If (5) holds, then for all \( u, v, 1 \leq i \leq n \) and \( \epsilon > 0 \) we have

\[
\Delta_S(u, v) \leq (K_A + \epsilon)u^{-\tau} \sum_{1 \leq i < j \leq n} A_{ij}(1 + \rho_{ij})^\tau \exp\left(-\frac{w^2}{1 + \rho_{ij}}\right)
\]

and

\[
\Delta_{S1}(u, v) \leq (K_A^\star + \epsilon)u^{-\tau} \sum_{1 \leq i < j \leq n} A_{ij}(1 + \rho_{ij})^\tau \exp\left(-\frac{w^2}{1 + \rho_{ij}}\right),
\]

where \( K_A = \frac{2}{\pi} c_A (\Gamma(\tau + 1))^2 \) and \( K_A = \frac{21 - \tau}{\tau} c_A \Gamma(\tau + 1) \).

We shall investigate below the more difficult case that the scaling random variable \( \mathbf{S} \) has distribution function with an infinite right endpoint. Motivated by the example of the exponential distribution in the Introduction, we shall assume that \( \mathbf{S} \) has tail behaviour similar to a Weibull distribution. Specifically, for given constants \( \alpha \in \mathbb{R}, c_B, L, p \in (0, \infty) \) suppose that

\[
\mathbb{P}(S > u) = (1 + o(1))c_B u^\alpha \exp(-Lu^p), \ u \to \infty
\]
is valid. The class of distribution functions satisfying (11) is quite large. More importantly, under (11) $SX$ has also a Weibull tail behaviour if $X$ is a $N(0,1)$ random variable independent of $S$, see e.g., [16]. We state next our second result for Weibull-type random scaling.

**Theorem 2.2.** Let $X, \tilde{X}, Y, \tilde{Y}, S, S_i, i \leq n$ be as above. If (11) holds, then for all $u, v_i, 1 \leq i \leq n$ and $\epsilon > 0$ we have

\begin{align}
\Delta s(u, v) & \leq (K_B + \epsilon)w^{\frac{4+2p}{2+\theta}}\sum_{1 \leq i < j \leq n} A_{ij}(1 + \rho_{ij})^{\frac{2n+2p}{2+\theta}} \\
& \times \exp \left(-2(1 + \rho_{ij})^{-\frac{2n+2p}{2+\theta}}Tw^{\frac{2p}{2n}}\right)
\end{align}

and

\begin{align}
\Delta s_1(u, v) & \leq (K_B + \epsilon)w^{\frac{4+2p}{2+\theta}}\sum_{1 \leq i < j \leq n} A_{ij}(1 + \rho_{ij})^{\frac{2n+2p}{2+\theta}} \\
& \times \exp \left(-2(1 + \rho_{ij})^{-\frac{2n+2p}{2+\theta}}Tw^{\frac{2p}{2n}}\right),
\end{align}

where $T = L^p \tau^p p^{-\frac{p}{\theta}} + (Lp)^{\frac{1}{2}} \pi^{\frac{1}{2}} 2^{-1}$, $K_B = 4c_B^2 \times (Lp)^{\frac{1}{2}} \pi^{\frac{1}{2}} (p + 2)^{-1}$ and $K_B' = 2w^{\frac{2}{2n}} \pi^{\frac{1}{2}} (p + 2)^{-1}$.

**Corollary 2.2.** Under the conditions of Theorem 2.2, for all $u$ large and some positive constants $Q$ we have

\begin{align}
\Delta s(u, v) & \leq Qw^{\frac{4n+2p}{2+\theta}}\sum_{1 \leq i < j \leq n} |\lambda_{ij}^{(1)}| \\
& \times \exp \left(-2(1 + |\lambda_{ij}^{(1)}|)^{-\frac{2n+2p}{2+\theta}}Tw^{\frac{2p}{2n}}\right)
\end{align}

and

\begin{align}
\Delta s_1(u, v) & \leq Qw^{\frac{4n+2p}{2+\theta}}\sum_{1 \leq i < j \leq n} |\lambda_{ij}^{(1)}| \\
& \times \exp \left(-2(1 + |\lambda_{ij}^{(1)}|)^{-\frac{2n+2p}{2+\theta}}Tw^{\frac{2p}{2n}}\right).
\end{align}

**Remark 2.1.**

a) Clearly, when $S$ is uniformly distributed on $(0,1)$ then condition (5) holds with $c_A = \tau = 1$. For this case we have two results, the one derived in the Introduction and that given by (7). We see that the bound obtained by (7) with $c_A = \tau = 1$ is better due to the term $w^{-4r}$.

b) Also for the case $S$ is a unit exponential random variable we have two bounds, one which holds for all values of $u, v_i, i \leq n$ and the asymptotic one given in Theorem 2.2. The bound implied by (12) with $c_B = 1, \alpha = 0, p = 1, L = 1$ is asymptotically better than that implied by (4).

### 3. Applications

An important contribution in extreme value theory concerned with maxima of triangular arrays of Gaussian random variables is [17]. Motivated by the findings of Hüsler and Reiss in 1989 (see [18]) the aforementioned contribution considered a triangular array of $N(0,1)$ random variables $\{X_n, i, n \geq 1\}$ such that for each $n$, $\{X_n, i, n \geq 1\}$ is a stationary Gaussian random sequence. Assume that $\varrho_{n,j} = \mathbb{E}(X_{n,j}X_{n+1,j})$ satisfies for any $j \geq 1$

\begin{align}
\lim_{n \to \infty} (1 - \varrho_{n,j}) \ln n = \delta_j \in (0, \infty), \quad \delta_0 = 0
\end{align}

and for each $n$, $\varrho_{n,j}$ decays fast enough as $j$ increases. Under some additional conditions (see Theorem 3.1 below) the deep contribution [17] shows that for the maxima $M_n = \max_{1 \leq i \leq n} X_{n,i}$

\begin{align}
\lim_{n \to \infty} P(M_n \leq a_n x + b_n) = \exp(-\vartheta \exp(-x)), \quad x \in \mathbb{R},
\end{align}

where

\begin{align}
a_n = (2 \ln n)^{-\frac{1}{2}}, \quad b_n = (2 \ln n)^{\frac{1}{2}} - \frac{1}{2} (2 \ln n)^{-\frac{1}{2}} (\ln \ln n + \ln 4\pi)
\end{align}

and

\begin{align}
\vartheta = P \left( E/2 + \sqrt{\delta_{k-1}W_k} \leq \delta_{k-1}, \quad \text{for all } k \geq 2 \right),
\end{align}

with $E$ a unit exponential random variable independent of $W_k$ and $\{W_k, k \geq 2\}$ being jointly Gaussian with zero means and covariances

\begin{align}
\mathbb{E}(W_iW_j) = \frac{\delta_{i-1} + \delta_{j-1} - \delta_{i-j-1}}{2\sqrt{\delta_{i-1}\delta_{j-1}}}.
\end{align}

The proof of (17) strongly relies on Berman’s inequality. Hence, our first application extends the result of [17] to triangular arrays of randomly scaled Gaussian random variables. In the following we investigate the effect of a comonotonic random scaling considering a bounded $S$ and thus $S = S_1$.

**Theorem 3.1.** Let $\{X_{n,i}, i, n \geq 1\}$ be a triangular array of standard Gaussian random variables defined as above satisfying (16), being further independent of the iid non-negative random variables $\{S_n, n \geq 1\}$ where $S_1$ satisfies (6). If there exist positive integers $r_n, l_n$ such that

\begin{align}
\lim_{n \to \infty} \frac{l_n}{r_n} = 0, \quad \lim_{n \to \infty} \frac{r_n}{n} = 0,
\end{align}

\begin{align}
\lim_{n \to \infty} \frac{n^2}{r_n} c_n^{-\tau} \sum_{j=l_n}^{n} |\varrho_{n,j}| (1 + |\varrho_{n,j}|)^{\tau} \exp \left(-\frac{c_n}{1 + |\varrho_{n,j}|}\right) = 0.
\end{align}

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with $c_n := 2\ln n - (2r + 1)\ln\ln n$ and further

$$
\lim\limsup_{n \to \infty} \sup_{j > m} \frac{c_n}{n} \frac{\ln n}{\ln\ln n} \left( 1 - \frac{\epsilon_n}{\ln\ln n} \right) = 0,
$$

then for the maxima $M_n = \max_{1 \leq i \leq n} S_n X_{n,i}$ the result in (17) holds with $\theta$ defined as above and

$$
a_n = (2\ln n)^{-1/2},
$$

$$
b_n = (2\ln n)^{1/2} + (2\ln n)^{-1/2} \times \left( \ln(\epsilon(2\pi)^{-1/2} \Gamma(1 + \tau)) - \frac{2\tau + 1}{2} \ln(\ln n + \ln 2) \right).
$$

Remark 3.1. Using similar arguments as in the proof of Theorem 3.1, the findings of the recent contribution [6] can also be extended by considering a randomly scaled Gaussian field on a lattice.

In our second application we consider scaled Gaussian random vectors where the scaling vector $S$ has independent components. Let $\{X_n, k = (X_{n,k}^{(1)}, X_{n,k}^{(2)}), 1 \leq k \leq n, n \geq 1\}$ be a triangular array of bivariate centered stationary Gaussian random vectors with unit-variance and correlation given by

$$
corr(X_n^{(i)}, X_n^{(j)}) = \lambda_0(n),
$$

$$
corr(X_n^{(i)}, X_n^{(j)}) = \lambda_{ij}(|k - l|, n),
$$

where $1 \leq k \neq l \leq n$ and $i, j \in \{1, 2\}$. Further, let $\{\hat{X}_n, k = (\hat{X}_{n,k}^{(1)}, \hat{X}_{n,k}^{(2)}), 1 \leq k \leq n, n \geq 1\}$ denote the associated iid triangular array of $\{X_n, k\}$, i.e., $\text{corr}(\hat{X}_{n,k}^{(1)}, \hat{X}_{n,k}^{(2)}) = \lambda_0(n)$ and $\text{corr}(\hat{X}_{n,k}^{(i)}, \hat{X}_{n,l}^{(j)}) = 0$, for $1 \leq k \neq l \leq n$ and $i, j \in \{1, 2\}$. Let $\{S_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be iid random variables being independent of $\{X_n, k = (X_{n,k}^{(1)}, X_{n,k}^{(2)}), 1 \leq k \leq n, n \geq 1\}$ and $\{\hat{X}_n, k = (\hat{X}_{n,k}^{(1)}, \hat{X}_{n,k}^{(2)}), 1 \leq k \leq n, n \geq 1\}$, respectively. Assume that the correlation $\lambda_0(n)$ satisfies

$$
\lim_{n \to \infty} \frac{b_n}{a_n} (1 - \lambda_0(n)) = 2\lambda^2 \quad \text{with} \quad \lambda \in [0, \infty],
$$

where

$$
a_n = \frac{1}{1 - F(b_n)} \int_{b_n}^{\infty} (1 - F(x)) dx, \quad b_n = F^{-1}\left(1 - \frac{1}{n}\right),
$$

with $F^{-1}$ the inverse of the df of $S_1, \hat{X}_{1,1}^{(1)}$. It is well-known (see e.g., [10]) that

$$
\lim\limsup_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P}\left( \max_{1 \leq k \leq n} S_{n,k}\hat{X}_{n,k}^{(1)} \leq u_n(x), \max_{1 \leq k \leq n} S_{n,k}\hat{X}_{n,k}^{(2)} \leq u_n(y) \right) - H_{\lambda}(x, y) \right| = 0,
$$

where $u_n(x) = a_n x + b_n$, with $a_n$ and $b_n$ defined in (18) and $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, provided that $\lim_{n \to \infty} \max_{u_n(x), b_n \leq u_n(x)} \lambda_{1}(k, n) \ln n = 0$ with $l_n = [n^\delta]$, $0 < \beta < (1 - \delta) / (1 + \delta)$ and $\delta = \max_{1 \leq k \leq n, n \geq 2} |\lambda_{1}(k, n)|$. Below we obtain a more general result for our 2-dimensional setup considering Weibull-type random scaling.

Theorem 3.2. Let $\{(X_{n,k}^{(1)}, X_{n,k}^{(2)}), 1 \leq k \leq n, n \geq 1\}$ be a bivariate triangular array of standard Gaussian random vectors defined as above. Let $\{S_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be iid
random variables being independent of \(\{(X_{n,k}^{(1)}, X_{n,k}^{(2)})\}, 1 \leq k \leq n, n \geq 1\). Suppose that the correlation \(\lambda_n(k)\) satisfy (23) with \(\lambda \in (0, \infty)\) and condition \(E\) holds. Let \(\beta\) be a constant satisfying \(0 < \beta < 2(1 + \sigma)^{-1} = 1\) with \(\sigma = \max_{1 \leq i \leq n, \kappa \geq 1} |\lambda_{ik}(k, n)| < 1\), and write \(\nu = \lceil n^\beta \rceil\). If (11) holds and the covariance function satisfies
\[
\lim_{n \to \infty} \max_{1 \leq i \leq n, 1 \leq k \leq n} |\lambda_{ik}(k, n)| \ln n = 0,
\]
then we have
\[
\lim_{n \to \infty} \sup_{1 \leq x_1, \cdots, x_n \leq 4, 1 \leq x_1, \cdots, x_n \leq 4} \mathbb{P} \left( -u_n(y_1) < m_{n,1}^{(1)}(\varepsilon_n) \leq M_{n,1}^{(1)}(\varepsilon_n) \leq u_n(x_1), \right. \\
- u_n(y_2) < m_{n,2}^{(1)}(\varepsilon_n) \leq M_{n,2}^{(1)}(\varepsilon_n) \leq u_n(x_2), \\
- u_n(y_3) < m_{n,3}^{(1)}(\varepsilon_n) \leq M_{n,3}^{(1)}(\varepsilon_n) \leq u_n(x_3), \\
- u_n(y_4) < m_{n,4}^{(2)}(\varepsilon_n) \leq M_{n,4}^{(2)}(\varepsilon_n) \leq u_n(x_4) \right|
\]
\[
= 0,
\]
where \(H_\lambda\) is defined in (24) and normalising constants \(a_n\) and \(b_n\) satisfy
\[
a_n = \frac{2 + p}{2p} T^{-\frac{2p}{2p}} (\ln n)^{-\frac{2p}{2p}},
\]
\[
b_n = \frac{(\ln n)^{\frac{2p}{2p}}} T + \frac{2 + p}{2p} T^{-\frac{2p}{2p}} (\ln n)^{-\frac{2p}{2p}} \times \left( \frac{\alpha}{p} \ln \ln n - \frac{\alpha}{p} \ln T + \ln \varpi_B \right)
\]
with \(T = 2^{-1} Q^2 + L Q^{-p}\), \(\varpi_B = c_B (2 + p)^{-\frac{1}{2}} Q^{-\alpha}\) and \(Q = (pL)^{1/(2 + p)}\).

4. PROOFS

Proof of Theorem 2.1 By the independence of \(\mathbf{S}\) and \((\mathbf{X}, \mathbf{Y})\) and the generalised Berman's inequality (see Theorem 2.1 in [22] and Lemma 11.1.2 in [21]), if (5) holds, then
\[
\Delta_S (u, v) \leq \frac{\pi}{2} (\Gamma(\tau + 1))^{2} (c_A + \varepsilon) w^{-4\tau} \sum_{1 \leq i < j \leq n} A_{ij} (1 + \rho_{ij})^{2\tau} \exp \left( -\frac{u^2}{2(1 + \rho_{ij})} \right).
\]

With similar arguments as above we have
\[
\Delta_S (u, v) \leq \frac{\pi}{2} (\Gamma(\tau + 1))^{2} (c_A + \varepsilon) w^{-4\tau} \sum_{1 \leq i < j \leq n} A_{ij} (1 + \rho_{ij})^{2\tau} \exp \left( -\frac{u^2}{2(1 + \rho_{ij})} \right).
\]

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\[
\sum_{1 \leq i < j \leq n} A_{ij} (1 + \rho_{ij})^T \exp \left( -\frac{w^2}{1 + \rho_{ij}} \right),
\]

hence the claim follows.

**Proof of Theorem 2.2** According to the independence of the scaling factors with the Gaussian random variables and the generalised Berman’s inequality (see Theorem 2.1 in [22] and Lemma 11.1.2 in [21]) again if (11) holds, then we have

\[
\Delta_s(u, v) = \int_{[0, \infty]^n} \left( \mathbb{P} \left( -\frac{v_i}{s} < X_i \leq \frac{u_i}{s}, 1 \leq i \leq n \right) - \mathbb{P} \left( -\frac{v_i}{s} < Y_i \leq \frac{u_i}{s}, 1 \leq i \leq n \right) \right) dG(s_1) \cdots dG(s_n)
\]

\[
\leq \frac{2}{\pi} \int_{[0, \infty]^n} \sum_{1 \leq i < j \leq n} \exp \left( -\frac{(w/s)^2 + (w/t)^2}{2(1 + \rho_{ij})} \right) dG(s_1) \cdots dG(s_n)
\]

\[
\Delta_s(u, v) \leq \frac{2}{\pi} \sum_{1 \leq i < j \leq n} \int_{[0, \infty]} \int_{[0, \infty]} \exp \left( -\frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s) dG(t),
\]

where \(\rho_{ij}\) and \(A_{ij}\) are defined in (1). Note that for \(1 \leq i, j \leq n\) and some positive constants \(c_1, c_2\), using similar arguments as in the proof of Theorem 2.1 in [16], we have

\[
\int_{\mathbb{R}^{2n}} \exp \left( -\frac{1}{2(1 + \rho_{ij})} \left( \frac{s}{s} \right)^2 \right) dG(s)
\]

\[
\approx c_B \int_{\mathbb{R}^{2n}} \exp \left( -\frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s)
\]

\[
\approx c_B \int_{\mathbb{R}^{2n}} \exp \left( -Ls^p - \frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) ds
\]

\[
\times \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{t}{t} \right)^2 \right) dt
\]

\[
\approx c_B \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s)
\]

\[
\times \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{t}{t} \right)^2 \right) dt
\]

\[
\approx \frac{c_B \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s)}{\int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s)}
\]

\[
\times \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{t}{t} \right)^2 \right) dt
\]

\[
\approx \frac{c_B \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s)}{\int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{w}{s} \right)^2 \right) dG(s)}
\]

\[
\times \int_{\mathbb{R}^{2n}} \exp \left( -Lp^q - \frac{1}{2(1 + \rho_{ij})} \left( \frac{t}{t} \right)^2 \right) dt
\]

as \(w \to \infty\). Hence for \(\epsilon > 0\) we have

\[
\Delta_s(u, v) \leq \frac{4(c_B + \epsilon)(Lp)^2 (1 - \alpha)}{p + 2} w^{2n+2p},
\]

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$x > 0$, with $F_1$ the distribution function of $S_nR$. According to Theorem 3.1 in [11] $F_1$ in the Gumbel max-domain of attraction and

$$\lim_{n \to \infty} \frac{\mathbb{P}(S_n X_{n,1} > t_n(y))}{\mathbb{P}(S_n X_{n,1} > u_n)} = e^{-y}, \quad \forall y \in \mathbb{R}.$$  

Hence, by Theorem 3.1 in [8]

$$\mathbb{P}(S_n X_{n,k} \leq u_n, k \in K | S_n X_{n,1} = t_n(y)) = \mathbb{P}(u_n(1 - x_{n,k-1})^{1/2} Z_n + \frac{x_{n,k-1}}{2} y \leq \delta_{k-1}, k \in K) \rightarrow \mathbb{P} \left( \sqrt{\delta_{k-1}} W_k + \frac{y}{2} \leq \delta_{k-1}, k \in K \right), \quad n \to \infty$$

uniformly on compact sets of $y$, where

$$\left( Z_{n,k}, k \in K \right) \overset{d}{=} R_{m,\tilde{B}_n} U_m,$$

with

$$\tilde{B}_n = \frac{\theta_n \otimes I_{n \otimes n}}{(1 - x_{n,k-1})(1 - x_{n,k-1})}, \quad i, j \in K$$

and $\{W_k, k \in K\}$ being jointly Gaussian with zero means and covariances

$$\mathbb{E}(W_i W_j) = \frac{\delta_{i-1} + \delta_{j-1} - \delta_{l-1}}{2\sqrt{\delta_{i-1} \delta_{j-1}}}, \quad i, j \in K.$$  

Since further

$$\mathbb{P}(S_n X_{n,k} \leq u_n, k \in K | S_n X_{n,1} = t_n(y)) = \int_0^1 p_{n,y} \mathbb{P}(S_n X_{n,1} \leq t_n(y)) \mathbb{P}(S_n X_{n,1} > u_n)$$

the proof is established by applying Lemma 4.4 in [8] (recall (26) and (27)). \hfill \square

**Proof of Theorem 3.1** According to (8), if $1 \leq k_1 < \cdots < k_s \leq n$ and $k = \min_{i \leq i \leq s}(k_{i+1} - k_i)$ then the joint distribution function $F_{k_1, \ldots, k_s}$ of $S_n X_{n,k_1}, \ldots, S_n X_{n,k_s}$ satisfies

$$\left| F_{k_1, \ldots, k_s}(u_n) - \prod_{i=1}^s \mathbb{P}(S_n X_{n,k_i} \leq u_n) \right| \leq Q u_n^{-2n} \sum_{i=k}^n \frac{(1 + \rho_{n,i})}{\sqrt{1 - \rho_{n,i}}} \exp \left( \frac{u_n^2}{2} \right).$$

Suppose now that $1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_p \leq n$ and $j_1 - i_p \geq t_n$. Identifying $\{k_1, \ldots, k_s\}$ in turn with $\{i_1, \ldots, i_p, j_1, \ldots, j_p\}$, we thus have

$$\left| F_{i_1, \ldots, i_p, j_1, \ldots, j_p}(u_n) - F_{i_1, \ldots, i_p}(u_n) F_{j_1, \ldots, j_p}(u_n) \right| \leq 3Q u_n^{-2n} \sum_{i=k}^n \frac{(1 + \rho_{n,i})}{\sqrt{1 - \rho_{n,i}}} \exp \left( \frac{u_n^2}{2} \right).$$

By Example 1 in [9] and Table 3.4.4 in [5] we have

$$\lim_{n \to \infty} n \mathbb{P}(S_n X_{n,1} \geq u_n(x)) = e^{-x}, \quad x \in \mathbb{R},$$

where $u_n(x) = a_n x + b_n$ with $a_n$ and $b_n$ defined in (22). Consequently, as $n \to \infty$

$$u_n^2(x) = 2 \ln n - (2r + 1) \ln \ln n + O(1).$$

Hence, in view of (19) and (20), Theorem 2.1 in [23] implies

$$\lim_{n \to \infty} \mathbb{P} \left( \max_{1 \leq i \leq n} S_n X_{n,i} \leq u_n(x) \right) - \exp \left( -n \mathbb{P}(S_n X_{n,1} > u_n(x)) \right) \mathbb{P} \left( \{ u_n | S_n X_{n,1} > u_n(x) \} \right) = 0.$$

Note that for $m \leq j \leq r_n$ we have

$$\mathbb{P} \left( W > u_n \left\{ \frac{1 - \rho_{n,j}}{1 + \rho_{n,j}} \right\} \frac{y}{\sqrt{1 - \rho_{n,j}}} \right) \leq \frac{Q \left( \frac{1 - \rho_{n,j}}{1 + \rho_{n,j}} \right) \left( \ln n \right)}{\sqrt{1 - \rho_{n,j}}}.$$

where $W$ is a $N(0,1)$ random variable. The claim can then be established by using similar arguments as in the proof of Theorem 2.1 in [17] making further use of (21) and Lemma 4.1. \hfill \square

Next, for some index sets $I_n \subset N$ we define

$$\tilde{M}(I_n, \varepsilon_n) := \begin{cases} \max \{ S_{n,k} X_{n,k}, k \in I_n, \varepsilon_n = 1 \}, & \text{if } \sum_{k \in I_n} \varepsilon_n \geq 1; \\ \inf \left\{ x | \mathbb{P} \left( S_{n,k} X_{n,k} \leq x \right) > 0 \right\}, & \text{otherwise}, \end{cases}$$

$$\tilde{m}(I_n, \varepsilon_n) := \begin{cases} \min \{ S_{n,k} X_{n,k}, k \in I_n, \varepsilon_n = 1 \}, & \text{if } \sum_{k \in I_n} \varepsilon_n \geq 1; \\ \inf \left\{ x | \mathbb{P} \left( S_{n,k} X_{n,k} \leq x \right) > 0 \right\}, & \text{otherwise}. \end{cases}$$

For simplicity, we write $\tilde{M}_n(\varepsilon_n) = \tilde{M}([1,2, \ldots, n], \varepsilon_n)$, $\tilde{M}(I_n) = \max \{ S_{n,k} X_{n,k}, k \in I_n \}$, $\tilde{M}_n = \max \{ S_{n,k} X_{n,k}, 1 \leq k \leq n \}$. Similarly we also define $\tilde{m}_n(\varepsilon_n), \tilde{m}(I_n), \tilde{m}_n$.  

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Lemma 4.2. Let \( \{(\hat{X}_{n,i}^{(1)}, \hat{X}_{n,i}^{(2)}), 1 \leq i \leq n, n \geq 1\} \) be a triangular array of centered stationary Gaussian random vectors defined as above with the correlation \( \lambda_0(n) \) satisfying (23) with \( \lambda \in (0, \infty) \). Further let \( \{S_{n,k}, 1 \leq k \leq n, n \geq 1\} \) be iid random variables being independent of \( \{(\hat{X}_{n,i}^{(1)}, \hat{X}_{n,i}^{(2)}), 1 \leq i \leq n, n \geq 1\} \) and satisfying (11). Then we have

\[
\begin{align*}
\lim_{n \to \infty} P \left( -u_n(y_1) < \hat{m}_n^{(1)} \leq \hat{M}_n^{(1)} \leq u_n(x_1), \\
-\hat{u}_n(y_2) < \hat{m}_n^{(2)} \leq \hat{M}_n^{(2)} \leq u_n(x_2) \right) = H_\lambda(x_1, x_2) H_\lambda(y_1, y_2).
\end{align*}
\]

Proof of Lemma 4.2 Our proof is similar to that of Theorem 2.1 in [14]. For any integer \( n \) we may write

\[
\begin{align*}
n (1 - P(n, x_1, x_2, y_1, y_2)) &= n P_1(n, x_1, x_2) + n P_2(n, y_1, y_2) \\
&- n P_3(n, x_1, y_2) - n P_4(n, y_1, x_2),
\end{align*}
\]

where

\[
\begin{align*}
P(n, x_1, x_2, y_1, y_2) := & \quad P \left( -u_n(y_1) < S_{n,1} \hat{X}_{n,1}^{(1)} \leq u_n(x_1), \\
&\quad -u_n(y_2) < S_{n,1} \hat{X}_{n,1}^{(2)} \leq u_n(x_2) \right), \\
P_1(n, x_1, x_2) := & \quad P \left( S_{n,1} \hat{X}_{n,1}^{(1)} > u_n(x_1) \right) + P \left( S_{n,1} \hat{X}_{n,1}^{(2)} > u_n(x_2) \right) \\
&\quad - P \left( S_{n,1} \hat{X}_{n,1}^{(1)} > u_n(x_1), S_{n,1} \hat{X}_{n,1}^{(2)} > u_n(x_2) \right), \\
P_2(n, y_1, y_2) := & \quad P \left( S_{n,1} \hat{X}_{n,1}^{(1)} \leq -u_n(y_1) \right) + P \left( S_{n,1} \hat{X}_{n,1}^{(2)} \leq -u_n(y_2) \right) \\
&\quad + P \left( S_{n,1} \hat{X}_{n,1}^{(1)} \leq -u_n(y_1), S_{n,1} \hat{X}_{n,1}^{(2)} \leq -u_n(y_2) \right), \\
P_3(n, x_1, y_2) := & \quad P \left( S_{n,1} \hat{X}_{n,1}^{(1)} > u_n(x_1), S_{n,1} \hat{X}_{n,1}^{(2)} \leq -u_n(y_2) \right), \\
P_4(n, y_1, x_2) := & \quad P \left( S_{n,1} \hat{X}_{n,1}^{(1)} \leq -u_n(y_1), S_{n,1} \hat{X}_{n,1}^{(2)} > u_n(x_2) \right).
\end{align*}
\]

The random vector \( (\hat{X}_{n,1}^{(1)}, \hat{X}_{n,1}^{(2)}) \) has the following stochastic representation

\[
(\hat{X}_{n,1}^{(1)}, \hat{X}_{n,1}^{(2)}) \overset{d}{=} (R \cos \theta, R \cos(\theta - \psi_n)),
\]

where \( R \) is a positive random variable being independent of the random variable \( \theta \) which is uniformly distributed in \( (-\pi, \pi) \) and \( \psi_n = \arccos(\lambda_0(n)) \). If \( S_{n,1} \) satisfy (11) and is independent of \( (\hat{X}_{n,1}^{(1)}, \hat{X}_{n,1}^{(2)}) \), using Laplace approximation (see e.g., [16]) we have that the distribution function of \( S_{n,1} R \) is in the max-domain of attraction of the Gumbel distribution. Hence, according to Remark 2.2 in [13] we have

\[
\lim_{n \to \infty} n P \left( S_{n,1} \hat{X}_{n,1}^{(1)} > u_n(x_1) \right) = e^{-x}, \quad x \in \mathbb{R},
\]

where \( u_n(x) = u_n + b_n \) with \( u_n \) and \( b_n \) defined in (25).

Moreover, by Theorem 2.1 in [7]

\[
\lim_{n \to \infty} n P_1(n, x_1, x_2) = \Phi \left( \frac{x_1 - x_2}{2\lambda} \right) e^{-x_2} + \Phi \left( \frac{x_2 - x_1}{2\lambda} \right) e^{-x_1},
\]

and since \( -S_{n,1} \hat{X}_{n,1}^{(1)} \leq S_{n,1} \hat{X}_{n,1}^{(2)} \),

\[
\lim_{n \to \infty} n P_2(n, y_1, y_2) = D(y_1, y_2).
\]

Since \( \lim_{n \to \infty} \lambda_0(n) = 1, \lim_{n \to \infty} \psi_n = 0 \) implying

\[
\lim_{n \to \infty} n P_3(n, x_1, y_2) = \lim_{n \to \infty} n P_4(n, y_1, x_2) = 0.
\]

Hence for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \)

\[
\lim_{n \to \infty} P \left( -u_n(y_1) < \hat{m}_n^{(1)} \leq \hat{M}_n^{(1)} \leq u_n(x_1), \\
-\hat{u}_n(y_2) < \hat{m}_n^{(2)} \leq \hat{M}_n^{(2)} \leq u_n(x_2) \right) = H_\lambda(x_1, x_2) H_\lambda(y_1, y_2),
\]

hence the proof is complete. \( \square \)

Lemma 4.3. Under the conditions of Lemma 4.2, if the indicator random variables \( \varepsilon_{n,i} = \{\varepsilon_{n,i}, 1 \leq i \leq n\} \) are independent of both \( \{(\hat{X}_{n,i}^{(1)}, \hat{X}_{n,i}^{(2)}), 1 \leq i \leq n\} \) and \( \{S_{n,i}, 1 \leq i \leq n\} \) and satisfying condition \( \mathcal{E} \), then

\[
\lim_{n \to \infty} \sup_{\varepsilon_{n,i} \in \{0,1\}} \frac{1}{n} \sum_{1 \leq i \leq n} \varepsilon_{n,i} \chi_{A_i} = 1, \quad \chi_{A_1} = 1, \chi_{A_2} = 0, \chi_{A_3} = 1, \chi_{A_4} = 0
\]

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\[ \left| \mathbb{P} \left( -u_n(y_1) < \hat{m}_n^{(1)}(\varepsilon_n) \leq \hat{M}_n^{(1)}(\varepsilon_n) \leq u_n(x_1) \right) - \mathbb{P} \left( -u_n(y_2) < \hat{m}_n^{(2)}(\varepsilon_n) \leq \hat{M}_n^{(2)}(\varepsilon_n) \leq u_n(x_2) \right) \right| \\
- \mathbb{P} \left( -u_n(y_3) < \hat{m}_n^{(1)}(\varepsilon_n) \leq \hat{M}_n^{(1)}(\varepsilon_n) \leq u_n(x_3) \right) - \mathbb{P} \left( -u_n(y_4) < \hat{m}_n^{(2)}(\varepsilon_n) \leq \hat{M}_n^{(2)}(\varepsilon_n) \leq u_n(x_4) \right) \\\- \mathbb{E} \left( H_n^{(1)}(x_1, x_2) H_n^{(2)}(y_1, y_2) H_n^{(3)}(x_3, x_4) H_n^{(4)}(y_3, y_4) \right) \right| = 0. \]

**Proof of Lemma 4.3** Using similar arguments as for the derivation of [19], let \( K_s = \{ j : (s-1)\nu + 1 \leq j \leq s\nu \}, 1 \leq s \leq l, \nu = \lfloor \frac{\mu}{2} \rfloor, x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \) and \( \beta_n = \{ \beta_{n,k} \mid 1 \leq k \leq n \} \) be a nonrandom triangular array consisting of 0’s and 1’s. For some random variable \( \eta \) such that \( 0 \leq \eta \leq 1 \) a.s., write

\[ B_{\mu,l} = \left\{ \omega : \eta(\omega) \in \left( 0, \frac{1}{2^l} \right], \mu = 0, \left( \frac{1}{2^l}, \frac{1}{2^l} + 1 \right), 0 < \mu < 2^l - 1 \right\} , \]

\[ B_{\mu,l},\beta_n = \{ \omega : \varepsilon_n(\omega) = \beta_{n,k}, 1 \leq k \leq n \} \cap B_{\mu,l}. \]

Set

\[ P(K_s, \beta_n, x, y) = \mathbb{P} \left( -u_n(y_1) < \hat{m}_n^{(1)}(K_s, \beta_n) \leq \hat{M}_n^{(1)}(K_s, \beta_n) \leq u_n(x_1) \right), \]

\[ -u_n(y_2) < \hat{m}_n^{(2)}(K_s, \beta_n) \leq \hat{M}_n^{(2)}(K_s, \beta_n) \leq u_n(x_2) \right), \]

\[ -u_n(y_3) < \hat{m}_n^{(1)}(K_s) \leq \hat{M}_n^{(1)}(K_s) \leq u_n(x_3) \right), \]

\[ -u_n(y_4) < \hat{m}_n^{(2)}(K_s) \leq \hat{M}_n^{(2)}(K_s) \leq u_n(x_4) \right) \]

and

\[ P(n, \beta_n, x, y) = \mathbb{P} \left( -u_n(y_1) < \hat{m}_n^{(1)}(\beta_n) \leq \hat{M}_n^{(1)}(\beta_n) \leq u_n(x_1) \right), \]

\[ -u_n(y_2) < \hat{m}_n^{(2)}(\beta_n) \leq \hat{M}_n^{(2)}(\beta_n) \leq u_n(x_2) \right), \]

\[ -u_n(y_3) < \hat{m}_n^{(1)} \leq \hat{M}_n^{(1)} \leq u_n(x_3) \right), \]

\[ -u_n(y_4) < \hat{m}_n^{(2)} \leq \hat{M}_n^{(2)} \leq u_n(x_4) \right) . \]

Using similar arguments as in the proof of Lemma 3.3 in [24] for \( n \) large we can choose a positive integer \( \tilde{\nu}_n \) such that \( 0 < \tilde{\nu}_n \leq \nu \) and \( \tilde{\nu}_n = o(n) \), by (29) we have

\[ \left| \mathbb{P} \left( -u_n(y_1) < \hat{m}_n^{(1)}(\varepsilon_n) \leq \hat{M}_n^{(1)}(\varepsilon_n) \leq u_n(x_1) \right) - \mathbb{P} \left( -u_n(y_2) < \hat{m}_n^{(2)}(\varepsilon_n) \leq \hat{M}_n^{(2)}(\varepsilon_n) \leq u_n(x_2) \right) \right| \]

\[ \leq (4l + 2)\tilde{\nu}_n \]

\[ \mathbb{P} \left( S_n, \hat{X}_{n,1}^{(1)} \leq -u_n(y_1) \right) + \mathbb{P} \left( S_n, \hat{X}_{n,1}^{(1)} > u_n(x_1) \right) \]

\[ + \mathbb{P} \left( S_n, \hat{X}_{n,1}^{(2)} \leq -u_n(y_2) \right) + \mathbb{P} \left( S_n, \hat{X}_{n,1}^{(2)} > u_n(x_2) \right) \]

\[ \rightarrow 0, \quad n \rightarrow \infty. \]

Note that

\[ 1 - \frac{\mu}{2^l} \Sigma_1 - \nu \left( 1 - \frac{\mu}{2} \right) \Sigma_2 \]

\[ + \left( \frac{\sum_{j \in K_s} \beta_{n,j}}{\nu} - \frac{\mu}{2^l} \right) \nu (\Sigma_2 - \Sigma_1) \]

\[ \leq P(K_s, \beta_n, x, y) \]

\[ \leq 1 - \frac{\mu}{2^l} \Sigma_1 - \nu \left( 1 - \frac{\mu}{2} \right) \Sigma_2 \]

\[ + \left( \frac{\sum_{j \in K_s} \beta_{n,j}}{\nu} - \frac{\mu}{2^l} \right) \nu (\Sigma_2 - \Sigma_1) + \nu \Sigma_3, \]

where

\[ \Sigma_1 = P_1(n, x_1, x_2) + P_2(n, y_1, y_2) \]

\[ - P_3(n, x_1, y_2) - P_4(n, y_1, x_2), \]

\[ \Sigma_2 = P_1(n, x_3, x_4) + P_2(n, y_3, y_4) \]

\[ - P_3(n, x_3, y_4) - P_4(n, y_3, x_4) \]

with \( P_i(n, z_1, z_2) \)’s defined in the proof of Lemma 4.2 and

\[ \Sigma_3 = \sum_{i,j=1,2}^{n} \sum_{l=2}^{\nu} \mathbb{P} \left( S_{n,1}, \hat{X}_{n,(\alpha+1)}^{(i)}, \hat{X}_{n,(\beta+1)}^{(j)} > u_n(x_i) \right) \]

\[ + \mathbb{P} \left( S_{n,1}, \hat{X}_{n,(\alpha+1)}^{(i)} > u_n(x_i) \right) \]

\[ + \mathbb{P} \left( S_{n,1}, \hat{X}_{n,(\alpha+1)}^{(i)} > u_n(y_i) \right) \]

\[ + \mathbb{P} \left( S_{n,1}, \hat{X}_{n,(\beta+1)}^{(j)} > u_n(y_j) \right) \]

\[ + \mathbb{P} \left( S_{n,1}, \hat{X}_{n,(\beta+1)}^{(j)} > u_n(y_j) \right) . \]

Since \( 0 \leq 1 - \frac{\mu}{2^l} \Sigma_1 - \nu (1 - \frac{\mu}{2}) \Sigma_2 \leq 1 \) applying Lemma 3 in [19] we obtain

\[ \sum_{\mu=0}^{l+1} \sum_{\beta_n \in \{0,1\}^n} \mathbb{E} \left( l \prod_{s=1}^{l} \mathbb{P}(K_s, \beta_n, x, y) \right) \]

\[ \leq \sum_{\mu=0}^{l+1} \sum_{\beta_n \in \{0,1\}^n} \mathbb{E} \left( l \prod_{s=1}^{l} \left( 1 - \frac{\nu}{2} n \Sigma_1 - \left( 1 - \frac{\mu}{2} \right) n \Sigma_2 \right) \right) \]

\[ \leq \sum_{\mu=0}^{l+1} \sum_{\beta_n} \mathbb{E} \left( l \left( \frac{\sum_{j \in K_s} \beta_{n,j}}{\nu} - \frac{\mu}{2^l} \right) \right) \left( n (\Sigma_1 - \Sigma_2) \right) \]

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\[ d(X,Y) = \inf \{ \varepsilon, \mathbb{P}(|X-Y| > \varepsilon) < \varepsilon \}. \]

Furthermore,

\[ 2^{l-1} \sum_{\mu=0, \beta_n \in [0,1]^n} \mathbb{E} \left[ \left| \prod_{s=1}^{l} \left[ 1 - \frac{\mu_n \Sigma_1 - (1 - \eta)n \Sigma_2}{l} \right] \right| (B_{\mu,l,\beta_n}) \right] \]

\[ \leq \frac{n(\Sigma_1 + \Sigma_2)}{2^{l-1}}. \]

By the fact that \( \lim_{n \to \infty} d \left( \frac{\Sigma_1}{\mu_n}, \eta \right) = 0 \) and utilising (29)–(32), by passing to limit for \( n \to \infty \) and then letting \( \nu \to \infty \) we obtain

\[ \left| \frac{1}{l} \left( 1 - \eta \right) \left( D(x_1, x_2) + D(y_1, y_2) \right) \right| \]

\[ \leq \frac{D(x_1, x_2) + D(y_1, y_2)}{2^{l-1}} + \frac{1}{l} \left( e^{-x_1} + e^{-y_1} + e^{-x_2} + e^{-y_2} \right)^2. \]

Next, letting \( l \to \infty \) implies

\[ \lim_{n \to \infty} \sup_{x_1, y_1, x_2, y_2 \in \mathbb{R}^{1,2,3,4}} \left| \mathbb{P}(n, \varepsilon_n, x, y) \right| \]

\[ - \mathbb{E} \left( H^1_\eta(x_1, x_2) H^1_\eta(y_1, y_2) H^{1-\eta}(x_3, x_4) H^{1-\eta}(y_3, y_4) \right) \]

\[ = 0, \]

hence the claim follows. \( \square \)

**Proof of Theorem 3.2** If (11) holds, by (12) for some positive constant \( \mathcal{Q} \) we have

\[ \left\{ \begin{array}{l}
- u_n(y_1) < \hat{m}^{(1)}(\varepsilon_n) \leq M^{(1)}(\varepsilon_n) \leq u_n(x_1), \\
- u_n(y_2) < \hat{m}^{(2)}(\varepsilon_n) \leq M^{(2)}(\varepsilon_n) \leq u_n(x_2), \\
- u_n(y_3) < \hat{m}^{(3)}(\varepsilon_n) \leq M^{(3)}(\varepsilon_n) \leq u_n(x_3), \\
- u_n(y_4) < \hat{m}^{(4)}(\varepsilon_n) \leq M^{(4)}(\varepsilon_n) \leq u_n(x_4), \\
- \mathcal{P} \left( - u_n(y_1) < \hat{m}^{(1)}(\varepsilon_n) \leq M^{(1)}(\varepsilon_n) \leq u_n(x_1), \\
- u_n(y_2) < \hat{m}^{(2)}(\varepsilon_n) \leq M^{(2)}(\varepsilon_n) \leq u_n(x_2), \\
- u_n(y_3) < \hat{m}^{(3)}(\varepsilon_n) \leq M^{(3)}(\varepsilon_n) \leq u_n(x_3), \\
- u_n(y_4) < \hat{m}^{(4)}(\varepsilon_n) \leq M^{(4)}(\varepsilon_n) \leq u_n(x_4) \right). 
\end{array} \right. \]


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