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IMMERSED SURFACES IN CUBED MANIFOLDS*

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Abstract. We investigate the separability of the canonical surface immersed in cubed manifolds of non-positive curvature, partially answering the hyperbolicity question of cubed 3-manifolds. We show that the canonical surface is separable if we assume that the degrees of all edges are even. Further, the general argument also gives the same result for higher-dimensional manifolds admitting a cubing of non-positive curvature. This main result is extended to some other structures, including flying-saucer and polyhedral decompositions as well as surgeries on certain link complements; these constructions provide examples of non-Haken manifolds that are virtually Haken.

1. Introduction. Every closed PL n-manifold M^n admits a decomposition into n-dimensional cubes by further subdivision of each simplex. Endowing each cube with the geometry of a regular Euclidean cube of edge length 3 induces a length-space metric on M^n. Such a metric has non-positive curvature (in the sense of Gromov) if and only if the link of each cell, at distance 1, has no closed geodesic in the induced spherical metric of length less than 2π (this is called the *vertex condition*). Let us denote each n-cube by I^n. Throughout the paper, the term "np cubed manifold" will mean a manifold with a cubing of non-positive curvature.

Each np cubed manifold M contains a canonical immersed submanifold K^{n-1}, possibly disconnected and non-orientable, which is also an np cubed manifold, and which is totally geodesic with respect to these polyhedral metrics. K^{n-1} is obtained as the union of the n (hyper)cubes I^{n-1} within each I^n which are parallel to and equidistant from all pairs of parallel opposite faces. These decompose I^n into 2^n subcubes, which we call "corners". We will refer to K^{n-1} as the *canonical (hyper)surface* of M.

For an np cubed closed 3-manifold M^3, Mosher [Mo] has proved that the fundamental group $\pi_1(M^3)$ satisfies exactly one of the following properties:

1. $\pi_1(M^3)$ is hyperbolic in the sense of Gromov; or
2. $\pi_1(M^3)$ is toroidal, that is, there exists a free abelian subgroup of rank 2.

Cubings of 3-manifolds arise naturally from polyhedral decompositions: a generic polyhedron is an abstract polyhedron whose combinatorial type corresponds to a connected graph on a 2-sphere for which all vertices are trivalent. *Nice* generic polyhedra are those without 2-gons or 3-gons.

Consider a 3-manifold M^3 decomposed as a union of generic polyhedra. Then, the dual polyhedral decomposition is also a decomposition into generic polyhedra. These two dual cell decompositions intersect in M^3, with complementary cells being the canonical cubical decomposition from either generic polyhedral decomposition.

This cubing induces a metric of non-positive curvature if all edges of one polyhedral decomposition have degree at least 4, all faces have degree at least 4, *and* the vertex condition above is satisfied. This requires that both the polyhedral decomposition and its dual to be decompositions of M^3 into nice generic polyhedra. We will assume that the vertex condition is satisfied in the following.

Call a polyhedral decomposition *nice* if all polyhedra and dual polyhedra are nice; then, the class of nice decompositions of 3-manifolds coincides with the class of np

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cubed 3-manifolds in the canonically induced metric since non-positive curvature for a cubed manifold necessitates all edges of degree at least 4, and all faces of a cube have degree (at least) 4. Conversely, a nice decomposition can be canonically decomposed into cubes. Similar descriptions hold for higher-dimensional cubings.

The construction of np cubed closed 3-manifolds admits a generalization to certain 3-manifolds with toral boundary components: identify pairs of faces of a collection of cubes, so as to obtain a 3-complex with all vertices having neighborhoods being either a cone on a sphere or on a torus. Delete all vertices of the latter type, to obtain an open 3-manifold M^3 which is naturally the interior of a compact manifold with toral boundary. We now further require that all edges meeting a toral vertex of the 3-complex have degree exactly 6, and moreover that every cube of the complex has exactly one toral vertex. We now give each 'cube-with-one-vertex-deleted' the geometry of a corner of a regular ideal hyperbolic 3-cube, the 'ideal vertex' being identified with the 'deleted vertex'. Many examples in which the resulting geometry is of strictly negative curvature are given in [ALR], and some of these will be discussed at greater length in Section 3. These examples yield 'almost-cubed' manifolds (by definition) by carrying out Dehn surgery on each cusp.

In this paper we will be primarily concerned with the canonical surface immersed in np cubed 3-manifolds M and the possibility of finding finite-degree covering spaces for M in which the immersion lifts to an embedding. This is motivated by the following conjecture [AR1]:

CONJECTURE 1.1. *Every np cubed atoroidal 3-manifold admits a complete hyperbolic structure.*

One way to attack this problem is to consider lifting the canonical surface, which we already know to be incompressible, as suggested by Aitchison and Rubinstein [AR1] in the following theorem. The proof of this theorem uses the rigidity result of Hass and Scott [HS] (which applies to all np cubed 3-manifolds) and a theorem of Culler and Shalen [CS] as well as Thurston's Uniformization Theorem.

THEOREM 1.2. *Suppose M is an np cubed, atoroidal 3-manifold. If there is an immersed incompressible surface which lifts to an embedded surface in some finite cover $\widehat{M}$ of M, then M is hyperbolic.*

We refer to such an immersed incompressible surface as *separable*, i.e., when there is a finite-degree cover of the manifold in which the surface lifts as an embedding. Such a 3-manifold is, by definition, virtually Haken.

CONJECTURE 1.3. *The canonical surface in an np cubed manifold is separable.*

If this is proved successfully, by Theorem 1.2, we will have proved Conjecture 1.1.

Two related results should be mentioned here. D. Long [Lo] has shown that every immersed closed surface *totally geodesic* in a closed orientable *hyperbolic* 3-manifold is separable. One may try to modify Long's proof for np cubed 3-manifolds since the canonical surface is obviously totally geodesic in the polyhedral structure of the np cubing. However, this appears to be difficult as the universal cover of an np cubed manifold is not geometrically uniform like $\mathbb{H}^3$. Long's proof depends heavily on the hyperbolic group structure, which cannot be used in the np cubed-manifold case.

The other approach is an attempt to show that the fundamental group of every np cubed manifold M is LERF, or subgroup-separable (see [Sc1] for a definition). This would imply that any incompressible (π_1-injective) submanifold S immersed in M lifts to an embedding in some finite cover of M [Sc1]. It is known that the fundamental groups of Seifert fiber spaces are LERF [Sc1], so in particular any immersed surface in a Seifert fiber space is separable. However, this approach turns out to be too optimistic; there are np cubed manifolds M with $\pi_1(M)$ non-LERF, and in fact some np cubed manifolds admit non-separable surfaces [Ma2] (although all such examples known so far are toroidal manifolds).

Clearly, the condition of LERF-ness is too strong for our purposes since all we need is *one* separable, incompressible surface in a given np cubed manifold. A natural starting point is to find a finite-degree cover where the canonical surface, whose existence and incompressibility we already know, lifts to an emdedded surface. Although it seems difficult at the moment to prove the existence of such a cover for every np cubed M, we will show a partial result in Section 2 and some improvements and applications in later sections as well. In fact, the argument we use applies to higher-dimensional np cubed manifolds and their canonical hypersurfaces as well.

2. Even-degree NP cubings. In dimension 2, the separability of the canonical curve in squared surfaces (i.e., 2-dimensional "np cubed" manifolds) is trivial since all surface groups are known to be LERF [Sc1]. But in [Ma1] it was explicitly shown that every component of the canonical curve in a squared surface lifts to an embedding in an appropriate *double* cover of the surface, provided that every vertex is of even degree. This condition on the degrees of codimension-2 simplices allows one to define a holonomy representation on the fundamental group, which can be used to define a regular covering space. This idea can be generalized to any finite dimension as follows to show the main result of this paper.

THEOREM 2.1. *Suppose M is an orientable np cubed manifold of dimension n, and let S be the canonical immersed hypersurface S (of dimension $n-1$). If all the codimension-2 faces of the cubing have even degrees, then there is a finite-degree covering space $\widehat{M}$ of M (of degree at most $n!$) to which S lifts as an embedding.*

Proof. The basic idea, quite intuitive, is to "unwrap" those elements of $\pi_1(M)$ which cause self-intersections (always at an angle of $\frac{\pi}{2}$) while leaving other elements untouched.

Let $\mathcal{O}_n$ be the orientation-preserving symmetry group of I^n, the n-cube. $\mathcal{O}_n$ is a semi-direct product $\mathbb{Z}_2^n \rtimes \mathcal{S}_n$, where $\mathbb{Z}_2^n$ refers to the direct product of n copies of $\mathbb{Z}_2$ and $\mathcal{S}_n$ is the symmetric group of n elements. This gives the short exact sequence

$$1 \to \mathbb{Z}_2^n \to \mathcal{O}_n \to \mathcal{S}_n \to 1.$$

Geometrically, $\mathbb{Z}_2^n$ is generated by n "flips," the rotations of the cube by π along the coordinate axes of $\mathbb{E}^n$. A nice way to see this structure of $\mathcal{O}_n$ is to "colour" the n canonical hypercubes of I^n by n distinct colours; observe that the 2^n elements of $\mathbb{Z}_2^n$ keep the colours invariant, and $\mathcal{O}_n/\mathbb{Z}_2^n$ is simply the permutations of the n colours.

Define a base point $p \in M$ as the center of one of the cubes, say C, and let L be the set of loops at p. Define the holonomy map $h\colon L \to \mathcal{O}_n$ in the usual way as follows. In the universal cover $\widetilde{M}$ of M, a loop $\lambda \in L$ represents some covering translation t, which translates C to some other n-cube tC. Now we can parallel-translate along the geodesic between p and tp to identify the canonical hypercubes I^{n-1} of the lift of S to $\widetilde{M}$ passing through p to those passing through tp. On the other hand, when C is matched to tC by the covering projection, these two sets of $(n-1)$-dimensional hypercubes are matched together. The composition of these two identifications gives an element of the symmetry group $\mathcal{O}_n$, which we define to be $h(\lambda)$.

Now, h is not well-defined on $\pi_1(S,p)$ since two distinct but homotopic loops could get mapped to different elements of $\mathcal{O}_n$; however, we can modify h and define $h_*\colon \pi_1(M,p) \to \mathcal{O}_n/\mathbb{Z}_2^n \cong \mathcal{S}_n$. Note that all we are doing here is to quotient out $\mathcal{O}_n$ by the normal subgroup $\mathbb{Z}_2^n$ generated by the rotations of the cube by the angle π.

To see that h_* is well-defined, we need to check that it does not depend on the choice of loop representing the same element of $\pi_1(M)$.

Now, the homotopy between two paths with the same endpoints in $\widetilde{M}$ can be deformed to be transverse to the skeleta of the cubing, meeting the $(n-1)$-skeleton transversely in arcs, the $(n-2)$-skeleton in points and disjoint from lower-dimensional skeleta. We can imagine moving the paths successively across one point of the $(n-2)$-skeleton at a time, so it suffices to see how the mapping changes in such a step.

Since we are assuming that each $(n-2)$-dimensional face has even degree, changing the path by moving across such a face induces an element of $\mathcal{O}_n$ which lies in the normal subgroup $\mathbb{Z}_2^n$ of the collection of flips. For the hypercube orthogonal to this face is clearly preserved and so are all the ones *parallel* to the face as well. Hence we get an induced, well-defined representation h_* of $\pi_1(M)$ to $\mathcal{S}_n$. Equivalently, one could show that all null-homotopic loops get mapped into $\mathbb{Z}_2^n$ by considering those loops going around a single $(n-2)$-dimensional face of M. In other words, the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{h} & \mathcal{O}_n \\ /\!\sim\big\downarrow & & \big\downarrow/\mathbb{Z}_2^n \\ \pi_1(M) & \xrightarrow{h_*} & \mathcal{S}_n \end{array}$$

The kernel of this representation defines an explicit $n!$-fold cover $\widehat{M}$, called the *coloured cover*, in which precisely the self-intersections of the canonical hypersurface (corresponding to those loops λ with $h_*([\lambda]) \neq 1$) "unwrap," since, in $\widehat{M}$, all the translations induce symmetries of I^n preserving all the canonical hypercubes forming a lift of S. Hence, the immersion of S lifts to an embedding in this covering space as required. □

Remark. If M is not orientable, just take the orientable double cover M' of M first. The local structure does not change, so we can still apply this theorem to M'.

Hence, we can conclude the following partial answer to Conjecture 1.1 by Theorem 1.2 and Theorem 2.1.

COROLLARY 2.2. *If M is an np cubed 3-manifold, where the degree of each edge is even, then M is virtually Haken. In particular, if M is also atoroidal, then M is hyperbolic.*

We can now look at a few examples. The first is a toroidal case, where M is a graph manifold. This is a specific example under the more general construction which appears in [Ma2].

EXAMPLE 2.3. Take the squared genus-2 surface shown in Figure 2.1, remove two disjoint squares, D_1 and D_2, and take the product of this with S^1. Assume that each edge has length 1 and that the length of S^1 is 4. Glue the two boundary tori (each consisting of $4 \times 4 = 16$ squares) together using the homeomorphism defined by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is the 90-degree twist on the tori so the horizontal and the vertical directions are switched. It is easy to verify that after the gluing in the construction, the closed np cubed manifold M has the property that the degree of each edge is even; in fact, the degrees are either 6 or 4. Therefore, by Theorem 2.1, the *canonical* surface is separable. It is interesting to point out that [Ma2] uses this construction to show that this manifold M contains a non-separable surface. Hence, this example indicates that the canonical surface is indeed a special incompressible surface.

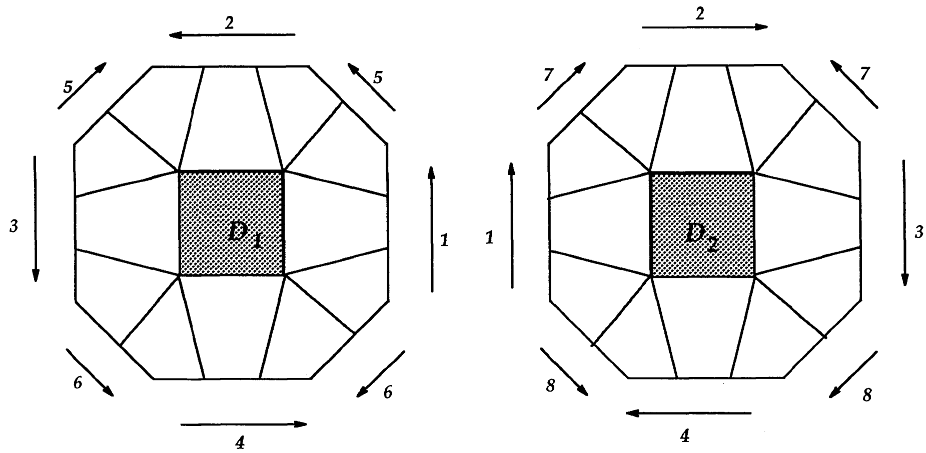

FIG. 2.1. *Squared Genus-2 Surface.*

EXAMPLE 2.4. An example with a hyperbolic ideal np cubing is the 5-fold covering space of the figure-8 knot complement. It consists of 2 cubes, each of which can be partitioned into 5 tetrahedra (see [Sk] and [AMR] for a description; it is based on the triangulation of Thurston [Th]). There is only one component for this canonical surface, consisting of 6 squares (the "Skinner surface" [Sk]), and this is the non-orientable surface of Euler characteristic -2. This np cubed manifold is not a closed one, but every edge is of degree 6, and the same argument applies here; therefore, the Skinner surface is separable. It is worth noting that this surface is totally geodesic in the hyperbolic metric of the figure-8 knot complement. Therefore, the result of Long [Lo] implies its separability. Here, however, the cubing structure has given a completely different explanation of why it is separable.

EXAMPLE 2.5. One may try to use the same argument when some of the edge degrees are odd. One classical example is the Weber-Seifert dodecahedral space X, which is known to be hyperbolic [WS]. Its np cubing structure, consisting of 20 cubes, is described in [AR1], and the canonical surface S_X of X is the orientable surface of genus 4 [Ma1]. One can think of this surface as the union of the 12 pentagons lying just underneath the 12 faces of the dodecahedron. As [WS] points out at the end of their paper, $H_1(X) = \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$, whereas $H_1(S_X)$ is 8 copies of $\mathbb{Z}$. Hence, if the induced map $f\colon H_1(S_X) \to H_1(X)$ is not surjective (say it misses one of the 3 generators), then there is a natural 5-fold cover, corresponding to the missed generator, in which some of the intersections of S_X may be eliminated. However, one can verify that the map f is indeed onto $H_1(X)$ (in fact, it is as surjective as possible, meaning that only 3 of the 8 generators of $H_1(S_X)$ are needed to make $Im(f) = H_1(X)$). Therefore, it appears that some new idea is necessary to handle odd-degree cases.

EXAMPLE 2.6. We now consider higher-dimensional examples. In dimensions 5 or higher there is a great deal known about aspherical manifolds due to the work of Farrell and Jones [FJ]. However, in dimension 4 very few results or examples have been found. Freedman's topological surgery methods [FQ] do not apply, so it is interesting to investigate any similarities between aspherical 4-manifolds and 3-manifolds.

A good class of examples is that of manifolds coming from Coxeter groups where the fundamental domain has no triangles. By this we mean that no 2-dimensional face of a fundamental polyhedron is a triangle. We can take the case where this polyhedron is non-compact of finite volume as well as where it is compact. There are many such examples in dimension 3, due to Andreev's Theorem, giving a complete characterization of such polyhedra in hyperbolic 3-space. (In higher dimensions much work has been done on constructing examples by the Russian school of Vinberg. See the references in [Vi].)

For example, there is an ideal hyperbolic polyhedron in dimension 4 which is the 24-cell, having 24 octahedral faces and dihedral angles $\frac{\pi}{2}$. So any 4-manifold formed by taking a torsion-free subgroup of finite index in the Coxeter group of reflections of this cell will have an np cubing satisfying the conditions of Theorem 2.1. One can then find the canonical complete np cubed hypersurface in such a manifold and there is a finite-sheeted covering (of degree 4!) where this lifts to an embedding. Many such examples have been investigated by McKenzie [Mc]. See also MacLachlan, Waterman, and Wielenberg [MWW] and Ratcliffe [Ra].

3. Examples from balanced links. In this section we consider a class of non-Haken manifolds to which our technique applies: we will identify some specific surgeries on a class of alternating links such that the resulting manifolds are virtually Haken. It has been very difficult so far to identify classes of manifolds known to be non-Haken: known classes include prescribed surgeries on 2-bridge knots (Hatcher and Thurston [HT]) and on once-punctured torus bundles ([CJR], [FH]).

Hass and Menasco [HM] analysed the 3-component link 8^3_4 (see Figure 3.1) in Rolfsen [Ro], showing that essentially all surgeries gave non-Haken 3-manifolds. They were motivated by the implications of the rigidity results of Aitchison, Lumsden and Rubinstein [ALR]: essentially all surgeries on this link yield manifolds which contain an immersed π_1-injective surface satisfying the 4-plane, 1-line condition of Hass and Scott [HS]. Such manifolds are then topologically rigid, and it was important to know that their results are a non-trivial extension of Waldhausen's rigidity results for Haken manifolds. The link 8^3_4 has been described in [AR3]: it contains totally geodesic closed surfaces and is arithmetic.

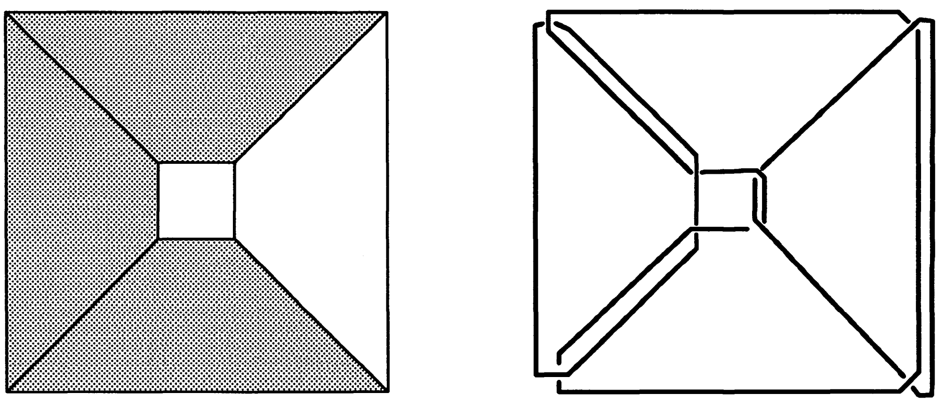

FIG. 3.1. *The link 8^3_4 arising from 2-colouring a cube.*

Suppose then that we denote by M_σ the closed manifold obtained by surgery on 8^3_4, with prescribed surgery given by $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_i \in \mathbb{Q} \cup \{\infty\}$ being the surgery prescription for the ith link component for some numbering of the components. We will show that many such M_σ are virtually Haken.

In fact, we will also produce infinitely many other closed manifolds which are virtually Haken: the question remains as to whether they are non-Haken in the first place. This latter question will not be addressed here.

We recall from [ALR] that a *balanced, alternating link* is a non-splittable, prime alternating link with an alternating projection for which:

- each vertex of the projection has exactly one of its 4 adjacent regions being a 2-gon; and
- all complementary planar regions have degree 4 or more.

If such a link has all planar regions which are *even degree* polygons, then the analysis of [ALR] shows that there is a combinatorial cubing of the 3-complex obtained by coning each cusp of $C(L) = S^3 - L$ to a distinct point, such that every cube has exactly one vertex as a cone point. Thus there is an 'almost' cubed structure of $C(L)$ to which the same analysis as in the preceding section can be applied.

DEFINITION 3.1. A balanced link with all planar regions of even degree will be called a *special balanced link.*

THEOREM 3.2. *The canonical surface of a special balanced link complement lifts to an embedding in a degree-6 covering space.*

Proof. The canonical surface lifts as an embedding in the coloured cover. □

In this cover, all surfaces can be given one of three colours, all colours appearing at each triple point. The same property then applies to the corresponding collection of preimage planes in the universal cover.

PROBLEM 3.3. Which surgeries $\sigma = (\sigma_1, \ldots, \sigma_k)$ on a k-component special balanced link L lift to the coloured cover?

THEOREM 3.4. *For each component L_i of L, essentially one third of surgeries σ_i can be lifted to the corresponding preimage cusps in the coloured cover.*

Proof. Consider a horosphere of the universal cover of the link complement. This is triangulated with the standard equilateral triangulation. The hexagonal structure dual to this is the view of the lifts to planes of the canonical surface. A 3-colouring of the planes, different colours at each triple point, leads to a determined 3-colouring of the hexagons of the horosphere. See Figure 3.2.

The cusp structure of a link component corresponds to a strip of $2n$ triangles, with an obvious choice of $\mathbb{Z} \oplus \mathbb{Z}$ generators: the meridian acts by shifting in the short/diagonal direction, the other generator being a shift along the long length of the strip (Figure 3.2). Three cases naturally occur: $n \equiv 0, 1$ or 2 *mod* 3. In each case, we can compare the permutations of the colourings of hexagons corresponding to the generators: If z is the cycle (rgb) (red - green - blue) for appropriately labelled colours, we obtain as images of the generators the pairs $(z, 1)$, (z, z) and (z, zz), respectively. The kernels can then be read off for the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_3 \subset S_3$. Surgery curves (p, q) with respect to this basis which lift to the cover corresponding to $\ker h_*$ are those in the kernel of this map and are respectively of the form $(3k, q)$, $(3k - q, q)$ or $(3k - 2q, q)$, with the extra condition of relative primality. Here k and q are otherwise arbitrary. □

The number n determining cusp shape is determined by counting crossings involving the component in question: one triangle occurs for each under or over arc at each crossing. For surgery coefficient assignments, it is easy then to check whether the covering of the complement extends to an unbranched or branched covering.

EXAMPLE 3.5. The link 8^3_4 appears in Figure 3.1: the 3 components K_i have cusps with $2n_i$ triangles, where $n_1 = 2,\ n_2 = 2,\ n_3 = 4$.

THEOREM 3.6. *Infinitely many allowed surgeries on the link 8^3_4 yield non-Haken manifolds. Each of these is thus virtually Haken, with a specific 6-fold Haken cover.*

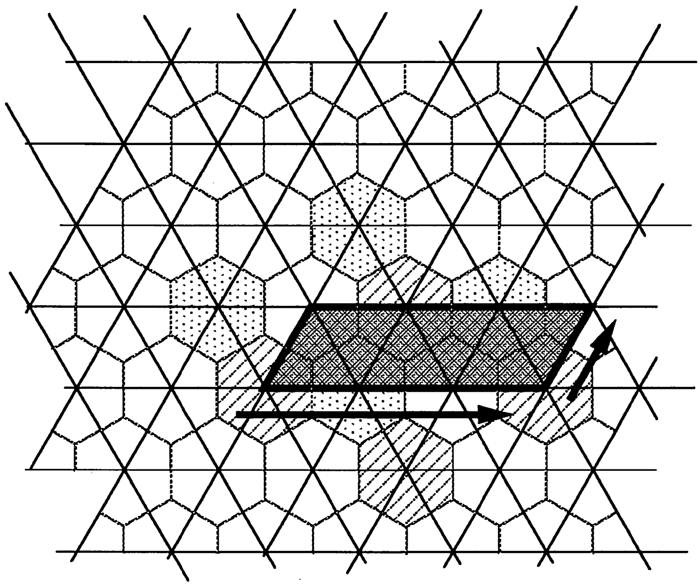

FIG. 3.2. *Coloured horosphere tessellation.*

Proof. We present a necessary modification of the argument of Hass and Menasco, by appealing to a more precise statement of Hatcher's result cited in their argument, to draw a stronger conclusion:

Hatcher [Ha] proved that if M is orientable, compact and irreducible, with ∂M a union of n tori, then the projective classes of curve systems in ∂M corresponding to incompressible, boundary-incompressible surfaces in M, form a dense subset of a finite collection of finite (projective) polyhedra, each projective-linearly embedded in the projective lamination space $\mathcal{PL}(\partial M) \cong S^{2n-1}$, and each of dimension less than n.

Dehn surgery coefficients provide a dense set of points of S^{2n-1}. Hass and Menasco argued that there were infinitely many Dehn surgeries on 8^3_4 corresponding to points not lying on the lower dimensional polyhedra, but which gave 3-manifolds which were topologically rigid by virtue of containing an immersed incompressible surface satisfying the 4-plane, 1-line condition used in the work of Hass and Scott.

The Dehn surgery coefficients corresponding to points on the lower dimensional polyhedra do not define a dense set in S^{2n-1}. Thus to conclude that there are infinitely many non-Haken manifolds which are virtually Haken, it suffices to argue that our allowed Dehn surgeries also form a dense set in $\mathcal{PL}(\partial M)$.

But this follows easily, since the allowed surgeries on each torus component form a $\mathbb{Z} \oplus \mathbb{Z}$ sublattice of all Dehn surgery possibilities on that torus; projectivising, we obtain the desired density in S^{2n-1}. □

Remark. We have not proved that all of these examples have canonical surfaces which are not totally geodesic with respect to the complete hyperbolic metric. Nonetheless, even if they are, we obtain more precise information than that provided by Long's theorem, in that we offer a precise cover to which the surfaces lift as embeddings. Moreover, the surfaces in the complement of the surgery solid tori exist in a hyperbolic manifold with incomplete metric, and we conjecture that generally the surfaces fail to be isotopic to totally geodesic surfaces after the complete metric of the link complement is deformed.

4. Some extensions in dimension 3. In this section we examine two interesting possible extensions of the result that some manifolds with an np cubing have a finite-sheeted covering where the canonical surface lifts to an embedding. In the first, decompositions of a manifold into polyhedra, which can be further subdivided into cubes, are examined. We will show that this more general class is equivalent to even-degree np cubings. In the second result, we will consider flying-saucer manifolds as discussed in [AR1] and show that if all the flying saucers are integral branched covers of the cube and all edges are of even degree, then the canonical surface is again separable. As a corollary, by a result of Paterson [Pa], it follows that the manifold has a geometric decomposition.

4.1. Polyhedral decompositions. Consider a decomposition of M into nice polyhedra. Call this decomposition *very nice* if all the edges and faces have even degree.

COROLLARY 4.1. *If M admits a very nice polyhedral decomposition, then M has a 6-fold Haken cover.*

This follows easily since such a decomposition induces an np cubing with all edges of even degree. We note that such a cubing itself is a very nice decomposition. Hence, the class of manifolds admitting very nice decompositions coincides with the class of manifolds admitting np cubings of even degree.

Remark. For very nice decompositions, the even-ness of edge degrees means that the faces of the decomposition extend antipodally across edges and hence define immersed π_1-injective surfaces away from vertices. These surfaces then have the same combinatorial structure as we expect to find from immersed totally geodesic surfaces in hyperbolic manifolds. Absence of branched points then arises from the combinatorial problem:

Suppose we have a triangulation of the 2-sphere with all vertices of even degree at least 4, and with every embedded 3-cycle in the graph being the boundary of a triangle. Then every *diagonal* geodesic should be embedded, where, by *diagonal geodesic* we mean a geodesic on the graph 1-skeleton that leaves by the antipodal edge at each vertex it enters.

4.2. Flying saucers. Recall that a flying saucer is obtained by dividing a cube out by the 3-fold rotation about a diagonal and then taking the branched covering of this "$\frac{1}{3}$ cube" along the image of the diagonal of order n, for some n at least 3 (see Figure 4.1 for $n = 6$). We say the flying saucer has degree $\frac{n}{3}$ over the cube. Note that the cone angle of a plane perpendicular to the diagonal is $\frac{2n\pi}{3}$ with the induced metric from the cube. If each n is divisible by 3, we say the flying saucers have integral degree. In this case the canonical surface in the cube projects to the $\frac{1}{3}$ cube and then lifts to the flying saucer as a new canonical immersed surface. Note that this surface will not be self transverse for $n = 3k$ for $k > 1$. It is not difficult to check that the canonical surface satisfies the 4-plane property but not necessarily the 1-line property. In particular as the 1-line property is required for an immersed incompressible surface to have a totally geodesic realisation in a hyperbolic 3-manifold, most such canonical surfaces cannot be of this type. Hence the separability result of Long [Lo] does not apply.

THEOREM 4.2. *Suppose that a compact 3-manifold has a flying saucer decomposition with all flying saucers having integral degrees and all edges having even degrees. Then there is a regular covering of degree 6 for which the canonical surface lifts to an*

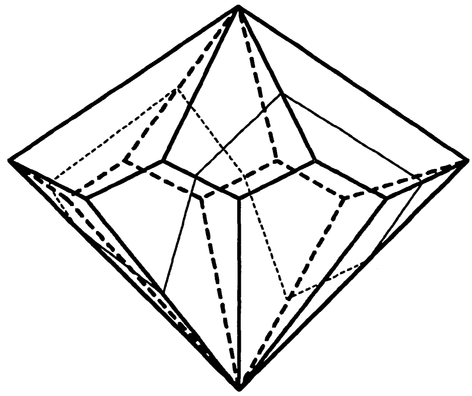

FIG. 4.1. *Flying saucer (of degree 2) and a canonical surface in it.*

embedding.

Proof. The argument is very similar to the np cubing case. We just need to note that the polyhedral metric coming from the flying saucers has cone angle $2k\pi$ at the diagonal so can still be viewed as piecewise Euclidean. Note that we can still 3-colour the canonical surfaces in this case (one of the three is shown in Figure 4.1). The holonomy along paths can be described as before. Observe that the composition of holonomies around a loop linking an edge still lies in the Klein 4-subgroup $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of the octahedral group $\mathcal{O}_3$; consequently, there is an $\mathcal{S}_3$ coloured cover where the canonical surface lifts to an embedding. □

EXAMPLE 4.3. An interesting example of a flying saucer manifold is of a hyperbolic surface bundle over a circle. However, here the canonical surface is either a collection of 3 embedded surface families or two surface families, one embedded and the other immersed. Start with a squared surface, e.g. of genus 2 and take the product with S^1 to give a cubing in an obvious way. Next we choose a sequence of diagonals in these cubes to fit together to form a collection of geodesic loops with the property that any essential vertical torus meets some of the loops. Now any branched covering where all these loops have branching degree at least 2 gives a hyperbolic bundle, as the manifold has a natural foliation by horizontal surfaces but the monodromy must be pseudo-Anosov as there are no essential tori. Now the canonical surface in this flying saucer decomposition splits into 3 surfaces. The reason is that in the original product of a surface and a circle, there are two or three surface families. One is the horizontal surface itself and the other is the product of the circle and the canonical curve in the squaring of the surface. If these loops form two embedded families, we get three embedded surface families which lift under the branched covering. If the curves have self intersections, then we get a single immersed family of surfaces. It is easy to choose the cubing to satisfy the conditions of the theorem in this case. So we have a quasi-Fuchsian "vertical" surface which is separable in a finite-sheeted cover. Note that here the canonical surface has a geometrically finite and a geometrically infinite part simultaneously.

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