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006

CONFORMAL MODULES*

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Abstract. In this paper we study a class of modules over infinite-dimensional Lie (super)algebras, which we call conformal modules. In particular we classify and construct explicitly all irreducible finite conformal modules over the Virasoro and the N=1 Neveu-Schwarz algebras, over the (super)current algebras and their semidirect sums.

0. Introduction. Conformal module is a basic tool for the construction of free field realizations of infinite-dimensional Lie (super)algebras in conformal field theory. This is one of the reasons to classify and construct such modules. In the present paper we solve this problem under the irreducibility assumption for the Virasoro and the Neveu-Schwarz algebras, for the (super)current algebras and their semidirect sums. Since complete reducibility does not hold for conformal modules, one has to discuss the extension problem. This problem is solved in [1].

The basic idea of our approach is to use three (more or less) equivalent languages. The first is the language of local formal distributions, the second is the language of modules over conformal algebras, and the third is the language of conformal modules over the extended annihilation subalgebras. The problem is solved using the third language by means of the crucial Lemma 3.1. Note that conformal modules over Lie algebras of Cartan type were studied in [7], where, in particular, a proof of Corollary 3.3 is contained.

This paper is organized as follows. In Section 1 the concepts of a Lie (super)algebra of formal distributions and of a conformal (super)algebra are recalled. They are more or less equivalent notions. In Section 2 we introduce conformal modules over a Lie (super)algebra of formal distributions and clarify their connections with modules over the corresponding conformal (super)algebra. We then show that modules over a conformal (super)algebra are in 1-1 correspondence with modules over the extended annihilation subalgebra of the associated Lie (super)algebra of formal distributions. At the end of Section 2 examples of conformal modules over the Virasoro, the (super)current and Neveu-Schwarz algebras and their various semidirect sums are constructed. In Section 3 we first prove the key lemma (Lemma 3.1) and with its help classify all irreducible finite conformal modules over the (extended) annihilation subalgebra of the above-mentioned Lie (super)algebras. The main result, stated in Theorem 3.2, which describes all finite irreducible modules over the conformal (super)algebras in question (hence all irreducible finite conformal modules over the corresponding Lie (super)algebras), is then immediate.

Roughly speaking, the main claim of the present paper is that all non-trivial modules over current, Virasoro, N=1 superconformal algebras and their semidirect sums are non-degenerate. For N>1 superconformal algebras interesting degeneracies occur in some non-trivial conformal modules. These effects are studied in [6].

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1. Preliminaries on local formal distributions and conformal superalgebras. A *formal distribution* (usually called a field by physicists) with coefficients in a complex vector space U is a generating series of the form:

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)} \in U$ and z is an indeterminate.

Two formal distributions $a(z)$ and $b(z)$ with coefficients in a Lie superalgebra $\mathfrak{g}$ are called (mutually) *local* if for some $N \in \mathbb{Z}_+$ one has:

$$(z-w)^N [a(z), b(w)] = 0. \tag{1.1}$$

Introducing the *formal delta function*

$$\delta(z-w) = z^{-1} \sum_{n\in\mathbb{Z}} (\frac{z}{w})^n,$$

we may write a condition equivalent to (1.1):

$$[a(z), b(w)] = \sum_{j=0}^{N} (a_{(j)}b)(w) \partial_w^{(j)} \delta(z-w), \tag{1.2}$$

(here $\partial_w^{(j)}$ stands for $\frac{1}{j!}\frac{\partial^j}{\partial w^j}$) for some formal distributions $(a_{(j)}b)(w)$ ([4], Theorem 2.3), which are uniquely determined by the formula

$$(a_{(j)}b)(w) = \mathrm{Res}_z (z-w)^j [a(z), b(w)]. \tag{1.3}$$

Formula (1.3) defines a $\mathbb{C}$-bilinear product $a_{(j)}b$ for each $j \in \mathbb{Z}_+$ on the space of all formal distributions with coefficients in $\mathfrak{g}$.

Note also that the space (over $\mathbb{C}$) of all formal distributions with coefficients in $\mathfrak{g}$ is a (left) module over $\mathbb{C}[\partial_z]$, where the action of ∂_z is defined in the obvious way, so that $\partial_z a(z) = \sum_n (\partial a)_{(n)} z^{-n-1}$, where $(\partial a)_{(n)} = -n a_{(n-1)}$.

The Lie superalgebra $\mathfrak{g}$ is called a *Lie superalgebra of formal distributions* if there exists a family $\mathfrak{F}$ of pairwise local formal distributions whose coefficients span $\mathfrak{g}$. In such a case we say that the family $\mathfrak{F}$ *spans* $\mathfrak{g}$. We will write $(\mathfrak{g}, \mathfrak{F})$ to emphasize the dependence on $\mathfrak{F}$.

The simplest example of a Lie superalgebra of formal distributions is the *current superalgebra* $\tilde{\mathfrak{g}}$ associated to a finite-dimensional Lie superalgebra $\mathfrak{g}$:

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

It is spanned by the following family of pairwise local formal distributions ($a \in \mathfrak{g}$):

$$a(z) = \sum_{n\in\mathbb{Z}} (a \otimes t^n) z^{-n-1}.$$

Indeed, it is immediate to check that

$$[a(z), b(w)] = [a, b](w)\delta(z-w).$$

The simplest example beyond current algebras is the (centerless) *Virasoro algebra*, the Lie algebra $\mathfrak{V}$ with basis L_n ($n \in \mathbb{Z}$) and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}.$$

It is spanned by the local formal distribution $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, since one has:

$$[L(z), L(w)] = \partial_w L(w)\delta(z-w) + 2L(w)\partial_w \delta(z-w). \tag{1.4}$$

The next important example is the semidirect sum $\mathfrak{V} \ltimes \tilde{\mathfrak{g}}$ with the usual commutation relations between $\mathfrak{V}$ and $\tilde{\mathfrak{g}}$:

$$[L_m, a\otimes t^n] = -na\otimes t^{m+n},$$

which is equivalent to

$$[L(z), a(w)] = \partial_w a(w)\delta(z-w) + a(w)\partial_w\delta(z-w).$$

Given a Lie superalgebra of formal distributions $(\mathfrak{g}, \mathfrak{F})$, we may always include $\mathfrak{F}$ in the minimal family $\mathfrak{F}^c$ of pairwise local distributions which is closed under ∂ and all products (1.3) ([4], Section 2.7). Then $\mathfrak{F}^c$ is a *conformal superalgebra* with respect to the products (1.3). Its definition is given below [4]:

A *conformal superalgebra* is a left ($\mathbb{Z}_2$-graded) $\mathbb{C}[\partial]$-module R with a $\mathbb{C}$-bilinear product $a_{(n)}b$ for each $n \in \mathbb{Z}_+$ such that the following axioms hold ($a, b, c \in R; m, n \in \mathbb{Z}_+$ and $\partial^{(j)} = \frac{1}{j!}\partial^j$):

(C0) $a_{(n)}b = 0$, for $n >> 0$,
(C1) $(\partial a)_{(n)}b = -na_{(n-1)}b$,
(C2) $a_{(n)}b = (-1)^{\mathrm{deg}a\mathrm{deg}b}\sum_{j=0}^{\infty}(-1)^{j+n+1}\partial^{(j)}(b_{(n+j)}a)$,
(C3) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{\infty}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + (-1)^{\mathrm{deg}a\mathrm{deg}b}b_{(n)}(a_{(m)}c)$.

Of course, conformal algebra coincides with its even part, i.e. $\mathrm{deg}a = 0$ for all $a \in R$ in this case. Note also the following consequence of (C1) and (C2):

(C2') $a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b$,

hence ∂ is a derivation of all products (1.3).

Conversely, assuming for simplicity (cf. Lemma 2.2b) that $R = \oplus_{i\in I}\mathbb{C}[\partial]a^i$ is a free (as a $\mathbb{C}[\partial]$-module) conformal superalgebra, we may associate to R a Lie superalgebra of formal distributions $(\mathfrak{g}(R), \mathfrak{F})$ with basis $a^i_{(m)}$ ($i \in I$, $m \in \mathbb{Z}$) and $\mathfrak{F} = \{a^i(z) = \sum_n a^i_{(n)}z^{-n-1}\}_{i\in I}$ with bracket (cf. (1.2)):

$$[a^i(z), a^j(w)] = \sum_{k\in\mathbb{Z}_+}(a^i_{(k)}a^j)(w)\partial_w^{(k)}\delta(z-w), \tag{1.5}$$

so that $\mathfrak{F}^c = R$.

Formula (1.5) is equivalent to the following commutation relations ($m, n \in \mathbb{Z}$; $i, j \in I$):

$$[a^i_{(m)}, a^j_{(n)}] = \sum_{k\in\mathbb{Z}_+}\binom{m}{k}(a^i_{(k)}a^j)_{(m+n-k)}. \tag{1.6}$$

It follows that the $\mathbb{C}$-span of all $a^i_{(n)}$ with $n \in \mathbb{Z}_+$ is a subalgebra of the Lie superalgebra $\mathfrak{g}(R)$. We denote this subalgebra by $\mathfrak{g}(R)_+$ and call it the *annihilation subalgebra.*

For example $\mathfrak{V}_+ = \sum_{j\geq -1} \mathbb{C}L_j$ and $\tilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[t]$.

The simplest examples of a conformal superalgebra is the *current conformal superalgebra* associated to a finite-dimensional Lie superalgebra $\mathfrak{g}$:

$$R(\tilde{\mathfrak{g}}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathfrak{g},$$

with the products defined by:

$$a_{(0)}b = [a,b], \quad a_{(j)}b = 0, \quad \text{for } j > 0,\ a, b \in \mathfrak{g},$$

and the *Virasoro conformal algebra* $R(\mathfrak{V}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} L$ with products (cf. (1.4)):

$$L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(j)}L = 0, \quad \text{for } j > 1.$$

Their semidirect sum is $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$ with additional non-zero products $L_{(0)}a = \partial a$ and $L_{(1)}a = a$, for $a \in \mathfrak{g}$. These examples are the conformal superalgebras associated to the Lie superalgebras of formal distributions described above.

The simplest superextension of the Virasoro algebra is the well-known (centerless) Neveu-Schwarz algebra $\mathfrak{N}$ which, apart from even basis elements L_n, has odd basis elements G_r, $r \in \frac{1}{2} + \mathbb{Z}$, with commutation relations:

$$[G_r, L_n] = (r - \frac{n}{2})G_{r+n}, \quad [G_r, G_s] = 2L_{r+s}.$$

The corresponding annihilation subalgebra in this case is

$$\mathfrak{N}_+ = \sum_{n\geq -1} \mathbb{C}L_n + \sum_{r \geq -\frac{1}{2}} \mathbb{C}G_r.$$

The conformal superalgebra, associated to $\mathfrak{N}$, is $R(\mathfrak{N}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} L + \mathbb{C}[\partial] \otimes_{\mathbb{C}} G$ with additional non-zero products:

$$L_{(0)}G = \partial G, \quad G_{(0)}L = \frac{1}{2}\partial G, \quad L_{(1)}G = G_{(1)}L = \frac{3}{2}G, \quad G_{(0)}G = 2L.$$

Other examples treated in this paper are *supercurrent algebras*

$$\tilde{\mathfrak{g}}_{\text{super}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}, \theta],$$

where θ is an odd indeterminate. The generating family $\mathfrak{F}$ of pairwise local formal distributions consists of currents $a(z)$ $(a \in \mathfrak{g})$ introduced above and supercurrents

$$a^{\theta}(z) = \sum_{n\in\mathbb{Z}} (a \otimes t^n\theta) z^{-n-1}.$$

Of course its associated conformal superalgebra is $R(\tilde{\mathfrak{g}}_{\text{super}}) = \mathbb{C}[\partial] \otimes (\mathfrak{g} \oplus \mathfrak{g}^{\theta})$, where $\mathfrak{g}^{\theta}$ is an identical copy of $\mathfrak{g}$, but with reversed parity. $R(\tilde{\mathfrak{g}}_{\text{super}})$ extends $R(\tilde{\mathfrak{g}})$ by the additional non-trivial product

$$a_{(0)}b^{\theta} = [a,b]^{\theta}, \quad a, b \in \mathfrak{g}.$$

The final example treated in this paper is the semidirect sum $\mathfrak{N} \ltimes \tilde{\mathfrak{g}}_{\text{super}}$, which is defined by letting $L_n = -t^n(t\frac{\partial}{\partial t} + \frac{n+1}{2}\theta\frac{\partial}{\partial\theta})$ and $G_r = -t^{r+\frac{1}{2}}(\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta})$ for $n \in \mathbb{Z}$ and $r \in \frac{1}{2}+\mathbb{Z}$. Its corresponding conformal superalgebra $R(\mathfrak{N} \ltimes \tilde{\mathfrak{g}}_{\text{super}}) = R(\mathfrak{N}) \ltimes R(\tilde{\mathfrak{g}}_{\text{super}})$ has the following additional non-trivial products:

$$L_{(0)}a^\theta = \partial a^\theta, \quad L_{(1)}a^\theta = \frac{1}{2}a^\theta, \quad G_{(0)}a^\theta = a, \quad G_{(0)}a = \partial a^\theta, \quad G_{(1)}a = a^\theta.$$

2. Preliminaries on conformal modules. Let $(\mathfrak{g}, \mathfrak{F})$ be a Lie superalgebra of formal distributions, and let V be a $\mathfrak{g}$-module. We say that a formal distribution $a(z) \in \mathfrak{F}$ and a formal distribution $v(z) = \sum_{n\in\mathbb{Z}} v_{(n)}z^{-n-1}$ with coefficients in V are *local* if

$$(z-w)^N a(z)v(w) = 0, \quad \text{for some } N \in \mathbb{Z}_+. \tag{2.1}$$

It follows from [4] Section 2.3 that (2.1) is equivalent to

$$a(z)v(w) = \sum_{j=0}^{N-1} (a_{(j)}v)(w)\partial_w^{(j)}\delta(z-w), \tag{2.2}$$

for some formal distributions $(a_{(j)}v)(w)$ with coefficients in V, which are uniquely determined by the formula

$$(a_{(j)}v)(w) = \text{Res}_z(z-w)^j a(z)v(w).$$

Formula (2.2) is obviously equivalent to

$$a_{(m)}v_{(n)} = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a_{(j)}v)_{(m+n-j)}. \tag{2.3}$$

EXAMPLE 2.1. Consider the following representation of the (centerless) Virasoro algebra in the vector space V with basis $v_{(n)}$, $n \in \mathbb{Z}$, over $\mathbb{C}$

$$L_m v_{(n)} = ((\Delta - 1)(m+1) - n)v_{(m+n)} + \alpha v_{(m+n+1)},$$

where $\Delta, \alpha \in \mathbb{C}$. In terms of formal distributions $L(z)$ and $v(z)$ this can be written as follows:

$$L(z)v(w) = (\partial + \alpha)v(w)\delta(z-w) + \Delta v(w)\partial_w\delta(z-w). \tag{2.4}$$

Hence $L(z)$ and $v(z)$ are local.

Suppose that V is spanned over $\mathbb{C}$ by the coefficients of a family E of formal distributions such that all $a(z) \in \mathfrak{F}$ are local with respect to all $v(z) \in E$. Then we call (V, E) a *conformal module over* $(\mathfrak{g}, \mathfrak{F})$.

The following is a representation-theoretic analogue (and a generalization) of Dong's lemma (see [4], Section 3.2).

LEMMA 2.1 [5]. *Let V be a module over a Lie superalgebra $\mathfrak{g}$, let $a(z)$ and $b(z)$ (respectively $v(z)$) be formal distributions with coefficients in $\mathfrak{g}$ (respectively in V).*

Suppose that all pairs (a,b), (a,v) and (b,v) are local. Then the pairs $(a_{(j)}b,v)$ and $(a,b_{(j)}v)$ are local for all $j\in\mathbb{Z}_+$.

This lemma shows that the family E of a conformal module (V,E) over $(\mathfrak{g},\mathfrak{F})$ can always be included in a larger family E^c which is still local with respect to $\mathfrak{F}$, hence to $\mathfrak{F}^c$, and such that $\partial E^c\subset E^c$ and $a_{(j)}E^c\subset E^c$ for all $a\in\mathfrak{F}$ and $j\in\mathbb{Z}_+$.

It is straightforward to check the following properties for $a,b\in\mathfrak{F}$ and $v\in E^c$:

$$[a_{(m)},b_{(n)}]v=\sum_{j=0}^{m}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}v, \tag{2.5}$$

$$(\partial a)_{(n)}v=[\partial,a_{(n)}]v=-na_{(n-1)}v. \tag{2.6}$$

(Here $[\cdot,\cdot]$ is the bracket of operators on E^c.) It follows from (2.5) (by induction on m) and (2.6) that $a_{(j)}E^c\subset E^c$ for all $a\in\mathfrak{F}^c$ and $j\in\mathbb{Z}_+$.

Thus any conformal module (V,E) over a Lie superalgebra of formal distributions $(\mathfrak{g},\mathfrak{F})$ gives rise to a module $M=E^c$ over the conformal superalgebra $R=\mathfrak{F}^c$, defined as follows. It is a (left) $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module equipped with a family of $\mathbb{C}$-linear maps $a\to a^M_{(n)}$ of R to $\mathrm{End}_{\mathbb{C}}M$, for each $n\in\mathbb{Z}_+$, such that the following properties hold (cf. (2.5) and (2.6)) for $a,b\in R$ and $m,n\in\mathbb{Z}_+$:

(M0) $a^M_{(n)}v=0$, for $v\in M$ and $n>>0$,

(M1) $[a^M_{(m)},b^M_{(n)}]=\sum_{j=0}^{m}\binom{m}{j}(a_{(j)}b)^M_{(m+n-j)}$,

(M2) $(\partial a)^M_{(n)}=[\partial,a^M_{(n)}]=-na^M_{(n-1)}$.

Conversely, suppose that a conformal superalgebra $R=\oplus_{i\in I}\mathbb{C}[\partial]a^i$ is a free $\mathbb{C}[\partial]$-module and consider the associated Lie superalgebra of formal distributions $(\mathfrak{g}(R),\mathfrak{F})$ (see Section 1). Let M be a module over the conformal superalgebra R and suppose (cf. Lemma 2.2b) that M is a free $\mathbb{C}[\partial]$-module with $\mathbb{C}[\partial]$-basis $\{v^\alpha\}_{\alpha\in J}$. This gives rise to a conformal module $V(M)$ over $\mathfrak{g}(R)$ with basis $v^\alpha_{(n)}$, where $\alpha\in J$ and $n\in\mathbb{Z}$, defined by (cf. (2.2)):

$$a^i(z)v^\alpha(w)=\sum_{j\in\mathbb{Z}_+}(a^i_{(j)}v^\alpha)(w)\partial_w^{(j)}\delta(z-w). \tag{2.7}$$

A conformal module (V,E) (respectively module M) over a Lie superalgebra of formal distributions $(\mathfrak{g},\mathfrak{F})$ (respectively over a conformal superalgebra R) is called *finite*, if E^c (respectively M) is a finitely generated $\mathbb{C}[\partial]$-module.

A conformal module (V,E) over $(\mathfrak{g},\mathfrak{F})$ is called *irreducible* if there is no non-trivial invariant subspace which contains all $v_{(n)}$, $n\in\mathbb{Z}$, for some non-zero $v\in E^c$. Clearly a conformal module is irreducible if and only if the associated module E^c over the conformal superalgebra $\mathfrak{F}^c$ is irreducible (in the obvious sense).

The above discussions, combined with the following lemma, reduce the classification of finite conformal modules over a Lie superalgebra of formal distributions $(\mathfrak{g},\mathfrak{F})$ to the classification of finite modules over the corresponding conformal superalgebra.

LEMMA 2.2 [5]. *(a) Let M be a module over a conformal superalgebra R and let $v\in M$ be such that $\partial v=\lambda v$ for some $\lambda\in\mathbb{C}$. Then v is an invariant, i.e. $R_{(m)}v=0$ for all $m\in\mathbb{Z}_+$. (b) Suppose that M is a finite module over a conformal superalgebra and suppose that M has no non-zero invariants. Then M is a free $\mathbb{C}[\partial]$-module.*

REMARK 2.1. Given a module M over a conformal superalgebra R, we may change its structure as a $\mathbb{C}[\partial]$-module by replacing ∂ by $\partial + A$, where A is an endomorphism over $\mathbb{C}$ of M which commutes with all $a^M_{(n)}$ (this will not affect axiom (M2)).

Note that the maps $a \to a_{(n)}$ of R to $\mathrm{End}_{\mathbb{C}} R$ defines an R-module, called the *adjoint module.*

Example 2.1 gives a 2-parameter family of (irreducible) conformal modules over the Virasoro algebra. Note also that the well-known family of graded Virasoro modules given by $L_m v_{(n)} = ((\Delta - 1)(m+1) - n + \alpha) v_{(m+n)}$ is conformal, but is finite if and only if $\alpha = 0$.

The following simple observation, which follows from definitions, is fundamental for representation theory of conformal superalgebras.

PROPOSITION 2.1. *Consider the Lie superalgebra of formal distributions* $(\mathfrak{g}(R), \mathfrak{F})$ *defined by (1.5) and let* $\mathfrak{g}(R)_+$ *be the annihilation subalgebra of* $\mathfrak{g}(R)$. *Denote by* $\mathfrak{g}(R)^+$ *the semidirect product of the* 1*-dimensional Lie subalgebra* $\mathbb{C}\partial$ *and the ideal* $\mathfrak{g}(R)_+$ *with the action of* ∂ *on* $\mathfrak{g}(R)_+$ *given by* $\partial(a^i_{(n)}) = -n a^i_{(n-1)}$. *Then a module* M *over the conformal superalgebra* R *is precisely a* $\mathfrak{g}(R)^+$*-module* M *(over* $\mathbb{C}$*) such that*

$$a^i_{(n)} v = 0, \quad \text{for each } v \in M \text{ and } n >> 0. \tag{2.8}$$

COROLLARY 2.1. *Let* $R = \oplus_{i \in I} \mathbb{C}[\partial] a^i$ *be a conformal superalgebra and* $M = \oplus_{j \in J} \mathbb{C}[\partial] v^j$ *be a free* $\mathbb{C}[\partial]$*-module. Then, given* $a^i_{(n)} v^j \in M$ *for all* $i \in I$, $j \in J$, $n \in \mathbb{Z}_+$, *which is* 0 *for* $n >> 0$, *we may extend uniquely the action of* $a^i_{(n)}$ *to all of* R *on* M *using (M2). Suppose that (M1) holds for all* $a = a^i_{(m)}$, $b = a^j_{(n)}$. *Then* M *is an* R*-module.*

Using Proposition 2.1 and Corollary 2.1, one can construct large families of finite modules over conformal superalgebras, and hence corresponding conformal modules over Lie superalgebras of formal distributions.

In conclusion we will list more examples of modules over conformal superalgebras. In Section 3 the irreducible ones listed below will turn out to exhaust the list of all irreducible finite modules over these conformal superalgebras.

EXAMPLE 2.2. Let $\mathfrak{V}_0 = \sum_{j \geq 0} \mathbb{C} L_j$ and consider a representation π of the Lie algebra $\mathfrak{V}_0$ in a finite-dimensional (over $\mathbb{C}$) vector space U. Let A be an endomorphism of U commuting with all $\pi(L_j)$ ($j \in \mathbb{Z}_+$). Then $\mathbb{C}[\partial] \otimes U$ is a finite module over the conformal algebra $R(\mathfrak{V})$ defined by the following formulas ($u \in U$):

$$L_{(0)} u = (\partial + A) u, \quad L_{(j)} u = \pi(L_{j-1}) u, \text{ for } j \geq 1.$$

For example we can take $\pi(L_0) = B$, where B is an endomorphism of U commuting with A. Then

$$L_{(0)} u = \partial u + A u, \quad L_{(1)} u = B u, \quad L_{(j)} u = 0, \ j \geq 1,$$

defines a finite module over the conformal algebra $R(\mathfrak{V})$, which we will denote by $M_{\mathfrak{V}}(A, B)$.

Translating back to the language of Lie algebras of formal distributions, Example 2.2 gives the following family of finite conformal modules over the Virasoro algebra in the space $U \otimes \mathbb{C}[t, t^{-1}]$ (we let, as usual, $u_{(n)} = u \otimes t^n$):

$$L_m u_{(n)} = (Au)_{(m+n+1)} - (m+n+1) u_{(m+n)} + \sum_{j=0}^{\infty} \binom{m+1}{j+1} (\pi(L_j) u)_{(m+n-j)}.$$

The above special case defined by a pair (A, B) of commuting operators in U is given by:

$$L_m u_{(n)} = (Au)_{(m+n+1)} + ((m+1)Bu - (m+n+1)u)_{(m+n)}.$$

We use the notation $M^c_{\mathfrak{V}}(A, B)$ for this $\mathfrak{V}$-module. The superscript c is to emphasize that it is a conformal module. We will use this convention freely throughout the text. Note that the module $M^c_{\mathfrak{V}}(A, B)$ is irreducible if and only if the $\mathfrak{V}_0$-module is irreducible and non-trivial, i.e. if and only if $\dim U = 1$ and $B = \Delta \neq 0$. We denote these $\mathfrak{V}$-modules by $M^c_{\mathfrak{V}}(\alpha, \Delta)$, $\alpha, \Delta \in \mathbb{C}$ and $\Delta \neq 0$. They are precisely those given by Example 2.1.

REMARK 2.2. (cf. Remark 2.1 and Example 2.2.) Suppose that the annihilation subalgebra $\mathfrak{g}_+$ of the Lie algebra $\mathfrak{g}^+ = \mathbb{C}[\partial] \ltimes \mathfrak{g}_+$, contains an element L_{-1} such that $\mathrm{ad}L_{-1} = \partial$ on $\mathfrak{g}_+$. Then $\mathfrak{g}^+$ is a direct sum of ideals $\mathbb{C}(\partial - L_{-1})$ and $\mathfrak{g}_+$. Hence in this case the study of modules over the corresponding conformal superalgebras (and thus of conformal modules) reduces to the study of modules over $\mathfrak{g}_+$ satisfying (2.8). This remark applies to all cases except for the current and the supercurrent algebras.

EXAMPLE 2.3. Consider the current Lie superalgebra $\tilde{\mathfrak{g}}$ and the associated conformal superalgebra $R(\tilde{\mathfrak{g}})$. Let π be a representation of $\tilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$ in a finite-dimensional vector space U, such that $(\mathfrak{g} \otimes t^n)U = 0$ for $n >> 0$. This defines on the space $U \otimes \mathbb{C}[t, t^{-1}]$ the structure of a conformal module over $\tilde{\mathfrak{g}}$ by the formula:

$$(a \otimes t^m)u_{(n)} = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (\pi(a \otimes t^j)u)_{(m+n-j)}, \quad a \in \mathfrak{g}, u \in U.$$

A special case of this construction is to take a finite-dimensional representation π of the Lie superalgebra $\mathfrak{g}$ in a finite-dimensional vector space U and extend it to $\tilde{\mathfrak{g}}_+$ by letting $\mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially. Then we have $(a \otimes t^m)u_{(n)} = (\pi(a)u)_{(m+n)}$. This $\tilde{\mathfrak{g}}$-module is denoted by $M^c_{\tilde{\mathfrak{g}}}(\pi)$. Translating back to the language of modules over the conformal algebra $R(\tilde{\mathfrak{g}})$ we obtain the free $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ with

$$a_{(0)}u = \pi(a)u, \quad a_{(n)}u = 0 \quad \text{for } n > 0, \text{ where } a \in \mathfrak{g}, u \in U.$$

We will denote this $R(\tilde{\mathfrak{g}})$-module by $M_{\tilde{\mathfrak{g}}}(\pi)$. It is irreducible if and only if π is irreducible. In this case we will denote the module by $M_{\tilde{\mathfrak{g}}}(\Lambda)$, where Λ is the highest weight of U.

EXAMPLE 2.4. Consider the N=1 Neveu-Schwarz algebra $\mathfrak{N}$ with associated conformal superalgebra $R(\mathfrak{N})$. Let $\mathfrak{N}_+$ denote the corresponding annihilation subalgebra. Consider a finite-dimensional representation (π, U) of $\mathfrak{N}_0$, the subalgebra of $\mathfrak{N}_+$ spanned by elements of non-negative modes. Let U^θ denote an identical copy of U with reversed parity. Then the following gives a structure of a module over $R(\mathfrak{N})$ on the free $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial] \otimes (U \oplus U^\theta)$:

$$L_{(0)}u = (\partial + A)u, \quad L_{(j)}u = \pi(L_{j-1})u, \quad L_{(0)}u^\theta = (\partial + A)u^\theta,$$

$$L_{(1)}u^\theta = (\pi(L_0)u)^\theta + \frac{1}{2}u^\theta, \quad L_{(j+1)}u^\theta = (\pi(L_j)u)^\theta + \frac{j+1}{2}\pi(G_{j-\frac{1}{2}})u, \quad G_{(0)}u = u^\theta,$$

$$G_{(j)}u = \pi(G_{j-\frac{1}{2}})u, \quad G_{(0)}u^\theta = (\partial u + Au), \quad G_{(j)}u^\theta = (\pi(G_{j-\frac{1}{2}})u)^\theta + 2\pi(L_{j-1})u,$$

where $j \geq 1$, $u \in U$, and A is an operator acting on U, commuting with all $\pi(\mathfrak{N}_0)$. In particular let $U = \mathbb{C}u$ be the one-dimensional $\mathfrak{N}_0$-module with action $L_0 u = \Delta u$, with

$0 \neq \Delta \in \mathbb{C}$, all other generators acting trivially. Then for $A = \alpha \in \mathbb{C}$ arbitrary, we obtain an irreducible module over $R(\mathfrak{N})$ denoted by $M_{\mathfrak{N}}(\alpha, \Delta)$. Hence as in the case of the Virasoro algebra we get a 2-parameter family of finite irreducible conformal modules denoted by $M^c_{\mathfrak{N}}(\alpha, \Delta)$. Explicitly $M^c_{\mathfrak{N}}(\alpha, \Delta) = (\mathbb{C}u + \mathbb{C}u^\theta) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$, with actions given by ($u_{(n)} = u \otimes t^n$, $u^\theta_{(n)} = u^\theta \otimes t^n$, $r \in \frac{1}{2} + \mathbb{Z}$ and $m, n \in \mathbb{Z}$):

$$\begin{aligned}
L_m u_{(n)} &= ((m+1)(\Delta - 1) - n) u_{(n+m)} + \alpha u_{(n+m+1)}, \\
L_m u^\theta_{(n)} &= ((m+1)(\Delta - \frac{1}{2}) - n) u^\theta_{(n+m)} + \alpha u^\theta_{(n+m+1)}, \\
G_r u_{(n)} &= u^\theta_{(r+n+\frac{1}{2})}, \\
G_r u^\theta_{(n)} &= ((r + \frac{1}{2})(2\Delta - 1) - n) u_{(r+n-\frac{1}{2})} + \alpha u_{(r+n+\frac{1}{2})}.
\end{aligned}$$

EXAMPLE 2.5. Consider the supercurrent algebra $\tilde{\mathfrak{g}}_{\text{super}}$. Let $R(\tilde{\mathfrak{g}}_{\text{super}})$, $\tilde{\mathfrak{g}}_{\text{super}+}$ be as usual. Let (π, U) be a finite-dimensional representation of $\tilde{\mathfrak{g}}_{\text{super}+}$. We obtain a module over $R(\tilde{\mathfrak{g}}_{\text{super}})$ as in the case of current algebra by setting for $n \geq 0$:

$$a_{(n)} u = \pi(a \otimes t^n) u, \quad a^\theta_{(n)} u = \pi(a \otimes t^n \theta) u, \quad a \in \mathfrak{g},\ u \in U.$$

Denote these modules by $M_{\tilde{\mathfrak{g}}_{\text{super}}}(\pi)$. In the special case when U is a finite-dimensional irreducible representation of $\mathfrak{g}$ of highest weight $\Lambda \neq 0$, extended to $\tilde{\mathfrak{g}}_{\text{super}+}$ trivially, the associated module over $R(\tilde{\mathfrak{g}}_{\text{super}})$ is irreducible and finite. We denote this module by $M_{\tilde{\mathfrak{g}}_{\text{super}}}(\Lambda)$.

The following theorem is immediate from the discussion in Section 2 and Theorem 3.2 proved in Section 3. Similar results hold for all other cases considered in this paper.

THEOREM 2.1. *Every non-trivial irreducible finite conformal module over the Virasoro algebra (respectively over the current superalgebra $\tilde{\mathfrak{g}}$) is isomorphic to $M^c_{\mathfrak{V}}(\alpha, \Delta)$ (respectively to $M^c_{\tilde{\mathfrak{g}}}(\pi)$ or its quotient), where Δ is a non-zero complex number (respectively π is a non-trivial finite-dimensional representation of $\mathfrak{g}$).*

3. The key lemma and classification of finite irreducible conformal modules. Let $(\mathfrak{g}, \mathfrak{F})$ be a Lie superalgebra of formal distributions. For each $N \in \mathbb{Z}_+$ let

$$\mathfrak{g}_N = \sum_{a \in \mathfrak{F}, n \geq N} \mathbb{C} a_{(n)}.$$

Suppose that $(\mathfrak{g}, \mathfrak{F})$ is *regular* [4], i.e. there exists a derivation ∂ of $\mathfrak{g}$ such that $\partial(a_{(n)}) = -n a_{(n-1)}$ for all $a \in \mathfrak{F}$ and $n \in \mathbb{Z}$. Obviously, $(\mathfrak{g}(R), \mathfrak{F})$, where R is a conformal superalgebra, is regular, hence all examples considered above are regular. Then $\mathfrak{g}_+ = \mathfrak{g}_0$ is the annihilation subalgebra, which is ∂-invariant and, due to Proposition 2.1, we have to study representations of the Lie superalgebra $\mathfrak{g}^+ = \mathbb{C}\partial \ltimes \mathfrak{g}_+$, called the *extended annihilation subalgebra.* This leads us to consider the following more abstract situation.

Let $\mathfrak{L}$ be a Lie superalgebra over $\mathbb{C}$ with a distinguished element ∂ and a descending sequence of subspaces $\mathfrak{L} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_n \supset \cdots$, such that $[\partial, \mathfrak{L}_k] = \mathfrak{L}_{k-1}$, for all $k > 0$. Let V be an $\mathfrak{L}$-module such that given any $v \in V$ there exists a non-negative integer N (depending on v) such that $\mathfrak{L}_N v = 0$. We will call such $\mathfrak{L}$-modules *conformal $\mathfrak{L}$-modules.* A conformal $\mathfrak{L}$-module is called *finite* if it is finitely generated as a $\mathbb{C}[\partial]$-module. Our goal is to describe irreducible finite conformal $\mathfrak{L}$-modules.

Let V be a conformal $\mathfrak{L}$-module. Let $V_n = \{v \in V | \mathfrak{L}_n v = 0\}$, and let N be the minimal non-negative integer such that $V_N \neq 0$ (it exists by definition). Let $U = V_N$. Now we can state the key lemma.

LEMMA 3.1. *Suppose that* $N \geq 1$. *Then* $\mathbb{C}[\partial]U = \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ *and therefore* $\mathbb{C}[\partial]U \cap U = U$. *In particular* U *is a finite-dimensional vector space if* V *is finite.*

Proof. Let L_a and R_a denote the left and right multiplication by the element a, respectively. Using $\mathrm{R}_a = \mathrm{L}_a - \mathrm{ad}a$ and the binomial formula, we get the following well-known formula in any associative algebra A,

$$ga^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j}(-\mathrm{ad}a)^j g, \quad a, g \in A.$$

Let $\{v_i\}$, $i \in I$, be a $\mathbb{C}$-linearly independent set of vectors in U generating the $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]U$. Suppose that $\sum_{i=1}^n p_i(\partial)v_i = 0$, where $p_i(\partial) \in \mathbb{C}[\partial]$, and not all $p_i(\partial) = 0$. Let m be the maximal degree of the $p_i(\partial)$'s. We write $p_i(\partial) = \sum_{j=0}^m c_{ij}\partial^j$, $c_{ij} \in \mathbb{C}$, so that we have $c_{im} \neq 0$, for some i. We have, since $N \geq 1$:

$$\begin{aligned}\mathfrak{L}_{N+m-1}\partial^k &= \sum_{j=0}^{k} \binom{k}{j} \partial^{k-j}(\mathrm{ad}\partial)^j(\mathfrak{L}_{N+m-1}) \\ &= \sum_{j=0}^{k} \binom{k}{j} \partial^{k-j}\mathfrak{L}_{N+m-1-j}.\end{aligned}$$

We have therefore

$$0 = \mathfrak{L}_{N+m-1}v = \sum_{i=1}^{n} c_{im}\mathfrak{L}_{N-1}v_i = \mathfrak{L}_{N-1}(\sum_{i=1}^{n} c_{im}v_i).$$

Since $\sum_{i=1}^n c_{im}v_i \neq 0$, this contradicts the minimality of N. Hence all the $p_i(\partial) = 0$, proving the lemma. $\square$

THEOREM 3.1. *Assume that* $[\mathfrak{L}_0, \mathfrak{L}_i] \subset \mathfrak{L}_i$ *for all* $i \geq 0$ *and that* $\mathfrak{L} = \mathbb{C}\partial + \mathbb{C}[\partial, \partial] + \mathfrak{L}_0$. *Let* V *be a non-trivial irreducible conformal* $\mathfrak{L}$*-module. Then either* $V \cong \mathrm{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$, *where* U *is an irreducible* $\mathfrak{L}_0$*-module (and* $\dim U < \infty$ *if* V *is finite), or else, provided that* $\mathfrak{L}_0$ *is an ideal of* $\mathfrak{L}$, V *can be a (non-trivial) finite-dimensional irreducible* $\mathfrak{L}/\mathfrak{L}_0$*-module.*

Proof. We continue to use the notation above. Clearly U is $\mathfrak{L}_0$-invariant. First assume that $N \geq 1$. By Lemma 3.1 we see that V contains an $\mathfrak{L}$-submodule isomorphic to $\mathrm{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$. Hence $V \cong \mathrm{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}} U$. Clearly U must be irreducible over $\mathfrak{L}_0$ in order for V to be irreducible. Conversely the same argument as in the proof of Lemma 3.1 shows that by applying $\mathfrak{L}_k$, for a suitable k, to a non-zero element of the form $\sum_{i=1}^n q_i(\partial)v_i$, where $q_i(\partial) \in \mathbb{C}[\partial]$ and $v_i \in U$, one obtains a non-zero element in U. This implies that such induced modules are irreducible.

Now suppose that $N = 0$. Let u be a non-zero vector in U. Then $U(\mathfrak{L})u = \mathbb{C}[\partial]U(\mathfrak{L}_0)u = \mathbb{C}[\partial]u$, and hence $V = \mathbb{C}[\partial]u$ by irreducibility of V. We consider two cases:

CASE 1. Suppose that $\mathfrak{L}_0$ is an ideal of $\mathfrak{L}$. Then $\mathfrak{L}_0$ acts trivially on V. Thus V is an irreducible $\mathbb{C}[\partial]$-module. It follows that if ∂ is even, then V is 1-dimensional, and if ∂ is odd with $[\partial, \partial] \neq 0$, V is either the trivial or a 2-dimensional $\mathfrak{L}/\mathfrak{L}_0$-module with $[\partial, \partial]$ acting as a non-zero scalar.

CASE 2. Suppose that $\mathfrak{L}_0$ is not an ideal of $\mathfrak{L}$. Then there exist $l_0, l'_0 \in \mathfrak{L}_0$ such that $[l_0, \partial] = \partial + l'_0$. An easy induction argument shows that, $l_0\partial^i u = i\partial^i u + f_i(\partial)u$ for $i \geq 1$, where $f_i(\partial)$ is a polynomial in ∂ of degree at most $i-1$ with zero constant term. Now if $v = \sum_{i=0}^n a_i\partial^i u$ is a non-zero vector in $\mathbb{C}[\partial] \otimes \mathbb{C}u$ with $a_0 \neq 0$, then $v - (\frac{1}{n})l_0 v$ is a non-zero vector of degree at most $n-1$. Thus proceeding this way we see that u is contained in the module generated by v. Therefore $\partial\mathbb{C}[\partial] \otimes \mathbb{C}u$ is the unique maximal submodule of $\mathbb{C}[\partial] \otimes \mathbb{C}u$ and hence V is the trivial module. □

We will now apply the theorem above to classify irreducible conformal modules over the Virasoro algebra, the current algebra and their semidirect product. In addition similar results can be obtained for the corresponding $N=1$ extended superalgebras by slightly modifying the arguments. In order to do so, the following lemma is useful.

LEMMA 3.2 [6]. *Let $\mathfrak{g}$ be a Lie superalgebra that is a semidirect product of a subalgebra $\mathfrak{a}$ and an ideal $\mathfrak{n}$. Let $\mathfrak{a}_0$ be an even subalgebra of $\mathfrak{a}$ such that $\mathfrak{n}$ is a completely reducible* ad$\mathfrak{a}_0$*-module with no trivial summand. Then $\mathfrak{n}$ annihilates a non-zero vector in any finite dimensional $\mathfrak{g}$-module V. In particular $\mathfrak{n}$ acts trivially in any irreducible finite-dimensional $\mathfrak{g}$-module.*

Proof. First, note that if $[a, b] = b$, then an eigenspace V_λ of b is a-invariant, hence $[a, b]_{|V_\lambda} = 0$, hence $\lambda = 0$, and therefore b is nilpotent on V.

Taking the image of $\mathfrak{g}$ in $\mathrm{End}_\mathbb{C}(V)$, we may assume that $\dim_\mathbb{C}\mathfrak{g} < \infty$, hence $\dim_\mathbb{C}\mathfrak{a}_0 < \infty$. By assumption we have $\mathfrak{n} = \oplus_\lambda \mathfrak{n}_\lambda$, where $\mathfrak{n}_\lambda$ is a non-trivial ad$\mathfrak{a}_0$-module with a non-zero highest weight λ. Hence there exists $a \in \mathfrak{a}_0$ and $b \in \mathfrak{n}_\lambda$ such that $[a, b] = b$, therefore b is nilpotent on V. Moreover all elements from the orbit $\mathrm{Ad}A_0 \cdot b$, where A_0 is the connected Lie group with Lie algebra $\mathfrak{a}_0$, are nilpotent on V. Since $\mathfrak{n}_\lambda$ is A_0-irreducible, this orbit spans $\mathfrak{n}_\lambda$, hence $\mathfrak{n}_\lambda$ is spanned by elements that are nilpotent on V. Thus $\mathfrak{n}$ is spanned by elements that are nilpotent on V, therefore from the version of Engel's theorem from [3] we deduce that there exists a $v \in V$, $v \neq 0$, annihilated by $\mathfrak{n}$. □

We are now in a position to classify finite conformal modules over the Virasoro, Neveu-Schwarz, current and the supercurrent algebras and their semidirect sums. Due to Section 2 we only need to classify finite modules over the corresponding (extended) annihilation subalgebras. For each of these Lie superalgebras of formal distributions, the corresponding annihilation subalgebras are of course the corresponding subalgebras defined on the line, instead of the circle. So we will use terminology like current algebras on the line etc. to denote the corresponding annihilation subalgebras.

The following corollaries follow immediately from Theorem 3.1 and Lemma 3.2.

COROLLARY 3.1. *Let $\mathfrak{g}$ be a direct sum of finite-dimensional simple Lie superalgebras. Let $\mathfrak{L} = \mathbb{C}\frac{d}{dt} \ltimes \mathfrak{g}[t]$, where $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is the current algebra on the line. Then every non-trivial irreducible finite conformal $\mathfrak{L}$-module is of the form $\mathrm{Ind}_{\mathfrak{g}[t]}^{\mathfrak{L}}U$, where U is a finite-dimensional non-trivial irreducible $\mathfrak{g}$-module or else it is the trivial $\mathfrak{g}[t]$-module on which $\frac{d}{dt}$ acts as a non-zero scalar.*

COROLLARY 3.2. *Let $\mathfrak{g}$ be a direct sum of finite-dimensional simple Lie superalgebras. Let $\mathfrak{L} = \mathbb{C}\frac{d}{dt} \ltimes \mathfrak{g}[t, \theta]$, where $\mathfrak{g}[t, \theta] = \mathfrak{g} \otimes \mathbb{C}[t, \theta]$. Then every non-trivial irreducible finite conformal $\mathfrak{L}$-module is of the form $\mathrm{Ind}_{\mathfrak{g}[t,\theta]}^{\mathfrak{L}}U$, where U is a finite dimensional non-trivial irreducible $\mathfrak{g}$-module or else it is the trivial $\mathfrak{g}[t, \theta]$-module on which $\frac{d}{dt}$ acts as a non-zero scalar.*

COROLLARY 3.3. *Let $\mathfrak{V}_+ = \sum_{k\geq -1} \mathbb{C}t^{k+1}\frac{d}{dt}$ be the Virasoro algebra on the line.*

Let $L_i = -t^{i+1}\frac{d}{dt}$ and let $\mathfrak{V}_0 = \sum_{k\geq 0}\mathbb{C}L_k$. Then any non-trivial irreducible finite conformal $\mathfrak{V}_+$-module is of the form $\mathrm{Ind}_{\mathfrak{V}_0}^{\mathfrak{V}_+}\mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is a non-trivial one-dimensional irreducible representation of $\mathfrak{V}_0$, on which L_0 acts as $\lambda \in \mathbb{C}^*$ and L_k act as 0 for all $k > 0$.

COROLLARY 3.4. *Let $\mathfrak{L} = \mathfrak{V}_+ \ltimes \mathfrak{g}[t]$ such that $[L_k, a\otimes t^n] = -na\otimes t^{n+k}$, $a \in \mathfrak{g}$. Let $\mathfrak{L}_0 = \mathfrak{V}_0 \ltimes \mathfrak{g}[t]$. Then every non-trivial irreducible finite conformal $\mathfrak{L}$-module is of the form $\mathrm{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}}U$, where U is a non-trivial irreducible $(\mathfrak{g}\oplus\mathbb{C}L_0)$-module with $\mathfrak{g}[t]t$ and L_k, $k > 0$, acting trivially.*

REMARK 3.1. Translating the modules over the annihilation subalgebra $\mathfrak{V}_+ \ltimes \mathfrak{g}[t]$ of Corollary 3.4 into the language of modules over conformal algebras we obtain a 3-parameter family of non-trivial modules over $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. We will denote these modules by $M_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\Lambda,\alpha,\Delta)$, where Λ is a dominant integral weight. Clearly when $\Lambda \neq 0$, $M_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\Lambda,\alpha,\Delta)$ is irreducible. When $\Lambda = 0$, $M_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(0,\alpha,\Delta)$, for $\Delta \neq 0$, is irreducible. They exhaust all irreducible finite modules over $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. The corresponding conformal modules over $\mathfrak{V}\ltimes\tilde{\mathfrak{g}}$ are irreducible and exhaust all non-trivial finite irreducible conformal modules.

For the Neveu-Schwarz algebra on the line we have the following description:

COROLLARY 3.5. *Let $\mathfrak{N}_+ = \sum_{n\geq -1}\mathbb{C}L_n + \sum_{r\geq -\frac{1}{2}}\mathbb{C}G_r$ be the Neveu-Schwarz algebra on the line. Let $\mathfrak{N}_0 = \sum_{n\geq 0}\mathbb{C}L_n + \sum_{r>0}\mathbb{C}G_r$. Then every non-trivial irreducible finite conformal $\mathfrak{N}_+$-module is of the form $\mathrm{Ind}_{\mathfrak{N}_0}^{\mathfrak{N}_+}\mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is a one dimensional irreducible representation of $\mathfrak{N}_0$, on which L_0 acts as the scalar $\lambda \in \mathbb{C}^*$ and L_k and G_r act trivially for $r, k > 0$.*

Proof. We define a filtration on the Lie superalgebra $\mathfrak{N}_+$ as follows: $\mathfrak{N}_i$ is the subalgebra spanned by $\{L_{\frac{i}{2}}, G_{\frac{i+1}{2}}, L_{\frac{i+2}{2}}, G_{\frac{i+3}{2}}, \cdots\}$, if i is even. If i is odd, then $\mathfrak{N}_i$ is spanned by the linearly independent vectors $\{G_{\frac{i}{2}}, L_{\frac{i+1}{2}}, G_{\frac{i+2}{2}}, L_{\frac{i+3}{2}}, \cdots\}$. We set $\partial = G_{-\frac{1}{2}}$ so that $[\partial,\partial] = 2L_{-1}$. We then have $\mathfrak{N}_+ = \mathbb{C}\partial + \mathbb{C}[\partial,\partial] + \mathfrak{N}_0$. Hence by Theorem 3.1 every non-trivial irreducible conformal module over $\mathfrak{N}_+$ is of the form $\mathrm{Ind}_{\mathfrak{N}_0}^{\mathfrak{N}_+}U$, where U is an irreducible $\mathfrak{N}_0$-module. Now we use Lemma 3.2 and the result follows. □

Using similar filtrations as in Corollary 3.5 one proves Corollaries 3.6 and 3.7 below. Although never used, Corollary 3.6 is stated for the sake of completeness.

COROLLARY 3.6. *Let $\mathfrak{g}$ be a direct sum of finite-dimensional simple Lie superalgebras. Let $\mathfrak{L} = (\mathbb{C}\frac{\partial}{\partial t} + \mathbb{C}(\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta})) \ltimes \mathfrak{g}[t,\theta]$. Then every irreducible non-trivial finite conformal $\mathfrak{L}$-module is either of the form $\mathrm{Ind}_{\mathfrak{g}[t,\theta]}^{\mathfrak{L}}U$, where U is a finite-dimensional irreducible representation of $\mathfrak{g}$ or it is an irreducible two dimensional representation of the subalgebra $\mathbb{C}\frac{\partial}{\partial t} + \mathbb{C}(\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta})$, on which $\frac{\partial}{\partial t}$ acts as a non-zero scalar and $\mathfrak{g}[t,\theta]$ acts trivially.*

COROLLARY 3.7. *Let $\mathfrak{L} = \mathfrak{N}_+ \ltimes \mathfrak{g}[t,\theta]$. Set $\mathfrak{L}_0 = \mathfrak{N}_0 \ltimes \mathfrak{g}[t,\theta]$. Then every non-trivial irreducible finite conformal $\mathfrak{L}$-module is of the form $\mathrm{Ind}_{\mathfrak{L}_0}^{\mathfrak{L}}U$, where U is a finite-dimensional non-trivial $\mathfrak{g}\oplus\mathbb{C}L_0$-module, on which L_k, G_r for all $k, r > 0$ and $\mathfrak{g}[t]t + \mathfrak{g}[t]\theta$ act trivially.*

REMARK 3.2. Corollary 3.7 gives a 3-parameter family of irreducible finite non-trivial modules over the conformal superalgebra $R(\mathfrak{N}) \ltimes R(\tilde{\mathfrak{g}}_{\mathrm{super}})$. We denote these modules by $M_{\mathfrak{N}\ltimes\tilde{\mathfrak{g}}_{\mathrm{super}}}(\Lambda,\alpha,\Delta)$, where Λ is a dominant integral weight of $\mathfrak{g}$, $\alpha, \Delta \in \mathbb{C}$. The conditions for irreducibility of these modules and their classification are as in Remark 3.1. The same holds for the corresponding conformal modules over $\mathfrak{N}\ltimes\tilde{\mathfrak{g}}_{\mathrm{super}}$.

Translating the above corollaries into the language of modules over conformal superalgebras (using Proposition 2.1 and Remark 2.2) we obtain the following

THEOREM 3.2. *Let $R(\mathfrak{V})$, $R(\tilde{\mathfrak{g}})$, $R(\mathfrak{N})$ and $R(\tilde{\mathfrak{g}}_{\text{super}})$ stand for the Virasoro, the current, the Neveu-Schwarz and the supercurrent conformal (super)algebras, respectively. Let $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$ and $R(\mathfrak{N}) \ltimes R(\tilde{\mathfrak{g}}_{\text{super}})$ denote their respective semidirect sums. Then the following is a complete list of their finite non-1-dimensional (over $\mathbb{C}$) irreducible modules:*

1. *$M_{\mathfrak{V}}(\alpha, \Delta)$ and $M_{\mathfrak{N}}(\alpha, \Delta)$, where $\alpha, \Delta \in \mathbb{C}$ with $\Delta \neq 0$.*
2. *$M_{\tilde{\mathfrak{g}}}(\Lambda)$ and $M_{\tilde{\mathfrak{g}}_{\text{super}}}(\Lambda)$, where Λ is a non-zero dominant integral weight.*
3. *$M_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\Lambda, \alpha, \Delta)$ and $M_{\mathfrak{N}\ltimes\tilde{\mathfrak{g}}_{\text{super}}}(\Lambda, \alpha, \Delta)$, where Λ is a non-zero dominant integral weight and $\alpha, \Delta \in \mathbb{C}$ or else if $\Lambda = 0$, then $\Delta \neq 0$.*

REMARK 3.3. Theorem 3.2 tells us that the family $M^c_{\mathfrak{N}}(\alpha, \Delta)$ (see Example 2.4 for explicit formulas) exhausts all finite irreducible non-trivial conformal modules over the Neveu-Schwarz algebra. Similarly Theorem 3.2 also classifies all finite irreducible non-trivial conformal modules of the semidirect sums.

It was shown in [2] that every semisimple finite conformal algebra is a direct sum of conformal algebras of the form $R(\mathfrak{V})$, $R(\tilde{\mathfrak{g}})$ and $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. The results of this section give a description of all finite irreducible modules over all finite semisimple conformal algebras. Namely we have the following

PROPOSITION 3.1. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ be a finite semisimple conformal algebra. Suppose that R_i is either $R(\mathfrak{V})$ or $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$ for some i. Let V be a finite irreducible module of the conformal algebra R whose restriction to R_i is non-trivial. Then the restriction of V to all R_j is trivial for $i \neq j$.*

Proof. Since R_i is either the conformal algebra $R(\mathfrak{V})$ or $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$, there exists $L^i_{(0)} \in R_i$ such that $[\partial - L^i_{(0)}, R_i] = 0$. Choose any $k \neq i$ and consider irreducible modules over the conformal algebra $R_i \oplus R_k$. By Proposition 2.1 we are to consider modules over $\mathbb{C}\partial \ltimes ((R_i)_+ \oplus (R_k)_+) \cong (R_i)_+ \oplus (\mathbb{C}(\partial - L^i_{(0)}) \ltimes (R_k)_+)$. Now a non-trivial irreducible $(R_i)_+$-module is free over $\mathbb{C}[L^i_{(0)}]$ and also a non-trivial irreducible $\mathbb{C}(\partial - L^i_{(0)}) \ltimes (R_k)_+$-module is free over $\mathbb{C}[\partial - L^i_{(0)}]$ by the above discussions. Thus their tensor product is free over $\mathbb{C}[L^i_{(0)}] \otimes \mathbb{C}[\partial - L^i_{(0)}]$, and hence cannot be finite over $\mathbb{C}[\partial]$. □

Proposition 3.1 and Theorem 3.2 imply immediately

THEOREM 3.3. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ be a finite semisimple conformal algebra, where each R_i is either simple or of the form $R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. Then R has a faithful finite module if and only if either all R_i are current conformal algebras or $n = 1$.*

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