THE REGULARITY OF $L^p$-SCALING FUNCTIONS

KA-SING LAU† AND MANG-FAI MA‡

Abstract. The existence of $L^p$-scaling function and the $L^p$-Lipschitz exponent have been studied in [Ji] and [LW] and a criterion is given in terms of a series of product of matrices. In this paper we make some further study of the criterion. In particular we show that for $p$ an even integer or for all $p \geq 1$ in some special cases, the criterion can be simplified to a computationally efficient form.

1. Introduction. The solution $f$ of a 2-scale dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n), \quad x \in \mathbb{R}$$

is called a scaling function. This class of functions has been studied in detail in recent literature in connection with wavelet theory [D] and constructive approximations [DGL]. The question of existence of continuous, $L^1$ and $L^2$ solutions was treated in Daubechies [D], Daubechies and Lagarias [DL1], Collela and Heil [CH], Eirola [E], Heil [H], and Micchelli and Prautzsch [MP]. The regularity of such solutions was studied, in addition to the above papers, in Cohen and Daubechies [CD], Daubechies and Lagarias [DL2,3], Herve [He], Lau, Ma and Wang [LWM] and Villemos [V1,2]. Also the existence of $L^p$-solutions has been characterized by Lau and Wang in [LW] and Jia [Ji].

In this paper, we will adopt the previous notations as in [CH], [DL1] and [LW]. Let $T_0 = [c_{2i-j-1}]_{1 \leq i, j \leq N}$ and $T_1 = [c_{2i-j}]_{1 \leq i, j \leq N}$ be the associated matrices of the coefficients $\{c_n\}$, i.e.,

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_4 & c_3 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N-1} \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ c_5 & c_4 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_N \end{pmatrix}.$$ 

It is known that if $\sum_{n=0}^{N} c_n = 2$, then $2$ is always an eigenvalue of $(T_0 + T_1)$. Furthermore, if $\sum c_{2n} = \sum c_{2n+1} = 1$, then $1$ is an eigenvalue of both $T_0$ and $T_1$. Let $v$ be a 2-eigenvector of $(T_0 + T_1)$ (which means a right eigenvector associated with the eigenvalue $2$) and let

$$\tilde{v} := (T_0 - I)v = (I - T_1)v.$$

In [LW] the following theorem was proved:

Theorem A. Suppose $1 \leq p < \infty$ and $\sum_{n=0}^{N} c_n = 2$. Then equation (1.1) has a nonzero compactly supported $L^p$-solution (notation: $L^p$-solution) if and only if there exists a 2-eigenvector $v$ of $(T_0 + T_1)$ satisfying

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{|J|=n} \|T_J \tilde{v}\|^p = 0.$$ 

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* Received March 20, 1997; accepted for publication June 25, 1997.
† Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (kslau@math.cuhk.edu.hk).
‡ Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260.
In [Ji], Jai studied the same existence question by means of a 'hat' function and obtained a similar criterion independently. Furthermore he showed that the existence of an $L^p$-solution implies that $\sum c_{2n} = \sum c_{2n+1} = 1$.

In this paper we will consider the regularity of the solution. We use the $L^p$-Lipschitz exponent to describe the regularity. It is defined by

$$\text{Lip}_p(f) = \liminf_{h \to 0^+} \frac{\ln \|\Delta_h f\|_p}{\ln h},$$

where $\Delta_h f(x) = f(x+h) - f(x)$. It is well known that for $1 \leq p < \infty$, if

$$\limsup_{h \to 0^+} \frac{1}{h} \|\Delta_h f\|_p < \infty$$

(which implies $\text{Lip}_p(f) = 1$), then $f'$ exists a.e. and is in $L^p$ and $f$ is the indefinite integral of $f'$. Recall that the $q$-Sobolev exponent of a function $f$ is defined as

$$\sup\{\alpha : \int (1 + |\xi|^\alpha) |\hat{f}(\xi)|^q d\xi < \infty\}.$$ 

For $p = q = 2$, the 2-Sobolev exponent equals to the $L^2$-Lipschitz exponent, and they are different when $p, q \neq 2$. In general the $L^p$-Lipschitz exponent describes the regularity of $f$ more accurately than the Sobolev exponent. The $q$-Sobolev exponent has been studied in [He]. The $L^p$-Lipschitz exponent (in a slightly different terminology) has been used to investigate the multifractal structure of scaling functions in [DL3] and [J1,2].

Let $H(\tilde{v})$ be the complex subspace spanned by $\{T_J \tilde{v} : J \text{ is a multi-index}\}$. (We use complex scalar because it will be more convenient to deal with the complex eigenvalues and eigenvectors.)

**Theorem B.** Suppose that $\sum c_{2n} = \sum c_{2n+1} = 1$ and either (i) $1$ is a simple eigenvalue of $T_0$ and $T_1$ or (ii) $H(\tilde{v}) = \{u \in \mathbb{C}^N : \sum_{i=1}^N u_i = 0\}$. Then for $f$ an $L^p$-solution of (1.1), $1 \leq p < \infty$,

$$\text{Lip}_p(f) = \liminf_{n \to \infty} \frac{\ln(2^{-n} \sum_{|J|=n} \|T_J \tilde{v}\|^p)}{p \ln(2^{-n})}.$$ (1.2)

We remark that Jia [Ji, Theorem 6.2] proved that the above formula by replacing $\|T_J \tilde{v}\|$ with $\|T_J \tilde{v}/H\|$ where $H$ denoted the hyperplane in (ii). Our special preference on $\|T_J \tilde{v}\|$ is that it allows us to calculate the sum in many cases (see Section 4 and 5). Even though there are some overlaps with Jia's result, we like to give a full proof of Theorem B because of completeness and the consistence of the development in the in the sequel.

To reduce the formula in Theorem B, we only consider the 4-coefficient dilation equation for simplicity. We show that if in addition $c_0 + c_3 = 1$, then

$$\text{Lip}_p(f) = \frac{\ln((|c_0|^p + |1-c_0|^p)/2)}{-p \ln 2},$$

(Proposition 4.3) and if $c_0 + c_3 = 1/2$, then

$$\text{Lip}_p(f) = \min\{1, \frac{\ln((|c_0|^p + |1/2-c_0|^p)/2)}{-p \ln 2}\}.$$ (Proposition 4.5). Note that the second case contains Daubechies scaling function $D_4$. The formula was actually obtained in [DL3] using a different method and assuming...
further \(1/2 < c_0 < 3/4\). For the general case we show that if \(p\) is an even integer, then

\[
\sum_{|J|=n} ||T_J v||^p = a W_p^n b
\]

for an auxiliary \((p + 1) \times (p + 1)\) matrix \(W_p\) depends only on the coefficients of the dilation equation and for some vectors \(a\) and \(b\) (Proposition 5.1). In particular for \(p = 2\), the matrix \(W_2\) is equivalent to the transition matrix used in [CD], [LW] and [V] for the existence of \(L^2\)-scaling function. By using (1.3) it is easy to show that the necessary and sufficient condition in Theorem A reduces to \(\rho(W_p) < 2\) and (1.2) becomes

\[
\operatorname{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p \ln 2}
\]

where \(\rho(W_p)\) is the spectral radius of \(W_p\) (Theorem 5.3).

The paper is organized as follows. In Section 2 we include some preliminary results concerning the eigen-properties of the matrices \(T_0\), \(T_1\) and \(T_0 + T_1\) that we need. We give a complete proof of Theorem B in Section 3. In Section 4, we will apply Theorem B to obtain explicit expressions for the two special cases described above. Finally in Section 5 we construct the matrix \(W_p\) in (1.3) and use the spectral radius of \(W_p\) to determine \(\operatorname{Lip}_p(f)\) when \(p\) is an even integer. We also make some remarks concerning extensions of the construction and discuss some unsolved questions.

2. Preliminaries. Throughout this paper, unless otherwise specified, we assume that \(1 \leq p < \infty\), \(c_n \in \mathbb{R}\), \(c_0, c_N \neq 0\) and \(\sum c_{2n} = \sum c_{2n+1} = 1\). For any \(k \geq 1\), let \(J = (j_1, \ldots, j_k)\), \(j_i = 0\) or \(1\), be the multi-index and \(|J|\) the length of \(J\). Let \(I_J = I_{(j_1, \ldots, j_k)}\) be the dyadic interval \([0, j_1 \cdots j_k, 0, j_1 \cdots j_k + 2^{-k})\). The matrix \(T_J\) represents the product \(T_{j_1} \cdots T_{j_k}\). If \(v\) is a 2-eigenvector of \((T_0 + T_1)\), it is clear that

\[
\frac{1}{2^k} \sum_{|J|=k} T_J v = \frac{1}{2^k} (T_0 + T_1)^k v = v.
\]

For any \(g \in L^p(\mathbb{R})\) with support in \([0, N]\), let \(g : [0, 1] \to \mathbb{R}^N\)

\[
g(x) = [g(x), g(x + 1), \ldots, g(x + (N - 1))]^t, \quad x \in [0, 1)
\]

be the vector-valued function representing \(g\) and let

\[
T g(x) = \begin{cases} 
T_0 g(2x) & \text{if } x \in [0, \frac{1}{2}), \\
T_1 g(2x - 1) & \text{if } x \in [\frac{1}{2}, 1).
\end{cases}
\]

It is easy to show that \(f\) is a solution of (1.1) if and only if \(f = T f\) [DL]. With no confusion, we use \(|| \cdot ||\) to denote the \(L^p\)-norm of \(g\) as well as the vector-valued function \(g\). Also for a vector \(u \in \mathbb{R}^n\), \(||u||\) will denote the \(\ell_p^n\)-norm in \(\mathbb{R}^N\).

Let \(g_I\) be the average \(|J|^{-1} \int_I g(x) \, dx\) of \(g\) on an interval \(I\).

**Proposition 2.1.** Let \(f\) be an \(L^p\)-solution of (1.1) and \(v = [f[0,1], \ldots, f[N-1,N]]^t\) be the vector defined by the average of \(f\) on the \(N\) subintervals. Then

(i) \(v\) is a 2-eigenvector of \((T_0 + T_1)\).

(ii) Let \(f_0(x) = v, x \in [0, 1]\), and let \(f_{n+1} = T f_n, n = 0, 1, \ldots, then

\[
f_n(x) = \sum_{|J|=n} (T_J v) x_{I_J}(x), \quad x \in [0, 1),
\]
and \( \| f_{n+1} - f_n \|_p^p = 2^{-n} \sum_{|J|=n} \| T_J \tilde{v} \|_p^p \).

(iii) \( \| f - f_n \|_p^p \leq c \frac{2^n}{2} \sum_{|J|=n} \| T_J \tilde{v} \|_p^p \) for some \( c > 0 \) and \( f_n \to f \) in \( L^p([0, 1], \mathbb{R}^N) \).

Proof. The proof of these statements can be found in [LW, Proposition 2.3, Lemma 2.4, 2.5, and Theorem 2.6]. In particular to prove the last identity in (ii), we observe that

\[
\| f_{n+1} - f_n \|_p^p = \frac{1}{2^{n+1}} \sum_{|J|=n} \left( \| T_J (T_0 - I) \tilde{v} \|_p^p + \| T_J (T_1 - I) \tilde{v} \|_p^p \right)
\]

\[
= \frac{1}{2^n} \sum_{|J|=n} \| T_J \tilde{v} \|_p^p .
\]

Let

\[
\alpha = \lim_{n \to \infty} \inf \frac{\ln(2^{-n} \sum_{|J|=n} \| T_J \tilde{v} \|_p^p)}{\ln(2^{-n})} .
\]

Then \( \alpha \) is the rate of convergence of \( 2^{-n} \sum_{|J|=n} \| T_J \tilde{v} \|_p^p \) to 0 in the sense that the sum is of order \( o(2^{-\beta n}) \) for any \( \beta < \alpha \). Let \( H(\tilde{v}) \) be the subspace (with complex scalar) spanned by \( \{ T_J \tilde{v} : J \text{ is a multi-index} \} \).

**Lemma 2.2.** Under the same conditions and notations as in Proposition 2.1, for any \( u \in H(\tilde{v}) \),

\[
\liminf_{n \to \infty} \frac{\ln(2^{-n} \sum_{|J|=n} \| T_J u \|_p^p)}{\ln(2^{-n})} \geq \alpha .
\]

Furthermore equality holds if \( H(u) = H(\tilde{v}) \).

Proof. Since \( H(\tilde{v}) \) is finite dimensional, it suffices to consider \( u = T_{J'} \tilde{v} \) for some \( J' \). Let \( |J'| = k \), then

\[
\frac{1}{2^n} \sum_{|J|=n} \| T_J u \|_p^p = \frac{1}{2^n} \sum_{|J|=n} \| T_J T_{J'} \tilde{v} \|_p^p \leq 2^k \frac{1}{2^{n+k}} \sum_{|J|=n+k} \| T_J \tilde{v} \|_p^p .
\]

It follows that

\[
\frac{\ln(2^{-n} \sum_{|J|=n} \| T_J u \|_p^p)}{\ln(2^{-n})} \geq \frac{\ln(2^k)}{\ln(2^{-n})} + \frac{\ln(2^{-n+k})}{\ln(2^{-n+k})} + \frac{\ln(2^{-n+k} \sum_{|J|=n+k} \| T_J \tilde{v} \|_p^p)}{\ln(2^{-n+k})} ,
\]

which implies the stated inequality. For the last statement we need only change the roles of \( u \) and \( \tilde{v} \) and make use of the inequality we just proved.

Let \( M = [c_{2i-j}]_{1 \leq i, j \leq N-1} \), i.e.,

\[
M = \begin{pmatrix}
  c_1 & c_0 & 0 & \ldots & 0 \\
  c_3 & c_2 & c_1 & \ldots & 0 \\
  c_5 & c_4 & c_3 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & c_{N-1}
\end{pmatrix}
\]

be the common submatrix of \( T_0 \) and \( T_1 \). If \( \sum c_{2n} = \sum c_{2n+1} = 1 \), then 1 is an eigenvalue of \( M \) and \( [1, 1, \ldots, 1] \) is the corresponding left 1-eigenvector. Let \( H = \{ u \in \mathbb{C}^N : \sum_{i=1}^N u_i = 0 \} \).

**Lemma 2.3.** There exist \( v_0, v_1 \notin H \) (i.e., \( \sum(v_0)_i = \sum(v_1)_i \neq 0 \)) such that (\( T_0 - I \))\( m \)\( v_0 = (T_1 - I)^m v_1 = 0 \) for some \( m > 0 \), and

(2.2) \( T_0 v_1 = T_1 v_0 \).
Remark. When $m = 1$, $v_0$ and $v_1$ are 1-eigenvectors of $T_0$ and $T_1$ respectively.

Proof. Let $E_\lambda = \{ u \in \mathbb{C}^{N-1} : (M - \lambda I)^m u = 0 \text{ for some } m > 0 \}$. Observe that for $\lambda \neq 1$ and for $u \in E_\lambda$,

$$0 = [1, 1, \ldots, 1] (M - \lambda I)^m u = (1 - \lambda)^m \sum_{i=1}^{N-1} u_i$$

for some $m > 0$, so that $\sum u_i = 0$. In view of $\mathbb{C}^{N-1} = E_1 \oplus \sum_{\lambda \neq 1} E_\lambda$, there exists $a \in E_1$ such that $\sum a_i \neq 0$. If 1 is a simple eigenvalue of $M$, $\dim E_1 = 1$ and hence the above $a$ is a 1-eigenvector of $M$. Let

$$(2.3)\quad v_0 := [0, a_1, \ldots, a_{N-1}]^t, \quad v_1 := [a_1, \ldots, a_{N-1}, 0]^t.$$ 

Then $v_0$ and $v_1$ are 1-eigenvectors of $T_0$ and $T_1$ respectively, and $v_0, v_1 \notin H$. Moreover, by the definitions of $T_0$ and $T_1$, we have

$$(2.4)\quad (T_0 v_1)_i = \sum c_{2i-j-1} a_j = \sum c_{2i-j} a_{j+1} = (T_1 v_0)_i.$$ 

so that $T_0 v_1 = T_1 v_0$. If 1 is not a simple eigenvalue of $M$, we let $m$ be the smallest positive integer so that $(M - I)^m a = 0$. Define $a^{(1)} = a, \ldots, a^{(m)} = (M - I)^{m-1} a$, and let

$$(2.5)\quad v_0^{(i)} = [0, a^{(i)}]^t \quad \text{and} \quad v_1^{(i)} = [a^{(i)}, 0]^t, \quad 1 \leq i \leq m.$$ 

Then $v_j^{(1)} \notin H$ and $v_j^{(m)}$ are eigenvectors of $T_j, j = 0, 1$ and

$$(2.6)\quad T_j v_j^{(i)} = v_j^{(i)} + v_j^{(i+1)}, \quad 1 \leq i \leq m-1, \quad j = 0, 1.$$ 

If we let $v_0 = v_0^{(1)}$ and $v_1 = v_1^{(1)}$, then a similar calculation like (2.4) implies that $T_0 v_1 = T_1 v_0$ again. □

Corollary 2.4. Let $v_0, v_1$ be chosen as in the proof of Lemma 2.3, Then

(i) $T_0 v_1^{(i)} = T_1 v_0^{(i)}$ for $1 \leq i \leq m$.
(ii) $T_j T_0^{k-1} v_0 = T_0 T_1^{k-1} v_1$ for $k \geq 1$.
(iii) $(T_0^n v_0)_1 = (T_1^n v_1)_N = 0$ and $(T_0^n v_0)_i = (T_1^n v_1)_{i-1}$ for $2 \leq i \leq N$.

Proof. (i) and (ii) follows directly from the same calculation as in the proof of the above lemma. The first identity in (iii) is a consequence of $(v_0)_1 = (v_1)_N = 0$ as in (2.3). For the second identity, if $v_0$ and $v_1$ are 1-eigenvectors of $T_0$ and $T_1$ respectively, (2.2) implies that

$$(T_0^n v_0)_i = (v_0)_i = (v_1)_{i-1} = (T_1^n v_1)_{i-1}.$$ 

For the general case we need only apply

$$T_j^n v_j^{(i)} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} v_j^{(i+1)} & \text{if } n < m \\ \sum_{i=0}^{m-1} \binom{n}{i} v_j^{(i+1)} & \text{if } n \geq m \end{cases}$$

which can be checked directly by using (2.6). □

Lemma 2.5. Let $v$ be a 2-eigenvector of $(T_0 + T_1)$ and $\tilde{v} = (T_0 - I)v$. Then $H(\tilde{v})$ is a subspace of $H$. Moreover, if (i) 1 is a simple eigenvalue of $T_0$ and $T_1$; or (ii) $H(\tilde{v}) = H$, then for $v_0, v_1 \notin H$ as defined in Lemma 2.3, there exists a constant $c$ such that

$$v = cv_0 + h_0 = cv_1 + h_1.$$
for some \( h_0, h_1 \in H(\tilde{v}) \).

Proof. Note that \([1, 1, \ldots, 1]^t\) is a left 1-eigenvector of \( T_0 \), so that \((T_0 - I)u \in H\) for every \( u \in \mathbb{C}^n \). In particular, \( \tilde{v} = (T_0 - I)v \) must be in \( H \). Also it is easy to show that \( H \) is invariant under \( T_0 \) and \( T_1 \), hence \( H(\tilde{v}) \) is a subspace of \( H \). Let \( v_0 \) and \( v_1 \) be as in Lemma 2.3, then \( a = \sum(v_0)_i = \sum(v_1)_i \neq 0 \). Let \( c = \sum_{i=1}^N v_i/a \), where \( v_i \)'s are the coordinates of \( v \) and let

\[
h_0 = v - cv_0 \quad \text{and} \quad h_1 = v - cv_1.
\]

By the choice of \( c \), we have \( h_0, h_1 \in H \) which implies case (ii) because \( H = H(\tilde{v}) \). In case (i), we observe that if 1 is a simple eigenvalue of \( T_0 \), then \( T_0 - I \) restricted on \( H \) is bijective; it is hence also bijective on the \((T_0 - I)\)-invariant subspace \( H(\tilde{v}) \). Consequently,

\[
\tilde{v} = (T_0 - I)v = (T_0 - I)(cv_0 + h_0) = (T_0 - I)h_0
\]

so that \( h_0 \) must be in \( H(\tilde{v}) \). The same proof holds for \( h_1 \). \( \square \)

3. Proof of Theorem B. Let \( f \) be an \( L^p \)-solution of (1.1) and let \( v = [f_{[0,1]}, \ldots, f_{[N-1,N]}]^t \) be the vector defined by the average of \( f \) over the \( N \)-subintervals (see Proposition 2.1), then \( v \) is a 2-eigenvector of \( (T_0 + T_1) \). Let

\[
f_n(x) = \sum_{|J|=n} (T_J v) \chi_J(x)
\]

and let \( f_n \) be the corresponding real valued function of \( f_n \) defined on \([0, N]\).

Lemma 3.1. For \( n \geq 1 \) and \( \ell \geq 0 \),

\[
\int_0^{1-2^{-n}} \|f_{n+\ell}(x + 2^{-n}) - f_{n+\ell}(x)\|^p \, dx = \frac{1}{2^{n+\ell}} \sum_{|J|=\ell} \left( \sum_{i=1}^n \sum_{|J|=n-i} \|T_J(T_0 T_0^{i-1} - T_0 T_1^{i-1})T_J v\|^p \right).
\]

Proof. We divide the interval \([0, 1 - 2^{-n}]\) into \( 2^n - 1 \) equal subintervals. For each subinterval, we further divide it into \( 2^\ell \) equal parts. In this way we have \( 2^\ell (2^n - 1) \) equal subintervals with length \( 2^{-(n+\ell)} \). For each such dyadic interval, we can write its binary representation with length \( 2^{n+\ell} \), say \( I(j_1, \ldots, j_n, j_1', \ldots, j_\ell') \). Since it is contained in \([0, 1 - 2^{-n}]\), at least one of the \( j_1, \ldots, j_n \) must equal 0. Suppose \( x \in I(j_1, \ldots, j_n, j_1', \ldots, j_\ell') \) with \( j_{n-i+1} \) as the last zero in \( \{j_1, \ldots, j_n\} \), i.e., \( x \in I(j_1, \ldots, j_{n-i}, 0, 1, \ldots, 1, j_1', \ldots, j_\ell') \), then \( x + 2^{-n} \in I(j_1, \ldots, j_{n-i+1}, 1, 0, 0, j_1', \ldots, j_\ell') \). It follows that

\[
f_{n+\ell}(x + 2^{-n}) - f_{n+\ell}(x) = T_{j_1} \cdots T_{j_{n-i}} T_i T_0 T_0^{i-1}(T_{j_1'} \cdots T_{j_\ell'} v) - T_{j_1} \cdots T_{j_{n-i}} T_0 T_1 T_1^{i-1}(T_{j_1'} \cdots T_{j_\ell'} v) = T_{j_1} \cdots T_{j_{n-i}} (T_i T_0^{i-1} - T_0 T_1^{i-1}) T_{j_1'} \cdots T_{j_\ell'} v.
\]

Since \( f_{n+\ell}(x + 2^{-n}) - f_{n+\ell}(x) \) is a constant function on each dyadic interval of size \( 2^{-(n+\ell)} \), an integration over the interval \([0, 1 - 2^{-n}]\) yields the lemma immediately. \( \square \)

We first give a lower bound estimate of \( \|A_{2^{-n}} f\| \).
PROPOSITION 3.2. For $n \geq 1$,

$$\|\Delta_{2^{-n}} f\|^p \geq \frac{2^{p-1}}{2^{n-1}} \sum_{|J| = n-1} \|T_J \tilde{v}\|^p.$$ 

Proof. Fix $n \geq 1$ and for any $\ell \geq 0$,

$$\|\Delta_{2^{-n}} f_{n+\ell}\|^p = \int_{-2^{-n}}^{N} |f_{n+\ell}(x + 2^{-n}) - f_{n+\ell}(x)|^p dx$$

$$\geq \int_{0}^{1-2^{-n}} |f_{n+\ell}(x + 2^{-n}) - f_{n+\ell}(x)|^p dx$$

$$= \frac{1}{2^{n+\ell}} \sum_{|J'| = \ell} \left( \sum_{i=1}^{n} \sum_{|J| = n-i} \|T_J (T_i T_0^{i-1} - T_0 T_1^{i-1}) T_J \tilde{v}\|^p \right)$$

(by Lemma 3.1)

$$\geq \frac{1}{2^{n+\ell}} \sum_{|J'| = \ell} \sum_{|J| = n-1} \|T_J (T_1 - T_0) T_J \tilde{v}\|^p$$

$$\geq \frac{1}{2^n} \sum_{|J| = n-1} \|T_J (T_1 - T_0) \tilde{v}\|^p$$

$$= \frac{1}{2^n} \sum_{|J| = n-1} \|T_J (T_1 - T_0) \tilde{v}\|^p$$

(by (2.1))

$$= 2^{p-1} \frac{1}{2^{n-1}} \sum_{|J| = n-1} \|T_J \tilde{v}\|^p$$ (use $(T_1 - T_0) \tilde{v} = -2\tilde{v}$).

The assertion now follows by letting $\ell \to \infty$. $\Box$

For the upper bound of $\|\Delta_h f\|$, we need an estimation of the integral of $|\Delta_h f_n(x)|$ near the integers $k = 0, \ldots, N$.

LEMMA 3.3. Under the same assumptions as in Lemma 2.5, for $n > 0$ and for $0 < h < 2^{-n}$,

$$\int_{E_n} |\Delta_h f_n(x)|^p dx \leq 2^p h \left( \|T_0^n h_0\|^p + \|T_1^n h_1\|^p \right)$$

where $E_n = \bigcup_{k=0}^{N} [k - 2^{-n}, k)$.

Proof. Since $f_n$ is a constant function on the dyadic intervals of size $2^{-n}$, we have

$$\int_{E_n} |\Delta_h f_n(x)|^p dx = \sum_{k=0}^{N} \int_{k-2^{-n}}^{k} |f_n(x + h) - f_n(x)|^p dx$$

$$= \sum_{k=0}^{N} \int_{k-h}^{k} |f_n(x + h) - f_n(x)|^p dx$$

$$= h \left( \|T_0^n \tilde{v}\|^p + \sum_{i=2}^{N} \|T_1^n \tilde{v}\|_{i-1}^p + \| - (T_1^n \tilde{v})_N \|_N^p \right).$$

Recall that $\tilde{v} = cv_0 + h_0 = cv_1 + h_1$ as in Lemma 2.5. Therefore, by Corollary 2.4(iii),

$$(T_0^n \tilde{v})_1 = c(T_0^n v_0)_1 + (T_0^n h_0)_1 = (T_0^n h_0)_1$$
and similarly \((T^n_1v)_N = (T^n_1h_1)_N\). Also for \(2 \leq i \leq N\), by Corollary 2.4(iii) again,
\[
(T^n_0v)_i - (T^n_1v)_{i-1} = c(T^n_0v_0)_i + (T^n_0h_0)_i - c(T^n_1v_1)_i - (T^n_1h_1)_{i-1}
\]
\[= (T^n_0h_0)_i - (T^n_1h_1)_{i-1}.
\]
We can continue the above estimation:
\[
\int_{E_n} |\Delta_h f_n(x)|^p \, dx = h \left( |(T^n_0h_0)_1|^p + \sum_{i=2}^{N} |(T^n_0h_0)_i - (T^n_1h_1)_{i-1}|^p + |(T^n_1h_1)_N|^p \right)
\]
\[\leq 2^p h \left( ||T^n_0h_0||^p + ||T^n_1h_1||^p \right)
\]
and complete the proof.

**Proposition 3.4.** Under the same assumptions as in Lemma 2.5, we have for \(0 < h < 2^{-n}\),
\[
||\Delta_h f_n||^p \leq \frac{2^{p+1}}{2^n} \left( \sum_{|J|=n} ||T_Jh_0||^p + \sum_{|J|=n} ||T_Jh_1||^p \right).
\]

**Proof.** Let \(E_n = \bigcup_{k=0}^{N} (k - 2^{-n}, k)\) and \(E_n = \bigcup_{k=0}^{N-1} (k, k + 1 - 2^{-n})\). Since \(f_n\) is supported by \([0, N]\), we have
\[
||\Delta_h f_n||^p = \int_{-2^{-n}}^{2^{-n}} |\Delta_h f_n(x)|^p \, dx
\]
\[= \int_{E_n} |\Delta_h f_n(x)|^p \, dx + \int_{E_n} |\Delta_h f_n(x)|^p \, dx
\]
:= \(I_1 + I_2\).

Lemma 3.3 implies that
\[
I_1 \leq 2^p h( ||T^n_0h_0||^p + ||T^n_1h_1||^p ).
\]
On the other hand, if we write \(I_2\) in the vector form, we have
\[
I_2 = \int_{0}^{1-2^{-n}} ||f_n(x+h) - f_n(x)||^p \, dx
\]
\[= h \sum_{k=1}^{n} \sum_{|J|=n-k} ||T_J(T_1T_0^{k-1} - T_0T_1^{k-1})v||^p.
\]
From Corollary 2.4(ii) we conclude that
\[
(T_1T_0^{k-1} - T_0T_1^{k-1})v = T_1T_0^{k-1}(cv_0 + h_0) - T_0T_1^{k-1}(cv_1 + h_1)
\]
\[= T_1T_0^{k-1}h_0 - T_0T_1^{k-1}h_1,
\]
and therefore
\[
I_2 \leq 2^p h \left( \sum_{k=1}^{n} \sum_{|J|=n-k} ||T_JT_1T_0^{k-1}h_0||^p + \sum_{k=1}^{n} \sum_{|J|=n-k} ||T_JT_0T_1^{k-1}h_1||^p \right)
\]
\[\leq 2^p h \left( \sum_{|J|=n} ||T_Jh_0||^p + \sum_{|J|=n} ||T_Jh_1||^p \right).
\]
The lemma then follows from the two estimates of \(I_1\) and \(I_2\).
We can now state and prove our main theorem of this section (i.e. Theorem B in Section 1).

**THEOREM 3.5.** Suppose that either (i) 1 is a simple eigenvalue of $T_0$ and $T_1$ or (ii) $H(\mathbf{v}) = \{ \mathbf{u} \in \mathbb{C}^N : \sum_{i=1}^N u_i = 0 \}$. If $f$ is a $L_p^p$-solution of (1.1), then

$$\text{Lip}_p(f) = \liminf_{n \to \infty} \frac{\ln(2^{-n} \sum_{|J|=n} \| T_J \mathbf{v} \|^p)}{p \ln(2^{-n})}.$$ 

**Proof.** As a direct consequence of Proposition 3.2, we have

$$\text{Lip}_p(f) \leq \liminf_{n \to \infty} \frac{\ln \| \Delta_{2^{-n}} f \|}{\ln(2^{-n})} \leq \liminf_{n \to \infty} \frac{\ln(2^{-n} \sum_{|J|=n} \| T_J \mathbf{v} \|^p)}{p \ln(2^{-n})}.$$ 

To prove the reverse inequality we first observe that $\| \Delta_h f \| \leq 2 \| f - f_n \| + \| \Delta_h f_n \|$. Proposition 2.1 (iii) and Proposition 3.4 imply that

$$\| \Delta_h f \|^p \leq C \left( 2^{-n} \sum_{|J|=n} \| T_J \mathbf{v} \|^p + 2^{-n} \sum_{|J|=n} \| T_J h_0 \|^p + 2^{-n} \sum_{|J|=n} \| T_J h_1 \|^p \right)$$

for some constant $C$ independent of $n$. Since $\mathbf{v}$, $h_0$, $h_1$ are all in $H(\mathbf{v})$, we can apply Lemma 2.2 to have the reverse inequality. \qed

4. **Lip$_p$(f) for some special cases.** For the 2-coefficient dilation equation $f(x) = f(2x) + f(2x - 1)$, the scaling function is $\chi_{[0,1]}$ and it is easy to calculate that $\text{Lip}_p(f) = 1/p$ from the definition.

**Proposition 4.1.** If $f$ is an $L_p^p$-solution of $f(x) = c_0 f(2x) + c_1 f(2x - 1) + c_2 f(2x - 2)$ with $c_0 + c_2 = 1$, $c_1 = 1$, and $c_0$, $c_2 \neq 0$, then

$$\text{Lip}_p(f) = \frac{\ln((|c_0|^p + |1 - c_0|^p)/2)}{-p \ln 2}.$$ 

**Proof.** In this case,

$$T_0 = \begin{pmatrix} c_0 & 0 \\ 1 - c_0 & 1 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 - c_0 \end{pmatrix},$$

and $\mathbf{v} = [c_0, c_0 - 1]^t$ is a 2-eigenvector of $(T_0 + T_1)$. Then

$$\tilde{\mathbf{v}} = (T_0 - I) \mathbf{v} = \left( \begin{array}{c} c_0(c_0 - 1) \\ -c_0(c_0 - 1) \end{array} \right) \neq 0.$$ 

Note that $\tilde{\mathbf{v}}$ is an $c_0$-eigenvector of $T_0$ and $(1-c_0)$-eigenvector of $T_1$. A straight-forward calculation yields

$$\frac{1}{2^n} \sum_{|J|=n} \| T_J \tilde{\mathbf{v}} \|^p = \frac{1}{2^n} \left( \sum_{k=0}^n \binom{n}{k} (|c_0|^p)^k (|1 - c_0|^p)^{n-k} \right) \| \tilde{\mathbf{v}} \|^p = \left( \frac{|c_0|^p + |1 - c_0|^p}{2} \right)^n \| \tilde{\mathbf{v}} \|^p.$$ 

This implies that

$$\text{Lip}_p(f) = \frac{\ln((|c_0|^p + |1 - c_0|^p)/2)}{-p \ln 2}. \quad \square$$

We now turn to the 4-coefficient dilation equation

$$f(x) = c_0 f(2x) + c_1 f(2x - 1) + c_2 f(2x - 2) + c_3 f(2x - 3)$$
with \( c_0 + c_2 = c_1 + c_3 = 1 \) and \( c_0, c_3 \neq 0 \). We first observe that

\[
T_0 = \begin{pmatrix}
  c_0 & 0 & 0 \\
  1 - c_0 & 1 - c_3 & c_0 \\
  0 & c_3 & 1 - c_0
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
  1 - c_3 & c_0 & 0 \\
  c_3 & 1 - c_0 & 1 - c_3 \\
  0 & 0 & c_3
\end{pmatrix}.
\]

The eigenvalues of \((T_0 + T_1)\) are 2, 1, and \((1 - c_0 - c_3)\), and the 2-eigenvector \( v \) is

\[
v = \begin{pmatrix}
  c_0(1 + c_0 - c_3) \\
  (1 + c_0 - c_3)(1 - c_0 + c_3) \\
  c_3(1 - c_0 + c_3)
\end{pmatrix}.
\]

Therefore

\[
\tilde{v} = (T_0 - I)v = \begin{pmatrix}
  c_0(c_0 - 1)(1 + c_0 - c_3) \\
  -c_0(c_0 - 1)(1 + c_0 - c_3) + c_3(c_3 - 1)(1 - c_0 + c_3) \\
  -c_3(c_3 - 1)(1 - c_0 + c_3)
\end{pmatrix}.
\]

Note that in Proposition 4.1, the computation can be made easier if \( \tilde{v} \) is an eigenvector of both \( T_0 \) and \( T_1 \). Here we have

**Lemma 4.2.** Let \( T_0 \) and \( T_1 \) be as in (4.2) and let \( v \) be the 2-eigenvector of \((T_0 + T_1)\) as in (4.3) and let \( \tilde{v} = (T_0 - I)v \). Then \( \tilde{v} \) is an eigenvector of both \( T_0 \) and \( T_1 \) (not necessary to the same eigenvalue) if and only if \( c_0 + c_3 = 1 \).

**Proof.** Suppose \( c_0 + c_3 = 1 \), then \( c_0 = c_1, c_2 = c_3 \), and (4.2) reduces to

\[
T_0 = \begin{pmatrix}
  c_0 & 0 & 0 \\
  1 - c_0 & c_0 & 0 \\
  0 & 1 - c_0 & 1 - c_0
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
  c_0 & c_0 & 0 \\
  1 - c_0 & 1 - c_0 & c_0 \\
  0 & 0 & 1 - c_0
\end{pmatrix},
\]

and \( \tilde{v} = [-2c_0^2c_3, 2c_0^2c_3 - 2c_0c_3^2, 2c_0c_3^2]^t \neq 0 \). By a direct calculation, \( \tilde{v} \) is a \( c_0 \)-eigenvector of \( T_0 \) and \((1 - c_0)\)-eigenvector of \( T_1 \).

Conversely, suppose \( \tilde{v} \) is an eigenvector of both \( T_0 \) and \( T_1 \). Let \( u_0 = [0, 1, -1]^t \) and \( u_1 = [1, -1, 0]^t \), then \( \tilde{v} = au_0 + bu_1 \) where \( a \) and \( b \) is determined by (4.4). By using \( u_0 \) and \( u_1 \) as a basis of the subspace \( H = \{ u \in \mathbb{C}^3 : \sum u_i = 0 \} \), we can rewrite \( T_0, T_1 \) (restricted on \( H \)) and \( \tilde{v} \) as follows:

\[
T_0 = \begin{pmatrix}
  1 - c_0 & c_3 \\
  0 & c_0 & c_0
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
  c_3 & 0 \\
  c_0 & 1 - c_0 - c_3
\end{pmatrix}, \quad \text{and} \quad \tilde{v} = \begin{pmatrix}
  a \\
  b
\end{pmatrix}.
\]

Note that \( T_0 \) has \( c_0 \) and \( 1 - c_0 - c_3 \) as eigenvalues while \( T_1 \) has \( c_3 \) and \( 1 - c_0 - c_3 \) as eigenvalues. We claim that \( \tilde{v} \) is an \( c_0 \)-eigenvector of \( T_0 \). For otherwise, \( \tilde{v} \) is an \((1 - c_0 - c_3)\)-eigenvector of \( T_0 \), then \( b \) must be zero and \( \tilde{v} = [a, 0]^t \). But this contradicts to the assumption that \( \tilde{v} \) is an eigenvector of \( T_1 \). Similarly, \( \tilde{v} \) must be a \( c_3 \)-eigenvector of \( T_1 \). Hence,

\[(T_0 + T_1)\tilde{v} = (c_0 + c_3)\tilde{v}.
\]

There are only three choices of the eigenvalues of \((T_0 + T_1)\): 2, 1 or \( 1 - c_0 - c_3 \). By a direct check we conclude that \( c_0 + c_3 = 1 \) is the only allowable case. \( \square \)

In view of Lemma 4.2 we can use the same technique as in Proposition 4.1 to prove the next proposition

**Proposition 4.3.** If \( f \) is an \( L_p^p \)-solution of (4.1) with the additional assumption that \( c_0 + c_3 = 1 \), then

\[
\text{Lip}_p(f) = \frac{\ln(|c_0|^p + |1 - c_0|^p)^{1/2}}{-p \ln 2}.
\]
In Figure 1, we draw the graphs of some scaling functions satisfying the assumption in the above proposition and their $L^p$-Lipschitz exponents. Note that if $\text{Lip}_p(f) = 1$ for all $1 \leq p < \infty$, then $f$ is differentiable almost everywhere and the derivative is in $L^p$ for all $1 \leq p < \infty$. This is the case for $c_0 = 0.5$ and is obvious from the graph of the corresponding scaling function. For the graph of $c_0 = 1.125$, we see that $\text{Lip}_p(f)$ is undefined for $p > 6$. Indeed $f \notin L^p(\mathbb{R})$, for $p > 6$, making use of the criterion in Theorem A.

We conclude this section by giving a formula of $\text{Lip}_p(f)$ with the coefficients satisfying $c_0 + c_3 = \frac{1}{2}$ instead of $c_0 + c_3 = 1$. It includes Daubechies scaling function $D_4$ which corresponds to $c_0 = (1 + \sqrt{3})/4$, $c_3 = (1 - \sqrt{3})/4$. This formula has been obtained in [DL3] using a different method and assuming in addition that $\frac{1}{2} < c_0 < \frac{3}{4}$. Here, we need an estimation on the product of two non-commutative matrices.

**Lemma 4.4.** Let $\beta_0, \beta_1 \in \mathbb{R}$. Let

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & \beta_0 \end{pmatrix} \text{ and } P_1 = \begin{pmatrix} 1 & 0 \\ \beta_0 & \beta_1 \end{pmatrix}.$$  

For any multi-index $J = (j_1, j_2, \ldots, j_n)$, we let $P_J = P_{j_1} \cdots P_{j_n}$. Then

$$P_J = \begin{pmatrix} 1 & 0 \\ \lambda_J & \mu_J \end{pmatrix}$$

where $\lambda_J = \beta_0(j_1 + j_2\beta_1 + \cdots + j_n(\beta_{j_{n-1}} \cdots \beta_{j_1}))$ and $\mu_J = \beta_j \beta_{j_{n-1}} \cdots \beta_{j_1}$. Let

$$\gamma = (|\beta_0|^p + |\beta_1|^p)/2.$$  

Then

$$2^{-n} \sum_{|J|=n} |\mu_J|^p = \gamma^n \quad \text{and} \quad 2^{-n} \sum_{|J|=n} |\lambda_J|^p \leq C n^p \max\{1, \gamma^n\}$$

for some constant $C > 0$ independent of $n$.

**Proof.** The explicit form of the product $P_J$ can easily be shown by induction. For the second part of the lemma, the first identity follows from

$$2^{-n} \sum_{|J|=n} |\mu_J|^p = 2^{-n} \sum_{j_1, \ldots, j_n=0,1} |\beta_{j_1} \cdots \beta_{j_n}|^p = \gamma^n.$$  

For the second identity we observe that

$$\left( \sum_{|J|=n} |\lambda_J|^p \right)^{1/p} = |\beta_0| \left( \sum_{j_1, \ldots, j_n=0,1} |j_1 + \sum_{i=2}^n j_i(\beta_{j_i-1} \cdots \beta_{j_1})|^p \right)^{1/p} \leq |\beta_0| \left( 2^{(n-1)/p} + \sum_{i=2}^n \left( \sum_{j_1, \ldots, j_{n-1}=0,1} |j_i(\beta_{j_{i-1}} \cdots \beta_{j_1})|^p \right)^{1/p} \right)$$

(by Minkowski inequality)

$$= |\beta_0| \left( 2^{(n-1)/p} + \sum_{i=2}^n (|\beta_0|^p + |\beta_1|^p)^{(i-1)/p} \right) \leq |\beta_0| 2^{(n-1)/p} \sum_{i=1}^n (\gamma^{1/p})^{i-1} \leq |\beta_0| n 2^{(n-1)/p} \max\{1, (\gamma^{1/p})^n\}.$$  

The last identity now follows. \qed
PROPOSITION 4.5. If \( f \) is the \( L_p \)-solution of (4.1) with the additional assumption
that \( c_0 + c_3 = \frac{1}{2} \), then

\[
(4.7) \quad \text{Lip}_p(f) = \min \left\{ 1, \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p \ln 2} \right\}.
\]

**Proof.** In this case, (4.2) reduces to

\[
T_0 = \begin{pmatrix}
c_0 & 0 & 0 \\
1 - c_0 & \frac{1}{2} + c_0 & 0 \\
0 & \frac{1}{2} - c_0 & 1 - c_0
\end{pmatrix}
\quad \text{and} \quad
T_1 = \begin{pmatrix}
\frac{1}{2} + c_0 & c_0 & 0 \\
\frac{1}{2} - c_0 & 1 - c_0 & \frac{1}{2} + c_0 \\
0 & 0 & 1 - c_0
\end{pmatrix}.
\]

Note that \( \mathbf{h} = [1, -2, 1]^t \) is a \( c_0 \)-eigenvector of \( T_0 \) and also a \( (\frac{1}{2} - c_0) \)-eigenvector of \( T_1 \). It is clear that

\[
2^{-n} \sum_{|J|=n} ||T_J\mathbf{h}||^p = 2^{-n} \left( |c_0|^p + \frac{1}{2} - c_0 \right)^n ||\mathbf{h}||^p.
\]

Since \( \mathbf{h} \in H(\tilde{v}) \), by Lemma 2.2, we have

\[
\text{Lip}_p(f) \leq \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p \ln 2}.
\]

Next observe that \( \mathbf{u} = [0, 1, -1]^t \) is a \( \frac{1}{2} \)-eigenvector of \( T_0 \) and \( T_1 \mathbf{u} = \frac{1}{2} \mathbf{u} + c_0 \mathbf{h} \). Therefore, by using \( \mathbf{u} \) and \( \mathbf{h} \) as a basis of the subspace \( H(\tilde{v}) \), we can rewrite \( T_0, T_1 \), restricted on \( H(\tilde{v}) \) in this case, and \( \tilde{v} \), as

\[
T_0 = \begin{pmatrix}
\frac{1}{2} & 0 & c_0 \\
0 & c_0 & \frac{1}{2} - c_0
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
\frac{1}{2} & 0 & c_0 \\
0 & c_0 & \frac{1}{2} - c_0
\end{pmatrix}, \quad \text{and} \quad \tilde{v} = \begin{pmatrix}
a \\
b
\end{pmatrix},
\]

where \( a = -\frac{1}{4} \) and \( b = \frac{3}{8}c_0(c_0 - 1)(\frac{1}{2} + 2c_0) \). Let \( \beta_0 = 2c_0 \) and \( \beta_1 = 1 - \beta_0 \). For \( J = (j_1, j_2, \ldots, j_n) \) we have \( T_J = \frac{1}{8^n} P_J \), so that

\[
||T_J\tilde{v}||^p = |a2^{-n}|^p + |2^{-n}(a\lambda_J + b\mu_J)|^p,
\]

where \( \lambda_J \) and \( \mu_J \) are defined as in Lemma 4.4. This implies \( ||T_J\tilde{v}||^p \geq |a2^{-n}|^p \) and \( \text{Lip}_p(f) \leq 1 \) by Theorem 3.5. Hence

\[
(4.8) \quad \text{Lip}_p(f) \leq \min \left\{ 1, \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p \ln 2} \right\}.
\]

On the other hand,

\[
||T_J\tilde{v}||^p \leq |a2^{-n}|^p + 2^p|a2^{-n}\lambda_J|^p + 2^p|b2^{-n}\mu_J|^p.
\]

By Lemma 4.4, we have

\[
2^{-n} \sum_{|J|=n} ||T_J\tilde{v}||^p \leq C n^p 2^{-pn} \max\{1, ((|\beta_0|^p + |\beta_1|^p)/2)^n\}
\]

\[
= C n^p \max\{2^{-pn}, ((|\beta_0/2|^p + |\beta_1/2|^p)/2)^n\}.
\]

Consequently we have the reverse inequality of (4.8) and completes the proof. \( \square \)

In figures 2a–e we again sketch some \( L^2 \)-scaling functions from Proposition 4.5 \((c_0 + c_3 = \frac{1}{2})\) and their \( L^p \)-Lipschitz exponents \( \text{Lip}_p(f) \) of \( p \). The case for \( c_0 = 0.25 \) corresponding to \( \chi_{[0,1]} * \chi_{[0,1]} \), it is differentiable and hence \( \text{Lip}_p(f) = 1 \) for all \( p \). The case corresponding to \( c_0 = 0.683... \) is the Daubechies scaling function \( D_4 \). From the
picture of Lip_p(f), D_4 has L^p-derivative for 1 \leq p < 2. It is known that for p = 2, D_4
is differentiable almost everywhere but the derivative is not in L^2. Also it is known
that the Hölder exponent of D_4 is 2 - \ln(1 + \sqrt{3})/\ln 2, which is the same number as
the formula in the proposition when p \to \infty.
5. Lip_p(f) when p is a positive even integer. The computation of Lip_p(f) in Section 4 depends on the existence of an eigenvector of both T_0 and T_1 (which may be associated with different eigenvalues). This technique cannot be used for the general case. In this section we show that if p is a positive even integer, then Lip_p(f) is related to the spectral radius of a matrix W_p whose entries are induced from the coefficients of the dilation equation. For simplicity, we only give the construction of W_p for the 4-coefficient dilation equation. It is not hard to extend this to the case with more coefficients.

In view of Theorem 3.5, we will first develop a simple expression for the sum \( 2^{-n} \sum |J|=n \|T_J \tilde{v}\|^p \) for \( p \) a positive even integer. Let \([0, 1, -1]^t\) and \([1, -1, 0]^t\) be a basis of \( H = \{ u \in \mathbb{C}^3 : \sum u_i = 0 \} \). Then \( T_0 \) and \( T_1 \) can be written as in (4.5). Let \( e_0 = [1, 0], e_1 = [0, 1] \). For a fixed \( u = [\alpha, \beta]^t \in H(\tilde{v}) \) (to be determined later), we define the vector \( a_n \) with the \( i \)-th entry by

\[
(a_n)_i = \sum_{|J|=n} (e_0 T_J u)^{p-i}(e_1 T_J u)^i, \quad i = 0, \ldots, p.
\]

If \( p \) is an even integer, then

\[
\sum_{|J|=n} \|T_J u\|^p = \sum_{|J|=n} (|e_0 T_J u|^p + |e_1 T_J u|^p)
\]

(5.1)

\[
= (a_n)_0 + (a_n)_p = \|(a_n)_0\| + \|(a_n)_p\|.
\]

Note that \( a_0 = [\alpha^p, \alpha^{p-1} \beta, \ldots, \alpha \beta^{p-1}, \beta^p]^t \). If we let \( d = 1 - c_0 - c_3 \), we have, in view of (4.5),

\[
e_0 T_0 = de_0 + c_3 e_1, \quad e_1 T_0 = c_0 e_1,
\]

\[
e_0 T_1 = c_3 e_0, \quad e_1 T_1 = c_0 e_0 + de_1,
\]

and hence

\[
(a_{n+1})_i = \sum_{|J|=n+1} (e_0 T_J u)^{p-i}(e_1 T_J u)^i
\]

\[
= \sum_{|J|=n} (e_0 T_0 T_J u)^{p-i}(e_1 T_0 T_J u)^i + \sum_{|J|=n} (e_0 T_1 T_J u)^{p-i}(e_1 T_1 T_J u)^i
\]

\[
= \sum_{|J|=n} ((de_0 + c_3 e_1) T_J u)^{p-i}(c_0 e_1 T_J u)^i
\]

\[
+ \sum_{|J|=n} ((c_3 e_0) T_J u)^{p-i}(c_0 e_0 + de_1 T_J u)^i
\]

\[
= \sum_{|J|=n} \left( \sum_{\ell=0}^{p-i} \binom{p-i}{\ell} d^{p-i-\ell}(e_0 T_J u)^{p-i-\ell} c_3^\ell (e_1 T_J u)^\ell \right) (c_0^i (e_1 T_J u)^i)
\]

\[
+ \sum_{|J|=n} \left( c_3^{p-i}(e_0 T_J u)^{p-i} \right) \left( \sum_{i=0}^i \binom{i}{\ell} c_0^{i-\ell}(e_0 T_J u)^{i-\ell} d^\ell (e_1 T_J u)^\ell \right)
\]

\[
= \sum_{\ell=0}^{p-i} \binom{p-i}{\ell} c_0^i c_3^{p-i-\ell} (a_n)_{i+\ell} + \sum_{i=0}^i \binom{i}{\ell} c_0^{i-\ell} c_3^{p-i-\ell} d^\ell (a_n)_{i+\ell}.
\]

Summarizing the above, we have
THE REGULARITY OF L^P-SCALING FUNCTIONS

PROPOSITION 5.1. For any integer p \geq 1, let W_p be the (p + 1) \times (p + 1) matrix defined by

\[(W_p)_{ij} = \begin{cases} \left(\frac{p-i}{p-j}\right)c_i c_j d^{p-j} & \text{for } 0 \leq i < j \leq p \\ c_i d^{p-i} + c_j d^i & \text{for } i = j \\ \left(\frac{i}{j}\right) c_i^{-j} c_j d^i & \text{for } 0 \leq j < i \leq p \end{cases}\]

where d = 1 - c_0 - c_3. Then

\[a_{n+1} = W_p a_n = W_p^{n+1} a_0\]

where \(a_0 = [\alpha^p, \alpha^{p-1} \beta, \ldots, \alpha^3 \beta^{-1}, \beta^p]^t\). In particular if p is an even integer, then

\[\sum_{|J|=n} ||T_J u||^p = [1, 0, 0, \ldots, 0, 1] W_p^n a_0.\]

The matrix \(W_p\) can be written as \(W_p = W_p^{(L)} + W_p^{(U)}\), where \(W_p^{(L)}\) and \(W_p^{(U)}\) are the lower and upper triangular part of \(W_p\), in a very symmetric manner. For example,

\[W_2^{(L)} = \begin{pmatrix} (0) c_3^2 & 0 & 0 \\ (0) c_0 c_3 & (1) c_3 d & 0 \\ (0) c_0^2 & (1) c_0 d & (2) d^2 \end{pmatrix}, \quad W_2^{(U)} = \begin{pmatrix} (2) d^2 & (2) c_3 d & (2) c_3^2 \\ 0 & (0) c_0 d & (1) c_0 c_3 \\ 0 & 0 & (0) c_0^2 \end{pmatrix}; \]

\[W_4^{(L)} = \begin{pmatrix} (0) c_4^2 & 0 & 0 & 0 & 0 \\ (0) c_0 c_4^2 & (1) c_3 d & 0 & 0 & 0 \\ (0) c_0^2 c_3^2 & (1) c_0 c_3 d & (2) c_3 d^2 & 0 & 0 \\ (0) c_0 c_3^3 & (1) c_0 c_3 d & (2) c_3 d^2 & (3) c_3 d^3 & 0 \\ (0) c_0^3 c_3 & (1) c_0 d & (2) c_0 d^2 & (3) c_0 d^3 & (4) d^4 \end{pmatrix}, \quad W_4^{(U)} = \begin{pmatrix} (4) d^4 & (4) c_3 d^3 & (4) c_3^2 d^2 & (4) c_3 d^3 & (4) c_3^2 \\ 0 & (4) c_0 d^3 & (3) c_0 c_3 d^2 & (2) c_0 c_3^2 d & (2) c_0 c_3^3 \\ 0 & 0 & (3) c_0 d^3 & (2) c_0 c_3^2 d & (2) c_0 c_3^3 \\ 0 & 0 & 0 & (1) c_0 d & (1) c_0 c_3 \\ 0 & 0 & 0 & 0 & (0) c_0 \end{pmatrix}; \]

Recall from basic linear algebra that if \(\rho(A)\) is the spectral radius of an \(N \times N\) matrix \(A\), then \(\rho(A) = \max\{|\lambda| : \lambda\ \text{is an eigenvalue of} \ A\}\) and

\[(5.2) \quad \lim_{n \to \infty} ||A^n||^{1/n} = \rho(A).\]

Let \(\lambda\) be any eigenvalue of \(A\), and \(E_\lambda = \{u \in \mathbb{C}^N : (A - \lambda I)^m u = 0\ \text{for some} \ m > 0\}\), then \(\mathbb{C}^N = E_\lambda \oplus Z\) for some \(A\)-invariant subspace \(Z\) of \(\mathbb{C}^N\). We say that \(u\) has a component in \(E_\lambda\) if \(u = u_\lambda + z\) with \(u_\lambda \neq 0\). It is clear that if \(u \in \mathbb{C}^N\) has a component in \(E_\lambda\), then there is a constant \(C > 0\) such that \(||A^n u|| \geq C|\lambda|^n\) for all \(n > 0\).

LEMMA 5.2. Let \(\lambda\) be the eigenvalue of \(W_p\) such that \(|\lambda| = \rho(W_p)\) and let \(E_\lambda\) be defined as above. Suppose \(\dim H(\tilde{v}) = 2\). Then there exists \(u = \alpha b_0 + \beta b_1 \in H(\tilde{v})\), where \(b_0 = [0, 1, -1]^t\), \(b_1 = [1, -1, 0]^t\), such that \(H(u) = H(\tilde{v})\) and the
corresponding \( a_0 = [\alpha, \alpha^{p-1}\beta, \ldots, \alpha^{p-1}\beta, \beta^p]^t \) has a component of \( E_\lambda \). For such \( u \), there is a constant \( C > 0 \) such that
\[
C \rho(W_p)^n \leq \sum_{|J|=n} ||T_J u||^p \quad \text{for all } n > 0.
\]

**Proof.** We choose \( p + 1 \) vectors \( u_i = \alpha_i b_0 + \beta_i b_1 \), \( i = 0, \ldots, p \), such that \( H(u_i) = H(\tilde{v}) \) and
\[
\alpha_i \beta_j - \alpha_j \beta_i \neq 0, \quad \text{for } i \neq j.
\]

Then the corresponding vectors
\[
\gamma_i = [\alpha_i^p, \alpha_i^{p-1}\beta_i, \ldots, \alpha_i^{p-1}\beta_i, \beta_i^p], \quad 0 \leq i \leq p
\]
form a basis of \( \mathbb{C}^{p+1} \) because the matrix with the vectors \( \gamma_i \) as rows is a Vandermonde matrix
\[
A = \begin{pmatrix}
\alpha_0^p & \alpha_0^{p-1}\beta_0 & \cdots & \alpha_0^{p-1}\beta_0 & \beta_0^p \\
\alpha_1^p & \alpha_1^{p-1}\beta_1 & \cdots & \alpha_1^{p-1}\beta_1 & \beta_1^p \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_p^p & \alpha_p^{p-1}\beta_p & \cdots & \alpha_p^{p-1}\beta_p & \beta_p^p
\end{pmatrix}
\]
and \( \det A = \prod_{0 \leq j, k \leq p}(\alpha_j \beta_k - \alpha_k \beta_j) \neq 0 \). Hence, one of the \( \gamma_i \)'s has a component of \( E_\lambda \). Let \( u \) be the corresponding \( u_i \) and the first part of the lemma follows. To prove the second part, we observe that for any \( J \) with \( |J| = n \) and \( 0 \leq j \leq p \),
\[
|e_0 T_J u|^{p-j}|e_1 T_J u|^j \leq \max\{|e_0 T_J u|^p, |e_1 T_J u|^p\} \leq |e_0 T_J u|^p + |e_1 T_J u|^p.
\]
Hence
\[
|(a_n)_j| = \sum_{|J|=n} |e_0 T_J u|^{p-j}|e_1 T_J u|^j
\]
\[
\leq \sum_{|J|=n} |e_0 T_J u|^p + \sum_{|J|=n} |e_1 T_J u|^p = |(a_n)_0| + |(a_n)_p|
\]
\[
= \sum_{|J|=n} ||T_J u||^p \quad \text{(by (5.1))}
\]
It follows that \( ||a_n||_1 = \sum_{j=0}^p |(a_n)_j| \leq p \sum_{|J|=n} ||T_J u||^p \). Since \( a_0 \) has a component of \( E_\lambda \), there exists a constant \( C > 0 \) such that
\[
C \rho(W_p)^n \leq ||W_p a_0||_1 = ||a_n||_1 \leq p \sum_{|J|=n} ||T_J u||^p.
\]

For the 4-coefficient dilation equation in (4.1), it is easy to check that \( \text{dim } H(\tilde{v}) = 0 \) if and only if \( (c_0, c_3) \in \{(0,0), (1,0), (0,1), (1,1)\} \), and the solutions are characteristic functions ([LW, Lemma 3.3]). Hence \( \text{Lip}_p(f) = 1/p \). Also \( \text{dim } H(\tilde{v}) = 1 \) if and only if \( c_0 + c_3 = 1 \) (Lemma 4.2), and in Proposition 4.3 we have given a formula of \( \text{Lip}_p(f) \) for this case. It remains to consider the case \( H(\tilde{v}) = 2 \), which will complete all the cases for all 4-coefficient scaling functions.

**Theorem 5.3.** Consider the 4-coefficient dilation equation in (4.1) with the assumption that \( \text{dim } H(\tilde{v}) = 2 \). For \( p \) a positive even integer, the equation has a (unique)
$L^p$-solution $f$ if and only if $\rho(W_p)/2 < 1$, and in this case

$$\text{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p \ln 2}.$$  

**Proof.** Note that for any $u \in H(\bar{v})$ and for any $\epsilon > 0$, we have for large $n$

$$\sum_{|J|=n} \|T_J u\|^p = |(a_n)_0| + |(a_n)_p| \quad \text{(by (5.1))}$$

$$\leq \|a_n\|_1 = \|W_p^n a_0\|_1 \quad \text{(by Proposition 5.1)}$$

$$\leq \|W_p^n\|_1 \|a_0\|_1$$

$$\leq \|a_0\|_1 (\rho(W_p) + \epsilon)^n. \quad \text{(by (5.2))}$$

If we choose $u \in H(\bar{v})$ as in Lemma 5.2, combining with Theorem A in Section 1, we have the first conclusion. The second assertion follows from Lemma 2.2, the estimation of $\sum_{|J|=n} \|T_J u\|^p$ from above and Lemma 5.2. $\Box$

Figure 3 shows the domain of $(c_0, c_3)$ for the existence of $L^p$-solutions for even integers using the above criterion $\rho(W_p)/2 < 1$. The curves are $\rho(W_p)/2 = 1$ corresponds to $p = 2, 4, 6, 10, 20, \text{ and } 40$. Note that when $p \to \infty$ the limit is the triangular region which is the approximate region plotted in [H] for the existence of continuous 4-coefficient scaling functions using the joint spectral radius. However, we are not able to prove this assertion yet, i.e., $\lim_{p \to \infty} \text{Lip}_p(f)$ is the Hölder exponent. Also we do not have a criterion for the existence of an $L^p$-solution.

Figure 4 is the graph of $\text{Lip}_4(f)$ plotted against the $(c_0, c_3)$-plane. It shows the overall picture of $\text{Lip}_4(f)$ for the 4-coefficient case. It looks similar to the graph of $\text{Lip}_2(f)$ plotted in [LMW].

We remark that if $c_0 > 0$, $c_3 > 0$, and $1 - c_0 - c_3 > 0$, then $T_0$ and $T_1$ in (4.5) are non-negative matrices. Also the vector $u$ in Lemma 5.2 can be chosen to be a positive vector. Hence (5.1) still holds if $p$ is a positive odd integer. Consequently, we have

**Proposition 5.4.** Consider the 4-coefficient dilation equation (4.1) with $c_0 > 0$, $c_3 > 0$, and $1 - c_0 - c_3 > 0$. Suppose $\dim H(\bar{v}) = 2$, then for $p$ a positive integer,

$$\text{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p \ln 2}.$$  

Without such positivity assumption on the coefficients, the expression in (5.3) does not necessarily give $\text{Lip}_p(f)$ for $p$ odd integers. For example Figure 5a is the graph of $-\ln(\rho(W_p)/2)/(p \ln 2)$, $p = 1, \cdots, 10$, of the scaling function corresponding to $c_0 = 0.5$ and $c_3 = -0.4$. The points bounce up and down but $\text{Lip}_p(f)$ should be convex in that region. Figure 5b corresponds to Daubechies scaling function $D_4$ ($c_3 < 0$). On this graph, the points are obtained by $-\ln(\rho(W_p)/2)/(p \ln 2)$ while the curve is $\text{Lip}_p(f)$ given by (4.7). It shows that for even integer $p$, they coincide. For odd integer $p$, the values obtained by (5.3) are different to $\text{Lip}_p(f)$ but surprisingly close (see Table 1). Also when $p \to \infty$, in our numerical and graphical experiments, the values obtained from $-\ln(\rho(W_p)/2)/(p \ln 2)$, $p$ odd integers, seems to converge to $\text{Lip}_p(f)$ rather rapidly.

Finally we remark that for the dilation equation with $N+1$ ($N > 3$) coefficients, if $\dim H(\bar{v}) = 1$ then $\bar{v}$ is an eigenvector of both $T_0$ and $T_1$, say $T_0 \bar{v} = a \bar{v}$ and $T_1 \bar{v} = b \bar{v}$.  

Then the same technique as in the proof of Proposition 4.1 yields

\[
\text{Lip}_p(f) = \frac{\ln((|a|^p + |b|^p)/2)}{-p\ln 2}
\]
THE REGULARITY OF $L^p$-SCALING FUNCTIONS

for $1 \leq p < \infty$. For the case $\dim H(\tilde{\nu}) \geq 2$, we can use a similar method to that in Proposition 5.1 to obtain a $(p+1)^{N-2} \times (p+1)^{N-2}$ square matrix $W_p$, and Theorem 5.3 and Proposition 5.4 will still hold for $\dim H(\tilde{\nu}) \geq 2$.

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