SMOOTHING OF ISOLATED HYPERSURFACE SINGULARITIES AND QUILLEN METRICS*

KEN-ICHI YOSHIKAWA[†]

Abstract. Using Bismut-Lebeau's theorem, we determine the singularity of Quillen metrics for smoothing of isolated hypersurface singularities and apply it to compute the asymptotic behavior of Ray-Singer analytic torsion for such degenerations.

0. Introduction. In [Q], Quillen introduced a metric on the determinant of cohomologies which is called Quillen metric today and calculated its curvature in some cases. After Quillen, in the celebrated work [B-G-S], Bismut-Gillet-Soulé generalized his result to arbitrary smooth morphism of Kähler manifolds and established the Riemann-Roch-Grothendieck formula at the differential form level. The purpose of this article is to study the boundary behavior of Quillen metrics. Namely, we investigate the behavior of Quillen metrics for a certain degenerating family of algebraic manifolds. Such a problem was treated by Wolpert ([W]) and also by Bismut-Bost ([B-B]) in the case of relative dimension one. (See also [T-T].) In [B 2], Bismut studies this problem when the family is ordinary singular and obtains a generalization of the result in [B-B]. We study the same problem for another type of degenerating family. When a family of complex manifold is given, the discriminant locus is the set of all points in the base space whose fiber is singular. In many families, generic point of the discriminant locus consists of those points whose fiber has only isolated hypersurface singularities. Therefore, it is natural to focus our consideration on the following degeneration.

DEFINITION. Let $\pi : X \to S = \{t \in \mathbb{C}; |t| < 1\}$ be a proper surjective holomorphic map of complex manifolds. The family (π, X, S) is said to be a smoothing of isolated hypersurface singularities (IHS), if π is of maximal rank outside of finite number of points in $X_0 := \pi^{-1}(0)$. In particular, X_0 has only isolated hypersurface singularities.

To state our theorem, fix notations. Let (π, X, S) be a smoothing of IHS, g_X a Kähler metric of X and $g_{X/S}$ the induced metric on TX/S. Let (E, h) be a holomorphic Hermitian vector bundle over X. Let $\lambda(E)$ be the determinant of cohomologies and $\| \cdot \|_Q$ its Quillen metric relative to $g_{X/S}$ and h (cf. §1).

MAIN THEOREM. *Suppose that (π, X, S) is projective over S. Then, $\|\cdot\|_Q$ extends to a singular Hermitian metric whose curvature current is*

$$c_1(\lambda(E), \| \cdot \|_Q) = \frac{(-1)^{n+1}}{(n+2)!} r(E)\mu(X_0)\delta_0 + \pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}$$

where $\dim X = n + 1$, $r(E) = rank\,E$, δ_0 the Dirac measure supported at 0 and $\mu(X_0)$ the total Milnor number of the singular fiber. Moreover, continuous part of the curvature current, namely $\pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}$, lies in $L^p_{loc}(S)$ for some $p > 1$ depending only on $Sing\,X_0$.

† Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-01, Japan (yosikawa@math.nagoya-u.ac.jp).

We apply Main Theorem to study the asymptotic behavior of analytic torsion in the case of smoothing of IHS. In the sequel of this section, we assume that (E, h) is the trivial line bundle with the trivial metric. One of the consequence of Main Theorem is that principal term of the asymptotic behavior of analytic torsion is determined by the total Milnor number of the singular fiber and determinant of period integrals. (For a related result, see [Fa].) To be precise, let us fix notations.

Let H be a relatively very ample line bundle over X and g_X the Kähler metric of X whose Kähler class coincides with $c_1(H)$. Let $\tau(X_t)$ be the Ray-Singer analytic torsion of the smooth fiber $(X_t, g|_{X_t})$. Fix a base point $t_0 \neq 0$. Let $H_n(X_{t_0}; \mathbb{Z})_{fr}$ be the free part of $H_n(X_{t_0}; \mathbb{Z})$ and fix its integral basis, say $\{a_1, \cdots, a_l\}$ where $l = \dim H^n(X_{t_0}, \mathbb{R})$. Let $\langle \cdot, \cdot \rangle$ be the intersection form on $H_n(X_{t_0}, \mathbb{Z})$ and $\Lambda := (\langle a_i, a_j \rangle)_{1 \leq i, j \leq l}$ the intersection matrix relative to above basis. For any $t \in S^*$, fix an integral basis $\{a_1(t), \cdots, a_l(t)\}$ of $H_n(X_t; \mathbb{Z})_{fr}$ whose intersection matrix is Λ. Let $\omega_{X/S} := \det(TX/S)^*$ be the relative canonical bundle of (π, X, S). Fix a $\mathcal{O}_S$-basis of $\pi_* \omega_{X/S}$, say $\{\omega_1, \cdots, \omega_m\}$ where $m = \dim H^n(X_t, \mathcal{O}_{X_{t_0}})$. Let $\Omega(t) := \left(\int_{a_i(t)} \omega_j(t) \right)_{1 \leq i \leq l, 1 \leq j \leq m}$ be the period matrix relative to $\{a_1, \cdots, a_l\}$ and $\{\omega_1, \cdots, \omega_m\}$.

THEOREM 0.1. *There exist* $r_1, \cdots, r_k \in \mathbb{Q}$ *such that as* $t \to 0$,

$$\tau(X_t) \sim \sum_{r=r_1, \cdots, r_k} \sum_{i,j \geq 0} \sum_{k \geq -h^{n,0} \cdot n} a_{rijk} |t|^r t^i \bar{t}^j (\log |t|)^k$$

where $h^{n,0} = h^{n,0}(X_t)$ $(t \neq 0)$. *Furthermore, there exists a constant* A *such that*

$$\lim_{t \to 0} \frac{\tau(X_t)}{|t|^{2 \frac{(-1)^n}{(n+2)!} \mu(Sing\, X_0)} \det \left({}^t\Omega(t) \Lambda \bar{\Omega}(t) \right)^{(-1)^n}} = \exp(A).$$

As an application of Main Theorem, we show in [Y2] that analytic torsion of smooth theta divisor is represented by the Siegel modular form characterizing the Andreotti-Mayer locus.

1. Determinant bundles and Quillen metrics. In this section, we recall some basic properties of Quillen metrics in the simplest setting. For the general treatment, see [S] and [F].

Let $\pi : X \to S$ be a proper smooth morphism of Kähler manifolds. Let (E, h) be a holomorphic Hermitian vector bundle over X. The determinant bundle $\lambda(E)$ is defined by the following formula:

$$(1.1) \qquad\qquad \lambda(E) := \bigotimes_{q \geq 0} (\det R^q \pi_* \mathcal{O}_X(E))^{(-1)^q}.$$

Let $g_{X/S}$ be a Kähler metric on the relative tangent bundle $TX/S := \ker \pi_*$. Namely $g_{X/S}|_{X_t}$ is Kähler for any fiber $X_t := \pi^{-1}(t)$. Via the Hodge theory, identify the fiber $\lambda_{X_t}(E_t) = \lambda(E)_t$ with the determinant of harmonic forms:

$$(1.2) \qquad \lambda(E)_t = \bigotimes_{q \geq 0} \left(\bigwedge^{max} H^q(X_t, E_t) \right)^{(-1)^q} \cong \bigotimes_{q \geq 0} \left(\bigwedge^{max} \mathcal{H}^{0,q}(X_t, E_t) \right)^{(-1)^q}$$

where $\mathcal{H}^{0,q}(X_t, E_t)$ stands for the harmonic $(0, q)$-forms with coefficients in E. Since $\mathcal{H}^{0,q}(X_t, E_t)$ carries the natural Hermitian structure by the integration of harmonic forms, so does $\lambda(E)_t$ via the identification (1.2). This metric is called the L^2-metric

of $\lambda(E)$ relative to $g_{X/S}$ and h, and is denoted by $\|\cdot\|_{L^2}$. Let $\square_t^{0,q}$ be the $\bar{\partial}$-Laplacian acting on E-valued $(0,q)$-forms on X_t and $\zeta_t^{0,q}(s)$ its zeta function. Then $\zeta_t^{0,q}(s)$ extends to a meromorphic function on the complex plane and is regular at $s = 0$.

DEFINITION 1.1. The Quillen metric of $\lambda(E)$ relative to $g_{X/S}$ and h is defined by

$$\|\cdot\|_Q^2(t) := \tau(X_t, E_t)\|\cdot\|_{L^2}^2(t)$$

where $\tau(X_t, E_t)$ is the Ray-Singer analytic torsion:

$$\tau(X_t, E_t) := \prod_{q \geq 0}(\det \square_t^{0,q})^{(-1)^q q}, \quad \det \square_t^{0,q} := \exp\left(-\frac{d}{ds}\bigg|_{s=0} \zeta_t^{0,q}(s)\right).$$

It is known that $\|\cdot\|_Q$ is a smooth Hermitian metric on $\lambda(E)$ if the morphism is smooth. For smooth Kähler morphisms, curvature and anomaly of the Quillen metric is computed by Bismut-Gillet-Soulé ([B-G-S]).

THEOREM 1.1. *Let* (π, X, S) *be a locally Kähler smooth morphism of complex manifolds. Let* $g_{X/S}$ *be a Kähler metric of* TX/S. *Let* $Td(TX/S, g_{X/S})$ *be the Todd form of* $(TX/S, g_{X/S})$ *and* $ch(E, h)$ *the Chern character of* (E, h). *Then,*

$$c_1(\lambda(\mathcal{O}_X), \|\cdot\|_Q) = \pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}$$

where $\alpha^{(p,p)}$ *stands for the* (p,p)-*part of the form* α.

THEOREM 1.2. *Let* $g_{X/S}$ *and* $g'_{X/S}$ *be Kähler metrics of* TX/S. *Let* $\|\cdot\|_Q$ *and* $\|\cdot\|'_Q$ *be the Quillen metrics of* $\lambda(E)$ *relative to* $g_{X/S}$ *and* $g'_{X/S}$ *respectively. Then,*

$$\log\left(\frac{\|\cdot\|'_Q}{\|\cdot\|_Q}\right)^2 = \pi_*(\widetilde{Td}(TX/S; g_{X/S}, g'_{X/S})ch(E, h))^{(0,0)}$$

where $\widetilde{Td}(TX/S; g_{X/S}, g'_{X/S})$ *is the Bott-Chern class of* TX/S *relative to the Todd form,* $g_{X/S}$ *and* $g'_{X/S}$.

THEOREM 1.3. *Let* h *and* h' *be Hermitian metrics of* E. *Let* $\|\cdot\|_Q$ *and* $\|\cdot\|'_Q$ *be the Quillen metrics of* $\lambda(E)$ *relative to* h *and* h'. *Then,*

$$\log\left(\frac{\|\cdot\|'_Q}{\|\cdot\|_Q}\right)^2 = \pi_*(Td(TX/S, g_{X/S})\widetilde{ch}(E, h))^{(0,0)}$$

where $\widetilde{ch}(E; h, h')$ *is the Bott-Chern class of* E *relative to the Chern character,* h *and* h'.

We also need a theorem of Bismut and Lebeau ([B-L]) which describes the Quillen norm of canonical section in terms of some secondary classes. (For the general cases, see [B-L].)

THEOREM 1.4. *Let* X *be a compact Kähler manifold and* $(Y, g_Y = g_X|_Y)$ *its smooth hypersurface with induced metric. Let* $L = [Y]$ *be the line bundle defined by* Y *and* s_Y *its canonical section, i.e.,* $(s_Y)_0 = [Y]$. *Let* $h_L = \|\cdot\|_L^2$ *be a Hermitian metric of* L *and* $g_{N_{Y/X}}$ *a Hermitian metric of* $N_{Y/X}$ *such that, on* Y, $\|ds_Y\|_{N_{Y/X}^* \otimes L_Y}^2 \equiv 1$ *where* $L_Y := L|_Y$ *and* $ds_Y \in H^0(Y, N_{Y/X}^* \otimes L)$. *Let* $\lambda_X(E \otimes L^{-1})$, $\lambda_X(E)$ *and* $\lambda_Y(E_Y)$ *be the determinant of cohomologies equipped with the Quillen metrics relative to* g_X, g_Y *and* $h_{E \otimes L^{-1}}, h_E, h_{E_Y}$. *Let* σ *be the canonical element of* $\lambda := \lambda_Y(E_Y) \otimes \lambda_X(E)^{-1} \otimes$

$\lambda_X(E \otimes L^{-1})$. *Then,*

$$\log \|\sigma\|^2_{\lambda,Q} = -\int_X Td(TX, g_X) Td^{-1}(L, h_L) ch(E, h) \log \|s\|^2_L$$

$$+ \int_Y Td^{-1}(N_{Y/X}, g_{N_{Y/X}}) \widetilde{Td}(\bar{\mathcal{E}}) ch(E_Y)$$

$$- \int_X Td(TX) R(TX) ch(E) + \int_Y Td(TY) R(TY) ch(E_Y)$$

where R is the Gillet-Soulé genus and $\widetilde{Td}(\bar{\mathcal{E}})$ is the Bott-Chern class relative to the Todd genus and $\bar{\mathcal{E}} = \{\mathcal{E}, (g_Y, g_{TX|_Y}, g_{N_{Y/X}})\} : 0 \to TY \to TX|_Y \to N_{Y/X} \to 0$.

Recall the definition of $\widetilde{Td}(\bar{\mathcal{E}})$. Let $\mathcal{E} : 0 \to E' \xrightarrow{i} E \xrightarrow{v} E'' \to 0$ be an exact sequence of vector bundles over a complex manifold M with Hermitian metrics $h_{E'}, h_E, h_{E''}$. Define a vector bundle over $M \times \mathbb{P}^1$ by

(1.3) $$\tilde{E} = \{(e, e'') \in E \oplus E''(-1); v(e) + \sigma_\infty \otimes e'' = 0\}$$

where $E =: pr_1^* E$, $E''(-1) := pr_1^* E'' \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$, σ_0 and σ_∞ the canonical section of $\mathcal{O}_{\mathbb{P}^1}(1)$ such that $(\sigma_0)_0 = 0$ and $(\sigma_\infty)_0 = \infty$. In the homogeneous coordinate, $\sigma_0 = z_0$ and $\sigma_\infty = z_1$ where $z = z_0/z_1$ is the inhomogeneous coordinate of $\mathbb{P}^1$. Let

(1.4) $$h'_{\tilde{E}}((e, e''), (e, e'')) := h_E(e, e) + h_{E''}(\sigma_0 \otimes e'', \sigma_0 \otimes e'')$$

be a Hermitian metric of $\tilde{E}$. Consider the following exact sequence of vector bundles on $M \times \mathbb{P}^1$: $0 \to E' \xrightarrow{i \oplus 0} \tilde{E} \xrightarrow{p_2} E''(-1) \to 0$ where $p_2 : \tilde{E} \ni (e, e'') \to e'' \in E''(-1)$. Then,

(1.5) $$\widetilde{Td}(\bar{\mathcal{E}}) = \int_{\mathbb{P}^1} \log |z|^2 \cdot Td(\tilde{E}, h'_{\tilde{E}}).$$

2. Families of hypersurfaces and relative Todd form.

Let S be a complex manifold and $D = \{z \in \mathbb{C}; |z| < 1\}$ be the unit disc. Through this section, S is not necessary the unit disc. Put $U := D^{n+1} \times S$. Let $F(z, s) \in \mathcal{O}(U)$ be a holomorphic function such that $F(\cdot, s)$ is not a constant function for any $s \in S$ where $z = (z_0, \cdots, z_n) \in D^{n+1}$. Set

(2.1) $$X := \{(z, s) \in U; F(z, s) = 0\} \subset U,$$

(2.2) $$p : U \ni (z, s) \to s \in S, \quad \pi := p|_X : X \to S,$$

(2.3) $$\Sigma := \{(z, s) \in U; d_{\mathbb{C}^n} F := \sum_{i=0}^n \frac{\partial F}{\partial z_i} dz^i = 0\}.$$

Let $T\mathbb{C}^{n+1} := \ker p_*|_X$ and $TX/S := \ker \pi_*$ be the vector bundles associated to p and π respectively defined on $U - \Sigma$. Put $N_{X/\mathbb{C}^{n+1}} := \ker p_*/\ker \pi_*$ and consider the following exact sequence:

(2.4) $$0 \longrightarrow TX/S \longrightarrow T\mathbb{C}^{n+1} \longrightarrow N_{X/\mathbb{C}^{n+1}} \longrightarrow 0.$$

Let $g_{\mathbb{C}^{n+1}} = \sum_{i=0}^n |dz_i|^2$ be the Euclidean metric of $T\mathbb{C}^{n+1}$ which gives the smooth orthogonal splitting $T\mathbb{C}^{n+1} = TX/S \oplus (TX/S)^\perp$. Put $g_{X/S} = g_{\mathbb{C}^{n+1}}|_{TX/S}$ for the induced metric of TX/S. $N_{X/\mathbb{C}^{n+1}}$ is equipped with the Hermitian metric $g_N = g_{(TX/S)^\perp}$ by the C^∞-identification $N_{X/\mathbb{C}^{n+1}} \cong (TX/S)^\perp$. Let $R_{X/S}$ be the curvature form of $(TX/S, g_{X/S})$ defined on $U - \Sigma$.

PROPOSITION 2.1. *On $U - \Sigma$, it holds that $[Td(R_{X/S})]^{(n+1,n+1)} \equiv 0$.*
For the proof, we need the followings.

LEMMA 2.1. *Let V be a vector space over $\mathbb{C}$. For any $A = (a_0, \cdots, a_n), B = (b_0, \cdots, b_n) \in V \otimes \mathbb{C}^{n+1}$ $(a_i, b_j \in V)$ and indeterminate x,*

$$\det(I + x{}^tB \wedge A) = (1 + x\, A \wedge {}^tB)^{-1}.$$

Here ${}^tB \wedge A = (b_i \wedge a_j) \in M(n, \wedge^*V)$ and $A \wedge {}^tB = \sum a_i \wedge b_j \in \wedge^*V$.

Proof. Since $(A \wedge {}^tB)^k = -Tr({}^tB \wedge A)^k$, $(A \wedge {}^tB)^l = 0$ $(l > \dim V)$, and

$$(2.5) \qquad \det(I - tX) = \exp\left(-\sum_{i=1}^{\infty} \frac{Tr(X^i)}{i} t^i\right)$$

for any square matrix X, we get the formula by putting $t = -x$ and $X = {}^tB \wedge A$. $\qquad \square$

LEMMA 2.2. *For any $F(x) = 1 + \sum_{i \geq 1} a_i x^i \in \mathbb{C}[[x]]$ and $A, B \in V \otimes \mathbb{C}^n$,*

$$\det F({}^tB \wedge A) = F(A \wedge {}^tB)^{-1}.$$

Proof. Put $F({}^tB \wedge A) = I + y{}^tB \wedge A$ and $y = \sum_{i \geq 1} a_i(A \wedge {}^tB)^{i-1}$. By Lemma 2.1, $\det F({}^tB \wedge A) = (1 + y\, A \wedge {}^tB)^{-1} = F(A \wedge {}^tB)^{-1}$. $\qquad \square$

Proof of Proposition 2.1. Let R_N be the curvature form of $(N_{X/\mathbb{C}^{n+1}}, g_N)$. Let $A \in A_U^{1,0}(Hom\,(TX/S, (TX/S)^\perp))$ be the second fundamental form of (2.3). By the Gauss-Codazzi equation ([Ko], I (6.1)), the curvature form of $(T\mathbb{C}^{n+1}, g_{\mathbb{C}^{n+1}})$ is represented by

$$(2.6) \qquad R_{T\mathbb{C}^{n+1}} = \begin{pmatrix} R_{X/S} - A^* \wedge A & -D'A^* \\ D''A & R_N - A \wedge A^* \end{pmatrix}.$$

Since $(T\mathbb{C}^{n+1}, g_{\mathbb{C}^{n+1}})$ is a flat vector bundle, we find $R_{T\mathbb{C}^{n+1}} \equiv 0$ which implies

$$(2.7) \qquad R_{X/S} = A^* \wedge A, \quad R_N = A \wedge A^*, \quad D''A = D'A^* = 0.$$

Applying Lemma 2.2 for $R_{X/S}$ and $F(x) = Td(x) = x/(1 - e^{-x})$, we get

$$(2.8) \qquad Td(R_{X/S}) = Td(R_N)^{-1}.$$

In particular,

$$(2.9) \qquad [Td(R_{X/S})]^{(n+1,n+1)} = (-1)^{n+1} \frac{1}{(n+2)!} \left(\frac{i}{2\pi} R_N\right)^{n+1}.$$

Consider the dual of (2.3):

$$(2.10) \qquad 0 \longrightarrow N^*_{X/\mathbb{C}^{n+1}} \longrightarrow \Omega^1_{\mathbb{C}^{n+1}} \longrightarrow \Omega^1_{X/S} \longrightarrow 0.$$

As $N^*_{X/\mathbb{C}^{n+1}}$ is generated by $d_{\mathbb{C}^{n+1}}F = \sum_{i=0}^{n} \frac{\partial F}{\partial z_i}(z, s)dz^i$, R_N is represented by

$$(2.11) \qquad R_N = -R^*_{N_{X/\mathbb{C}^{n+1}}} = -\bar{\partial}\partial \log\|d_{\mathbb{C}^{n+1}}F\|^2, \quad \|d_{\mathbb{C}^{n+1}}F\|^2 = \sum_{i=0}^{n} \left|\frac{\partial F}{\partial z_i}(z, s)\right|^2.$$

Let $\nu : U - \Sigma \to \mathbb{P}^n$ be the Gauss map:

$$(2.12) \qquad \nu : U - \Sigma \ni (z, s) \to \left[\frac{\partial F}{\partial z_0}(z, s) : \cdots : \frac{\partial F}{\partial z_n}(z, s)\right] \in \mathbb{P}^n.$$

By (2.11), we find

$$(2.13) \qquad \frac{i}{2\pi} R_N = \nu^* \omega_{\mathbb{P}^n}, \quad \omega_{\mathbb{P}^n} = \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2,$$

which combined with (2.9) yields

$$(2.14) \qquad [Td(R_{X/S})]^{(n+1,n+1)} = \frac{(-1)^{n+1}}{(n+2)!} \nu^* \omega_{\mathbb{P}^n}^{n+1} = 0. \qquad \square$$

3. Estimates of anomaly. Let (π, X, S) be the smoothing of IHS as before. Let g_X and g'_X be Kähler metrics of X. The induced metric on TX/S are denoted by $g_{X/S}$ and $g'_{X/S}$. Let E be a holomorphic vector bundle over X and h, h' its Hermitian metrics. We prove the following theorem in this section.

THEOREM 3.1. *There exists* $\alpha > 0$, $r_1, \cdots, r_a \in \mathbb{Q}_{\geq 0}$ *and* $p > 1$ *depending only on* $Sing X_0$ *such that*

(1) $\pi_*(\widetilde{Td}(TX/S; g_{X/S}, g'_{X/S}) ch(E, h))^{(0,0)}$ *lies in* $C^\alpha(S)$ *and admits the following asymptotic expansion as* $t \to 0$:

$$\pi_*(\widetilde{Td}(TX/S; g_{X/S}, g'_{X/S}) ch(E, h))^{(0,0)} \sim \sum_r \sum_{i,j \geq 0} \sum_{0 \leq k \leq n} a_{rijk} |t|^r t^i \bar{t}^j (\log |t|)^k,$$

(2) $\pi_*(Td(TX/S, g_{X/S}) ch(E, h))^{(1,1)}$ *lies in* $L^p_{loc}(S)$,

(3) $\pi_*(Td(TX/S, g_{X/S}) \widetilde{ch}(E; h, h'))^{(0,0)}$ *lies in* $C^\alpha(S)$ *and admits the following asymptotic expansion as* $t \to 0$:

$$\pi_*(Td(TX/S, g_{X/S}) \widetilde{ch}(E; h, h'))^{(0,0)} \sim \sum_r \sum_{i,j \geq 0} \sum_{0 \leq k \leq n} b_{rijk} |t|^r t^i \bar{t}^j (\log |t|)^k.$$

(4) $\pi_*(Td(TX/S, g_{X/S}) \widetilde{ch}(E, h))^{(1,1)}$ *lies in* $L^p_{loc}(S)$.

For the proof, we need the following theorem due to Barlet ([Ba]).

THEOREM 3.2. *Let* X *be a complex manifold of dimension* $n+1$ *and* K *a compact subset. Let* $\phi \in \Omega_0^{n,n}(K)$ *be a smooth* (n,n)-*form supported in* K. *Let* $f : X \to S$ *be a holomorphic function. Then,* $f_*(\phi)$ *is a Hölder continuous function on* S. *Furthermore, there exists* $r_1, \cdots, r_k \in \mathbb{Q} \cap [0, 2)$ *such that, as* $t \to 0$,

$$f_*(\phi)(t) \sim \sum_{r=r_1, \cdots, r_k} \sum_{i,j \geq 0} \sum_{0 \leq k \leq n} a_{rijk} |t|^r t^i \bar{t}^j (\log |t|)^k.$$

Proof of Theorem 3.1. Since the proof of (3) and (4) is similar to that of (1) and (2), we only prove the former. Furthermore, we assume E is trivial and $h \equiv 1$ for simplicity because the problem is of local nature and the proof of the case $h \not\equiv 1$ is similar. Let $p : \mathbb{P}^\vee(TX) \to X$ be the projective bundle whose fiber at x, say $\mathbb{P}^\vee(TX)_x = \mathbb{P}^\vee(T_x X)$, is the set of all hyperplanes in $T_x X$. Let F_X be the universal vector bundle over $\mathbb{P}^\vee(TX)$. Let H_x be a hyperplane in $T_x X$ and $[H_x]$ the point in $\mathbb{P}^\vee(T_x X)$ corresponding to H_x. Then, $(F_X)_{(x,[H_x])} = H_x$. By the construction, we obtain the following exact sequence of vector bundles over $\mathbb{P}^\vee(TX)$:

$$(3.1) \qquad 0 \longrightarrow F_X \longrightarrow p^* TX \longrightarrow N_X \longrightarrow 0$$

where $N_X = \mathcal{O}_{\mathbb{P}^\vee(TX)}(1)$. Define Hermitian metrics on F_X by $g_{F_X} := p^* g_X|_{F_X}$ and $g'_{F_X} := p^* g'_X|_{F_X}$. Let $\nu : TX/S \to F_X$ be the Gauss map:

$$(3.2) \qquad \nu : T_x X/S \ni (x, v) \to (x, [T_x X/S], v) \in F_{X,(x,[T_x X/S])}.$$

The underlying map from X to $\mathbb{P}^\vee(TX)$ is also denoted by ν. Then, $\nu : X \to \mathbb{P}^\vee(TX)$ is a rational map which is holomorphic on $X - Sing\,X_0$. From the universal property of F_X, it follows that $(TX/S, g_{X/S}) = \nu^*(F_X, g_{F_X})$. In particular, we get
(3.3)
$$Td(TX/S, g_{X/S}) = \nu^* Td(F_X, g_{F_X}), \quad \widetilde{Td}(TX/S; g_{X/S}, g'_{X/S}) = \nu^* \widetilde{Td}(F_X; g_{F_X}, g'_{F_X}).$$

Due to Hironaka, there exists a proper modification $f : Y \to X$ such that
(1) $f : Y - f^{-1}(Sing\,X_0) \to X - Sing\,X_0$ is an isomorphism,
(2) $(\pi \circ f)^{-1}(0) = \sum_i k_i D_i$ $(k_i > 0)$ is a divisor with simple normal crossing,
(3) There exists a holomorphic map $\phi : Y \to \mathbb{P}^\vee(TX)$ such that $\phi = \nu \circ f$ on $Y - f^{-1}(Sing\,X_0)$.
(1) Put $q := \pi \circ f : Y \to S$. It follows from (3.3) that

(3.4) $$\pi_*(\widetilde{Td}(TX/S; g_{X/S}, g'_{X/S}))^{(0,0)} = q_*(\phi^* \widetilde{Td}(F_X; g_{F_X}, g'_{F_X}))^{(0,0)}.$$

Since $\phi^* \widetilde{Td}(F_X; g_{F_X}, g'_{F_X})$ is a smooth (n,n)-form on Y, we get the assertion by Theorem 3.2.
(2) By the assumption, for any $w \in Y$, there exist local coordinates $(U_w, (w_0, \cdots, w_n))$ such that $q(w) = w_0^{k_0} \cdots w_l^{k_l}$ $(k_i \geq 1, l \leq n)$. Set

(3.5) $$\tau := \frac{1}{l} \sum_{i=1}^l \frac{1}{k_i} (-1)^{i-1} w_i dw_0 \wedge \cdots \wedge dw_{i-1} \wedge dw_{i+1} \wedge \cdots \wedge dw_n.$$

Let ρ_{U_w} be a smooth function supported in U_w. Since $\phi^* Td(TX/S, g_{X/S})$ is a smooth $(n+1, n+1)$-form on U_w, there exists $h \in C_0^\infty(U_w)$ such that

(3.6) $$\rho_{U_w} \phi^* Td(TX/S, g_{X/S}) = h\, dw_0 \wedge d\bar{w}_0 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n.$$

Since $q^*(dt/t) \wedge \tau = dw_0 \wedge \cdots \wedge dw_n$, we get

(3.7)
$$q_*(\rho_{U_w} \phi^* Td(TX/S, g_{X/S}))^{(1,1)}(t) = q_*(h\, dw_0 \wedge d\bar{w}_0 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n)(t)$$
$$= \frac{dt \wedge d\bar{t}}{|t|^2} \int_{q^{-1}(t) \cap U_w} h\, \tau \wedge \bar{\tau}.$$

By Theorem 3.2, $\pi_*(\rho_{U_w} Td(TX/S, g_{X/S}))^{(1,1)}$ has the asymptotic expansion
(3.8)
$$q_*(\rho_{U_w} \phi^* Td(TX/S, g_{X/S}))^{(1,1)}(t) \sim \frac{dt \wedge d\bar{t}}{|t|^2} \sum_r \sum_{i,j \geq 0} \sum_{0 \leq k \leq n} c_{rijk} |t|^r t^i \bar{t}^j (\log|t|)^k.$$

Put $S_\epsilon = \{t \in S; |t| < \epsilon\} \Subset S$. Since
(3.9)
$$\left| \int_{S_\epsilon} q_*(\rho_{U_w} \phi^* Td(TX/S, g_{X/S}))^{(1,1)} \right| = \left| \int_{q^{-1}(S_\epsilon)} \rho_{U_w} \phi^* Td(TX/S, g_{X/S}))^{(1,1)} \right| < \infty,$$

$c_{rijk} \neq 0$ if and only if $r + i + j > 0$. In particular, there exists $p > 1$ such that $\pi^*(Td(TX/S, g_{X/S}))^{(1,1)} \in L_{loc}^p(S)$. This proves (2). $\square$

4. An analytic characterization of Milnor number.
Let D be the unit disc in $\mathbb{C}$. We prove the following theorem in this section.

THEOREM 4.1. *Let $f(z)$ be a holomorphic function defined on the polydisc D^{n+1} centered at $0 \in \mathbb{C}^{n+1}$ such that the origin is an isolated critical point of f. Set*

$Y_t := D^{n+1} \cap f^{-1}(t)$ *for the fiber of* f. *Let* $\mu(f)$ *be the Milnor number of* $(Y_0, 0)$. *Then,*

$$\int_{Y_t} (\frac{i}{2\pi}\partial\bar{\partial}\log\|df\|^2)^n \log\|df\|^2 = \mu(f)\log|t|^2 + O(1) \quad (t \to 0).$$

Proof. At first, let us consider the case that $f(z)$ has non-degenerate singularity at $z = 0$. There exist local coordinates $(U, (w_0, \cdots, w_n))$ around the critical point of f such that $f(w) = w_0^2 + \cdots + w_n^2$. Let $g_1 := \sum_{i=0}^{n}|dw_i|^2$ be a flat Kähler metric on U. Various norms relative to g_1 is denoted by $\|\cdot\|_1$. By Stokes's theorem,

$$
\begin{aligned}
I(t) &:= \int_{Y_t} \left(\frac{i}{2\pi}\partial\bar{\partial}\log\|df\|^2\right)^n \log\|df\|^2 \\
(4.1) \qquad &= \int_{Y_t} \left(\frac{i}{2\pi}\partial\bar{\partial}\log\|df\|_1^2\right)^n \log\|df\|_1^2 \\
&\quad + \int_{Y_t} \log\frac{\|df\|^2}{\|df\|_1^2} \cdot \sum_{i+j=n}(\frac{i}{2\pi}\partial\bar{\partial}\log\|df\|^2)^i(\frac{i}{2\pi}\partial\bar{\partial}\log\|df\|_1^2)^j + O(1)
\end{aligned}
$$

where the $O(1)$-term comes from the integral along the boundary. Let $q : V \to D^g$ be the blowing-up at 0. Then, $q^*\log\frac{\|df\|^2}{\|df\|_1^2}$ extends to a smooth function on V. Therefore, the second integral of (4.1) is a continuous function by Theorem 3.2. Since $\|df\|_1^2(w) = \|w\|^2$, we get

$$(4.2) \qquad I(t) = \int_{w_0^2+\cdots+w_n^2=t} (\frac{i}{2\pi}\partial\bar{\partial}\log\|w\|^2)^n \log\|w\|^2 + O(1) = \log|t|^2 + O(1).$$

This proves the case of non-degenerate critical point.

Consider the general case. Let $F(z, s)$ be a one parameter deformation of $f(z)$ defined on D^{n+2} such that $F(z, 0) = f(z)$. Put $f_s(z) := F(z, s)$. Let $Y_{(t,s)} := f_s^{-1}(t)$ be the fiber of the map $\pi : (z, s) \to (f_s(z), s)$. Let Δ be the divisor of D^{n+2} over which the fiber of π is singular. We may assume that f_s $(s \neq 0)$ has only finitely many non-degenerate critical points by choosing $F(x, s)$ as the Morsification of $f(z)$. Let ρ be a cut-off function on D^{n+2} such that $\rho \equiv 1$ on a neighborhood of 0, say W, and $\rho \equiv 0$ on a neighborhood of ∂D^{n+2}. Put

$$(4.3) \qquad \phi(t, s) := \int_{Y_{(t,s)}} \rho \cdot (\frac{i}{2\pi}\partial\bar{\partial}\log\|dF\|^2)^n \log\|dF\|^2, \quad \chi := \frac{i}{2\pi}\partial\bar{\partial}\phi.$$

Then,

$$(4.4) \quad \chi = \frac{i}{2\pi}\int_{Y_{(t,s)}-W} \rho \cdot (\frac{i}{2\pi}\partial\bar{\partial}\log\|dF\|^2)^{n+1} + \int_{Y_{(t,s)}} \partial\rho \wedge F_1 + \bar{\partial}\rho \wedge F_2 + \partial\bar{\partial}\rho \wedge F_3$$

for some smooth forms F_1, F_2, F_3 because $(\partial\bar{\partial}\log\|dF\|^2)^{n+1} \equiv 0$ on W (cf. (2.14)). Thus χ extends to a smooth real (1,1)-form over D^2. Since $\phi|_\gamma = \log|t|^2 + \psi$ ($\psi \in C^\infty(\gamma)$) for any holomorphic curve γ which intersects transversally to Δ by (4.2), applying the argument of [B-B, §10.2], it follows that

$$(4.5) \qquad\qquad \frac{i}{2\pi}\partial\bar{\partial}\phi = \delta_\Delta - \chi$$

in the sense of current over D^2. Since the multiplicity of Δ at 0 is $\mu(f)$ (cf. [Lo]),

restricting (4.5) to the disc $D \times \{0\}$, we get

$$(4.6) \qquad \frac{i}{2\pi} \partial \bar{\partial}(\phi|_{D \times \{0\}}) = mult_0 \Delta \cdot \delta_0 - \chi|_{D \times \{0\}} = \mu(f)\delta_0 - \chi|_{D \times \{0\}}.$$

The assertion follows from (4.6).   □

5. An explicit formula for the Bott-Chern form. Let $X = D^{n+1}$ be the polydisc and $Y = \{z \in X; \theta(z) = 0\}$ its smooth hypersurface defined by $\theta \in \mathcal{O}(X)$ through this section. Consider the following exact sequence of vector bundles over Y:

$$(5.1) \qquad \mathcal{E} : 0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow N_{Y/X} \longrightarrow 0.$$

Let $g_X = \sum_{i=0}^n |dz_i|^2$ be the flat metric on TX and g_Y the induced metric on TY. Let g_N be a Hermitian metric of $N_{Y/X}$ induced by the C^∞-identification $(TY)^\perp \cong N_{Y/X}$. Let g'_N be another metric on N such that $g'_{N*}(d\theta, d\theta) \equiv 1$ where $g'_{N*} := (g'_N)^{-1}$ and $d\theta$ the generator of N^*. Set $\bar{\mathcal{E}} := (\mathcal{E}, (g_{TY}, g_X, g'_N))$. We prove the following theorem in this section.

THEOREM 5.1.

$$\widetilde{Td}(\bar{\mathcal{E}}) = \frac{1 - Td(\nu^* c_1(H))}{\nu^* c_1(H)} \log \|d\theta\|^2 + \int_0^1 \log \frac{1-s}{s} \frac{d}{ds} \frac{Td^{-1}(s\,\nu^* c_1(H))}{\nu^* c_1(H)} ds.$$

where $\nu : D^{n+1} \ni z \to \left(\frac{\partial \theta}{\partial z_0}(z) : \cdots : \frac{\partial \theta}{\partial z_n}(z) \right) \in \mathbb{P}^n$ is the Gauss map relative to θ.

Before proving Theorem 5.1, let us verify that $\bar{\mathcal{E}}$ satisfies Bismut's condition (A) (cf. [B1]). Put $L = [Y]$ for the line bundle on X defined by the divisor Y and $L_Y := L|_Y$ its restriction to Y. As L has the canonical section s such that $(s)_0 = Y$, $s_1 := \theta^{-1}s$ is a nowhere vanishing section trivializing L. Let h_L be the Hermitian metric of L such that $h_L(s,s) = |\theta|^2$ and h_{L_Y} its restriction to L_Y. Clearly $h_L(s_1, s_1) \equiv 1$ and the Chern form of (L, h_L) vanishes on X. Now the normal bundle $N_{Y/X}$ is identified with L_Y by the following map:

$$(5.2) \qquad \otimes d\theta : N_{Y/X} \ni \left[\frac{\partial}{\partial z_i} \right] \to \frac{\partial \theta}{\partial z_i} s_1 \in L_Y.$$

Let $p : N_{Y/X} \to Y$ be the projection. By the construction, the following is clear.

LEMMA 5.1. $\otimes d\theta : (p^* L_Y^{-1}, p^* h_{L_Y}^{-1}) \to (p^* N_{Y/X}, p^* g'_{N*})$ is an isometry. Namely, $\|d\theta \otimes s_1\|'_{N^* \otimes L_Y} \equiv 1$ where $\| \cdot \|'_{N^* \otimes L_Y} := g'_{N*} \otimes h_{L_Y}$.

In the sequel, identify $s_1 = 1$. We compute $\widetilde{Td}(\bar{\mathcal{E}})$ by using the formula (1.5). The Gauss map induces the following metric on L_Y:

$$(5.3) \qquad \nu^* h_H \left(\frac{\partial \theta}{\partial z_i}, \frac{\partial \theta}{\partial z_i} \right) = \frac{|\frac{\partial \theta}{\partial z_i}|^2}{\|d\theta\|^2}, \quad \|d\theta\|^2 := \sum_{i=0}^n \left| \frac{\partial \theta}{\partial z_i} \right|^2$$

where h_H is the standard metric of $H = \mathcal{O}_{\mathbb{P}^n}(1)$. Its Chern form is given by

$$(5.4) \qquad c_1(L, \nu^* h_H) = \nu^* c_1(H), \quad c_1(H) = \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2.$$

Define $\widetilde{TX}$ and $g'_{\widetilde{TX}}$ in the same manner as (1.3) and (1.4).

LEMMA 5.2. Put $u := 1/z$ for the coordinate of $\mathbb{P}^1$ near ∞. Then,

$$g'_{\widetilde{TX}} = g_{TY} \oplus \left(1 + \frac{\|d\theta\|^2}{|u|^2} \right) g_N$$

Proof. By (1.4),

$$(5.5) \qquad g'_{\widetilde{TX}} = g_Y \oplus (g_N + \frac{1}{|u|^2} g'_N).$$

The assertion follows from the following:

$$(5.6) \qquad \frac{g'_N}{g_N} = \frac{\|d\theta\|^2}{\|d\theta\|'^2_{N^* \otimes L_Y}} = \|d\theta\|^2. \qquad \square$$

Consider the following exact sequence of Hermitian vector bundles over $\mathbb{P}^n$:

$$(5.7) \qquad 0 \longrightarrow E \longrightarrow \mathbb{C}^{n+1} \longrightarrow H = \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

where E is the universal bundle over $\mathbb{P}^n$, i.e., $E_u = \{z \in \mathbb{C}^{n+1}; \sum_{i=0}^n u_i z_i = 0\}$ for $u \in \mathbb{P}^n$. $\mathbb{C}^{n+1}$ is equipped with the standard Euclidean metric and E and H are equipped with the induced metric from (5.7). Then, the following is clear by the definition:

$$(5.8) \qquad (TY, g_{TY}) = \nu^*(E, h_E), \quad (N_{Y/X}, h_N) = \nu^*(H, h_H),$$

which combined with Lemma 5.2 yields the following lemma.

LEMMA 5.3. *Put* $\phi^2 := \|d\theta\|^2$. *Then,*

$$g'_{\widetilde{TX}} = \nu^* h_E \oplus \left(1 + \frac{\phi^2}{|u|^2}\right) \nu^* h_H.$$

Put $a(u,z) := 1 + |u|^{-2} \phi^2(z)$. Let A be the second fundamental form of the exact sequence (5.7). As $(\mathbb{C}^{n+1}, h_{\mathbb{C}^{n+1}})$ is flat, we have the following formula by the Gauss-Codazzi equation.

LEMMA 5.4.

$$R_{\widetilde{TX}} = \begin{pmatrix} (1 - \frac{1}{a})\nu^* R_E & \partial \log a \wedge \nu^* A^* \\ -\frac{1}{a}\bar{\partial} \log a \wedge \nu^* A & (1 - \frac{1}{a})\nu^* R_H + \bar{\partial}\partial \log a \end{pmatrix}$$

where $R_{\widetilde{TX}}$, R_E *and* R_H *are curvatures of* $(\widetilde{TX}, h'_{\widetilde{TX}})$, (E, h_E) *and* (H, h_H).
Set

$$(5.9) \quad v := \phi u, \quad B := \nu^* A, \quad x := \partial \log \frac{\phi}{|v|^2}, \quad y := \frac{|v|^2}{1 + |v|^2} \bar{\partial} \log \frac{|v|^2}{\phi}, \quad \omega := \bar{\partial}\partial \phi.$$

Then, similar to (2.7), we get $B \wedge B^* + \omega = \nu^* R_H + \omega = 0$. By Lemma 5.4, we have the following formula.

PROPOSITION 5.1. *Put* $T := \begin{pmatrix} B^* \wedge B & x \wedge B^* \\ y \wedge B & xy \end{pmatrix}$. *Then,*

$$R := R_{\widetilde{TX}} = \frac{1}{1 + |v|^2} \cdot T.$$

LEMMA 5.5. *Set* $T_\epsilon = \begin{pmatrix} B^* \wedge B & \epsilon x \wedge B^* \\ \epsilon y \wedge B & \epsilon^2 xy \end{pmatrix} = \begin{pmatrix} B^* \\ \epsilon y \end{pmatrix} (B \quad -\epsilon x)$. *Then,*

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} Tr\, T_\epsilon^k = 0, \quad \frac{1}{2}\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} Tr\, T_\epsilon^k = (-1)^{k-1} k\, xy\, \omega^{k-1}.$$

Proof. As in Lemma 2.1, we get

$$(5.10) \qquad Tr\, T_\epsilon^k = (-1)^{k-1}(\epsilon^2\, xy + \omega)^k.$$

The assertion follows from (5.10). □

LEMMA 5.6. *Let $F(X)$ be a $GL(n+1,\mathbb{C})$-invariant polynomial. Then,*

$$\int_{\mathbb{P}^1} F\left(\frac{i}{2\pi}R\right) = \frac{2\pi i}{\omega}\left\{F\left(\frac{i}{2\pi}\begin{pmatrix}\nu^* R_E & 0 \\ 0 & 0\end{pmatrix}\right) - F(0)\right\}.$$

Proof. Let $F(x_1, \cdots, x_{n+1})$ be a polynomial such that $F(X) = F(Tr\,X, \cdots, Tr\,X^{n+1})$. By Lemma 5.5, we find

(5.11)
$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(t\,T_\epsilon) = 0, \quad \left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} F(t\,T_\epsilon) = \frac{2txy}{\omega} \cdot \frac{d}{dt}F(Tr(t\,T_0), \cdots, Tr(t\,T_0)^{n+1}).$$

Since $x^2 = y^2 = 0$, the coefficients of ϵ^k in the Taylor expansion of $F(t \cdot T_\epsilon)$ vanishes for $k > 2$ and we have

(5.12)
$$F(t\,T_\epsilon) = F(t\,T_0) + t\frac{xy}{\omega}\frac{d}{dt}F(t\,T_0)\epsilon^2.$$

Set $t = \frac{i}{2\pi}/(1 + |v|^2)$ and $\epsilon = 1$. Since

(5.13)
$$t\,T_1 = \frac{i}{2\pi}R, \quad t\,T_0 = \frac{i}{2\pi}\frac{1}{1+|v|^2}\begin{pmatrix}\nu^* R_E & 0 \\ 0 & 0\end{pmatrix}, \quad txy = \omega_{\mathbb{P}^1},$$

where $\omega_{\mathbb{P}^1}$ is the Fubini-Study form of $\mathbb{P}^1$, we obtain

(5.14)
$$\int_{\mathbb{P}^1} F\left(\frac{i}{2\pi}R\right) = \int_{\mathbb{P}^1} \omega_{\mathbb{P}^1}\frac{2\pi i}{\omega}\left.\frac{d}{dt}\right|_{t=\frac{1}{1+|v|^2}} F\left(t\frac{i}{2\pi}\begin{pmatrix}\nu^* R_E & 0 \\ 0 & 0\end{pmatrix}\right)$$

$$= \frac{2\pi i}{\omega}\int_0^1 \frac{d}{dx}F\left(x\frac{i}{2\pi}\begin{pmatrix}\nu^* R_E & 0 \\ 0 & 0\end{pmatrix}\right)dx$$

$$= \frac{2\pi i}{\omega}\left\{F\left(\frac{i}{2\pi}\begin{pmatrix}\nu^* R_E & 0 \\ 0 & 0\end{pmatrix}\right) - F(0)\right\}. \quad □$$

Proof of Theorem 5.1. Putting $F(X) = Td(X)$ in Lemma 5.6, it follows from Lemma 2.2

(5.15)
$$\int_{\mathbb{P}^1} Td\left(\frac{i}{2\pi}R\right) = \frac{Td^{-1}(\nu^* c_1(H)) - 1}{\nu^* c_1(H)}.$$

In the same manner as (5.14), we obtain

(5.16)
$$\int_{\mathbb{P}^1} \log|v|^2 Td\left(\frac{i}{2\pi}R\right) = \frac{1}{\nu^* c_1(H)}\int_0^1 ds \log\frac{1-s}{s}\frac{d}{ds}Td^{-1}(s\,\nu^* c_1(H)).$$

By (1.5) and (5.9), we have

(5.18)
$$\widetilde{Td}(\bar{\mathcal{E}}) = -\log\phi\int_{\mathbb{P}^1} Td\left(\frac{i}{2\pi}R\right) - \int_{\mathbb{P}^1} \log|v|^2 Td\left(\frac{i}{2\pi}R\right),$$

which combined with (5.15) and (5.16) yields the desired formula. □

6. Proof of main theorem for globalizable families.

DEFINITION 6.1. Let E be a holomorphic vector bundle on a complex manifold X. The pair (X, E) is said to be *globalizable*, if X is embeddable into a projective algebraic manifold of the *same dimension* to which E extends as a coherent sheaf.

Let us use the same notations as in section 0.

THEOREM 6.1. *Suppose that the pair (X, E) is globalizable. Then, $\| \cdot \|_Q$ is a singular Hermitian metric on $\lambda(E)$ whose curvature current is*

$$c_1(\lambda(E), \| \cdot \|_Q) = \frac{(-1)^{n+1}}{(n+2)!} r(E)\mu(Sing\, X_0)\delta_0 + \pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}$$

where $\pi_*(Td(TX/S)ch(E, h))^{(1,1)}$ lies in $L^p_{loc}(S)$ for some $p > 1$.

For the proof, we need the following Lemma.

LEMMA 6.1. *Let D be the unit disc, $\chi \in L^\infty(D)$ and $f \in C^\infty(D)$. Suppose that $\Delta\chi = f$ on $D - \{0\}$ in the sense of distribution. Then $\chi \in C^\infty(D)$ and above equality holds on D in the classical sense.*

Proof. Since $f \in C^\infty(D)$, there exists $g \in C^\infty(D)$ such that $\Delta g = f$. Then $\Delta(\chi - g) = 0$ on $D - \{0\}$ in the sense of distribution. As $\chi - g$ is bounded on a neighborhood of 0, it extends to a harmonic function on D by the Riemann extension theorem. Therefore χ is smooth on D and satisfies $\Delta\chi = f$. $\square$

Proof of Theorem 6.1. The proof is similar to that of Bismut [B2]. At first, we treat the case that $X \subset X'$ and E extends to a vector bundle E' over X'. By Theorem 1.2, 1.3 and 3.1, we may asssume that the curvatures of (TX, g_X) and (E, h_E) vanish on a neighborhood of $Sing\, X_0$ and that g_X (resp. h_E) extends to a Kähler (resp. Hermitian) metric $g_{X'}$ (resp. $h_{E'}$) on TX' (resp. E'). Since $\{X_t\}_{t \in S}$ is a family of divisors in X', $L := \{L_t\}_{t \in S}$ ($L_t := [X_t]$) is a holomorphic family of line bundles over X' and defines a line bundle over $X' \times S$. Fix a smooth function h on S such that $h(u) = |u|^2$ for $|u| < \frac{1}{4}$ and $h(u) = 1$ for $|u| > \frac{1}{2}$. Let $s \in H^0(X' \times S, L)$ be the canonical section of L such that $(s_t)_0 = X_t$ where $s_t := s|_{X' \times \{0\}}$. Define a Hermitian metric h_{L_t} on L_t such that $h_{L_t}(s_t, s_t)(z) = h(\pi(z) - t)$ for $z \in X$ and $h_{L_t}(s_t, s_t) \equiv 1$ for $z \in X' - X$. Then, the family of metrics $h_L := \{h_{L_t}\}$ becomes a smooth Hermitian metric on L. Put $\tilde{s}_t(z) := (\pi(z) - t)^{-1}s_t(z)$. Then $\tilde{s}_t$ is also a holomorphic section on a neighborhood of X_0 and trivializes L_t over X_t. Moreover we get $h_{L_t}(\tilde{s}_t, \tilde{s}_t) \equiv 1$ on X_t. Let $d\pi$ be the generator of $N^*_{X_t/X}$ for any $t \in S - \{0\}$. Let $g'_{N^*,t}$ be the Hermitian metric of $N^*_{X_t/X}$ defined by $g'_{N^*,t}(d\pi, d\pi) \equiv 1$. Put $g'_{N,t} := (g'_{N^*,t})^{-1}$. Then on X_0 holds the following equality by the construction of h_{L_t} and $g'_{N^*,t}$:

$$(6.1) \qquad\qquad \|d\pi \otimes \tilde{s}_t\|_{g'_{N^*,t} \otimes L_t} \equiv 1.$$

Let $\sigma_0 \in \Gamma(S, \lambda_X(E))$ be a holomorphic section such that $\sigma(t) \neq 0$ for any $t \in S$. Let $p_i := pr_i$ be the projection to the i-th factor in $X' \times S$. Regarding $X' \times S$ as the trivial family over S, we obtain line bundles $\lambda_{X' \times S}(p_1^*E')$ and $\lambda_{X' \times S}(p_1^*E' \otimes L^{-1})$ over S. By the construction of L, $\lambda := \lambda_X(E) \otimes \lambda_{X' \times S}(p_1^*E')^{-1} \otimes \lambda_{X' \times S}(p_1^*E' \otimes L)$ is canonically trivial and has a canonical section σ. Let $\sigma_1 \in \Gamma(S, \lambda_{X' \times S}(p_1^*E'))$ and $\sigma_2 \in \Gamma(S, \lambda_{X' \times S}(p_1^*E' \otimes L^{-1}))$ be nowhere vanishing sections such that

$$(6.2) \qquad\qquad \sigma_0 \otimes \sigma_1^{-1} \otimes \sigma_2 = \sigma.$$

Let $\| \cdot \|_{Q1}$ and $\| \cdot \|_{Q2}$ be the Quillen metrics of $\lambda_{X' \times S}(p_1^*E')$ and $\lambda_{X' \times S}(p_1^*E' \otimes L^{-1})$ relative to $(g_{X'}, h_{E'})$ and $(g_{X'}, h_{E'} \otimes h_L^{-1})$ respectively. By (6.1), we may apply Theorem 1.4 to $X_t \subset X'$ to obtain

$$(6.3) \qquad\qquad \log\|\sigma_0\|^2(t)_Q = \left[\iint_{X_t} \widetilde{Td}(\bar{\mathcal{E}}_t)ch(E', h_E)\right]^{(0,0)} + \phi(t),$$

where

$$(6.4) \quad \phi(t) = \log \|\sigma_1\|_{Q1}^2(t) - \log \|\sigma_2\|_{Q2}^2(t)$$
$$- \int_{X'} Td(TX', g_{X'})Td^{-1}(L_t, h_{L_t})ch(E', h_{E'}) \log \|s_t\|_{L_t}^2 + Const.$$

and $\bar{\mathcal{E}}$ is an exact sequence of holomorphic Hermitian vector bundles:

$$(6.5) \quad \bar{\mathcal{E}}_t : 0 \longrightarrow (TX_t, g_{X_t}) \longrightarrow (TX|_{X_t}, g_X) \longrightarrow (N_{X_t/X'}, g'_{N,t}) \longrightarrow 0.$$

Since $\log \|\sigma_1\|_{Q1}^2$ and $\log \|\sigma_2\|_{Q2}^2$ are smooth function on S by Theorem 1.1, ϕ is a continuous function on S. Let U_p be a small neighborhood of $p \in Sing\,X_0$ in X. As $\widetilde{Td}(\bar{\mathcal{E}}_t)$ depends smoothly on $t \in S$ over $X - \cup_{p \in Sing\,X_0} U_p$ and $(E', h_{E'})$ is flat on each U_p, we get

$$(6.6) \quad \log \|\sigma_0\|_Q^2(t) = r(E) \sum_{p \in Sing\,X_0} \left[\int_{X_t \cap U_p} \widetilde{Td}(\bar{\mathcal{E}}_t) \right]^{(0,0)} + \psi(t)$$

for some $\psi \in C^0(S)$. Put $f_p := \pi|_{U_p}$. By Theorem 4.1 and 5.1,

$$(6.7)$$
$$\int_{X_t \cap U_p} \widetilde{Td}(\bar{\mathcal{E}}_t) = \int_{X_t \cap U_p} \frac{1 - Td^{-1}(\nu_{f_p}^* c_1(H))}{\nu_{f_p}^* c_1(H)} \log \|df_p\|^2 + c(n) \int_{X_t \cap U_p} \nu_{f_p}^* c_1(H)^n$$
$$= \frac{(-1)^{n+1}}{(n+2)!} \int_{X_t \cap U_p} \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|df_p\|^2 \right)^n \log \|df_p\|^2 + O(1)$$
$$= \frac{(-1)^{n+1}}{(n+2)!} \mu(p) \log |t|^2 + O(1)$$

which together with (6.6) yields

$$(6.8) \quad \log \|\sigma_0\|_Q^2(t) = \frac{(-1)^{n+1}}{(n+2)!} r(E)\mu(p) \log |t|^2 + \chi(t), \quad \chi \in L^\infty(S).$$

By Theorem 1.1, it holds on $S - \{0\}$ that

$$(6.9) \quad \frac{i}{2\pi} \bar{\partial}\partial \chi = \pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}.$$

Since $\pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}$ extends to a smooth (1,1)-form over S by Proposition 2.1 and $\chi \in L^\infty(S)$, we find by (6.9) and Lemma 6.1 that $\chi \in C^\infty(S)$. The assertion follows from (6.8), (6.9) and smoothness of χ.

Next, we treat the general case where the extension E' of E is no more a vector bundle. Since X' is a projective manifold, there exists a resolution of E' by vector bundles $\{E_i\}_{1 \le i \le N}$ over X':

$$(6.10) \quad 0 \to E_N \to E_{N-1} \to \cdots \to E_1 \to E' \to 0.$$

Here we identify vector bundles with corresponding locally free sheaves. Fix Hermitian metrics $\{h_i\}$ on each E_i. As every E_i is a vector bundle, we can apply Theorem 6.1 for each (E_i, h_i) over X. Since E' is a vector bundle over X, by an argument using anomaly formula ([B-G-S, Theorem 0.3]), we also get the theorem for (E, h). $\qquad \square$

7. Approximation by algebraic families and proof of main theorem. We prove Main Theorem in this section. For simplicity, we assume that $Sing\,X_0$ consists of one point i.e., $Sing\,X_0 = \{0\}$. (General case is proved similarly.)

Step 1. Fix an embedding $i : X \hookrightarrow \mathbb{P}^N \times S$, $\pi = pr_2 \circ i$ and define the polarization $\mathcal{O}_X(1) := i^* pr_1^* \mathcal{O}_{\mathbb{P}^N}(1)$. Since X is projective over S, it is enough to show the theorem under the assumption that the vector bundle E is a quotient of some $\mathcal{O}_X(-1)^{\oplus l}$: $\psi : \mathcal{O}_X(-1)^{\oplus l} \to E \to 0$. Let $P_s(m)$ be the Hilbert polynomial of X_s ($s \in S$) with respect to $\mathcal{O}_X(1)$. As X is flat over S, $P(m) = P_s(m)$ is constant in $s \in S$. Let $T := Hilb_{\mathbb{P}^N}^P$ be the Hilbert scheme of subvarieties with the Hilbert polynomial $P(m)$ and $p_1 : Y \to T$ the universal family over T. Then, $Y \subset T \times \mathbb{P}^N$ and Y_t is a subvariety of $\mathbb{P}^N$ corresponding to $t \in Hilb_{\mathbb{P}^N}^P$ (cf [G]). Set $\mathcal{O}_Y(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_Y$. Let $\phi_s(m)$ be the Hilbert polynomial of $\mathcal{O}_{X_s}(E_s)$ with respect to $\mathcal{O}_{X_s}(1)$. Since $\mathcal{O}_X(E)$ is flat over S, $\phi(m) = \phi_s(m)$ is constant in $s \in S$. Let $Q = Quot^\phi(\mathcal{O}_Y(-1)^{\oplus l}/Y/T)$ be the Quot-scheme of coherent $\mathcal{O}_Y$-modules which are quotient of $\mathcal{O}_Y(-1)^{\oplus l}$ with the Hilbert polynomial $\phi(m)$. Let $p_2 : Q \to T$ be the projection. Put $Y_Q := Y \times_T Q$ and $q : Y_Q \to Q$ for the projection to the second factor. The universal quotient sheaf over Y_Q is denoted by F (cf. [G]): $\Psi : \mathcal{O}_{Y_Q}(-1)^{\oplus l} \to F \to 0$.

Let $\gamma : S \to Q$ be the holomorphic map defined by $\gamma(s) = (X_s, E_s) \in Q$. Then, by the universality of the Hilbert scheme and the Quot-scheme, we get the following identification:

$$(7.1) \qquad X = Y_Q \times_\gamma S, \quad E = \gamma^* F.$$

Since T and Q are projective varieties, we can approximate the holomorphic curve $\gamma : S \to Q$ by a sequence of algebraic curves due to Demailly, Lempert and Schiffman ([D-L-S], [L]).

LEMMA 7.1. *There exist a sequence of algebraic curves $\{C_m\}_{m \geq 1}$ with distinguished open subsets $\{U_m\}_{m \geq 1}$ isomorphic to S, i.e., $i_m : S \cong U_m$, and a sequence of holomorphic maps $\gamma_m : C_m \to Q$ such that, under the identification of U_m with S via i_m,*
(1) $\gamma_m(t) = \gamma(t) + O(t^{m+1})$,
(2) γ_m converges to γ uniformly on $S(\frac{1}{2})$.

Put $X_m := Y_Q \times_{\gamma_m} S$, $\pi_m := pr_2|_{X_m}$ and $E_m := \gamma_m^* F$. Then, the pair (X_m, E_m) is globalizable for any m. To show this, put $Z_m := Y_Q \times_{\gamma_m} C_m$, $E_m' := \gamma_m^* F$. Let $f_m : \tilde{Z}_m \to Z_m$ be a resolution of singularity and put $E_m'' := f_m^* E_m'$. Since X is non-singular, so is X_m by the construction. Thus, we may assume that f_m is an isomorphism restricted to $f_m^{-1}(X_m)$. As Y_Q is projective, so is $\tilde{Z}_m$. Furthermore, as F is a coherent sheaf over Y_Q, so is E_m'' over $\tilde{Z}_m$. Clearly, $(\tilde{Z}_m, E_m'')$ is a globalization of (X_m, E_m).

Step 2. Let $(Y_Q, 0)$ and $(Q, \gamma(0))$ be the germs of analytic spaces at 0 and $\gamma(0)$ respectively. Since $q : Y_Q \to Q$ is a flat morphism, $q : (Y_Q, 0) \to (Q, \gamma(0))$ becomes a deformation of isolated hypersurface singularity $(X_0, 0)$. Since $(X_0, 0)$ is an IHS, its Kuranishi space $Def(X_0, 0)$ is non-singular and $Def(X_0, 0) = (\mathbb{C}^\mu, 0)$ where μ is the Milnor number of $(X_0, 0)$. Let $w = (w_0, \cdots, w_n)$ be the coordinates of $(\mathbb{C}^{n+1}, 0)$ and $t = (t_1, \cdots, t_\mu)$ of $Def(X_0, 0)$. The semi-universal deformation of $(X_0, 0)$ is constructed as follows (cf. [Lo, Chap.6]). Let $f(w)$ be a polynomial in w-variables such that $\mathcal{O}_{X_0, 0} \cong \mathbb{C}\{w\}/(f(w))$ and $\{\phi_1(w), \cdots, \phi_\mu(w)\}$ a basis of the $\mathbb{C}$-vector space $\mathbb{C}\{w\}/(\partial f/\partial w_0, \cdots, \partial f/\partial w_n)$. Then,

$$(7.2) \qquad (\mathcal{X}, 0) := \{(w, t) \in \mathbb{C}^{n+1} \times Def(X_0, 0); f(w) + \sum_{i=1}^{\mu} t_i \phi_i(w) = 0\}$$

with the projection map $pr_2 : (\mathcal{X}, 0) \to Def(X_0, 0)$ becomes the semi-universal de-

formation. By the versality, there exists a small neighborhood of 0 in Y_Q, a small neighborhood V of $\gamma(0)$ in Q, a holomorphic map $J : (V, \gamma(0)) \to Def(X_0, 0)$ and an embedding $j : (U, 0) \hookrightarrow (\mathcal{X}, 0) \times_J (V, \gamma(0))$ such that

(1) $q((U, 0)) = (V, \gamma(0))$, $q|_U = pr_2 \circ j$,

(2) $j : (X_0 \cap U, 0) \hookrightarrow (\mathcal{X}_0, 0)$ is an embedding.

Since $(\mathcal{X}, 0) \times_J (V, \gamma(0)) \subset (\mathbb{C}^{n+1}, 0) \times (V, \gamma(0))$, we may consider j to be an embedding: $j : (U, 0) \hookrightarrow (\mathbb{C}^{n+1}, 0) \times (V, \gamma(0))$.

Step 3. Let us construct a good metric on the relative tangent bundle TY_Q/Q. Since $Y_Q \subset \mathbb{P}^N \times Q$, there exists another embedding $i : (U, 0) \hookrightarrow (\mathbb{P}^N, 0) \times (V, \gamma(0))$. Fix an embedding $k : (V, \gamma(0)) \hookrightarrow (\mathbb{C}^M, 0)$. Let $u = (u_1, \cdots, u_M)$ be the coordinates of $\mathbb{C}^M$. Then, the embedding induces a Kähler metric on V by $ds_V^2 := k^* ds_{\mathbb{C}^M}^2$ where $ds_{\mathbb{C}^M}^2 = \partial\bar{\partial}\|u\|^2$ is the Euclidean metric of $\mathbb{C}^M$. By the embeddings i and j, U is eqquiped with the following Kähler metrics:

$$(7.3) \qquad g_1 := i^*(ds_{\mathbb{P}^N}^2 + ds_V^2), \quad g_2 := j^*(ds_{\mathbb{C}^{n+1}}^2 + ds_V^2)$$

where $ds_{\mathbb{P}^N}^2$ is the Fubini-Study metric. Since g_1 and g_2 are induced by embeddings into some ambient spaces, they are quasi-isometric to each other. Namely, there exists a constant $C > 0$ such that it holds on U

$$(7.4) \qquad C^{-1} g_1 \leq g_2 \leq C g_1.$$

Let $z = (z_1, \cdots, z_N)$ be the inhomogeneous coordinates of $\mathbb{P}^N$ centered at 0. Let $B(r)$ be the metric balll of radius r centered at 0 in $\mathbb{P}^N$ and $U(r) := U \cap (B(r) \times V)$. Fix $\delta > 0$ such that $U(\delta) \subset U$. Let $\rho_1(r) \in C^\infty(\mathbb{R})$ such that $\rho_1(r) \equiv 0$ for $r \leq \delta_1$, $\rho_1(r) = r$ for $r \geq \delta_2$, $\rho_1'(r) \geq 0$ and $\rho_1''(r) \geq 0$ for any $r \in \mathbb{R}$ where $0 < \delta_1 < \delta_2 < \delta$. Let $\rho_2(r)$ be a smooth cut-off function such that $\rho_2(r) \equiv 1$ for $r \leq \delta_2$ and $\rho_2(r) \equiv 0$ for $r \geq \delta$. Define a potential function $\Phi_\epsilon \in C^\infty(U)$ by

$$(7.5) \quad \Phi_\epsilon := \epsilon\, i^* \rho_2(\log(1 + \|z\|^2))\, j^*(\|w\|^2 + \|u\|^2) + i^*\{\rho_1(\log(1 + \|z\|^2)) + \|u\|^2\}.$$

Since $ds_{\mathbb{P}^N}^2 = \partial\bar{\partial}\log(1 + \|z\|^2)$ and $ds_{\mathbb{C}^{n+1}}^2 = \partial\bar{\partial}\|w\|^2$, we find by (7.4) that $\partial\bar{\partial}\Phi_\epsilon > 0$ on U if ϵ is sufficiently small. By the construction,

$$(7.6) \qquad \partial\bar{\partial}\Phi_\epsilon|_{TY_Q/Q}(x) = \begin{cases} \epsilon\, j^* ds_{\mathbb{C}^{n+1}}^2 |_{TY_Q/Q} & (x \in U(\delta_1)) \\ i^* ds_{\mathbb{P}^N}^2 |_{TY_Q/Q} & (x \in U \setminus U(\delta_0)). \end{cases}$$

Therefore, we can define a Hermitian metric $g_{Y_Q/Q}$ on TY_Q/Q by

$$(7.7) \qquad g_{Y_Q/Q}(x) = \begin{cases} \partial\bar{\partial}\Phi_\epsilon|_{TY_Q/Q}(x) & (x \in U) \\ i^* ds_{\mathbb{P}^N}^2 |_{TY_Q/Q}(x) & (x \in U \setminus U(\delta_0)). \end{cases}$$

By Proposition 2.1 and (7.6), we get the following.

LEMMA 7.2. *It holds on $U(\delta_1)$ that $[Td(TY_Q/Q, g_{Y_Q/Q})]^{(n+1,n+1)} \equiv 0$. Moreover, $[Td(TY_Q/Q, g_{Y_Q/Q})]^{(n+1,n+1)}$ extends to a smooth $(n+1, n+1)$-form on a neighborhood of X_0.*

Step 4. Let us construct a good metric on F. Since F is locally free along X_0, so is on a neighborhood of X_0 in Y_Q because locally freeness is an open condition. In particular, there exists a hermitian metric h_F on $F|_{q^{-1}(V)}$ whose curvature vanishes on U if U and V are chosen small enough.

Step 5. Let $g_m := \gamma_m^* g_{Y_Q/Q}$ and $h_m := \gamma_m^* h_F$ be the induced Hermitian metrics on TX_m/S and E_m. Since it is enough to show Main Theorem for some Kähler metric of X and some Hermitian metric of E, we may assume that TX/S and E are

equipped with the restriction of $g_{Y_Q/Q}$ and h_F. Let $\lambda = \det Rp_*F$ be the determinant of cohomologies, $\|\cdot\|_{Q,m}$ the Quillen metric of $\lambda|_{X_m}$ relative to g_m and h_m. Let $\sigma \in \Gamma(U, \lambda)$ be a holomorphic section such that $\sigma(z) \neq 0$ for any $z \in U$. Applying Theorem 6.1 for $\{(X_m, g_m), (E_m, h_m)\}$, we get the following formula.
(7.8)

$$c_1(\gamma_m^*\lambda|_{X_m}, \|\cdot\|_{Q,m}) = \frac{(-1)^{n+1}}{(n+2)!}\mu(X_0)r(E)\delta_0 + \pi_*(Td(TX_m/S, g_m)ch(E_m, h_m))^{(1,1)}.$$

By (7.10) and Lemma 7.2, there exist a family of smooth functions $\{\chi_m\} \subset C^\infty(S)$ such that, on S,

(7.9) $$\log\|\gamma_m^*\sigma\|_{Q,m}^2(t) = \frac{(-1)^n}{(n+2)!}\mu(X_0)r(E)\log|t|^2 + \chi_m(t).$$

From Lemma 7.1 and the construction of metrics $g_{Y_Q/Q}$ and h_F, it follows that for any $t \in S - \{0\}$,

(7.10) $$\|\gamma_m^*\sigma\|_{Q,m}(t) \to \|\gamma^*\sigma\|_Q(t) \quad (m \to \infty),$$

because $\log\|\sigma\|_Q^2$ is a smooth function on $V \setminus \Delta$ where Δ is the discriminant locus of $q: Y_Q \to Q$. In particular, the family $\{\chi_m\}$ is uniformly bounded on $\partial S(\frac{1}{2})$. By (7.9) and Theorem 1.1, we have

(7.11) $$\frac{i}{2\pi}\bar{\partial}\partial\log\chi_m = \pi_*(Td(TX_m/S, g_m)ch(E_m, h_m))^{(1,1)}.$$

Since $\{\pi_*(Td(TX_m/S, g_m)ch(E_m, h_m))^{(1,1)}\}$ is a family of smooth $(1,1)$-forms on $S(\frac{1}{2})$ converging uniformly to $\pi_*(Td(TX/S, g_{X/S})ch(E, h))^{(1,1)}$ by Lemma 7.2 and Step 4, it follows from the elliptic regularity theory (cf. [G-T, Theorem 8.6]) that $\{\chi_m\}$ is an uniformly bounded equi-continuous family on $S(\frac{1}{2})$. Put

(7.12) $$\chi(t) := \log\|\gamma^*\sigma\|_Q^2(t) - \frac{(-1)^n}{(n+2)!}\mu(X_0)r(E)\log|t|^2.$$

From (7.10) together with the Ascoli-Arzelá Theorem, it follows that $\lim\chi_m = \chi$ uniformly on $S(\frac{1}{2})$ and $\chi \in C^\infty(S(\frac{1}{2}))$. Main Theorem follows from (7.12) and the smoothness of χ. $\Box$

8. Smoothing of IHS and analytic torsion. We prove Theorem 0.1 in this section. The same notations as in section 0 are used through this section. First, we study the behavior of the L^2-metric on $\lambda(\mathcal{O}_X)$.

Let H be the relatively very ample line bundle over X. We may assume that $X \subset \mathbb{P}^N \times S$, $H = pr_1^*\mathcal{O}_{\mathbb{P}^N}(1)$ and $\pi = pr_2|_X$ where pr_i is the projection to the i-th factor. Since $Sing\,X_0$ is a finite set, we can choose generic hyperplane sections $H_1, \cdots, H_n$ such that $Sing\,X_0 \notin H_i$ and $Sing(X_0 \cap H_1 \cap \cdots \cap H_i) = \emptyset$ for any $i = 1, \cdots, n$. Put $X^q := X \cap H_1 \cap \cdots \cap H_{n-q}$, $\pi^q := \pi|_{X^q}$ and $X_t^q := X^q \cap X_t = X_t \cap H_1 \cap \cdots \cap H_q$. Then, we may also assume that $\pi^q: X^q \to S$ is a smooth morphism for any $q > 0$ by the construction. As was shown by Steenbrink ([St]), $R^q\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module for any $q \geq 0$. Choose a $\mathcal{O}_S$-basis $\omega_1^q, \cdots, \omega_{m_q}^q$ of $R^q\pi_*\mathcal{O}_{X_t}$. Put $\sigma^q := \wedge_{i=1}^{m_q}\omega_i^q \in \det R^q\mathcal{O}_{X_t}$ and $\sigma := \otimes_{q=0}^n(\sigma^q)^{(-1)^q} \in \lambda(\mathcal{O}_X)$. By Dolbeault's theorem, there exists $\bar{\partial}$-closed $(0,q)$-form ϕ_i^q representing ω_i^q defined on $X_\epsilon := \pi^{-1}(S_\epsilon)$ where $S_\epsilon = \{t \in S; |t| \leq \epsilon\}$. Put $\phi_i^q(t) := \phi_i^q|_{X_t}$ and $\psi_i^q(t) := \phi_i^q|_{X_t^q}$. Let $\mathbb{H}\phi_i^q(t)$ and $\mathbb{H}\psi_i^q(t)$

be their harmonic representatives. Then, $\overline{\mathbb{H}\phi_i^q(t)}$ (resp. $\overline{\mathbb{H}\psi_i^q(t)}$) is a holomorphic q-from on X_t (resp. X_t^q). In particular, $\mathbb{H}\phi_i^q(t)|_{X_t^q}$ is also a harmonic form on X_t^q.

LEMMA 8.1.

$$\mathbb{H}\phi_i^q(t)|_{X_t^q} = \mathbb{H}\psi_i^q(t).$$

Proof. Let $[\mathbb{H}\phi_i^q(t)]$ (resp. $[\mathbb{H}\psi_i^q(t)]$) be the Dolbeaut cohomology class. Then, we get the assertion by

$$(8.1) \qquad [\mathbb{H}\phi_i^q(t)|_{X_t^q}] = [\mathbb{H}\phi_i^q(t)]|_{X_t^q} = [\phi_i^q]|_{X_t^q} = [\mathbb{H}\psi_i^q(t)]$$

because both $\mathbb{H}\phi_i^q(t)|_{X_t^q}$ and $\mathbb{H}\psi_i^q(t)$ are harmonic forms on X_t^q. $\quad\square$

PROPOSITION 8.1. $\|\sigma^q\|_{L^2}^2$ *is a positive C^∞-function on S for any $q < n$.*

Proof. Let g_t be the Kähler form of $g_X|_{X_t}$ and $[g_t]$ its Kähler class. By the assumption, we get $[g_t] = c_1(H)|_{X_t}$ in the de Rham cohomology group. From the definition of L^2-metric and Lemma 8.1, it follows that

$$
(8.2)
\begin{aligned}
\|\sigma^q\|_{L^2}^2(t) &= \left| \det\left(\int_{X_t} \mathbb{H}\phi_i^q(t) \wedge \overline{\mathbb{H}\phi_j^q}(t) \wedge g_t^{n-q} \right)_{1\le i,j\le m_q} \right| \\
&= \left| \det\left(\int_{X_t} \mathbb{H}\phi_i^q(t) \wedge \overline{\mathbb{H}\phi_j^q}(t) \wedge c_1(H)^{n-q} \right)_{1\le i,j\le m_q} \right| \\
&= \left| \det\left(\int_{X_t^q} \mathbb{H}\psi_i^q(t) \wedge \overline{\mathbb{H}\psi_j^q}(t) \right)_{1\le i,j\le m_q} \right|.
\end{aligned}
$$

Therefore, it is enough to show that $[\phi_1^q]|_{X_0^q}, \cdots, [\phi_{m_q}^q]|_{X_0^q}$ are linearly independent in $H^q(X_0^q, \mathcal{O}_{X_0^q})$ to prove the proposition. Since H_1 is an ample divisor of X_0, $r_1 : H^q(X_0, \mathcal{O}_{X_0}) \to H^q(X_0^1, \mathcal{O}_{X_0^1})$ is an isomorphism for $q < n - 1$ and an injection for $q = n - 1$ where r_1 is the restriction map. Inductively, we find that the restriction map $r_q : H^q(X_0, \mathcal{O}_{X_0}) \to H^q(X_0^q, \mathcal{O}_{X_0^q})$ is an injection for any $q < n$. In particular, $[\phi_1^q]|_{X_0^q}, \cdots, [\phi_{m_q}^q]|_{X_0^q}$ are linearly independent in $H^q(X_0^q, \mathcal{O}_{X_0^q})$ because so are $[\phi_1^q]|_{X_0}, \cdots, [\phi_{m_q}^q]|_{X_0}$ in $H^q(X_0, \mathcal{O}_{X_0})$ by the linearly independence of $\omega_1^q, \cdots, \omega_{m_q}^q$ at $t = 0$. $\quad\square$

Let us study the case $q = n$. Write $\phi_i^n = \tau_i$ and $m_n = m$. Let $\omega_1, \cdots, \omega_m$ be the basis of $\omega_{X/S}$ as in section 0. We can choose τ_i such that

$$(8.3) \qquad \int_{X_t} \omega_i \wedge \tau_j = \delta_{ij}.$$

PROPOSITION 8.2.

$$\left| \det\left(\int_{X_t} \mathbb{H}\tau_i(t) \wedge \mathbb{H}\bar{\tau}_j(t) \right)_{1\le i,j\le m} \right| = \det\left({}^t\Omega(t)\Lambda\bar{\Omega}(t) \right)^{-1}.$$

Proof. Write $\mathbb{H}\tau_i(t) = \sum_{j=1}^m a_{ij}(t)\bar{\omega}_j(t)$ for some (m, m)-matrix $A(t) := (a_{ij}(t))$. Then we have

$$(8.4)$$
$$\left| \det\left(\int_{X_t} \mathbb{H}\tau_i(t) \wedge \mathbb{H}\bar{\tau}_j(t) \right)_{1\le i,j\le m} \right| = |\det A(t)|^2 \cdot \left| \det\left(\int_{X_t} \omega_i(t) \wedge \bar{\omega}_j(t) \right)_{1\le i,j\le m} \right|.$$

By (8.3), we find

$$(8.5) \qquad A(t) = \left(\int_{X_t} \omega_i(t) \wedge \bar{\omega}_j(t) \right)^{-1}.$$

Let $\{a_1(t), \cdots, a_l(t)\}$ be the basis of $H_n(X_t, \mathbb{Z})_{fr}$ as in section 0. By the Küneth formula for $\Delta_t \subset X_t \times X_t$ where Δ_t is the diagonal set, it holds in $H_n(X_t, \mathbb{Z})_{fr}$

$$(8.6) \qquad [\Delta_t] = \sum_{i,j=1}^{m} \lambda_{ij}\, pr_1^* a_i(t) \wedge pr_2^* a_j(t) + E$$

where E is the term containing an element of $pr_i^* H_q(X_t, \mathbb{Z})_{fr}$ $(q \neq n)$ and $\Lambda = (\lambda_{ij})$ is the intersection form on $H_n(X_t, \mathbb{Z})_{fr}$. By (8.6), we get

$$(8.7) \qquad \begin{aligned} \int_{X_t} \omega_i(t) \wedge \bar{\omega}_j(t) &= \int_{\Delta_t} pr_1^* \omega_i(t) \wedge pr_2^* \bar{\omega}_j(t) \\ &= \sum_{i,j=1}^{m} \lambda_{ij} \int_{a_i(t)} \omega_i(t) \overline{\int_{a_j(t)} \omega_j(t)} \end{aligned}$$

which together with (8.4) and (8.5) yields the assertion. $\qquad\square$

PROPOSIITON 8.3 ([M]). *The following asymptotic expansion holds as $t \to 0$:*

$$\int_{a_i(t)} \omega_j(t) \sim \sum_{r=r_1,\cdots,r_k} \sum_{i \geq 0} \sum_{j=0}^{p} a_{rij} t^{r+i} (\log t)^j$$

where $r = r_1, \cdots, r_k \in \mathbb{Q}_{>-1}$ and $p = \dim H^0(X_t, \Omega_{X_t}^n)$.

Proof of Theorem 0.1. Let $\|\cdot\|_Q$ be the Quillen metric of $\lambda(\mathcal{O}_X)$ relative to $g_{X/S}$. By the Definition 1.1 and Proposition 8.2, there exists $f \in C^\infty(S)$ such that

$$(8.8) \qquad \|\sigma\|_Q^2(t) = \exp(f(t))\, \tau(X_t) \det\left({}^t\Omega(t)\Lambda\bar{\Omega}(t)\right)^{(-1)^{n+1}}.$$

Let g_X' be a Kähler metric of X such that g_X' is Euclidean flat on a neighborhood of $Sing\, X_0$ and $[g_X'] = [g_X]$ in $H_{dR}^2(X, \mathbb{R})$. Let $g_{X/S}'$ be the induced metric on TX/S and $\|\cdot\|_Q'$ the Quillen metric of $\lambda(\mathcal{O}_X)$ relative to $g_{X/S}'$. By Main Theorem for $(\lambda(\mathcal{O}_X), \|\cdot\|_Q')$, there exists a smooth $(1,1)$-form χ on S such that, in the sense of current,

$$(8.9) \qquad \frac{i}{2\pi} \bar{\partial}\partial \log(\|\sigma\|_Q')^2 = \frac{(-1)^{n+1}}{(n+2)!} \mu(X_0)\delta_0 + \chi$$

because $\pi_*(Td(TX/S, g_{X/S}'))^{(1,1)}$ is a smooth $(1,1)$-form on S by Proposition 2.1, which together with Theorem 1.2 and Theorem 3.1 (1) yields

$$(8.10) \qquad \log \|\sigma\|_Q^2(t) = \frac{(-1)^n}{(n+2)!} \mu(X_0) \log|t|^2 + F(t)$$

where $F \in C^\alpha(S)$ with some $\alpha > 0$. Moreover $F(t)$ admits the following asymptotic expansion:

$$(8.11) \qquad F(t) \sim \sum_{r=r_1,\cdots,r_l} \sum_{i,j \geq 0} \sum_{k=0}^{n} a_{rijk} |t|^r t^i \bar{t}^j (\log|t|)^k$$

where $r_1, \cdots, r_l \in \mathbb{Q}_+$, and so does $\exp(F(t))$. The theorem follows from (8.8), (8.10), (8.11) and Proposition 8.3. $\quad\square$

THEOREM 8.1. *Suppose that $Sing\, X_0$ consists of rational singularities. Then, there exists a constant A such that*

$$\lim_{t \to 0} \frac{\tau(X_t)}{|t|^{2\frac{(-1)^n}{(n+2)!}\mu(Sing\, X_0)}} = \exp(A).$$

Proof. From the rationality of $Sing\, X_0$, it follows that any $\omega_i(0)$ is square integrable over X_0 (cf. [Bu]) and $\det\left(\int_{X_0} \omega_i(0) \wedge \bar{\omega}_j(0)\right) \neq 0$ because $\{\omega_1(0), \cdots, \omega_m(0)\}$ is a basis of $H^0(X_0, \Omega_{X_0}^n)$ over $\mathbb{C}$. Therefore, by Theorem 0.1, it is enough to show that

$$(8.12) \qquad \lim_{t \to 0} \int_{X_t} \omega_i(t) \wedge \bar{\omega}_j(t) = \int_{X_0} \omega_i(0) \wedge \bar{\omega}_j(0).$$

For simplicity, we assume $Sing\, X_0 = \{o\}$. Let $(U, (z_0, \cdots, z_n))$ be a coordinate neighborhood around o and $f \in \mathcal{O}(U)$ the local defining equation of (X_0, o), i.e., $X_0 \cap U = \{z \in U; f(z) = 0\}$. As π is of maximal rank on $X - U$, it is clear

$$(8.13) \qquad \lim_{t \to 0} \int_{X_t - U} \omega_i(t) \wedge \bar{\omega}_j(t) = \int_{X_0 - U} \omega_i(0) \wedge \bar{\omega}_j(0).$$

By the definition of relative canonical form, there exists $\eta_i \in \mathcal{O}(U)$ such that $df \wedge \omega_i = \eta_i dz_0 \wedge \cdots \wedge dz_n$.

Let $p: Y \to X$ be a good embedded resolution of f, Y_0 the proper transform of X_0 and $p^{-1}(0) = E_1 \cup \cdots \cup E_k$ the exceptional divisors such that $Y_0 \cup E_1 \cup \cdots E_k$ is a divisor with normal crossing on Y. Let $Y_t = p^{-1}(X_t)$. Since $Sing\, X_0$ is rational, it follows from [v.S-St, (3.1) Theorem] that

$$(8.14) \qquad p^*\left(\frac{df}{f}\right) \wedge p^*\omega_i = p^*\left(\eta_i \frac{dz_0 \wedge \cdots \wedge dz_n}{f}\right) \in \Omega_{p^{-1}(U)}^{n+1}(\log Y_0).$$

Let $x \in Y_0$, and $(V, (w_0, \cdots, w_n))$ be a local coordinates around x such that $Y_0 \cap V = \{w \in V; w_0 = 0\}$ and $p^*f(w) = w_0 w_1^{d_1} \cdots w_n^{d_n}$ $(d_i \in \mathbb{Z}_{\geq 0})$. By (8.14), there exists $a_i(w) \in \mathcal{O}(V)$ such that

$$(8.15) \qquad p^*\left(\eta_i \frac{dz_0 \wedge \cdots \wedge dz_n}{f}\right)\bigg|_V = a_i(w) \frac{dw_0}{w_0} \wedge dw_1 \wedge \cdots \wedge dw_n.$$

By (8.14) again, we get

$$(8.16) \qquad p^*\omega_i = a_i(w)\, dw_1 \wedge \cdots \wedge dw_n + p^*df \wedge \xi_i$$

for some $\xi_i \in \Omega_V^{n-1}(*Y_0 + *E_1 + \cdots + *E_k)$ which yields
(8.17)

$$\lim_{t \to 0} \int_{Y_t \cap V} p^*\omega_i(t) \wedge \overline{p^*\omega_j(t)} = \int_{Y_0 \cap V} a_i(w)\overline{a_j(w)}\, dw_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n$$

$$= \int_{Y_0 \cap V} p^*\omega_i(0) \wedge \overline{p^*\omega_j(0)}.$$

Similarly, for any $x \in E_i$, there is a neighborhood W such that

$$(8.18) \qquad \lim_{t \to 0} \int_{Y_t \cap W} p^*\omega_i(t) \wedge \overline{p^*\omega_j(t)} = \int_{Y_0 \cap W} p^*\omega_i(0) \wedge \overline{p^*\omega_j(0)}$$

which together with (8.13) and (8.17) yields (8.12).   □

REMARK. By (8.12), we find that -1 is a simple root of the b-function for IHS and any other root is strictly smaller than -1.

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