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COMPARISON OF ALGEBRAIC AND SYMPLECTIC GROMOV-WITTEN INVARIANTS*

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1. Introduction. As Witten suggested in [W1], [W2], the GW-invariants for a symplectic manifold X are multi-linear maps

$$\gamma^X_{A,g,n} : H^*(X;\mathbb{Q})^{\times n} \times H^*(\overline{\mathfrak{M}}_{g,n};\mathbb{Q}) \longrightarrow \mathbb{Q}, \tag{1.1.1}$$

where $A \in H_2(X,\mathbb{Z})$ is any homology class, n, g are non-negative integers and $\overline{\mathfrak{M}}_{g,n}$ is the moduli space of stable n-pointed genus g curves. The basic idea of defining such invariants is to enumerate holomorphic maps from Riemann surfaces to X. To illustrate this, we let X be a smooth projective manifold and form the moduli space $\mathfrak{M}_{g,n}(X,A)$ of all holomorphic maps $f:\Sigma\to X$ from smooth n-pointed Riemann surfaces $(\Sigma;x_1,\dots,x_n)$ to X such that $f_*([\Sigma])=A$. $\mathfrak{M}_{g,n}(X,A)$ is a quasi-projective scheme and its expected dimension can be calculated using the Riemann-Roch theorem. We will further elaborate the notion of expected dimension later. For the moment we will denote it by $r_{\exp}$. Note that it depends implicitly on the choice of X, A, g and n. When $r_{\exp}=0$, then $\mathfrak{M}_{g,n}(X,A)$ is expected to be discrete. If $\mathfrak{M}_{g,n}(X,A)$ is discrete, then the degree of $\mathfrak{M}_{g,n}(X,A)$, considered as a 0-cycle, is the GW-invariant of X. We remark that we will ignore the issue of non-trivial automorphism groups of maps in $\mathfrak{M}_{g,n}(X,A)$ in the introduction. When $r_{\exp}>0$, then $\mathfrak{M}_{g,n}(X,A)$ is expected to have pure dimension $r_{\exp}$. If it does, then we pick n subvarieties of X, say $V_1,\dots,V_n$, so that their total codimension is $r_{\exp}$. We then form a subscheme of $\mathfrak{M}_{g,n}(X,A)$ consisting of maps f of which $f(x_i)\in V_i$. This subscheme is expected to be discrete. It it does, then its degree is the GW-invariant of X. Put them together, we obtain the GW-invariants $\gamma^X_{A,g,n}$ of X. This is similar to the construction of the Donaldson polynomial invariants for 4-manifolds.

Here are the two big *ifs* in carry out this program.

Question I: Whether the moduli scheme $\mathfrak{M}_{g,n}(X,A)$ has pure dimension $r_{\exp}$?

Question II: Whether the subschemes of $\mathfrak{M}_{g,n}(X,A)$ that satisfy certain incidence relations have the expected dimensions?

Similar to Donaldson polynomial invariants, the affirmative answer to the above two questions are in general not guaranteed. One approach to overcome this difficulty, beginning with Donaldson's invariants of 4-manifolds, is to "deform" the moduli problems and hope that the answers to the "deformed" moduli problems are affirmative. In the case of GW-invariants, one can deform the complex structure of the smooth variety X to not necessary integrable almost complex structure J and study the same moduli problem by replacing holomorphic maps with pseudo-holomorphic maps. This was investigated by Gromov [Gr], and Ruan [Ru] in which he constructed certain GW-invariants of rational type for semi-positive symplectic manifolds. The first mathematical theory of GW-invariants came from the work of Ruan and the second author, in which they found that the right set up of GW-invariants for semi-positive

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manifolds can be provided by using the moduli of maps satisfying non-homogeneous Cauchy-Riemann equations. In this set up, they constructed the GW-invariants of all semi-positive symplectic manifolds and proved fundemantal properties of these invariants. All Fano-manifolds and Calabi-Yau manifolds are special examples of semi-positive symplectic manifolds.

Attempts to push this to cover general symplectic manifolds so far have failed. New approaches are needed in order to get a hold on the GW-invariants of general varieties (or symplectic manifolds). The first step is to convert the problem of counting mappings, which essentially is homology in nature, into the frame work of cohomology theory of the moduli problem. More precisely, we first compactify the moduli space $\mathfrak{M}_{g,n}(X,A)$ to, say, $\overline{\mathfrak{M}}_{g,n}(X,A)$. We require that the obvious evaluation map

$$e : \mathfrak{M}_{g,n}(X,A) \longrightarrow X^n$$

that sends $(f;\Sigma;x_1,\dots,x_n)$ to $(f(x_1),\dots,f(x_n))$ extends to

$$\bar{e} : \overline{\mathfrak{M}}_{g,n}(X,A) \longrightarrow X^n.$$

We further require that if $\mathfrak{M}_{g,n}(X,A)$ has pure dimension r_{exp}, then $\overline{\mathfrak{M}}_{g,n}(X,A)$ supports a fundamental class

$$[\overline{\mathfrak{M}}_{g,n}(X,A)] \in H_{2r_{\text{exp}}}(\overline{\mathfrak{M}}_{g,n}(X,A);\mathbb{Q}).$$

Then the GW-invariants of X are multi-linear maps

$$\gamma^X_{A,g,n} : H^*(X)^{\times n} \times H^*(\overline{\mathfrak{M}}_{g,n}) \longrightarrow \mathbb{Q} \tag{1.1.2}$$

that send (α,β) to

$$\gamma^X_{A,g,n}(\alpha,\beta) = \int_{[\overline{\mathfrak{M}}_{g,n}(X,A)]} \bar{e}^*(\alpha)\cup\pi^*(\beta),$$

where $\pi : \overline{\mathfrak{M}}_{g,n}(X,A) \to \overline{\mathfrak{M}}_{g,n}$ is the forgetful map. Note that in such cases the GW-invariants are defined without reference to the answer to *question II.*

Even when the answer to *question I* is negative, we can still define the GW-invariants if a virtual moduli cycle

$$[\overline{\mathfrak{M}}_{g,n}(X,A)]^{\text{vir}} \in H_{2r_{\text{exp}}}(\overline{\mathfrak{M}}_{g,n}(X,A);\mathbb{Q})$$

can be found that function as the fundamental cycle $[\overline{\mathfrak{M}}_{g,n}(X,A)]$ should the dimension of $\overline{\mathfrak{M}}_{g,n}(X,A)$ is r_{exp}. In this case, we simply define $\gamma^X_{A,g,n}$ as before with $[\overline{\mathfrak{M}}_{g,n}(X,A)]$ replaced by $[\overline{\mathfrak{M}}_{g,n}(X,A)]^{\text{vir}}$.

The standard compactification of $\mathfrak{M}_{g,n}(X,A)$ is the moduli space of stable morphisms from n-pointed genus g curves, possibly nodal, to X of the prescribed fundamental class. This was first studied for pseudo-holomorphic maps by Parker and Wolfson [PW] and in algebraic geometry by Kontsevich [Ko]. Because points of the compactification $\overline{\mathfrak{M}}_{g,n}(X,A)$ are maps f whose domains have n-marked points $x_1,\dots,x_n$, the evaluation map e extends canonically to $\bar{e}$ that sends such map f to $(f(x_1),\dots,f(x_n))$.

The virtual moduli cycles $[\overline{\mathfrak{M}}_{g,n}(X,A)]^{\text{vir}}$ for projective variety X were first constructed by the authors. Their idea is to construct a virtual normal cone embedded in a vector bundle based on the obstruction theory of stable morphisms [LT1]. An alternative construction of such cones was achieved by Behrend and Fantechi [BF, Be]. For general symplectic manifolds, such virtual moduli cycles were constructed by the authors, and independently, by Fukaya and Ono [FO, LT2]. Shortly after them, Siebert [Si] and Ruan [Ru2] gave different constructions of such virtual moduli cycles. Both Siebert and Ruan's approach relied on constructing global, finite-dimensional resolutions of so the called cokernel bundles (cf. [Si] and [Ru2], Appendix).

However, one question remains to be investigated. Namely, if X is a smooth projective variety then on one hand we have the algebraically constructed GW-invariants, and on the other hand, by viewing X as a symplectic manifold using the Kähler form on X, we have the GW-invariants constructed using analytic method. These two approaches are drastically different. One may expect, although far from clear, that for smooth projective varieties the algebraic GW-invariants and their symplectic counterparts are identical.

The main goal of this paper is to prove what was expected is indeed true.

THEOREM 1.1. *Let X be any smooth projective variety with a Kähler form ω. Then the algebraically constructed GW-invariants of X coincide with the analytically constructed GW-invariants of the symplectic manifold (X^{top}, ω).*

The proof of this paper was outlined in [LT3]. During the preparation of the paper, we learned from B. Siebert that he was able to prove a similar result.

We now outline the proof of our Comparison Theorem. We begin with a few words on the algebraic construction of the virtual moduli cycle. Let $w \in \overline{\mathfrak{M}}_{g,n}(X, A)$ be any point associated to the stable morphism $f : \Sigma \to X$. It follows from the deformation theory of stable morphisms that there is a complex $\mathcal{C}_w$, canonical up to quasi-isomorphisms, such that its first cohomology $\mathcal{H}^1(\mathcal{C}_w)$ is the space of the first order deformations of the map w, and its second cohomology $\mathcal{H}^2(\mathcal{C}_w)$ is the obstruction space to deformations of the map w. Let φ_w be a Kuranishi map of the obstruction theory to deformations of w. Note that φ_w is the germ of a holomorphic map from a neighborhood of the origin $o \in \mathbb{C}^{m_1}$ to $\mathbb{C}^{m_2}$, where $m_i = \dim \mathcal{H}^i(\mathcal{C}_w)$. Let $\hat{o}$ be the formal completion of $\mathbb{C}^{m_1}$ along o and let $\hat{w}$ be the subscheme of $\hat{o}$ defined by the vanishing of φ_w. Note that $\hat{w}$ is isomorphic to the formal completion of $\overline{\mathfrak{M}}_{g,n}(X, A)$ along w (Here as before we will ignore the issue of automorphism groups of maps in $\overline{\mathfrak{M}}_{g,n}(X, A)$). This says that "near" w the scheme $\overline{\mathfrak{M}}_{g,n}(X, A)$ is a "subset" of $\mathbb{C}^{m_1}$ defined by the vanishing of m_2-equations. Henceforth, if these equations are in general position, then $\dim \hat{w} = m_1 - m_2$, which is the expected dimension r_{exp} we mentioned before. The case where $\overline{\mathfrak{M}}_{g,n}(X, A)$ has dimension bigger than r_{exp} is exactly when the vanishing locus of these m_2-equations in φ_w do not meet properly near o. Following the excess intersection theory of Fulton and MacPherson [Fu], the "correct" cycle should come from first constructing the normal cone $C_{\hat{w}/\hat{o}}$ to $\hat{w}$ in $\hat{o}$, which is canonically a subcone of $\hat{w} \times \mathbb{C}^{m_2}$, and then (topologically) intersect the cone with the zero section of $\hat{w} \times \mathbb{C}^{m_2} \to \hat{w}$. To make sense of this we need to construct a pair consisting of a global cone of a vector bundle over $\overline{\mathfrak{M}}_{g,n}(X, A)$ so that its restriction to $\hat{w}$ is the pair $C_{\hat{w}/\hat{o}} \subset \hat{w} \times \mathbb{C}^{m_2}$. This requires patching these cones together to form a global cone. The main difficulty in doing so comes from the fact that the dimensions $\mathcal{H}^2(\mathcal{C}_w)$ may vary as w vary while only $\dim \mathcal{H}^1(\mathcal{C}_w) - \dim \mathcal{H}^2(\mathcal{C}_w)$ is a topological number. This makes the cones $C_{\hat{w}/\hat{o}}$ to sit inside bundles of varying ranks. To overcome this difficulty, the authors come with the idea of finding a global $\mathbb{Q}$-vector bundle E_2 over $\overline{\mathfrak{M}}_{g,n}(X, A)$ and a subcone N of E_2 such that near fibers over w, the cone N is a (canonical) fattening of the cone $C_{\hat{w}/\hat{o}}$ (See section 3 or [LT1] for more details). In the end, we let j be the zero section of E_2 and let j^* be the Gysin map

$$A_*E_2 \longrightarrow A_*\overline{\mathfrak{M}}_{g,n}(X,A),$$

where A_* denotes the Chow-cohomology group with rational coefficient (see [Fu, Vi]). Then the algebraic virtual moduli cycle is

$$[\overline{\mathfrak{M}}_{g,n}(X,A)]^{\mathrm{vir}} = j^*([N]) \in A_{r_{\exp}}\overline{\mathfrak{M}}_{g,n}(X,A).$$

Now let us recall briefly the analytic construction of the GW-invariants of symplectic manifolds. Let (X,ω) be any smooth symplectic manifold with J a tamed almost complex structure. For A, g and n as before, we can form the moduli space of J-holomorphic maps $f:\Sigma\to X$ where Σ are n-pointed genus g smooth Riemann surfaces such that $f_*([\Sigma]) = A$. We denote this space by $\mathfrak{M}_{g,n}(X,A)^J$. It is a finite dimensional topological space. As before, we compactify it to include all J-holomorphic maps whose domains are possibly with nodal singularities. We denote the compactified space by $\overline{\mathfrak{M}}_{g,n}(X,A)^J$. To proceed, we embed $\overline{\mathfrak{M}}_{g,n}(X,A)^J$ inside an ambient space $\mathbf{B}$ and realize it as the vanishing locus of a section of a "vector bundle". Without being precise, the space $\mathbf{B}$ is the space of all *smooth* maps from possibly nodal n-pointed Riemann surfaces to X, the fiber of the bundle over f are all $(0,1)$-forms over domain(f) with values in f^*T_X and the section of the bundle is the one that sends f to $\bar\partial f$. We denote this bundle by $\mathbf{E}$ and the section by Φ. It follows from the construction that $\Phi^{-1}(0)$ is homeomorphic to $\overline{\mathfrak{M}}_{g,n}(X,A)^J$. In this setting, defining the GW-invariants of (X,ω) is essentially about constructing the Euler class of $[\Phi:\mathbf{B}\to\mathbf{E}]$. Such construction is not obvious since $\mathbf{B}$ is an infinite dimensional topological space. Although at each $w\in\Phi^{-1}(0)$ the formal differential $d\Phi(w):T_w\mathbf{B}\to\mathbf{E}_w$ is Fredholm, which has real index $2r_{\exp}$, the conventional perturbation scheme does not apply directly since near maps in $\mathbf{B}$ whose domains are singular the space $\mathbf{B}$ is not smooth and $\mathbf{E}$ does not admit local trivializations. To overcome this difficulty, in [LT2] the authors introduced the notion of weakly $\mathbb{Q}$-Fredholm bundles, showed that $[\Phi:\mathbf{B}\to\mathbf{E}]$ is a weakly $\mathbb{Q}$-Fredholm bundle and that any weakly $\mathbb{Q}$-Fredholm bundle admits an Euler class. Let

$$e[\Phi:\mathbf{B}\to\mathbf{E}] \in H_{2r_{\exp}}(\mathbf{B};\mathbb{Q})$$

be the Euler class. Since the evaluation map of $\mathfrak{M}_{g,n}(X,A)^J$ extends to an evaluation map $\mathbf{e}:\mathbf{B}\to X^n$, the Euler class, which will also be referred to as the symplectic virtual cycle of $\overline{\mathfrak{M}}_{g,n}(X,A)^J$, defines a multi-linear map $\gamma^{X,J}_{A,g,n}$ as in (1.1.1). $\gamma^{X,J}_{A,g,n}$ are symplectic GW-invariants.

We will review the notion of weakly smooth Fredholm bundles in section 2. Here to say the least, $[\Phi:\mathbf{B}\to\mathbf{E}]$ is weakly Fredholm means that near each point of $\Phi^{-1}(0)$ we can find a finite rank subbundle $\mathbf{V}$ of $\mathbf{E}$ such that $U=\Phi^{-1}(\mathbf{V})$ is a smooth finite dimensional manifold, $V:=\mathbf{V}|_U$ is a smooth vector bundle and the lift $\phi:U\to V$ of Φ is smooth. (Note that the rank of V may vary but $\dim_{\mathbb{R}} U - \operatorname{rank}_{\mathbb{R}} V = 2r_{\exp}$). For such finite dimensional models $[\phi:U\to V]$, which are called weakly smooth approximations, we can perturb ϕ slightly to obtain ϕ' so that $\phi'^{-1}(0)$ are smooth manifolds in U. Note $\dim\phi'^{-1}(0) = 2r_{\exp}$. To construct the Euler class, we first cover a neighborhood of $\Phi^{-1}(0)$ in $\mathbf{B}$ by finitely many such approximations that satisfy certain compatibility condition. We then perturb each section in the approximation and obtain a collection of locally closed $\mathbb{Q}$-submanifolds of $\mathbf{B}$ of dimension $2r_{\exp}$. By imposing certain compatibility condition on the perturbations, this collection of $\mathbb{Q}$-submanfolds patch together to form a $2r_{\text{exp}}$-dimensional cycle in $\mathbf{B}$, which represents a homology class in $H_{2r_{\text{exp}}}(\mathbf{B};\mathbb{Q})$. This is the Euler class of $[\Phi:\mathbf{B}\to\mathbf{E}]$.

Now we assume that X is a smooth projective variety and ω is a Kähler form of X. Let J be the complex structure of X. Then $\overline{\mathfrak{M}}_{g,n}(X,A)$ is homeomorphic to $\overline{\mathfrak{M}}_{g,n}(X,A)^J$. The two GW-invariants $\gamma^X_{A,g,n}$ and $\gamma^{X,J}_{A,g,n}$ will be identical if the homology classes $[\overline{\mathfrak{M}}_{g,n}(X,A)]^{\text{vir}}$ and $e[\Phi:\mathbf{B}\to\mathbf{E}]$ are identical. Here we view $[\overline{\mathfrak{M}}_{g,n}(X,A)]^{\text{vir}}$ as a class in $H_*(\mathbf{B};\mathbb{Q})$ using

$$\overline{\mathfrak{M}}_{g,n}(X,A)\sim_{\text{homeo}}\overline{\mathfrak{M}}_{g,n}(X,A)^J\subset\mathbf{B}.$$

To illustrate why these two classes are equal, let us examine the following simple model. Let Z be a compact smooth variety and let E be a holomorphic vector bundle over Z with a holomorphic section s. There are two ways to construct the Euler class of E. One is to perturb s to a smooth section r so that the graph of r is transversal to the zero section of E. The Euler class of E is the homology class in $H_*(Z;\mathbb{Q})$ of $r^{-1}(0)$. This is the topological construction of the Euler class of E. The algebraic construction is as follows. Let t be a large scalar and let Γ_{ts} be the graph of ts in the total space of E. Since s is an algebraic section, it follows that the limit

$$\Gamma_{\infty s}=\lim_{t\to\infty}\Gamma_{ts}$$

is a complex dimension $\dim Z$ cycle supported on union of subvarieties of E. We then let r be a smooth section of E in general position and let $\Gamma_{\infty s}\cap\Gamma_r$ be their intersection. Its image in Z defines a homology class in $H_*(\mathbf{Z};\mathbb{Q})$. It is a classical result that this also defines the Euler class $e(E)$.

Back to our construction of GW-invariants, the analytic construction of GW-invariants, which was based on perturbations of sections in the finite dimensional models (weakly smooth approximations) $[\phi:U\to V]$, is clearly a generalization of the topological construction of the Euler classes of vector bundles. As to the algebraic construction of GW-invariants, it is based on a cone in a $\mathbb{Q}$-vector bundle over $\overline{\mathfrak{M}}_{g,n}(X,A)$. Comparing to the algebraic construction of the Euler class of $E\to Z$, what is missing is the section s of which the cone is the limit of the graphs of its dilations. Following [LT1], the cone $\Gamma_{\infty s}$ only relies on the restriction of s to an "infinitesimal" neighborhood of $s^{-1}(0)$ in Z, and can also be reconstructed using the Kuranishi maps of the obstruction theory to deformations of points in $s^{-1}(0)$ induced by the defining equation $s=0$. Along this line, to each finite dimensional model $[\phi:U\to V]$ we can form a cone $\Gamma_{\infty\phi}=\lim\Gamma_{t\phi}$ in $V|_{\phi^{-1}(0)}$. Hence to show that the two virtual moduli cycles coincide, it suffices to establish a relation, similar to quasi-isomorphism of complexes, between the cone N constructed based the obstruction theory of $\overline{\mathfrak{M}}_{g,n}(X,A)$ and the collection $\{\Gamma_{\infty\phi}\}$. This is reduced to showing that the obstruction theory to deformations of maps in $\overline{\mathfrak{M}}_{g,n}(X,A)$ is identical to the obstruction theory to deformations of elements in $\phi^{-1}(0)$ induced by the defining equation ϕ. This identification of two obstruction theories follows from the canonical isomorphism of the Cěch cohomology and the Dolbeault cohomology of vector bundles.

The layout of the paper is as follows. In section two, we will recall the analytic construction of the GW-invariants of symplectic manifolds. We will construct the Euler class of $[\Phi:\mathbf{B}\to\mathbf{E}]$ in details using the weakly smooth approximations constructed in [LT2]. In section three, we will construct a collection of holomorphic weakly smooth approximations for projective manifolds. The proof of the Comparisom Theorem will occupy the last section of this paper.

2. Symplectic construction of GW invariants. The goal of this section is to review the symplectic construction of the GW-invariants of algebraic varieties. We will emphasize on those parts that are relevant to our proof of the Comparison Theorem. In this section, we will use the standard notation in real differential geometry.

We begin with the symplectic construction of GW-invariants. Let X be a smooth complex projective variety, and let $A \in H_2(X, \mathbb{Z})$ and let g, $n \in \mathbb{Z}$ be fixed once and for all. We recall the notion of stable C^l-maps [LT2, Definition 2.1].

DEFINITION 2.1. *An n-pointed stable map is a collection $(f; \Sigma; x_1 \ldots, x_n)$ consisting of an n-pointed connected prestable complex curve $(\Sigma; x_1, \ldots, x_n)$ possibly with normal crossing singularity and a smooth map $f : \Sigma \to X$ such that every rational component of Σ that contains at most two nodal and marked points altogether has non-trivial image under f.*

For convenience, we will abbreviate $(f; \Sigma; x_1, \ldots, x_n)$ to $(f; \Sigma; \overrightarrow{x})$. Often, we will use $\mathcal{C}$ to denote an arbitrary stable map and use $f_{\mathcal{C}}$ and $\Sigma_{\mathcal{C}}$ to denote its corresponding mapping and domain. Two stable maps $(f; \Sigma; \overrightarrow{x})$ and $(f'; \Sigma'; \overrightarrow{y})$ are said to be equivalent if there is an isomorphism $\rho: \Sigma \to \Sigma'$ such that $f' \circ \rho = f$ and $\rho(\overrightarrow{x}) = \overrightarrow{y}$. When $(f; \Sigma; \overrightarrow{x}) \equiv (f'; \Sigma'; \overrightarrow{y})$, such a ρ is called an automorphism of $(f; \Sigma; \overrightarrow{x})$.

We let $\mathbf{B}$ be the space of equivalence classes $[\mathcal{C}]$ of C^l-stable maps $\mathcal{C}$ such that the arithmetic genus of $\Sigma_{\mathcal{C}}$ is g and $f_{\mathcal{C}*}([\Sigma]) = A \in H_2(X; \mathbb{Z})$. Note that $\mathbf{B}$ was denoted by $\bar{\mathcal{F}}^l_A(X, g, n)$ in [LT2]. Over $\mathbf{B}$ there is a generalized bundle $\mathbf{E}$ defined as follows. Let $\mathcal{C}$ be any stable map and let $\tilde{f} : \tilde{\Sigma} \to X$ be the composite of f with $\pi : \tilde{\Sigma} \to \Sigma$, where $\tilde{\Sigma}$ is the normalization of Σ. We define $\Lambda^{0,1}_{\mathcal{C}}$ to be the space of all C^{l-1}-smooth sections of $(0, 1)$-forms of $\tilde{\Sigma}$ with values in $\tilde{f}^* T_X$. Assume $\mathcal{C}$ and $\mathcal{C}'$ are two equivalent stable maps, then there is a canonical isomorphism $\Lambda^{0,1}_{\mathcal{C}'} \cong \Lambda^{0,1}_{\mathcal{C}}$. We let $\Lambda^{0,1}_{[\mathcal{C}]}$ be $\Lambda^{0,1}_{\mathcal{C}} / \operatorname{Aut}(\mathcal{C})$. Then the union

$$\mathbf{E} = \bigcup_{[\mathcal{C}] \in \mathbf{B}} \Lambda^{0,1}_{[\mathcal{C}]}$$

is a fibration over $\mathbf{B}$ whose fibers are finite quotients of infinite dimensional linear spaces. There is a natural section

$$\Phi : \mathbf{B} \longrightarrow \mathbf{E}$$

defined as follows. For any stable map $\mathcal{C}$ with mapping $f : \Sigma \to X$, we define $\Phi(\mathcal{C})$ to be the image of $\bar{\partial} f \in \Lambda^{0,1}_{\mathcal{C}}$ in $\Lambda^{0,1}_{[\mathcal{C}]}$. Obviously, for $\mathcal{C} \sim \mathcal{C}'$ we have $\Phi(\mathcal{C}) = \Phi(\mathcal{C}')$. Thus Φ descends to a map $\mathbf{B} \to \mathbf{E}$, which we still denote by Φ.

From now on, we will denote by $\mathfrak{M}_{g,n}(X, A)$ the moduli scheme of stable moprhisms $f : C \to X$ from n-pointed genus g curves C (possibly with nodal singularities) to X with $f_*([C]) = A$.

LEMMA 2.2. *The vanishing locus of Φ is canonically homeomorphic to the underlying topological space of $\mathfrak{M}_{g,n}(X, A)$.*

Proof. A stable C^l-stable map $\mathcal{C}$ in $\mathbf{B}$ is in the vanishing locus of Φ if and only if $f_{\mathcal{C}}$ is holomorphic. Since $\Sigma_{\mathcal{C}}$ is compact, $\mathcal{C}$ is the underlying analytic map of a stable morphism. This induces a canonical map $\Phi^{-1}(0) \to \mathfrak{M}_{g,n}(X, A)^{\mathrm{top}}$. It is easy to see that it is bijective and continuous. This proves the lemma. □

To discuss the smoothness of Φ, we need the local uniformizing charts of $\Phi : \mathbf{B} \to \mathbf{E}$ near $\Phi^{-1}(0)$. Let $w \in \mathbf{B}$ be any point represented by the stable map $(f_0; \Sigma_0; \overrightarrow{x})$ with automorphism group G_w. We pick integers $r > 0$ and smooth ample divisor H with $[H] \cdot [A] = r$ such that any point $x \in f_0^{-1}(H)$ is contained in the smooth locus of Σ_0 and

$$\operatorname{Im}(df_0(x)) + T_{f_0(x)}H = T_{f_0(x)}X. \tag{2.2.1}$$

Now let $\mathbf{U} \subset \mathbf{B}$ be a sufficiently small neighborhood of $w \in \mathbf{B}$ and let $\tilde{\mathbf{U}}$ be the collection of all $(\mathcal{C}; \overrightarrow{z})$ where $\mathcal{C} \in \mathbf{U}$ and $\overrightarrow{z} = (z_{n+1}, \cdots, z_{n+r})$ of which the z_i's are an ordering of the discrete set of $f_0^{-1}(H)$. By making $\mathbf{U}$ sufficiently small, we can assume without loss of generality that for all $\mathcal{C} \in \mathbf{U}$ we have $\operatorname{card}(f_0^{-1}(H)) = r$.[1] Let $\pi_{\mathbf{U}} : \tilde{\mathbf{U}} \to \mathbf{U}$ be the projection that sends $(\mathcal{C}; \overrightarrow{z})$ to $\mathcal{C}$. Clearly, G_w acts on $\pi_{\mathbf{U}}^{-1}(w)$ canonically by permuting their $(n+r)$-marked points. Namely, for any $\sigma \in G_w$ and $\mathcal{C} \in \pi_{\mathbf{U}}^{-1}(w)$ with marked points $z_1, \ldots, z_{n+r}$, $\sigma(\mathcal{C})$ is the same map with the marked points $\sigma(z_1), \ldots, \sigma(z_{n+r})$. In particular, we can view G_w as a subgroup of the permutation group S_{n+r}. Note that if H is in general position then elements in $\tilde{\mathbf{U}}$ have no non-trivial automorphisms and have distinct marked points. Let $G_{\tilde{\mathbf{U}}} = G_w$. Since fibers of $\pi_{\mathbf{U}}$ are invariant under $G_{\tilde{\mathbf{U}}}$, $\pi_{\mathbf{U}}$ induces a map $\tilde{\mathbf{U}}/G_{\tilde{\mathbf{U}}} \to \mathbf{U}$, which is obviously a covering[2] if $\mathbf{U}$ is sufficiently small. $\pi_{\mathbf{U}} : \tilde{\mathbf{U}} \to \mathbf{U}$ is called a local uniformizing chart of $w \in \mathbf{B}$. Next, we look at the bundle $\mathbf{E}$. Let $\tilde{\mathbf{U}} \to \mathbf{U}$ be as before. We let

$$\mathbf{E}_{\tilde{\mathbf{U}}} = \bigcup_{\mathcal{C} \in \tilde{\mathbf{U}}} \Lambda_{\mathcal{C}}^{0,1}$$

and let $\Phi_{\tilde{\mathbf{U}}} : \tilde{\mathbf{U}} \to \mathbf{E}_{\tilde{\mathbf{U}}}$ be the section that sends $(\mathcal{C}, \overrightarrow{z})$ to $\bar{\partial} f_{\mathcal{C}}$. Then $\Phi_{\tilde{\mathbf{U}}}$ is $G_{\tilde{\mathbf{U}}}$-equivariant and $\Phi|_{\mathbf{U}} : \mathbf{U} \to \mathbf{E}|_{\mathbf{U}}$ is the descent of $\Phi_{\tilde{\mathbf{U}}}/G_{\tilde{\mathbf{U}}} : \tilde{\mathbf{U}}/G_{\tilde{\mathbf{U}}} \to \mathbf{E}_{\tilde{\mathbf{U}}}/G_{\tilde{\mathbf{U}}}$. Note that fibers of $\mathbf{E}_{\tilde{\mathbf{U}}}$ over $\tilde{\mathbf{U}}$ are linear spaces. In the following, we will call $\Lambda = (\tilde{\mathbf{U}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \Phi_{\tilde{\mathbf{U}}}, G_{\tilde{U}})$ a uniformizing chart of $(\mathbf{B}, \mathbf{E}, \Phi)$ over $\mathbf{U}$.

In case $\mathbf{V} \subset \mathbf{U}$ is an open subset, we let $\tilde{\mathbf{V}} = \pi_{\mathbf{U}}^{-1}(\mathbf{V})$, let $G_{\tilde{\mathbf{V}}} = G_{\tilde{\mathbf{U}}}$, let $\mathbf{E}_{\tilde{\mathbf{V}}} = \mathbf{E}_{\tilde{\mathbf{U}}}|_{\tilde{\mathbf{V}}}$ and let $\Phi_{\tilde{\mathbf{V}}} = \Phi_{\tilde{\mathbf{U}}}|_{\tilde{\mathbf{V}}}$. The data $\Lambda' = (\tilde{\mathbf{V}}, \mathbf{E}_{\tilde{\mathbf{V}}}, \Phi_{\tilde{\mathbf{V}}}, G_{\tilde{\mathbf{V}}})$ also forms a uniformizing chart of $(\mathbf{B}, \mathbf{E}, \Phi)$. We call it the *restriction* of the original chart to $\mathbf{V}$, and denote it by $\Lambda|_{\mathbf{V}}$. We can also construct uniformizing charts by pull back. Let $G_{\tilde{\mathbf{V}}}$ be a finite group acting effectively on a topological space $\tilde{\mathbf{V}}$, let $G_{\tilde{\mathbf{V}}} \to G_{\tilde{\mathbf{U}}}$ be a homomorphism and $\varphi : \tilde{\mathbf{V}} \to \tilde{\mathbf{U}}$ be a $G_{\tilde{\mathbf{V}}}$-equivariant map so that $\tilde{\mathbf{V}}/G_{\tilde{\mathbf{V}}} \to \tilde{\mathbf{U}}/G_{\tilde{\mathbf{U}}}$ is a local covering map. Then we set $\mathbf{E}_{\tilde{\mathbf{V}}} = \varphi^* \mathbf{E}_{\tilde{\mathbf{U}}}$ and $\Phi_{\tilde{\mathbf{V}}} = \varphi^* \Phi_{\tilde{\mathbf{U}}}$. The data $\Lambda' = (\tilde{\mathbf{V}}, \mathbf{E}_{\tilde{\mathbf{V}}}, \Phi_{\tilde{\mathbf{V}}}, G_{\tilde{\mathbf{V}}})$ is also a uniformizing chart. We will call Λ' the *pull back* of Λ, and denote it by $\varphi^* \Lambda$. In the following, we will denote the collection of all uniformizing charts of $(\mathbf{B}, \mathbf{E}, \Phi)$ by $\mathfrak{C}$.

The collection $\mathfrak{C}$ has the following compatibility property. Let

$$\Lambda_i = (\tilde{\mathbf{U}}_i, \mathbf{E}_{\tilde{\mathbf{U}}_i}, \Phi_{\tilde{\mathbf{U}}_i}, G_{\tilde{\mathbf{U}}_i}), \quad i = 1, \ldots, k,$$

be a collection of uniformizing charts in $\mathfrak{C}$ over $\mathbf{U}_i \subset \mathbf{B}$ respectively. Let $p \in \cap_{i=1}^k \mathbf{U}_i$ be any point. Then there is a uniformizing chart $\Lambda = (\tilde{\mathbf{V}}, \mathbf{E}_{\tilde{\mathbf{V}}}, \Phi_{\tilde{\mathbf{V}}}, G_{\tilde{\mathbf{V}}})$ over $\mathbf{V} \subset \cap^k \mathbf{U}_i$

[1]In case X is a symplectic manifold, then we should use locally closed real codimension 2 submanifold instead of H, as did in [LT2]. Here we use this construction of uniformizing charts because it is compatible to the construction of atlas of the stack $\mathfrak{M}_{g,n}(X, A)$ in algebraic geometry.

[2]In this paper we call $p : A \to B$ a covering if p is a covering projection [Sp] and $\#(p^{-1}(x))$ is independent of $x \in B$. We call $p : A \to B$ a local covering if $p(A)$ is open in B and $p : A \to p(A)$ is a covering.

with $p \in \mathbf{V}$ such that there are homomorphisms $G_{\tilde{\mathbf{V}}} \to G_{\tilde{\mathbf{U}}_i}$ and equivariant local covering maps $\varphi_i : \tilde{\mathbf{V}} \to \pi_{\tilde{\mathbf{U}}_i}^{-1}(\mathbf{V}) \subset \tilde{\mathbf{U}}_i$ compatible with $\tilde{\mathbf{V}} \to \mathbf{V}$ and $\pi_{\tilde{\mathbf{U}}_i}^{-1}(\mathbf{V}) \to \mathbf{V} \subset \mathbf{U}_i$, such that $\varphi_i^*(\mathbf{E}_{\tilde{\mathbf{U}}_i}, \Phi_{\tilde{\mathbf{U}}_i}) \cong (\mathbf{E}_{\tilde{\mathbf{V}}}, \Phi_{\tilde{\mathbf{V}}})$. In this case, we say Λ is *finer than* $\Lambda_i|_{\mathbf{V}}$.

The main difficulty in constructing the GW invariants in this setting is that the smoothness of $(\tilde{\mathbf{U}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \Phi_{\tilde{\mathbf{U}}})$ is unclear when $\mathbf{U}$ contains maps whose domains are singular. To overcome this difficulty, the authors introduced the notion of generalized Fredholm bundles in [LT2]. The main result of [LT2] is the following theorem that is fundamental to the construction of the GW invariants of symplectic manifolds.

THEOREM 2.3. *The data $[\Phi : \mathbf{B} \to \mathbf{E}]$ is a generalized oriented Fredholm V-bundle of relative index $2r_{\text{exp}}$, where $r_{\text{exp}} = c_1(X) \cdot A + n + (n-3)(1-g)$ is half of the virtual (real) dimension of $\Phi^{-1}(0)$.*

THEOREM 2.4. *For any generalized oriented Fredholm V-bundle $[\Phi : \mathbf{B} \to \mathbf{E}]$ of relative index r, we can assign to it an Euler class $e([\Phi : \mathbf{B} \to \mathbf{E}])$ in $H_r(\mathbf{B}; \mathbb{Q})$ that satisfies all the expected properties of the Euler classes.*

As explained in the introduction, the pairing of the Euler class of $[\Phi : \mathbf{B} \to \mathbf{E}]$ with the tautological topological class provides the symplectic version of the GW invariants of X. Further, the Comparison Theorem we set out to prove amounts to compare this Euler class with the image of the virtual moduli cycle $[\mathfrak{M}_{g,n}(X, A)]^{\text{vir}}$ in $H_r(\mathbf{B}; \mathbb{Q})$ via the inclusion $\mathfrak{M}_{g,n}(X, A)^{\text{top}} \subset \mathbf{B}$. In the remainder part of this section, we will list the properties of $[\Phi : \mathbf{B} \to \mathbf{E}]$ that are relevant to the construction of its Euler class. This list is essentially equivalent to saying that $[\Phi : \mathbf{B} \to \mathbf{E}]$ is a generalized oriented Fredholm V-bundle. After that, we will construct the Euler class of $[\Phi : \mathbf{B} \to \mathbf{E}]$ in details.

We begin with the notion of weakly smooth structure. A local smooth approximation of $[\Phi : \mathbf{B} \to \mathbf{E}]$ over $\mathbf{U} \subset \mathbf{B}$ is a pair $(\Lambda, \mathbf{V})$, where $\Lambda = (\tilde{\mathbf{U}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \Phi_{\tilde{\mathbf{U}}}, G_{\tilde{\mathbf{U}}})$ is a uniformizing chart over $\mathbf{U}$ and $\mathbf{V}$ is a finite equi-rank $G_{\tilde{\mathbf{U}}}$-vector bundle over $\tilde{\mathbf{U}}$ that is a $G_{\tilde{\mathbf{U}}}$-equivariant subbundle of $\mathbf{E}_{\tilde{\mathbf{U}}}$ such that

$$U := \Phi_{\tilde{\mathbf{U}}}^{-1}(\mathbf{V}) \subset \tilde{\mathbf{U}}$$

is an equi-dimensional smooth manifold,

$$V := \mathbf{V}|_U$$

is a smooth vector bundle and the lifting $\phi_{\mathbf{V}} : U \to V$ of $\Phi_{\tilde{\mathbf{U}}}|U$ is a smooth section. An orientation of $(\Lambda, \mathbf{V})$ is a $G_{\tilde{\mathbf{V}}}$-invariant orientation of the real line bundle $\wedge^{\text{top}}(TU) \otimes \wedge^{\text{top}}(V)^{-1}$ over U. We call $\operatorname{rank} V - \dim U$ the index of $(\Lambda, \mathbf{V})$ (We remind that all ranks and dimensions in this section are over reals). Now assume that $(\Lambda', \mathbf{V}')$ is another weakly smooth structure of identical index over $\mathbf{U}' \subset \mathbf{B}$. We say that $(\Lambda', \mathbf{V}')$ is finer than $(\Lambda, \mathbf{V})$ if the following holds.

(1). The restriction $\Lambda'|_{\mathbf{U}' \cap \mathbf{U}}$ is finer than $\Lambda|_{\mathbf{U}' \cap \mathbf{U}}$;

(2). If we let $\varphi : \pi_{\tilde{\mathbf{U}}'}^{-1}(\mathbf{U}' \cap \mathbf{U}) \to \pi_{\tilde{\mathbf{U}}}^{-1}(\mathbf{U}' \cap \mathbf{U})$ be the covering map then $\varphi^* \mathbf{V} \subset \varphi^* \mathbf{E}_{\tilde{\mathbf{U}}} \equiv \mathbf{E}_{\tilde{\mathbf{U}}'}|_{\pi_{\tilde{\mathbf{U}}'}^{-1}(\mathbf{U}' \cap \mathbf{U})}$ is a subbundle of $\mathbf{V}'|_{\pi_{\tilde{\mathbf{U}}'}^{-1}(\mathbf{U}' \cap \mathbf{U})}$;

(3). For any $w \in U' \cap \Phi_{\tilde{\mathbf{U}}'}^{-1}(0)$ the homomorphism $T_w U' \to \big(V'/\varphi^* V\big)|_w$ induced by $d\phi_{\mathbf{V}'}(w) : T_w U' \to V'|_w$ is surjective, and the map $\phi_{\mathbf{V}'}^{-1}(\varphi^* \mathbf{V}) \to U$ induced by φ is a local diffeomorphism between smooth manifolds. Note that the last condition implies that if we identify $T_{\varphi(w)} U$ with $T_w \phi_{\mathbf{V}'}^{-1}(\varphi^* \mathbf{V}) \subset T_w U'$, then the induced homomorphism

$$T_wU'/T_{\varphi(w)}U \longrightarrow (V'/\varphi^*V)|_w \tag{2.2.2}$$

is an isomorphism. In case both $(\Lambda, \mathbf{V})$ and $(\Lambda', \mathbf{V}')$ are oriented, then we require that the orientation of $(\Lambda, \mathbf{V})$ coincides with that of $(\Lambda', \mathbf{V}')$ based on the isomorphism

$$\wedge^{\mathrm{top}}(T_wU') \otimes \wedge^{\mathrm{top}}(V'|_w)^{-1} \cong \wedge^{\mathrm{top}}(T_{\varphi(w)}U) \otimes \wedge^{\mathrm{top}}(V|_{\varphi(w)})^{-1} \tag{2.2.3}$$

induced by (2.2.2).

Now let $\mathfrak{A} = \{(\Lambda_i, \mathbf{V}_i)\}_{i\in\mathcal{K}}$ be a collection of oriented smooth approximations of $(\mathbf{B}, \mathbf{E}, \Phi)$. In the following, we will denote by $\mathbf{U}_i$ the open subsets of $\mathbf{B}$ such that Λ_i is a smooth chart over $\mathbf{U}_i$. We say $\mathfrak{A}$ covers $\phi^{-1}(0)$ if $\phi^{-1}(0)$ is contained in the union of the images of $\mathbf{U}_i$ in $\mathbf{B}$.

DEFINITION 2.5. *An index r oriented weakly smooth structure of $(\mathbf{B}, \mathbf{E}, \Phi)$ is a collection $\mathfrak{A} = \{(\Lambda_i, \mathbf{V}_i)\}_{i\in\mathcal{K}}$ of index r oriented smooth approximations such that $\mathfrak{A}$ covers $\Phi^{-1}(0)$ and that for any $(\Lambda_i, \mathbf{V}_i)$ and $(\Lambda_j, \mathbf{V}_j)$ in $\mathfrak{A}$ with $p \in \mathbf{U}_i \cap \mathbf{U}_j$, there is a $(\Lambda_k, \mathbf{V}_k) \in \mathfrak{A}$ such that $p \in \mathbf{U}_k$ and $(\Lambda_k, \mathbf{V}_k)$ is finer than $(\Lambda_i, \mathbf{V}_i)$ and $(\Lambda_j, \mathbf{V}_j)$.*

Let $\mathfrak{A}'$ be another index r oriented weakly smooth structure of $(\mathbf{B}, \mathbf{E}, \Phi)$. We say $\mathfrak{A}'$ is finer than $\mathfrak{A}$ if for any $(\Lambda, \mathbf{V}) \in \mathfrak{A}$ over $\mathbf{U} \subset \mathbf{B}$ and $p \in \mathbf{U} \cap \Phi^{-1}(0)$, there is a $(\Lambda', \mathbf{V}') \in \mathfrak{A}'$ over $\mathbf{U}'$ such that $p \in \mathbf{U}'$ and $(\Lambda', \mathbf{V}')$ is finer than $(\Lambda, \mathbf{V})$. We say that two weakly smooth structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are equivalent if there is a third weakly smooth structure that is finer than both $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

PROPOSITION 2.6 ([LT2]). *The tuple $(\mathbf{B}, \mathbf{E}, \Phi)$ constructed at the beginning of this section admits a canonical oriented weakly smooth structure of index $2r_{\mathrm{exp}}$.*

We remark that the construction of such a weakly smooth structure is the core of the analytic part of [LT2].

In the following, we will use the weakly smooth structure of $[\Phi : \mathbf{B} \to \mathbf{E}]$ to construct its Euler class. The idea of the construction is as follows. Given a local smooth approximation $(\Lambda, \mathbf{V})$ over $\mathbf{U} \subset \mathbf{B}$, it associates a finite dimensional model consisting of a smooth manifold U, a vector bundle V over U and a smooth section $\phi_{\mathbf{V}} : U \to V$. Following the topological construction of the Euler classes, we perturb $\phi_{\mathbf{V}}$ to a new section $\tilde{\phi}_{\mathbf{V}} : U \to V$ so that $\tilde{\phi}_{\mathbf{V}}$ is transversal to the zero section of V. With special care, the currents $\{\tilde{\phi}_{\mathbf{V}}^{-1}(0)\}$ patch together to form a well-defined cycle in $\mathbf{B}$. The Euler class of $[\Phi : \mathbf{B} \to \mathbf{E}]$ is the homology class represented by this cycle. As is clear from this description, the main difficulty of this construction is how to make sure that the currents $\{\tilde{\phi}_{\mathbf{V}}^{-1}(0)\}$ patch together.

Let $\mathfrak{A} = \{(\Lambda_\alpha, \mathbf{V}_\alpha)\}_{\alpha\in\mathcal{K}}$ be the weakly smooth structure provided by Proposition 2.6. For convenience, for any $\alpha \in \mathcal{K}$ we will denote the corresponding uniformizing chart Λ_α by $(\tilde{\mathbf{U}}_\alpha, \tilde{\mathbf{E}}_\alpha, \tilde{\Phi}_\alpha, G_\alpha)$. Accordingly, we will denote the projection $\pi_{\mathbf{U}_\alpha} : \tilde{\mathbf{U}}_\alpha \to \mathbf{U}_\alpha$ by π_α, denote $\tilde{\Phi}_\alpha^{-1}(\mathbf{V}_\alpha)$ by U_α, denote $\mathbf{V}_\alpha|_{U_\alpha}$ by V_α and denote the lifting of $\tilde{\Phi}_\alpha|_{U_\alpha} : U_\alpha \to \tilde{\mathbf{E}}_\alpha|_{U_\alpha}$ by $\phi_\alpha : U_\alpha \to V_\alpha$. Without loss of generality, we can assume that for any approximation $(\Lambda_\alpha, \mathbf{V}_\alpha) \in \mathfrak{A}$ over $\mathbf{U}_\alpha$ and any $\mathbf{U}' \subset \mathbf{U}_\alpha$, the restriction $(\Lambda_\alpha, \mathbf{V}_\alpha)|_{\mathbf{U}'}$ is also a member in $\mathfrak{A}$. In the following, we call subsets $S \subset \mathbf{U}_\alpha$ *symmetric* if $S = \pi_\alpha^{-1}(\pi_\alpha(S))$.

Our first step is to pick a covering data for $\Phi^{-1}(0) \subset \mathbf{B}$ provided by the following covering lemma.

LEMMA 2.7 ([LT2]). *There is a finite collection $\mathcal{L} \subset \mathcal{K}$ and a total ordering of $\mathcal{L}$ of which the following holds: the set $\Phi^{-1}(0)$ is contained in the union of $\{U_\alpha\}_{\alpha\in\mathcal{L}}$;*

for any pair $\alpha < \beta \in \mathcal{L}$ the approximation $(\Lambda_\beta, \mathbf{V}_\beta)$ is finer than the approximation $(\Lambda_\alpha, \mathbf{V}_\alpha)$.

Proof. The lemma is part of Proposition 2.2 in [LT2]. It is proved there by using the stratified structures of $(\mathbf{B}, \mathbf{E}, \Phi)$. Here we will give a direct proof of this by using the definition of smooth approximations and by assuming $\Phi^{-1}(0)$ is triangulable. This is the case when X is projective. Let k be the real dimension of $\Phi^{-1}(0)$. To prove the lemma, we construct $k+1$ subsets $\mathcal{L}_k, \dots, \mathcal{L}_0 \subset \mathcal{K}$ such that for each $\alpha \in \cup_{i=0}^k \mathcal{L}_i$ there is an open symmetric subset $\mathbf{U}'_\alpha \Subset \mathbf{U}_\alpha$ such that in addition to $U'_\alpha = U_\alpha \cap \pi_\alpha^{-1}(\mathbf{U}'_\alpha) \Subset U_\alpha$ we have: (1) for each $i \le k$ the set

$$\mathbf{Z}_i = \Phi^{-1}(0) - \cup_{j \ge i} \cup_{\alpha \in \mathcal{L}_j} \mathbf{U}'_\alpha \tag{2.2.4}$$

is a triangulable space whose dimension is at most $i-1$; (2) For any pair of distinct $(\alpha, \beta) \in \mathcal{L}_i \times \mathcal{L}_j$ with $i \le j$, the restriction $(\Lambda_\alpha, \mathbf{V}_\alpha)|_{\mathbf{U}'_\alpha \cap \mathbf{U}'_\beta}$ is finer than $(\Lambda_\beta, \mathbf{V}_\beta)|_{\mathbf{U}'_\alpha \cap \mathbf{U}'_\beta}$. We will construct $\mathcal{L}_i$ inductively, starting from $\mathcal{L}_k$. We first pick a finite $\mathcal{L}_k \subset \mathcal{K}$ so that $\cup_{\alpha \in \mathcal{L}_k} \mathbf{U}_\alpha \supset \Phi^{-1}(0)$. This is possible since $\Phi^{-1}(0)$ is compact. Since $\Phi^{-1}(0)$ is triangulable, we can find a symmetric $\mathbf{U}'_\alpha \subset \mathbf{U}_\alpha$ for each $\alpha \in \mathcal{L}_k$ so that $\{\mathbf{U}'_\alpha\}_{\alpha \in \mathcal{L}_k}$ is disjoint, $U'_\alpha \Subset U_\alpha$ and the $\mathbf{Z}_k$ defined in (2.2.4) is trangulable with dimension at most $k-1$. Now we assume that we have found $\mathcal{L}_k, \dots, \mathcal{L}_i$ as desired. Then for each $x \in \mathbf{Z}_i$ we can find a neighborhood $\mathbf{O}$ of $x \in \mathbf{B}$ such that for any $\alpha \in \cup_{j \ge i} \mathcal{L}_j$ either $x \in \mathbf{U}_\alpha$ or $\mathbf{O} \cap \mathbf{U}'_\alpha = \emptyset$. Let $\mathcal{I}_x$ be those α in $\cup_{j \ge i} \mathcal{L}_j$ such that $x \in \mathbf{U}_\alpha$. Then by the property of $\mathfrak{A}$ there is a $\beta \in \mathcal{K}$ so that $(\Lambda_\beta, \mathbf{V}_\beta)$ is finer than $(\Lambda_\alpha, \mathbf{V}_\alpha)|_{\mathbf{U}'_\alpha}$ for all $\alpha \in \mathcal{I}_x$. Without loss of generality, we can assume that $\mathbf{U}_\beta \subset \mathbf{O}$. Then $(\Lambda_\beta, \mathbf{V}_\beta)$ is finer than $(\Lambda_\alpha, \mathbf{V}_\alpha)$ for all $\alpha \in \cup_{j \ge i} \mathcal{L}_j$. Since $\mathbf{Z}_i$ is compact, we can cover it by finitely many such $(\Lambda_\beta, \mathbf{V}_\beta)$'s, say indexed by $\mathcal{L}_{i-1} \subset \mathcal{K}$. On the other hand, since $\mathbf{Z}_i$ is triangulable with dimension at most $i-1$, we can find symmetric $\mathbf{U}'_\alpha \subset \mathbf{U}_\alpha$ for each $\alpha \in \mathcal{L}_{i-1}$ so that $U'_\alpha \Subset U_\alpha$ for $\alpha \in \mathcal{L}_{i-1}$ and $\mathbf{Z}_i - \cup_{\alpha \in \mathcal{L}_{i-1}} \mathbf{U}'_\alpha$ is trianglable with dimension at most $i-2$. This way, we can find the set $\mathcal{L}_k, \dots, \mathcal{L}_0$ as desired. In the end, we simply put $\mathcal{L} = \cup_{i=0}^k \mathcal{L}_i$. We give it a total ordering so that whenever $\alpha \in \mathcal{L}_i$, $i \ge j$ and $\beta \in \mathcal{L}_j$ then $\alpha \le \beta$. This proves the Lemma. □

We now fix such a collection $\mathcal{L}$ once and for all. Since $\mathcal{L}$ is totally ordered, in the following we will replace the index by integers that range from 1 to $\#(\mathcal{L})$ and use k to denote an arbitrary member of $\mathcal{L}$. Let $\{\phi_k : U_k \to V_k\}_{k \in \mathcal{L}}$ be the corresponding collection of finite dimensional models. We now build the comparison data into the collection $\{U_k\}_{k \in \mathcal{L}}$ and $\{V_k\}_{k \in \mathcal{L}}$. We denote the composite $U_k \to \mathbf{U}_k \to \mathbf{B}$ by ι_k. For any pair $k \ge l$, we set $U_{k,l} = \iota_k^{-1}(\iota_l(U_l))$, which is contained in U_k. Then there is a canonical map and a canonical vector bundle inclusion

$$f_k^l : U_{k,l} \to U_l \quad \text{and} \quad (f_k^l)^*(V_l) \xrightarrow{\subset} V_k|_{U_{k,l}} \tag{2.2.5}$$

that is part of the data making $(\Lambda_k, \mathbf{V}_k)$ finer than $(\Lambda_l, \mathbf{V}_l)$. Note that $U_{k,l} \subset U_k$ is a locally closed submanifold, $f_k^l(U_{k,l})$ is open in U_l and $f_k^l : U_{k,l} \to f_k^l(U_{k,l})$ is a covering map. Because of the compatibility condition, for any $k > l > m$ if $U_{k,l} \cap U_{k,m} \ne \emptyset$ then $f_k^l(U_{k,l} \cap U_{k,m}) \subset U_{l,m}$ and

$$f_l^m \circ f_k^l = f_k^m : U_{k,l} \cap U_{k,m} \longrightarrow U_m. \tag{2.2.6}$$

Further, restricting to $U_{k,l} \cap U_{k,m}$ the pull back

$$(f_k^m)^*(V_m)|_{U_{k,l} \cap U_{k,m}} = (f_k^l)^*(f_l^m)^*(V_m)|_{U_{k,l} \cap U_{k,m}} \subset V_k|_{U_{k,l} \cap U_{k,m}}. \tag{2.2.7}$$

In the following, we will use $\mathfrak{U}$ to denote the collection of data $\{(U_{k,l}, f_k^l)\}$ and use $\mathfrak{V}$ to denote the data $\{(V_k, (f_k^l)^*)\}$. We will call the pair $(\mathfrak{U}, \mathfrak{V})$ a good atlas of the weakly smooth structure $\mathfrak{A}$ of $[\Phi : \mathbf{B} \to \mathbf{E}]$. For technical reason, we need the notion of *pre-compact sub-atlas* of $(\mathfrak{R}, \mathfrak{V})$.

DEFINITION 2.8. *Let $\{S_k\}_{k\in\mathcal{L}}$ be a collection of symmetric open subsets $S_k \Subset U_k$ such that $\{S_k\}$ still covers $\Phi^{-1}(0)$. We let $S_{k,l} = (f_k^l)^{-1}(S_l) \cap S_k$, let $W_k = V_k|_{S_k}$ and let g_k^l and $(g_k^l)^*$ be the restriction to $S_{k,l}$ of f_k^l and $(f_k^l)^*$ respectively. Then $(\mathfrak{S}, \mathfrak{W})$, where $\mathfrak{S} = \{(S_{k,l}, g_k^l)\}$ and $\mathfrak{W} = \{(W_k, (g_k^l)^*)\}$, is also a good atlas of $[\Phi : \mathbf{B} \to \mathbf{E}]$. We call $(\mathfrak{S}, \mathfrak{W})$ a precompact sub-atlas of $(\mathfrak{R}, \mathfrak{V})$, and denote it in short by $\mathfrak{S} \Subset \mathfrak{R}$.*

To describe the relation between ϕ_k's, we need to introduce the notion of regular extension. Let M be a manifold and $M_0 \subset M$ be a locally closed submanifold. Let $V \to M$ be a smooth vector bundle and $V_0 \to M_0$ a subbundle of $V|_{M_0}$. We assume that both (M, V) and (M_0, V_0) are oriented. Let $\Gamma_M(V)$ be the space of smooth sections of V over M and $h \in \Gamma_M(V)$ be a section. We say that h is an extension of $h_0 \in \Gamma_{M_0}(V_0)$ if the restriction of h to M_0 is identical to h_0 under the inclusion $V_0 \subset V|_{M_0}$. We say h is a regular extension of h_0 if further for any $x \in M_0$ the homomorphism

$$dh(x) : T_xM/T_xM_0 \longrightarrow (V/V_0)|_x \tag{2.2.8}$$

is an isomorphism and the orientation of (M, V) and (M_0, V_0) are compatible over M_0 based on the isomorphism (2.2.8).

DEFINITION 2.9. *A collection $\mathfrak{h} := \{h_k\}_{k\in\mathcal{L}}$, where $h_k \in \Gamma_{U_k}(V_k)$, is called a (resp. regular) section of $\mathfrak{V}$ if for any pair $k \geq l \in \mathcal{L}$ the section h_k is a (resp. regular) extension of h_l.*

In the following, we will use $\mathfrak{h} : \mathfrak{U} \to \mathfrak{V}$ to denote a smooth section with $\mathfrak{h}$ understood to be $\{h_k\}_{k\in\mathcal{L}}$. We set $\mathfrak{h}^{-1}(0)$ to be the collection $\{h_k^{-1}(0)\}$ and set $\iota(\mathfrak{h}^{-1}(0))$ to be the union of $\iota_k(h_k^{-1}(0))$ in $\mathbf{B}$. We say that $\mathfrak{h}^{-1}(0)$ is proper if $\iota(\mathfrak{h}^{-1}(0))$ is compact. Without loss of generality, we can assume that $\dim U_k > 0$ for all $k \in \mathcal{L}$. We say that $\mathfrak{h}$ is transversal to the zero section $\mathbf{0} : \mathfrak{U} \to \mathfrak{V}$ if $\mathfrak{h}$ is a regular section and if for any $k \in \mathcal{L}$ the graph Γ_{h_k} of h_k is transversal to the 0 section of V_k in the total space of V_k.

We now give a useful criterion for the properness of $\mathfrak{h}^{-1}(0)$.

LEMMA 2.10. *Let the notation be as before. Then $\mathfrak{h}^{-1}(0)$ is proper if and only if there are symmetric open subsets $U_k' \Subset U_k$ for each $k \in \mathcal{L}$ such that*

$$\bigcup_{k\in\mathcal{L}} \iota_k(h_k^{-1}(0)) \subset \bigcup_{k\in\mathcal{L}} \iota_k(U_k')$$

and such that for each $k \in \mathcal{L}$,

$$h_k^{-1}(0) \cap (U_k - U_k') \subset \Big(\bigcup_{l<k} (f_k^l)^{-1}(U_l)\Big) \cup \Big(\bigcup_{l>k} f_k^l(U_{k,l}')\Big). \tag{2.2.9}$$

Proof. We first assume that $\mathbf{Z} = \cup_{k\in\mathcal{L}} \iota_k(h_k^{-1}(0))$ is compact. Then since $\{U_k\}_{k\in\mathcal{L}}$ covers $\mathbf{Z}$ and since $\dim U_k > 0$, for each $k \in \mathcal{L}$ we can find symmetric $U_k' \Subset U_k$ so that $\{U_k'\}_{k\in\mathcal{L}}$ still covers $\mathbf{Z}$. Obviously, this implies (2.2.9). Conversely, if we have found $U_k' \subset U_k$ as stated in the lemma, then $\{\mathrm{cl}(\iota_k(U_k')) \cap \mathbf{Z}\}$ will cover $\mathbf{Z}$, where $\mathrm{cl}(A)$ is the closure of A. Since $\mathrm{cl}(\iota_k(U_k')) \cap \mathbf{Z}$ are compact and since $\mathbf{Z} \cap \mathrm{cl}(\iota_k(U_k'))$ is closed in $\mathrm{cl}(\iota_k(U_k'))$, $\mathbf{Z}$ is compact as well. This proves the lemma. □

LEMMA 2.11. *Let $\phi : \mathfrak{U} \to \mathfrak{V}$ be the collection $\{\phi_k\}$, $\phi_k \in \Gamma_{U_k}(V_k)$, induced by $\{\tilde{\Phi}_k\}_{k\in\mathcal{L}}$. Then ϕ is a regular section with proper vanishing locus.*

Proof. This is equivalent to the fact that $[\Phi : \mathbf{B} \to \mathbf{E}]$ is a weakly Fredholm V-bundle, which was introduced and proved in [LT2]. □

Now let $\mathfrak{h} : \mathfrak{U} \to \mathfrak{V}$ be a regular section such that $\mathfrak{h}$ is transversal to the zero section and $\mathfrak{h}^{-1}(0)$ is proper. We now show that the data $\{h_k^{-1}(0)\}$ descends to a closed oriented current in $\mathbf{B}$ with rational coefficients supported on a stratified set whose boundary is 0. (It is closed in the sense that its boundary current is 0). In particular, it defines a singular homology class in $H_*(\mathbf{B}, \mathbb{Q})$.

Recall that for each $k \in \mathcal{L}$ the associated group G_k acts on U_k such that U_k/G_k is a covering of $\iota_k(U_k)$. We let m_k be the product of the order of G_k with the number of the sheets of the covering $U_k/G_k \to \iota_k(U_k)$. Note that then $U_{k,l} \to f_k^l(U_{k,l})$ is an m_k/m_l-fold covering. Because h_k is a regular extension of $(f_k^l)^*(h_l)$, $(f_k^l)^*(h_l)^{-1}(0)$ is an open submanifold of $h_k^{-1}(0)$ with identical orientations. Hence $\iota_k(h_k^{-1}(0))$ and $\iota_l(h_l^{-1}(0))$ patch together to form a stratified subset, and consequently the collection $\{\iota_k(h_k^{-1}(0))\}_{k\in\mathcal{L}}$ patch together to form a stratified subset, say $\mathbf{Z}$, in $\mathbf{B}$. Now we assign multiplicities to open strata of $\mathbf{Z}$. Let $\mathbf{O}_k = \iota_k(h_k^{-1}(0))$. Since $\mathbf{O}_k \subset \mathbf{Z}$ is an open subset, we can assign multiplicities to $\mathbf{O}_k$ so that as oriented current $[\mathbf{O}_k] = \iota_*(\frac{1}{m_k}[h_k^{-1}(0)])$, where $[h_k^{-1}(0)]$ is the current of the oriented manifold $h_k^{-1}(0)$ with multiplicity one. Here the orientation of $h_k^{-1}(0)$ is the one induced by the orientation of (U_k, V_k). Using the fact that $U_{k,l} \to f_k^l(U_{k,l})$ is a covering with m_k/m_l sheets, the assignments of the multiplicities of $\mathbf{O}_k$ and $\mathbf{O}_l$ over $\iota_k(U_k) \cap \iota_l(U_l)$ coincide. Therefore $\mathbf{Z}$ is an oriented stratified set of pure dimension with rational multiplicities. We let $[\mathbf{Z}]$ be the corresponding current. It remains to check that $\partial[\mathbf{Z}] = 0$ as current. Clearly, $\partial[\mathbf{Z} \cap \mathbf{O}_k] \subset \mathrm{cl}(\mathbf{O}_k) - \mathbf{O}_k$. Since $\{\mathbf{O}_k \cap \mathbf{Z}\}$ is an open covering of $\mathbf{Z}$, $\partial[\mathbf{Z}] = 0$ if $\mathbf{Z}$ is compact. But this is what we have assumed in the first place. This proves that $[\mathbf{Z}]$ is a closed current. We denote the induced homology cycle by

$$[\mathfrak{h}^{-1}(0)] \in H_*(\mathbf{B}, \mathbb{Q}).$$

In the remainder of this section, we will perturb the section $\phi : \mathfrak{U} \to \mathfrak{V}$ to a new section so that it is transversal to the zero section and its vanishing locus is compact. The current defined by the vanishing locus of the perturbed section will define the Euler class of $[\Phi : \mathbf{B} \to \mathbf{E}]$.

We begin with a collection $\mathcal{S} = \{S_l\}_{l\in\mathcal{L}}$ of symmetric open $S_k \Subset U_k$ such that $\{\iota_l(S_l)\}$ cover $\iota(\phi^{-1}(0))$. For technical reason, we assume that for each $k \in \mathcal{L}$ the boundary ∂S_k, which is defined to be $\mathrm{cl}(S_k) - S_k$ in U_k, is a smooth manifold of dimension $\dim S_k - 1$. By slightly altering S_k if necessarily, we can and do assume that S_k is symmetric, ∂S_k is transversal to $U_{k,l}$ along $\partial S_k \cap (f_k^l)^{-1}(\mathrm{cl}(S_l))$ for all $l < k$. This is possible because G_k is a finite group and $U_k/G_k \to \tau_k(U_k)$ is a finite covering. We call such $\mathcal{S}$ satisfying the transversality condition on its boundary. Following the convention, we set $S_{k,l} = (f_k^l)^{-1}(S_l) \cap S_k$. We now construct a collection of (closed) tubular neighborhoods of $S_{k,l}$ in U_k. We fix the index k and consider the closed submanifold (with boundary) $\Sigma_l := \mathrm{cl}(S_{k,l}) \subset U_k$. Because of the transversality condition on ∂S_l and on ∂S_k, we can find a D^h-bundle $p_l : T_l \to \Sigma_l$, where D^h is the closed unit ball in $\mathbb{R}^h$ and $h = \dim U_k - \dim U_{k,l}$, and a smooth embedding

$\eta_l : T_l \to U_k$ of which the following two conditions hold: First, the restriction of η_l to the zero section $\Sigma_l \subset T_l$ is the original embedding $\Sigma_l \subset U_k$; Secondly

$$\eta_l(p_l^{-1}(\Sigma_l \cap \partial S_k)) \subset \partial S_k \quad \text{and} \quad \eta_l(p_l^{-1}(S_{k,l})) \subset S_k. \tag{2.2.10}$$

For any $0 < \epsilon < 1$, we let $T_l^\epsilon \subset T_l$ be the closed ϵ-ball subbundle of T_l. By abuse of notation, in the following we will not distinguish T_l^ϵ from its image $\eta_l(T_l^\epsilon)$ in U_k. We will call T_l^ϵ the ϵ-tubular neighborhood of Σ_l in U_k. One property we will use later is that if $U_{k,l} \cap U_{k,l'} \neq \emptyset$ for $l' < l < k$, then $U_{k,l} \cap U_{k,l'}$ is an open subset of $U_{k,l'}$, and hence for $0 < \epsilon \ll 1$ we have

$$\Sigma_{l'} \cap T_l^\epsilon \subset \Sigma_{l'} \cap \Sigma_l \quad \text{for} \quad 0 < \epsilon \ll 1. \tag{2.2.11}$$

Now consider $\Sigma_l \subset U_k$. Since T_l is a disk bundle over Σ_l, it follows that we can extend the subbundle $(f_k^l)^*(V_l)|_{\Sigma_l} \subset V_k|_{\Sigma_l}$ to a smooth subbundle of $V_k|_{T_l}$, denoted by $F_l \subset V_k|_{T_l}$. We then fix an isomorphism and the inclusion

$$p_l^*\big((f_k^l)^*(V_l)|_{\Sigma_l}\big) \cong F_l \subset V_k|_{T_l}. \tag{2.2.12}$$

In this way, we can extend any section ζ of $(f_k^l)^*(V_l)|_{\Sigma_l}$ to a section of $V_k|_{T_l}$ as follows. We first let $\zeta' : T_l \to F_l$ be the obvious extension using the isomorphism (2.11). We then let $\zeta_{\text{ex}} : T_l \to V_k|_{T_l}$ be the induced section using the inclusion $F_l \subset V_k|_{T_l}$. We will call ζ_{ex} the standard extension of ζ to T_l. We fix a Riemannian metric on U_k and a metric on V_k. For any section ζ as before, we say that ζ is sufficiently small if its C^2-norm is sufficiently small. We now state a simple but important observation.

LEMMA 2.12. *Let the notation be as before. Then there is an $\epsilon > 0$ such that for any section $g : T_l \to F_l \subset V_k|_{T_l}$ such that $\| g \|_{C^2} < \epsilon$, the section $h_k|_{T_l} + g$ is non-zero over $T_l^\epsilon - \Sigma_l$.*

Proof. This follows immediately from the fact that Σ_l is compact and that for any $x \in U_{k,l}$ the differential

$$dh_k(x) : T_x U_k / T_x U_{k,l} \longrightarrow \big(V_k/(f_k^l)^*(V_l)\big)|_x$$

is an isomorphism. □

We now state and prove the main proposition of this section.

PROPOSITION 2.13. *Let $\mathfrak{h} : \mathfrak{U} \to \mathfrak{V}$ be a regular section with $\mathfrak{h}^{-1}(0)$ proper, let $\mathfrak{U}' \Subset \mathfrak{U}$ be a good sub-atlas, let $\mathfrak{V}'$ be the restriction of $\mathfrak{V}$ to $\mathfrak{U}'$ and let $\mathfrak{h}'$ be the the restriction of $\mathfrak{h}$ to $\mathfrak{U}'$. We assume that the vanishing locus of $\mathfrak{h}'$ is proper. Then there is a smooth family of regular sections $\mathfrak{g}(t) : \mathfrak{U}' \to \mathfrak{V}'$, $t \in [0,1]$, such that*

$$\bigcup_{t \in [0,1]} \iota\big(\mathfrak{g}(t)^{-1}(0)\big) \times \{t\} \subset \mathrm{B} \times [0,1] \tag{2.2.13}$$

is compact, that $\mathfrak{g}(0) = \mathfrak{h}'$ and that $\mathfrak{g}(1)$ is transversal to the zero section of $\mathfrak{V}'$.

Proof. We keep the notion used in Definition 2.8. We assume $\mathfrak{U}'$ is given by open subsets $\{U_k'\}$. We will construct the perturbation over U_1' and then successively extends it to the remainder of $\{U_k'\}$. Let k be any positive integer no bigger than $\#(\mathcal{L}) + 1$. We now state the induction hypothesis $\mathbf{H}_k$:

Condition $\mathbf{H}_k$: For each $l \leq k$, there is a symmetric open subsets $S_l \subset U_l$ such that $U_l' \Subset S_l \Subset U_l$, and a smooth family of sufficiently small sections $e_l(t) \in \Gamma_{U_l}(V_l)$, $l \leq k$ that satisfy the following condition:
(1) $e_l(0) \equiv 0$;
(2) Let $\mathbf{h}_l(t) = h_l + e_l(t) \in \Gamma_{U_l}(V_l)$, then for any $l < m < k$ the section $\mathbf{h}_m(t)|_{S_m}$ is a regular extension of $(f_m^l)^*(\mathbf{h}_l(t))|_{S_{m,l}}$;
(3) For any $l < k$, the section $\mathbf{h}_l(1)$ is transversal to the zero section of V_l over S_l;
(4) For any $l < k$ and $t \in [0,1]$,

$$\mathbf{h}_l(t)^{-1}(0) \cap (S_l - U_l') \subset \big(\bigcup_{i \leq l} (f_l^i)^{-1}(S_i)\big) \bigcup \big(\bigcup_{m \geq l} f_m^l(U_{m,l}')\big). \tag{2.2.14}$$

Clearly, the condition $\mathbf{H}_1$ is automatically satisfied. Now assume that we have found $\{S_l\}_{l<k}$ and $\{e_l\}_{l<k}$ required by the condition $\mathbf{H}_k$. We will demonstrate how to find e_k and a new sequence of open subsets $\{S_l'\}_{l \leq k}$ so that the condition $\mathbf{H}_{k+1}$ will hold for $\{e_l\}_{l \leq k}$ and $\{S_l'\}_{l \leq k}$.

We pick a symmetric $S_k \Subset U_k$ such that $U_k' \Subset S_k \Subset U_k$ and that it satisfies the tranversality condition on its boundary. We continue to use the notation developed earlier. In particular, we let Σ_l be the closure of $S_{k,l}$, let T_l be the (closed) tubular neighborhood of $\Sigma_l \subset U_k$ with the projection $p_l : T_l \to \Sigma_l$ and let F_l be the subbundle of $V_k|_{T_l}$ with the isomorphism (2.2.12). Let $\zeta_l(t)$ be the standard extension of $(f_k^l)^*(e_l(t))|_{\Sigma_l} \in \Gamma_{\Sigma_l}(V_k)$ to T_l. Note that $h_k|_{T_l} + \zeta_l(t)$ is a regular extension of $(f_k^l)^*(\mathbf{h}_l(t))|_{\Sigma_l}$. Because $\{\mathbf{h}_l\}_{l<k}$ satisfies condition $\mathbb{H}_k$, for $l < m < k$ and $x \in \Sigma_l \cap \Sigma_m$ we have $(f_k^l)^*(\mathbf{h}_l(t))(x) = (f_k^m)^*(\mathbf{h}_m(t))(x)$, from (2.2.7). Now let

$$A_l = p_l^{-1}\big((f_k^l)^{-1}(S_l)\big) - \bigcup_{l<m<k} p_m^{-1}\big((f_k^m)^{-1}(\mathrm{cl}\,(U_m'))\big)$$

and let

$$B_l = \mathrm{cl}\,(U_{k,l}') - \bigcup_{k>m>l} (f_k^m)^{-1}(S_m).$$

Note that $\{A_l\}_{l<k}$ covers $\mathrm{Int}\big(\cup_{l<k} T_l\big)$, that $B_l \Subset A_l$ and that $\{B_l\}_{l<k}$ is a collection of compact subsets of U_k. Now let $\epsilon > 0$ be sufficiently small. We choose a collection of non-negative smooth functions $\{\rho_l\}_{l<k}$ that obeys the requirement that $\mathrm{supp}(\rho_l) \Subset \mathrm{Int}(A_l \cap T_l^\epsilon)$, that $\rho_l \equiv 1$ in a neighborhood of B_l and that $\sum_{l<k} \rho_l \equiv 1$ in a neighborhood of $\cup_{l<k} \mathrm{cl}\,(U_{k,l}')$. This is possible because the last set is compact and is contained in $\mathrm{Int}(\cup_{l<k} A_l)$. We set

$$\zeta(t) = \sum_{l<k} \rho_l \cdot \zeta_l(t).$$

Now we claim that for each $l < k$ the section $h_k + \zeta(t)$ is a regular extension of $(f_k^l)^*(\mathbf{h}_l(t))$ in a neighborhood of $\mathrm{cl}\,(U_{k,l}')$. Let x be any point in $\mathrm{cl}\,(U_{k,l}')$. We first consider the case where x is contained in B_m for some $m \geq l$. Let $y = f_k^m(x)$, then $y \in S_m$. Then restricting to a sufficiently small neighborhood of x the section $h_k + \zeta(t)$ is equal to $h_k + \zeta_m(t)$. Since $h_k + \zeta(t)$ is a regular extension of $(f_k^m)^*(\mathbf{h}_m(t))$ near x and since $\mathbf{h}_m(t)$ is a regular extension of $(f_m^l)^*(\mathbf{h}_l(t))$ in a neighborhood of $y \in S_m$, $h_k + \zeta(t)$ is a regular extension of $(f_k^l)^*(\mathbf{h}_l(t))$ near x. We next consider the case where x is not contained in any of the B_m's. Let Λ be the set of all $m > l$ such that

$x \in (f_k^m)^{-1}(S_m)$. Then for any $m < k$ that is not in Λ, $\rho_m \equiv 0$ in a neighborhood of x. Here we have used the fact that $\Sigma_m \cap T_l^\epsilon \subset \Sigma_m \cap \Sigma_l$ for $0 < \epsilon \ll 1$ (see (2.2.11)). On the other hand, by induction hypothesis for each $m \in \Lambda$ the section $h_k + \zeta_m(t)$ is a regular extension of $(f_k^l)^*(\mathbf{h}_l(t))$ near x. Therefore since $\sum_{m\in\Lambda} \rho_m \equiv 1$ near x, in a small neighborhood of x

$$h_k + \zeta(t) = \sum_{m\in\Lambda} \rho_m \cdot (h_k + \zeta_m(t))$$

is also a regular extension of $(f_k^l)^*(\mathbf{h}_l(t))$.

Our last step is to extend $\zeta(t)$ to U_k. We let $e_k(t)$ be a smooth family of sufficiently small sections of V_k such that $e_k(0) \equiv 0$, that the restriction of $e_k(t)$ to a neighborhood of $\cup_{l<k} \operatorname{cl}\left((f_k^l)^{-1}(U_l')\right)$ is $\zeta(t)$ and such that the section $\mathbf{h}_k(1)$ is transversal to the zero section in a neighborhood of $\operatorname{cl}(U_k')$ in S_k. The last condition is possible because $h_k + \zeta(1)$ is transversal to the zero section in a neighborhood of $\cup_{l<k} \operatorname{cl}(U_{k,l}')$. Therefore, by possibly replace S_l with $S_l' \Subset S_l$ for $l < k$ if necessary, we can find an $S_k' \Subset S_k$ satisfying $U_k' \Subset S_k'$ such that the induction hypothesis $\mathbf{H}_k$ holds for $\{e_l\}_{l\leq k}$ and $\{S_l'\}_{l\leq k}$, except possibly the condition (4).

We now show that the (4) of $\mathbf{H}_k$ holds as well. We only need to check the inclusion (2.2.14) for $l = k$. Let $S_k' \Subset S_k$ be a symmetric open subset such that $U_k' \Subset S_k' \Subset S_k$. By Lemma 2.10

$$\mathbf{h}_k^{-1}(0) \cap (\operatorname{cl}(S_k') - U_k') \subset \Big(\bigcup_{i<k} (f_k^i)^{-1}(S_i)\Big) \bigcup \Big(\bigcup_{i>k} f_i^k(U_{i,k}')\Big). \tag{2.2.15}$$

Now let

$$D_1 = \mathbf{h}_k^{-1}(0) \bigcap (\operatorname{cl}(S_k') - U_k') \bigcap \Big(\bigcup_{i<k} (f_k^i)^{-1}(U_i')\Big)$$

and let

$$D_2 = \mathbf{h}_k^{-1}(0) \bigcap (\operatorname{cl}(S_k') - U_k') \bigcap \Big(\bigcup_{i>k} f_i^k(U_{i,k}')\Big).$$

Since $\mathbf{h}_k(t)$ are small perturbations of h_k, we can assume that $\mathbf{h}_k(t)$ are chosen so that for any $t \in [0,1]$ the left hand side of (2.2.15) is contained in the union of a neighborhood V_1 of D_1 and a neighborhood V_2 of D_2. We remark that if we choose $\{e_l\}_{l\leq k}$ so that their C^2-norms are sufficiently small, then we can make V_1 and V_2 arbitrary close to D_1 and D_2 respectively. Then by Lemma 2.12 the vanishing locus of $\mathbf{h}_k(t)$ inside V_1 is contained in $\cup_{i<k} (f_k^i)^{-1}(S_i)$. On the other hand, since $\cup_{i\geq k} f_i^k(U_{i,k}')$ is open, it contains V_2 since D_2 is compact and $V_2 \supset D_2$ is sufficiently small. Thus we have inclusion (2.2.14) if we replace S_k by S_k'. This proves the inclusion (2.2.14).

By induction we have found $\{S_k\}_{k\in\mathcal{L}}$ and $\{e_k(t)\}_{k\in\mathcal{L}}$ that satisfy the condition $\mathbf{H}_k$ for $k = \#(\mathcal{L}) + 1$. Now let $\mathbf{g}_l(t) = \mathbf{h}_l(t)|_{U_l'}$. Then $\mathfrak{g}(t) = \{\mathbf{g}_l(t)\}_{l\in\mathcal{L}}$ satisfies the condition of the proposition. Note that the left hand side of (2.2.13) is compact because it is closed and is contained in the union of compact sets $\{\iota_k(\operatorname{cl}(U_k'))\}_{k\in\mathcal{L}}$. This proves the proposition. □

Let $\mathfrak{g}(t)$ be the perturbation constructed by Proposition 2.13 with $\mathfrak{h} = \phi$, where ϕ is given in Lemma 2.7. We define the Euler class of $[\Phi : \mathbf{B} \to \mathbf{E}]$ to be the homology class in $H_*(\mathbf{B}; \mathbb{Q})$ represented by the current $[\mathfrak{g}(1)^{-1}(0)]$. In the remainder of this section, we will show that this class is independent of the choice of the chart $\mathfrak{U}$ and the perturbation $\mathfrak{g}$.

PROPOSITION 2.14. *Let the notation be as before. Then the homology class* $[\mathfrak{g}(1)^{-1}(0)] \in H_*(\mathbf{B};\mathbb{Q})$ *so constructed is independent of the choice of perturbations.*

Proof. First, we show that if we choose two perturbations $\mathfrak{g}_1(t)$ and $\mathfrak{g}_2(t)$ based on identical sub-atlas $\mathfrak{U}' \Subset \mathfrak{U}$ as stated in Proposition 2.13, then as homology class we have $[\mathfrak{g}_1(1)^{-1}(0)] = [\mathfrak{g}_2(1)^{-1}(0)]$. To prove this, all we need is to construct a family of perturbations $\mathfrak{g}_s(t)$, where $s \in [0,1]$, that satisfies conditions similar to that of the perturbations constructed in Proposition 2.13. Since then the current

$$\bigcup_{s\in[0,1]} \iota(\mathfrak{g}_s(1)^{-1}(0)) \times \{s\} \subset \mathbf{B} \times [0,1]$$

is a homotopy between the currents $\mathfrak{g}_0(1)^{-1}(0)_{\mathrm{cur}}$ and $\mathfrak{g}_1(1)^{-1}(0)_{\mathrm{cur}}$. The construction of $\mathfrak{g}_s(t)$ is parallel to the construction of $\mathfrak{g}(t)$ in Proposition 2.13 by considering the data over $\{U_k \times [0,1]\}_{k\in\mathcal{L}}$.

Next, we show that the cycle $[\mathfrak{g}(1)^{-1}(0)]$ does not depend on the choice of $\mathfrak{U} \Subset \mathfrak{U}$. Let $\mathfrak{U}_1 \Subset \mathfrak{U}$ and $\mathfrak{U}_2 \Subset \mathfrak{U}$ be two good sub-atlas and let $\mathfrak{g}_1(t)$ and $\mathfrak{g}_2(t)$ are two perturbations subordinate to $\mathfrak{U}_1$ and $\mathfrak{U}_2$ respectively. Clearly, we can choose a sub-atlas $\mathfrak{U}_0 \Subset \mathfrak{U}$ such that $\mathfrak{U}_1 \subset \mathfrak{U}_0$ and $\mathfrak{U}_2 \subset \mathfrak{U}_0$. Let $\mathfrak{g}_0(t)$ be a perturbation given by Proposition 2.13 subordinate to $\mathfrak{U}_0$. Then $\mathfrak{g}_0(t)$ is also subordinate to $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Hence by the previous argument

$$[\mathfrak{g}_1(1)^{-1}(0)] = [\mathfrak{g}_0(1)^{-1}(0)] = [\mathfrak{g}_2(1)^{-1}(0)] \in H_*(\mathbf{B};\mathbb{Q}).$$

It remains to show that the class $[\mathfrak{g}(1)^{-1}(0)]$ does not depend on the choice of the good atlas $\mathfrak{U}$. For this, it suffices to show that for any two good atlas $\mathfrak{U}$ and $\mathfrak{U}'$ so that $\mathfrak{U}$ is finer than $\mathfrak{U}'$ the respective perturbations $\mathfrak{g}(t)$ and $\mathfrak{g}'(t)$ gives rise to identical homology classes $[\mathfrak{g}(1)^{-1}(0)] = [\mathfrak{g}'(1)^{-1}(0)]$. This is clear since a perturbation data of $\mathfrak{g}'(1)$ can be extended to a perturbation $\mathfrak{h}(t)$ over $\mathfrak{U}$, by repeating the construction in Proposition 2.13. Therefore, by the previous argument,

$$[\mathfrak{g}'(1)^{-1}(0)] = [\mathfrak{h}(1)^{-1}(0)] = [\mathfrak{g}(1)^{-1}(0)] \in H_*(\mathbf{B};\mathbb{Q}).$$

This proves the Proposition. □

3. Analytic charts. The goal of this section is to construct a collection of analytic local smooth approximations $(\Lambda, \mathbf{V})$. We call a smooth approximation $(\Lambda, \mathbf{V})$ analytic if the resulting finite dimensional model $\phi_{\mathbf{V}} : U \to V$ has the property that U is a complex manifold, V is a holomorphic vector bundle and $\phi_{\mathbf{V}}$ is a holomorphic section. In the next section we will show that such $\phi_{\mathbf{V}}$'s are Kuranishi maps, and hence the cones $\lim_{t\to\infty} \Gamma_{t\phi}$ are the virtual cones constructed in [LT1].

We will use the standard notation in complex geometry in this section. For instance, if M is a complex manifold, we will denote by T_xM the complex tangent space of M at x. We will use complex dimension throughout this section, thus the dimension of a real manifold takes value in $\mathbb{Z}[\frac{1}{2}]$. We will use the words analytic and holomorphic interchangeably in this section as well.

We begin with the construction of analytic local smooth approximations. Let $w \in \mathbf{B}$ be any point representing a holomorphic stable map $f : \Sigma \to X$ with n-marked points. We pick a smooth complex hypersurface H of X in general position and let $k =$

$[H]\cdot[A]$ where $A = f_*([\Sigma])$. Then we form a uniformizing chart $\Lambda = (\tilde{\mathbf{U}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \Phi_{\tilde{\mathbf{U}}}, G_{\tilde{\mathbf{U}}})$ of w over $\mathbf{U} \subset \mathbf{B}$ as constructed after Lemma 2.2. Here as usual we assume that $\mathbf{U}$ is sufficiently small so that all stable maps in $\mathbf{U}$ intersect H transversally and positively. Note that the correspondence by forgetting the last k-marked points is the projection $\pi_{\mathbf{U}}: \tilde{\mathbf{U}} \to \mathbf{U}$. Let $\mathcal{Y}$ over $\tilde{\mathbf{U}}$ be the universal (continuous) family of curves with $(n+k)$-marked sections and let $\mathcal{F}: \mathcal{Y} \to X$ be the universal map.

We let $\pi: \tilde{\mathbf{U}} \to \mathfrak{M}_{g,n+k}$ be the tautological map induced by the family $\mathcal{Y}$ with its marked sections. Here $\mathfrak{M}_{g,n+k}$ is the moduli space of $(n+k)$-pointed stable curves of genus g. Without loss of generality, we can assume that all fibers of $\mathcal{Y}$ with the marked points have trivial automorphism groups. It follows that $\mathfrak{M}_{g,n+k}$ is smooth near $\pi(\tilde{\mathbf{U}})$. As in section 1, we view $G_{\tilde{\mathbf{U}}}$ as a subgroup of S_{n+k}. Then $G_{\tilde{\mathbf{U}}}$ acts on $\mathfrak{M}_{g,n+k}$ by permuting the $(n+k)$-marked points of the curves in $\mathfrak{M}_{g,n+k}$, and the map $\pi: \tilde{\mathbf{U}} \to \mathfrak{M}_{g,n+k}$ is $G_{\tilde{\mathbf{U}}}$-equivariant. Now let $O \subset \mathfrak{M}_{g,n+k}$ be a smooth $G_{\tilde{\mathbf{U}}}$-invariant open neighborhood of $\pi(\tilde{\mathbf{U}}) \subset \mathfrak{M}_{g,n+k}$ and let $p: \mathcal{X} \to O$ be the universal family of stable curves over O with $(n+k)$ marked sections. It follows that the $G_{\tilde{\mathbf{U}}}$-action on O lifts to $\mathcal{X}$. For convenience, we let $\mathcal{X} \times_O \tilde{\mathbf{U}}$ be the topological subspace of $\mathcal{X} \times \tilde{\mathbf{U}}$ that is the preimage of $\Gamma_\pi \subset O \times \tilde{\mathbf{U}}$ under $\mathcal{X} \times \tilde{\mathbf{U}} \to O \times \tilde{\mathbf{U}}$, where $\Gamma_\pi \subset O \times \tilde{\mathbf{U}}$ is the graph of $\pi: \tilde{\mathbf{U}} \to O$. Since fibers of $\mathcal{Y}$ have trivial automorphism groups, there is a canonical $G_{\tilde{\mathbf{U}}}$-equivariant isomorphism $\mathcal{Y} \cong \mathcal{X} \times_O \tilde{\mathbf{U}}$ as family of pointed curves. Let $\pi_{\mathcal{X}}$ and $\pi_{\tilde{\mathbf{U}}}$ be the first and the second projection of $\mathcal{X} \times_O \tilde{\mathbf{U}}$.

Next, we let $(\mathcal{X}_n, O_n; \Sigma, p_n, \varphi_n)$ be a semi-universal family of the n-pointed curve Σ. Namely, $\mathcal{X}_n$ is a (holomorphic) family of pointed prestable curves over the pointed smooth complex manifold $p_n \in O_n$; The dimension of O_n is equal to $\dim_{\mathbb{C}} \operatorname{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma)$, where $D \subset \Sigma$ is the divisor of the n-marked points of Σ; $\varphi_n: \Sigma \to \mathcal{X}_n|_{p_n}$ is an isomorphism of Σ with the fiber of $\mathcal{X}_n$ over p_n as n-pointed curve; The Kuranishi map $T_{p_n} O_n \to \operatorname{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma)$ of the family $\mathcal{X}_n$ is an isomorphism. Note that $G_{\tilde{\mathbf{U}}}$ acts canonically on Σ. We let $\Pi_n(\mathcal{X})$ be the family of curves over O that is derived from $\mathcal{X}$ by discarding the last k-marked sections of $\mathcal{X}$. We let $B = \pi_{\mathbf{U}}^{-1}(w)$. We then fix a $G_{\tilde{\mathbf{U}}}$-equivariant isomorphism

$$\coprod_{z \in B} \Pi_n(\mathcal{X})|_z \longrightarrow B \times \Sigma \tag{3.3.1}$$

over B. Let $\operatorname{Aut}_{p_n}(\mathcal{X}_n)$ be the group of automorphisms of $\mathcal{X}_n$ that are fibers preserving, that keep the fiber $\mathcal{X}_n|_{p_n}$ invariant and fix the n-sections of $\mathcal{X}_n$. Possibly after shrinking O_n if necessary, we can assume that there is a homomorphism $\rho: G_{\tilde{\mathbf{U}}} \to \operatorname{Aut}_{p_n}(\mathcal{X}_n)$ such that for any $\sigma \in G_{\tilde{\mathbf{U}}}$ the $\rho(\sigma)$ action on $\mathcal{X}_n|_{p_n}$ is exactly the σ action on Σ via the isomorphism φ_n. Finally, possibly after shrinking $\mathbf{U}$ and O, we can pick a $G_{\tilde{\mathbf{U}}}$-equivariant holomorphic map $\varphi: O \to O_n$ such that $\varphi(B) = p_n$ and that there is a $G_{\tilde{\mathbf{U}}}$-equivariant isomorphism of n-pointed curves $\tilde{\varphi}: \mathcal{X} \to O \times_{O_n} \mathcal{X}_n$ that extends the isomorphism (3.3.1). We remark that the reason for doing this is to ensure that the smooth approximation we are about to construct is $G_{\tilde{\mathbf{U}}}$-equivariant.

Next, we let l be an integer to be specified later and let $W_i \subset \Sigma$, $i = 1, \ldots, l$, be l disjoint open disks away from the marked points and the nodal points of Σ. We assume that $\cup_{i=1}^l W_i$ is $G_{\tilde{\mathbf{U}}}$-invariant and that for any $\sigma \in G_{\tilde{\mathbf{U}}}$ whenever $\sigma(W_i) = W_i$ then $\sigma|_{W_i} = \mathbf{1}_{W_i}$. By shrinking $\mathbf{U}$, O and O_n if necessary, we can find disjoint open subsets $\mathcal{W}_{n,i} \subset \mathcal{X}_n$ such that $\cup_{i=1}^l \mathcal{W}_{n,i}$ is $G_{\tilde{\mathbf{U}}}$-invariant and $\mathcal{W}_{n,i} \cap \Sigma = W_i$. Let $\pi_O: \mathcal{W}_{n,i} \to O$ be the projection. Without loss of generality, we can assume that there is a holomorphic map $\pi_{W_i}:\mathcal{W}_{n,i}\to W_i$ so that

$$\coprod(\pi_O,\pi_{W_i}):\bigcup_{i=1}^{l}\mathcal{W}_{n,i}\longrightarrow O\times(\bigcup_{i=1}^{l}W_i)$$

is a biholomorphism. For convenience, for each i we will fix a biholomorphism between W_i and the unit disk in $\mathbb{C}$, and will denote by $W_i^{1/2}$ the open disk in W_i of radius $1/2$. We let $\mathcal{W}_i$ be the disjoint open subsets of $\mathcal{Y}$ defined by

$$\mathcal{W}_i=\mathcal{W}_{n,i}\times_{O_n}\tilde{\mathbf{U}}\subset\mathcal{X}_n\times_{O_n}\tilde{\mathbf{U}}\cong\mathcal{X}\times_O\tilde{\mathbf{U}}\cong\mathcal{Y}.$$

We will call W_i and $\mathcal{W}_i$ the distinguished open subsets of Σ and $\mathcal{Y}$ respectively. Without loss of generality, we can assume that $\cup_{i=1}^{l}\mathcal{W}_i$ is disjoint from the $(n+k)$-sections of $\mathcal{Y}$. We also assume that there are holomorphic coordinate charts $\Theta_i\subset X$ so that $\mathcal{F}(\mathcal{W}_i)\subset\Theta_i$. We let $(w_{i,1},\dots,w_{i,m})$, where $m=\dim X$, be the coordinate variable of Θ_i and let $\theta_i=\partial/\partial w_{i,1}$. For each i we pick a nontrivial $(0,1)$-form γ_i on W_i with $\operatorname{supp}(\gamma_i)\Subset W_i^{1/2}$. We demand further that if there is a $\sigma\in G_{\tilde{\mathbf{U}}}$ so that $\sigma(W_i)=W_j$ then $\Theta_i=\Theta_j$ as coordinate chart and $\sigma^*(\gamma_i)=\gamma_j$. We then let σ_i be the $(0,1)$-form over $\mathcal{W}_i$ with values in $\mathcal{F}^*(T_X)|_{\mathcal{W}_i}$ that is the product of the pull back of γ_i via $\pi_{W_i}:\mathcal{W}_i\to W_i$ with $\mathcal{F}^*(\theta_i)|_{\mathcal{W}_i}$, and let $\tilde{\sigma}_i$ be the section over $\mathcal{Y}$ that is the extension of σ_i by zero. Obviously, $\tilde{\sigma}_i$ is a section of $\mathbf{E}_{\tilde{\mathbf{U}}}$, and $(\tilde{\sigma}_1,\dots,\tilde{\sigma}_l)$ is linearly independent fiberwise. Hence it spans a complex subbundle of $\mathbf{E}_{\tilde{\mathbf{U}}}$, denoted by $\mathbf{V}$. It follows from the construction that $\mathbf{V}$ is $G_{\tilde{\mathbf{U}}}$-equivariant.

As in the previous section, we let $U=\Phi_{\tilde{\mathbf{U}}}^{-1}(\mathbf{V})$, let $V=\mathbf{V}|_U$ and let $\phi:U\to W$ be the lifting of $\Phi_{\tilde{\mathbf{U}}}|_U:U\to\mathbf{E}_{\tilde{\mathbf{U}}}|_U$. The remainder task of this section is to show that we can choose W_i, γ_i and Θ_i so that U admits a canonical complex structure and the section ϕ is holomorphic when W is endowed with the holomorphic structure so that the basis $\tilde{\sigma}_1|_U,\dots,\tilde{\sigma}_l|_U$ are holomorphic.

To specify our choice of W_i, γ_i and Θ_i, we need first to define the Dolbeault cohomology of holomorphic vector bundles over singular curves. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_\Sigma$-modules and let E be the associated vector bundle, namely, $\mathcal{O}_\Sigma(E)=\mathcal{E}$. We let $\Omega^0_{\text{cpt}}(E)$ be the sheaf of smooth sections of E that are holomorphic in a neighborhood of $\operatorname{Sing}(\Sigma)$ and let $\Omega^{0,1}_{\text{cpt}}(E)$ be the sheaf of smooth sections of $(0,1)$-forms with values in E that vanish in a neighborhood of $\operatorname{Sing}(\Sigma)$. Let

$$\bar{\partial}:\Gamma(\Omega^0_{\text{cpt}}(E))\longrightarrow\Gamma(\Omega^{0,1}_{\text{cpt}}(E)) \tag{3.3.2}$$

be the complex that send $\varphi\in\Omega^0_{\text{cpt}}(E)$ to $\bar{\partial}(\varphi)$. Since φ is holomorphic near nodes of Σ, $\bar{\partial}(\varphi)$ vanishes near nodes of Σ as well. This shows that the above complex is well defined. We define the Dolbeault cohomology $H^0_{\bar{\partial}}(E)$ and $H^1_{\bar{\partial}}(E)$ to be the kernel and the cokernel of $\bar{\partial}$.

LEMMA 3.1. *Let $H^i(\mathcal{E})$ be the Cěch cohomology of the sheaf $\mathcal{E}$. Then there are canonical isomorphisms $H^0_{\bar{\partial}}(E)\cong H^0(\mathcal{E})$ and $\Psi:H^{0,1}_{\bar{\partial}}(E)\cong H^1(\mathcal{E})$.*

Proof. The proof is identical to the proof of the classical result that the Dolbeault cohomology is isomorphic to the Cěch cohomology for smooth complex manifolds. Obviously, $H^0_{\bar{\partial}}(E)$ is canonically isomorphic to $H^0(\mathcal{E})$. We now construct Ψ. We first cover Σ by open subsets $\{A_i\}$ so that the intersection of any of its subcollection is contractible. Now let φ be any global section in $\Omega^{0,1}_{\text{cpt}}(E)$. Then over each A_i we can find a smooth $\eta_i\in\Gamma_{A_i}(\Omega^0_{\text{cpt}}(E))$ such that $\bar{\partial}\eta_i=\varphi|_{A_i}$. Clearly, the class in $H^1(\mathcal{E})$

represented by the cocycle $[\eta_{ij}]$, where $\eta_{ij} = \eta_i|_{A_i\cap A_j} - \eta_j|_{A_i\cap A_j}$, is independent of the choice of η_i, and thus defines a homomorphism $\Gamma(\Omega^{0,1}_{\mathrm{cpt}}(E)) \to H^1(\mathcal{E})$. It is routine to check that it is surjective and its kernel is exactly $\mathrm{Im}(\bar\partial)$. Therefore, we have $H^{0,1}_{\bar\partial}(E) \cong H^1(\mathcal{E})$. Clearly this isomorphism does not depend on the choice of the covering $\{A_i\}$. This proves the lemma. □

For any $z \in \tilde{\mathbf{U}}$, we denote by $\tilde\sigma_i(z)$ the restriction of $\tilde\sigma_i$ to the fiber of $\mathcal{Y}$ over z. We now choose the l open disks $W_i \subset \Sigma$, the $(0,1)$-forms γ_i on W_i and the coordinate charts $\Theta_i \subset X$ such that for any $\tilde w \in \pi_U^{-1}(w)$ the collection $\tilde\sigma_1(\tilde w), \dots, \tilde\sigma_l(\tilde w)$ spans $H^{0,1}_{\bar\partial}(f^*T_X)$. This is certainly possible if we choose l large because of the freedom in choosing W_i, Θ_i and γ_i. We fix once and for all such choices of W_i, Θ_i and γ_i. We then let $\mathcal{W}_i \subset \mathcal{Y}$, $\mathbf{V} \to \mathbf{E}_{\tilde{\mathbf{U}}}$ and $U = \Phi_{\tilde{\mathbf{U}}}^{-1}(\mathbf{V})$ be the objects constructed before according to this choice of W_i, γ_i and Θ_i.

We now demonstrate that this choice gives to the desired analytic smooth approximations. Let $\mathcal{Y}_U \to U$ be the restriction to $U \subset \tilde{\mathbf{U}}$ of the family $\mathcal{Y} \to \tilde{\mathbf{U}}$ with the marked sections and let $F: \mathcal{Y}_U \to X$ be the associated map. We also fix a smooth function η_i over W_i so that $\bar\partial\eta_i = \gamma_i$. We next extend the collection $\{W_i\}_{i=1}^l$ to an open covering $\{W_i\}_{i=1}^L$ so that the intersection of any subcollection of $\{W_i\}$ are contractible and $W_i^{1/2} \cap W_j^{1/2} = \emptyset$ for all $i \le l$ and $j > l$. For convenience, we agree that $\eta_j = 0$ for $j > l$.

We now fix an $\tilde w \in U$ over w.

Lemma 3.2. *There is a constant R such that for any Čech 1-cocycle $[\tau_{ij}]$, where $\tau_{ij} \in \Gamma_{W_i\cap W_j}(f^*T_X)$, there are constants a_i and holomorphic sections $\zeta_i \in \Gamma_{W_i}(f^*T_X)$ for $i = 1, \dots, L$ such that*

$$(\zeta_j + a_j\eta_j)|_{W_j\cap W_i} - (\zeta_i + a_i\eta_i)|_{W_j\cap W_i} = \tau_{ji}$$

and

$$\sum_{i=1}^{L}(\|\zeta_i\|_{L_2} + |a_i|) \le R(\sum_{i,j}^{L}\|\tau_{ij}\|_{L_2}).$$

Proof. The existence of $\{a_i\}$ and $\{\zeta_i\}$ follows from the fact that the images of $\tilde\sigma_1(\tilde w), \dots, \tilde\sigma_l(\tilde w)$ spans $H^{0,1}_{\bar\partial}(f^*T_X)$ and that $H^{0,1}_{\bar\partial}(f^*T_X)$ is isomorphic to $H^1(f^*T_X)$. The L^2 estimate is routine, using the harmonic theory on the normalization of Σ. We will skip the details here. □

We let $\Gamma(\Omega^{0,1}_{\mathrm{cpt}}(f^*T_X))^\dagger$ be the quotient of $\Gamma(\Omega^{0,1}_{\mathrm{cpt}}(f^*T_X))$ by the linear span of $\tilde\sigma_1(\tilde w), \dots, \tilde\sigma_l(\tilde w)$. Because $\{\tilde\sigma_i(\tilde w)\}_{i=1}^L$ is invariant under the automorphism group of the stable map f, $\Gamma(\Omega^{0,1}_{\mathrm{cpt}}(f^*T_X))^\dagger$ is independent of the choice of $\tilde w \in \pi_U^{-1}(w)$. We let

$$\bar\partial^\dagger: \Gamma(\Omega^{0}_{\mathrm{cpt}}(f^*T_X)) \to \Gamma(\Omega^{0,1}_{\mathrm{cpt}}(f^*T_X))^\dagger$$

be the induced complex. We define $H^0_{\bar\partial}(f^*)^\dagger$ and $H^{0,1}_{\bar\partial}(f^*T_X)^\dagger$ be the kernel and the cokernel of the above complex.

Corollary 3.3. *Let the notation be as before. Then* $\dim H^{0,1}_{\bar\partial}(f^*T_X)^\dagger = 0$ *and* $\dim H^0_{\bar\partial}(f^*T_X)^\dagger = \deg(f^*T_X) + m(1-g) + l$.

Proof. The vanishing of $H^{0,1}_{\bar\partial}(f^*T_X)^\dagger$ follows from the surjectivity of $\bar\partial^\dagger$ which follows from the fact that $\tilde{\sigma_1}(\tilde w),\cdots\tilde{\sigma_l}(\tilde w)$ spans $H^{0,1}_{\bar\partial}(f^*T_X)$. The second identity follows from

$$\begin{aligned} & \dim H^0_{\bar\partial}(f^*T_X)^\dagger - \dim H^{0,1}_{\bar\partial}(f^*T_X)^\dagger \\ = \; & \dim H^0_{\bar\partial}(f^*T_X) - \dim H^{0,1}_{\bar\partial}(f^*T_X) + l = \chi(f^*T_X) + l \end{aligned}$$

and the Riemann-Roch theorem. □

Next, we will describe the tangent space of U at $\tilde w$. By the smoothness result of [LT2], we know that U is a smooth manifold of (complex) dimension $r_{\exp}$. As before, we let $D\subset\Sigma$ be the divisor of the first n-marked points of $\tilde w$. Since f is holomorphic, $df^\vee$ is a homomorphism of the sheaves $f^*\Omega_X\to\Omega_\Sigma$. We let

$$\mathcal{D}^\bullet_{\tilde w} = [f^*\Omega_X \xrightarrow{\alpha} \Omega_\Sigma(D)]$$

be the induced complex indexed at -1 and 0. We first define the extension space $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde w},\mathcal{O}_\Sigma)^\dagger$ and then show that it is canonically isomorphic to $T_{\tilde w}U$. We begin with the notation of associated sheaves.

DEFINITION 3.4. *Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_\Sigma$-modules that is locally free away from the nodal points of Σ and let E be the holomorphic vector bundle over Σ^0, where Σ^0 is the smooth locus of Σ, such that $\mathcal{O}_{\Sigma^0}(E)=\mathcal{E}|_{\Sigma^0}$. We define the associated sheaf $\mathcal{E}^{\mathcal{A}}$ to be the sheaf so that the germ of $\mathcal{E}^{\mathcal{A}}$ at a nodal point $p\in\Sigma$ (resp. a smooth point $p\in\Sigma^0$) is the germ of $\mathcal{E}$ at p (resp. the germ of $\Omega^0(E)$ at p). Here $\Omega^0(E)$ is the sheaf of smooth sections of E.*

The set $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde w},\mathcal{O}_\Sigma)^\dagger$ consists of equivalence classes of pairs (v_1,v_2): The data v_1 is an element in $\operatorname{Ext}^1(\Omega_\Sigma(D),\mathcal{O}_\Sigma)$ that defines an exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \xrightarrow{\varphi_1} \mathcal{B} \xrightarrow{\varphi_2} \Omega_\Sigma(D) \longrightarrow 0. \tag{3.3.3}$$

Note that $\mathcal{B}$ is locally free over Σ^0. The data v_2 is a homomorphism $f^*\Omega_X\to\mathcal{B}^{\mathcal{A}}$ such that, first of all, the diagram

$$\begin{tikzcd} & & f^*\Omega_X \arrow[r, equal] \arrow[d, "v_2"'] & f^*\Omega_X \arrow[d, "df^\vee"'] & \\ 0 \arrow[r] & \mathcal{O}^{\mathcal{A}}_\Sigma \arrow[r, "\varphi_1"] & \mathcal{B}^{\mathcal{A}} \arrow[r, "\varphi_2"] & \Omega_\Sigma(D)^{\mathcal{A}} \arrow[r] & 0 \end{tikzcd} \tag{3.3.4}$$

is commutative, where the lower sequence is induced by (3.3.3). Secondly, since v_2 is holomorphic near nodes of Σ and since $df^\vee$ is holomorphic, the differential $\bar\partial v_2$ vanishes near nodes of Σ and thus lifts to a global section β of $\Omega^{0,1}_{\mathrm{cpt}}(f^*T_X)$. We require that $\beta\in\operatorname{Span}\{\tilde\sigma_i(\tilde w): i=1,\cdots,l\}$.

The equivalence relation of such pairs are the usual equivalence relation of the diagram (3.3.4). Namely, two pairs (v_1,v_2) and (v_1',v_2') with the associated data $\{\mathcal{B},\varphi_i\}$ and $\{\mathcal{B}',\varphi_i'\}$ are equivalent if there is an isomorphism $\eta:\mathcal{B}\to\mathcal{B}'$ so that $\eta\circ\varphi_1=\varphi_1'$, $\varphi_2=\varphi_2'\circ\eta$ and $\eta\circ v_2=v_2'$.

LEMMA 3.5. *Let the notation be as before. Then $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde w},\mathcal{O}^{\mathcal{A}}_\Sigma)^\dagger$ is canonically a complex vector space of complex dimension $r_{\exp}$.*

Proof. The fact that $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde w},\mathcal{O}^{\mathcal{A}}_\Sigma)^\dagger$ forms a complex vector space can be established using the usual technique in homological algebra. For instance, if $r\in$

$\mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}^A_\Sigma)^\dagger$ is represented by $\{\mathcal{B}, \varphi_i, v_2\}$ shown in the diagram (3.3.4), then for any complex number a the element ar is represented by the same diagram with φ_1 replaced by $a\varphi_1$. We now prove that

$$\dim \mathrm{Ext}^1_\Sigma(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}^A_\Sigma)^\dagger = r_{\exp}. \tag{3.3.5}$$

Clearly, the following familiar sequence is exact in this case:

$$\mathrm{Ext}^0(\Omega_\Sigma(D), \mathcal{O}_\Sigma) \xrightarrow{\alpha} H^0_{\bar\partial}(f^*T_X)^\dagger$$

$$\longrightarrow \mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger \longrightarrow \mathrm{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma) \longrightarrow H^{0,1}_{\bar\partial}(f^*T_X)^\dagger.$$

Since f is stable, α is injective. Hence (3.3.5) follows from Corollary 3.3 and the Riemann-Roch theorem. This proves the lemma. □

Recall that if Z is a variety (or a scheme) then the Zariski tangent space $T_\xi Z$ is the space of morphisms $\mathrm{Spec}\,\mathbb{C}[t]/(t^2) \to Z$ that send the only closed point of $\mathrm{Spec}\,\mathbb{C}[t]/(t^2)$ to ξ. In the following, we will imitate this construction to construct the space of $\mathbb{C}$-tangents of U at $\tilde{w}$. We still denote by $W_1, \ldots, W_l$ the l-distinguished open subsets of Σ and let $\{W_i\}_{i=1}^L$ be an extension of $\{W_i\}_{i=1}^l$ to an open covering of Σ specified before Lemma 3.2. We assume that there are coordinate charts Θ_i of X such that $f(W_i) \subset \Theta_i$. By abuse of notation, we will fix the embedding $\Theta_i \subset \mathbb{C}^m$ and view any map to Θ_i as a map to $\mathbb{C}^m$. We let $\iota_i : \Theta_i \to X$ be the tautological inclusion and let

$$g_{ij} : \iota_j^{-1}(\iota_i(\Theta_i)) \to \iota_i^{-1}(\iota_j(\Theta_j)) \subset \Theta_i \subset \mathbb{C}^m$$

be the transition functions of X.

We first introduce the notion of pre-$\mathbb{C}$-tangents of U at $\tilde{w}$. Intuitively, a pre-$\mathbb{C}$-tangent is the scheme analogue of a morphism $\mathrm{Spec}\,\mathbb{C}[t]/(t^2) \to U$ should U be a scheme.

DEFINITION 3.6. *A pre-$\mathbb{C}$-tangent ξ of U at $\tilde{w}$ consists of a flat family of prestable curves $\pi_T : C_T \to T$ over an open neighborhood T of $0 \in \mathbb{C}$, an open covering $\{\tilde{W}_i\}_{i=1}^L$ of C_T and smooth maps $\tilde{f}_i : \tilde{W}_i \to \Theta_i$ of which the following holds: (1) The fiber of C_T over $0 \in T$, denoted by C_0, is isomorphic to Σ as n-pointed curve; (2) The open covering $\{\tilde{W}_i\}$ has the property that $\tilde{W}_i \cap C_0 = W_i$ for all i and that for $i \le l$ there is a holomorphic map $\pi_{W_i} : \tilde{W}_i \to W_i$ whose restriction to $W_i \subset \tilde{W}_i$ is the identity and that $(\pi_T, \pi_{W_i}) : \tilde{W}_i \to T \times W_i$ is a biholomorphism; (3) Let $\bar\partial_0$ (resp. $\bar\partial_i$) be the $\bar\partial$-differential with respect to the holomorphic variable of T (resp. W_i) using the biholomorphism (π_T, π_{W_i}) and let γ_i and θ_i be the $(0,1)$-forms and the vector fields chosen before. Then we require that for $i > l$ the maps $\tilde{f}_i : \tilde{W}_i \to \Theta_i$ are holomorphic and for $i \le l$ we have $\bar\partial_0(\tilde{f}_i) = 0$ and*

$$\bar\partial_i(\tilde{f}_i) = \pi_T^*\varphi_i \cdot \pi_{W_i}^*(\gamma_i \cdot f^*(\theta_i)), \tag{3.3.6}$$

where φ_i are holomorphic functions over T; (4) If we let z_0 be the holomorphic variable of $\mathbb{C} \supset T$ then we require that

$$\tilde{f}_{ji} := \tilde{f}_i - g_{ij} \circ \tilde{f}_j : \tilde{W}_{ij} \longrightarrow \mathbb{C}^m \tag{3.3.7}$$

is divisible by z_0^2, where $\tilde{W}_{ij}$ is a neighborhood of $W_i \cap W_j$ in $\tilde{W}_i \cap \tilde{W}_j$ over which $\tilde{f}_{ji}$ is well-defined.

We denote the set of all pre-$\mathbb{C}$-tangents by $T^{\mathrm{pre}}_{\tilde{w}}U$.

We next define a canonical map

$$T^{\mathrm{pre}}_{\tilde{w}}U \longrightarrow \mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger. \tag{3.3.8}$$

Let ξ be any pre-$\mathbb{C}$-tangent given by the data above. By the theory of deformation of n-pointed curves [LT1, section 1], the analytic family C_T defines canonically an exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{B} \longrightarrow \Omega_\Sigma(D) \longrightarrow 0 \tag{3.3.9}$$

associated to an extension class $v_1(\xi) \in \mathrm{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma)$. Note that away from the nodes of Σ and the support of D the sheaf $\mathcal{B}$ is canonically isomorphic to $\Omega_{C_T} \otimes_{\mathcal{O}_{C_T}} \mathcal{O}_{C_0}$. Because $\tilde{f}_i : \tilde{W}_i \to \Theta_i$ are holomorphic for $i > l$, it follows from [LT1] that there are canonical homomorphisms of sheaves $u_i : f^*\Omega_X|_{W_i} \to \mathcal{B}|_{W_i}$ such that

$$\begin{tikzcd}
 & & f^*\Omega_X|_{W_i} \arrow[r, equal] \arrow[d, "u_i"] & f^*\Omega_X|_{W_i} \arrow[d, "df^\vee|_{W_i}"] & \\
0 \arrow[r] & \mathcal{O}^{\mathcal{A}}_{W_i} \arrow[r] & \mathcal{B}^{\mathcal{A}}|_{W_i} \arrow[r] & \Omega_{W_i}(D)^{\mathcal{A}} \arrow[r] & 0
\end{tikzcd} \tag{3.3.10}$$

is commutative, where the lower sequence is induced by (3.3.9). Now we look at the open set W_i for $i \leq l$. By our construction, W_i is smooth and is away from the $(n+k)$-marked points of $\tilde{w}$. Thus $\mathcal{B}^{\mathcal{A}}|_{W_i}$ is canonically isomorphic to the sheaf of smooth sections of $TC_T|_{W_i}$, denoted $\Omega^0(TC_T|_{W_i})$, and the dual of $d\tilde{f}_i$ defines canonically a homomorphism $u_i : f^*\Omega_X|_{W_i} \to \mathcal{B}^{\mathcal{A}}|_{W_i}$ that makes the above diagram commutative. Because of the condition (3.3.6), the lift of $\bar{\partial} u_i$ to $\Omega^{0,1}(f^*T_{W_i})$ is a constant multiple of $\tilde{\sigma}_i(\tilde{w})|_{W_i}$. Further, because of the condition that $\tilde{f}_{ji}$ is divisible by z_0^2, the collection $\{u_i\}_{i=1}^L$ patch together to form a homomorphism $v_2(\xi) : f^*\Omega_X \to \mathcal{B}^{\mathcal{A}}$ that makes the diagram (3.3.4) commutative. Hence $(v_1(\xi), v_2(\xi))$ defines an element in $\mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$, which is defined to be the image of ξ.

We remark that in this construction we have only used the fact that the stable map associated to $\tilde{w}$ is holomorphic, that the domain Σ of $\tilde{w}$ has l distinguished open subsets W_i with $(0,1)$-forms $\tilde{\sigma}_i(\tilde{w})$. Because for any $\eta \in U$ its domain Σ_η also has l distinguished open subsets $\mathcal{W}_i \cap \Sigma_\eta$ and forms $\tilde{\sigma}(\eta)$, we can define the extension group $\mathrm{Ext}^1(\mathcal{D}^\bullet_\eta, \mathcal{O}_{\Sigma_\eta})^\dagger$, the space of pre-$\mathbb{C}$-tangents of U at η and the analogous canonical map as in (3.3.8) if the map f_η of η is holomorphic.

To justify our choice of $\mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$, we will construct, to each $v \in \mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$, a pre-$\mathbb{C}$-tangent $\xi^v \in T^{\mathrm{pre}}_{\tilde{w}}U$ so that the image of ξ^v under (3.3.8) is v. Let $v = (v_1, v_2)$ be any element in $\mathrm{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$ defined by the diagram (3.3.4). Let $T \subset \mathbb{C}$ be a neighborhood of 0 and let C_T be an analytic family of n-pointed curves so that $C_0 \cong \Sigma$ and the Kuranishi map $T_0\mathbb{C} \to \mathrm{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma)$ send 1 to v_1. For instance, we can take C_T be the pull back of $\mathcal{X}_n$ via an analytic map $(T, 0) \to (\mathcal{O}_n, p_n)$. We let $\{W_i\}_{i=1}^L$ be a covering of Σ as before and let $\{\tilde{W}_i\}_{i=1}^L$ be a covering of C_T that are the pull back of $\mathcal{W}_{n,i}$. Note that for $i \leq l$, they come with biholomorphisms $\tilde{W}_i \cong W_i \times T$. Let Θ_i be open charts of X as before with $f(W_i) \Subset \Theta_i$. For $i > l$, since the restriction of (3.3.4) to W_i is analytic, we can find analytic $\tilde{f}_i : \tilde{W}_i \to \Theta_i$, possibly after shrinking T if necessary, such that $\tilde{f}_i$ are related to $v_2|_{W_i}$ as to how u_i are related to $v_2(\xi)|_{W_i}$ before. By analytic analogue of deformation theory (see [LT1]) such $\tilde{f}_i$ do exist. For $i \leq l$, since W_i are smooth and $\mathcal{B}^{\mathcal{A}}|_{W_i}$ are the sheaves $\Omega^0(T^*C_T|_{W_i})$, we simply let $\tilde{f}_i$ be smooth so that in addition to $\tilde{f}_i$ satisfying the condition on pre-$\mathbb{C}$-tangents we require that $v_2|_{W_i}$ coincide with the dual of $d\tilde{f}_1|_{W_i}$. Note that $(C_T, \{\tilde{f}_i\})$ will be a pre-$\mathbb{C}$-tangent if $\tilde{f}_{ji}$ in (3.3.7) is divisible by $\pi_T^*(z_0^2)$. But this is true because for any $p \in W_i \cap W_j$, the differential $d\tilde{f}_i(p)$ and $d\tilde{f}_j(p)$ from T_pC_T to $T_{f(p)}X$ are identical. We let the so constructed pre-$\mathbb{C}$-tangent be ξ^v. Of course ξ^v is not unique. It is obvious from the construction that the image of ξ^v under (3.3.8) is v. We remark that it follows from the construction that for any complex number $c \neq 0$ the pull back of $(C_T, \{\tilde{f}_i\})$ under $L_c: \mathbb{C} \to \mathbb{C}$ defined by $L_c(z_0) = cz_0$ is a pre-$\mathbb{C}$-tangent, say ξ^{cv}, whose image under (3.3.8) is cv. Combining this with the work in [LT2], it is to easy to see that the real tangent space of U at $\tilde{w}$ is canonically isomorphic to the underlying real vector space of $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$.

In the following, we will give U an analytic coordinate and show that the complex tangent space of U at $\tilde{w}$ is canonically isomorphic to $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$. Let $r = \dim U$, which is $r_{\exp} + l = \dim \operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$. We fix a $\mathbb{C}$-isomorphism $T_0\mathbb{C}^r \cong \operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$. Composed with the canonical

$$\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger \to \operatorname{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma),$$

we obtain a homomorphism

$$T_0\mathbb{C}^r \longrightarrow \operatorname{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma). \tag{3.3.11}$$

Let $\mathcal{X}_n$ over O_n be the semi-universal family of the n-pointed curve Σ given before. We let S be a neighborhood of $0 \in \mathbb{C}^r$ and let $\varphi: S \to O_n$ be a holomorphic map with $\varphi(0) = 0$ such that

$$d\varphi(0): T_0S \equiv T_0\mathbb{C}^r \longrightarrow T_{p_n}O_n \cong \operatorname{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma)$$

is the homomorphism (3.3.11). We let $\pi_S: C_S \to S$ be the family of n-pointed curves over S that is the pull back of $\mathcal{X}_n$. Note that $C_S|_0$, denoted by C_0, is canonically isomorphic to Σ.

We keep the open covering $\{W_i\}_{i=1}^L$ of Σ ($\cong C_0$) chosen before. We now select an open covering $\{\tilde{W}_i\}_{i=1}^L$ of a neighborhood of $C_0 \subset C_S$ so that $\tilde{W}_i \cap C_0 = W_i$. For $i \leq l$, we let $\tilde{W}_i$ be the pull back of $\mathcal{W}_{n,i} \subset \mathcal{X}_n$. For $i > l$ and W_i smooth, we choose $\tilde{W}_i$ so that there is a holomorphic map $\pi_{W_i}: \tilde{W}_i \to W_i$ so that the restriction of π_{W_i} to $W_i \subset \tilde{W}_i$ is the identity map. For $i > l$ such that W_i contains a nodal point, we assume that $\tilde{W}_i$ is biholomorphic to the unit ball in $\mathbb{C}^{r+1}$ so that $W_i \subset \tilde{W}_i$ is defined by $w_1w_2 = 0$ and $w_i = 0$ for $i \geq 3$, where (w_i) are the coordinate variables of $\mathbb{C}^{r+1}$, and the restriction of π_S to $\tilde{W}_i$ is given by

$$(w_1, \cdots, w_{r+1}) \mapsto (w_1w_2, w_2, \cdots, w_{r+1}) \in \mathbb{C}^r.$$

The upshot of this is that if h is a holomorphic function on W_i, then we can extend it canonically to $\tilde{W}_i$ as follows. In case W_i is smooth, then the extension of h is the composite of $\tilde{W}_i \to W_i$ with h; In case W_i is singular, then φ has a unique expression

$$a + w_1h_1(w_1) + w_2h_2(w_2),$$

where $a \in \mathbb{C}$ and h_1, h_2 are holomophic. We then let its extension be the holomorphic function on $\tilde{W}_i$ that has the same expression. We fix the choice of $\{W_i\}$ and $\{\tilde{W}_i\}$. Without loss of generality, we can assume that there are coordinate charts $\Theta_i \subset X$ so that $f(W_i) \Subset \Theta_i$. For $i \leq l$ the charts Θ_i are the charts we have chosen before.

Our construction of local holomorphic charts of U is parallel to the original construction of Kodaira-Spencer of semi-universal family of deformation of holomorphic structures without obstructions. To begin with, possibly after shrinking $\tilde{W}_i$ if necessary we can assume that the map $f|_{W_i}: W_i \to \Theta_i$ can be extended to a holomorphic $F_{0,i}: \tilde{W}_i \to \Theta_i$ (recall f is holomorphic). To proceed, for $i > l$ we let $\mathcal{A}(\tilde{W}_i, \Theta_i)$ be the space of holomorphic maps from $\tilde{W}_i$ to $\mathbb{C}^m$. For $i \leq l$, by using the isomorphism $\tilde{W}_i \cong W_i \times S$ any smooth function $\varphi: \tilde{W}_i \to \mathbb{C}^m$ can be expressed in terms of its m components $\varphi_j(z, \xi)$, $j = 1, \ldots, m$, where $z = (z_1, \cdots, z_r)$ and ξ are holomorphic coordinates of S and W_i respectively. We define $\mathcal{A}(\tilde{W}_i, \Theta_i)$ to be the set of smooth maps $\varphi: \tilde{W}_i \to \mathbb{C}^m$ so that

$$\begin{cases} \bar{\partial}_{z_k}\varphi_j = 0 \quad \text{for} \quad k = 1, \ldots, r \quad \text{and} \quad j = 1, \ldots, m; \\ \bar{\partial}_{\xi}\varphi_j = 0 \quad \text{for} \quad j \geq 2 \quad \text{and} \quad \bar{\partial}_{\xi}\varphi_1 = c\sigma_i' \quad \text{for some} \quad c \in \mathbb{C}, \end{cases}$$

where σ_i' is the $(0,1)$-form taking values in $\varphi^*\mathbb{C}^n$ corresponding to the form σ_i using the canonical embedding $\Theta_i \subset \mathbb{C}^n$. Note that $\mathcal{A}(\tilde{W}_i, \Theta_i)$ are $\mathcal{O}_S$-modules. With this understanding, if we let $\mathcal{I} \subset \mathcal{O}_S$ be the ideal sheaf of $0 \in S$, then we denote by $\mathcal{I}^q\mathcal{A}(\tilde{W}_i, \Theta_i)$ the image of $\mathcal{I}^q \otimes_{\mathcal{O}_S} \mathcal{A}(\tilde{W}_i, \Theta_i)$ in $\mathcal{A}(\tilde{W}_i, \Theta_i)$.

In the following, we will construct a sequence of maps $F_{k,i} \in \mathcal{A}(\tilde{W}_i, \Theta_i)$ indexed by $k \geq 1$ and $1 \leq i \leq L$ that satisfy the following inductive hypothesis:

I$_1$: For each i, $F_{k+1,i} - F_{k,i} \in \mathcal{I}^k\mathcal{A}(\tilde{W}_i, \Theta_i)$;

I$_2$: The restriction $F_{1,i}|_{W_i}: W_i \to \mathbb{C}^m$ factor through $\Theta_i \subset \mathbb{C}^m$ and $\iota_i \circ (F_{1,i}|_{W_i}): W_i \to X$ is identical to $f|_{W_i}: W_i \to X$;

I$_3$: In a neighborhood $\tilde{W}_{ij}$ of $W_i \cap W_j$ in $\tilde{W}_i \cap \tilde{W}_j$ over which the map

$$F_{k,ij} = g_{ij} \circ F_{k,j} - F_{k,i}: \tilde{W}_{ij} \to \mathbb{C}^m \tag{3.3.12}$$

is well defined, $F_{k,ij} \in \mathcal{I}^k\mathcal{H}(\tilde{W}_{ij}, \mathbb{C}^m)$, where $\mathcal{H}(\tilde{W}_{ij}, \mathbb{C}^m)$ is the $\mathcal{O}_S$-module of holomorphic maps from $\tilde{W}_{ij}$ to $\mathbb{C}^m$;

I$_4$: For any vector $\eta \in \mathbb{C}^r$, we let $L_\eta: \mathbb{C} \to \mathbb{C}^r$ be the unique $\mathbb{C}$-linear map so that $L_\eta(1) = \eta$ and let η^{pre} be the pre-$\mathbb{C}$-tangent associated to the pull back of $(C_S, \{F_{2,i}\})$ under L_η. Using the standard isomorphism $T_0S \equiv T_0\mathbb{C}^r \cong \mathbb{C}^r$, we obtain a map

$$T_0S \longrightarrow \operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger \tag{3.3.13}$$

that send $\eta \in T_0S$ to the image of η^{pre} under (3.3.8). We require that this map is the isomorphism (3.3.11).

For $k = 1$ we simply let $F_{1,i}$ be the standard extension of $f|_{W_i}: W_i \to \Theta_i$ to $\tilde{W}_i \to \Theta_i$. We now construct $\{F_{2,i}\}$. We let π_1 and π_2 be the first and the second projection of $\mathbb{C}^r \times \Sigma$ where we view $\mathbb{C}^r$ as the total space of $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$. It follows from the definition of the extension group that there is a universal diagram

$$\begin{tikzcd} & & \pi_2^* f^* \Omega_X \arrow[r, equal] \arrow[d, "\nu_2"] & \pi_2^* f^* \Omega_X \arrow[d, "\pi_2^*(df^\vee)"] & \\ 0 \arrow[r] & \pi_2^* \mathcal{O}_\Sigma^{\mathcal{A}} \arrow[r] & \tilde{\mathcal{B}}^{\mathcal{A}} \arrow[r] & \pi_2^* \Omega_\Sigma(D)^{\mathcal{A}} \arrow[r] & 0 \end{tikzcd} \tag{3.3.14}$$

such that its restriction to fibers of $\mathbb{C}^r \times \Sigma$ over $\xi \in \mathbb{C}^r$ are the diagrams (3.3.4) associated to $\xi \in \operatorname{Ext}^1(\mathcal{D}^\bullet_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$. By deformation theory of pointed curves, for any smooth point $p \in \Sigma$ the vector space $\tilde{\mathcal{B}} \otimes k(p)$ is canonically isomorphic to the cotangent space $T_p^*C_S$. By applying the construction of $\xi^v \in T_{\tilde{w}}^{\text{pre}}U$ from a single vector

$v \in \operatorname{Ext}^1(\mathcal{D}^{\bullet}_{\tilde{w}}, \mathcal{O}_\Sigma)^\dagger$ to the family version, we can construct the family $\{F_{2,i}\}$ as required. We will leave the details to the readers.

Now we show that we can successively construct $F_{k,i}$ that satisfies the four conditions **I**'s above. Assume that for some $k \geq 2$ we have constructed $\{F_{k,i}\}$ that satisfies the four conditions above. Let $\tilde{W}_{ij}$ be the neighborhood of $W_{ij} = W_i \cap W_j \subset \tilde{W}_i \cap \tilde{W}_j$ so that (3.3.12) is well-defined. Then by the condition $\mathbf{I}_3$ above, $F_{k,ij} \in \mathcal{I}^k\mathcal{H}(\tilde{W}_{ij}, \mathbb{C}^m)$. Let $I = (i_1, \dots, i_r)$ be any length k multiple index, namely, $i_j \geq 0$ and $\sum i_j = k$. As usual, we will denote by ∂^I the symbol $\partial^k/\partial z_1^{i_1} \cdots \partial z_r^{i_r}$ and by z^I the term $z_1^{i_1} \cdots z_r^{i_r}$. Then because of the condition $\mathbf{I}_3$ above, $\varphi_{I,ij} = \partial^I F_{k,ij}|_{W_{ij}}$ is a holomorphic section of $f^*T_X|_{W_{ij}}$ using the standard isomorphism

$$TX|_{\Theta_i} \cong T\Theta_i \cong \Theta_i \times \mathbb{C}^m,$$

and the collection $[\varphi_{I,ij}]$ defines a Cěch 1-cocycle of f^*T_X. We let $\{\phi_{I,i}\}$, where $\phi_{I,i} = \zeta_i + a_i\eta_i$, be the collection provided by Lemma 3.2. Using the standard isomorphism $TX|_{\Theta_i} \cong \Theta_i \times \mathbb{C}^m$, we can view each $\phi_{I,i}$ as a map $W_i \to \mathbb{C}^m$. We let $\tilde{\phi}_{I,i}: \tilde{W}_i \to \mathbb{C}^m$ be the standard extension of $\phi_{I,i}$ and let $G_{I,i} = \pi_S^*(z^I)\tilde{\phi}_{I,i}$. Clearly, $\partial^I G_{I,i} = \phi_{I,i}$. Now we let

$$F_{k+1,i} = F_{k,i} + \sum_{\ell(I)=k} G_{I,i}.$$

It is direct to check that the collection $\{F_{k+1,i}\}$ satisfies the conditions $\mathbf{I}_1 - \mathbf{I}_4$ before. Finally, by the estimate in Lemma 3.2, by shrinking $\tilde{W}_i$ if necessary we can assume $\lim_k F_{k,i}$ converges over $\tilde{W}_i$. Let $F_{\infty,i}$ be its limit. Because $f(W_i) \Subset \Theta_i$, by shrinking $\tilde{W}_i$ further if necessary we can assume $F_{\infty,i}(\tilde{W}_i) \subset \Theta_i \subset \mathbb{C}^m$. It follows that we can find a neighborhood $S^0 \subset S$ of $0 \in S$ such that $\pi_S^{-1}(S^0) \subset \cup_i \tilde{W}_i$. Finally, because $F_{\infty,i}$ is analytic near W_i for $i > l$ and is analytic in S direction using $\tilde{W}_i \cong W_i \times S$ for $i \leq l$, the condition $\mathbf{I}_3$ implies that the collection $F_{\infty,i}|_{\tilde{W}_i \cap \pi_S^{-1}(S^0)}$ defines a map

$$F_S : \pi_S^{-1}(S^0) \longrightarrow X$$

which is holomorphic away from the union of $\tilde{W}_1, \dots, \tilde{W}_l$. Further, for each $i \leq l$ if we let ξ_i be a holomorphic variable of W_i and let π_{W_i} and π_{S^0} be the first and the second projection of $\tilde{W}_i \cap \pi_S^{-1}(S^0) \cong W_i \times S^0$, then

$$\frac{\partial}{\partial \bar{\xi}_i}\big(F_S|_{\tilde{W}_i \cap \pi_S^{-1}(S^0)}\big)d\bar{\xi}_i = \pi_{S^0}^*(\varphi_i)\pi_{U_i}^*(\gamma_i)F_S^*(\theta_i)|_{\tilde{W}_i} \tag{3.3.15}$$

for a holomorphic function φ_i over S^0. Finally, we let $k = [H] \cdot [A]$ and let Z be the subset of

$$\pi_S^{-1}(S^0) \times_S \cdots \times_S \pi_S^{-1}(S^0) \qquad (k \text{ times})$$

consisting of $(s; x_{n+1}, \dots, x_{n+k})$ such that $s \in S^0$ and $x_{n+1}, \dots, x_{n+k}$ are distinct points in $\pi_S^{-1}(s)$ that lie in $F_s^{-1}(H)$. Note that if we choose $\mathbf{U}$ to be small enough, then $F_s^{-1}(H)$ has exactly k points. Let C_Z be the family of $(n+k)$-pointed curves over Z so that its domain is the pull back of C_S via $Z \to S$, its first n-marked sections is the pull back of the n-marked sections of C_S and its last k-sections of the fiber of C_Z over $(s; x_{n+1}, \dots, x_{n+k})$ is $x_{n+1}, \dots, x_{n+k}$. Coupled with the pull back of F_S, denoted $F_Z : C_Z \to X$, we obtain a family of stable (continuous) maps from $(n+k)$-pointed curves to X.

Let $\eta : Z \to \tilde{\mathbf{U}}$ be the tautological map. Note $\eta(0) = \tilde{w}$. We claim that $\eta(Z) \subset U$. Indeed, let $z \in Z$ be any point and let C_z be the domain of z. It follows from our construction that the l distinguished open subsets $\{\tilde{W}_i \cap C_z\}_{i=1}^{l}$ coincide with the pull-back of the l-distinguished open subset $\{W_i\}$. We denote them by $\{W_{z,i}\}_{i=1}^{l}$. Further, $f_z = F_Z|_{C_z}$ is holomorphic away from $\cup_{i=1}^{l} W_{z,i}$ and $\bar{\partial} f_z|_{U_{z,i}}$ is a constant multiple of $\gamma_i \cdot f_z^*(\theta_i)$. Hence the value of the section $\Phi_{\tilde{U}} : \tilde{\mathbf{U}} \to \mathbf{E}_{\tilde{\mathbf{U}}}$ at $\eta(z)$ is contained in the subspace $\mathbf{V}_{\eta(z)} \subset \mathbf{E}_{\tilde{\mathbf{U}}}|_{\eta(z)}$. This shows that $\eta(z) \in U$. Recall $U = \Phi_{\tilde{\mathbf{U}}}(\mathbf{V})$.

PROPOSITION 3.7. *The induced map $\eta : Z \to U$ is a local diffeomorphism near $0 \in Z$.*

Proof. This follows immediately from the proof of the basic Lemma in [LT2]. Indeed, from [LT2] and the previous construction we know that both U and Z are smooth and of identical dimensions. Thus to prove that η is a local diffeomorphism near $0 \in Z$ all we need to show is that $d\eta(0) : T_0 Z \to T_{\tilde{w}} U$ is a isomorphism. But this follows from the construction of Z and η. This proves the Proposition. □

By shrinking S^0 if necessary, we can assume that $\eta : Z \to U$ is a local diffeomorphism. By further shrinking Z if necessary, we can assume that η is one-to-one and $\eta(Z) \subset U$ is invariant under $G_{\tilde{\mathbf{U}}}$. The data $\eta : Z \to U$ is the analytic coordinate of $\tilde{w} \in U$ we want. For convenience, we will view Z as an open subset of U.

PROPOSITION 3.8. *Let V' be the restriction of V to Z endowed with the holomorphic structure so that $\tilde{\sigma}_1|_Z, \dots, \tilde{\sigma}_l|_Z$ is a holomorphic frame. Then $\phi' \equiv \phi_{\mathbf{V}}|_Z : Z \to V'$ is holomorphic.*

Proof. This follows immediately from (3.3.15). □

Let $\xi \in \phi_{\mathbf{V}}^{-1}(0) \cap Z$ be any point and let $f : C \to X$ be the associated stable map with D the divisor of its first n-marked points. We assume f is holomorphic. Then there is a canonical exact sequence of vector spaces

$$\operatorname{Ext}^1(\Omega_C(D), \mathcal{O}_C) \longrightarrow H^1(f^*T_X) \longrightarrow \operatorname{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C) \longrightarrow 0,$$

where $\mathcal{D}^\bullet = [f^*\Omega_X \to \Omega_C(D)]$, which is induced by the short exact sequence of complexes

$$0 \longrightarrow [0 \to \Omega_C(D)] \longrightarrow [f^*\Omega_X \to \Omega_C(D)] \longrightarrow [f^*\Omega_X \to 0] \longrightarrow 0.$$

From our construction, we see that the vector space $T_\xi U$ is canonically isomorphic to $\operatorname{Ext}^1(\mathcal{D}^\bullet, \mathcal{O}_C^A)^\dagger$. Hence the differential $d\phi_{\mathbf{V}}(\xi) : T_\xi U \to V|_\xi$ induces an exact sequence of vector spaces

$$\operatorname{Ext}^1(\mathcal{D}^\bullet, \mathcal{O}_C^A)^\dagger = T_\xi U \overset{d\phi_{\mathbf{V}}(z)}{\longrightarrow} V|_\xi \longrightarrow \operatorname{Coker}(d\phi_{\mathbf{V}}(z)) \longrightarrow 0.$$

Note that there are canonical homomorphisms

$$\operatorname{Ext}^1(\mathcal{D}^\bullet, \mathcal{O}_C^A)^\dagger \to \operatorname{Ext}^1(\Omega_C(D), \mathcal{O}_C) \quad \text{and} \quad V|_\xi \to H_{\bar{\partial}}^{0,1}(f^*T_X) \cong H^1(f^*T_X).$$

LEMMA 3.9. *There is a canonical isomorphism δ (as shown below) that fits into* *the commutative diagram of exact sequences*

$$\begin{array}{ccccccc}
\mathrm{Ext}^1(\mathcal{D}^\bullet,\mathcal{O}_C)^\dagger & \xrightarrow{d\phi_{\mathbf{V}}(\xi)} & V|_\xi & \longrightarrow & \mathrm{Coker}(d\phi_{\mathbf{V}}(z)) & \longrightarrow & 0\\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \delta} & & \\
\mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C) & \xrightarrow{\psi} & H^1(f^*T_X) & \longrightarrow & \mathrm{Ext}^2(\mathcal{D}^\bullet,\mathcal{O}_C) & \longrightarrow & 0.
\end{array}$$

Proof. It suffices to show that the left square is commutative. We first describe ψ. Let $\alpha\in\mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C)$ that defines the exact sequence

$$0\longrightarrow\mathcal{O}_C\longrightarrow\mathcal{B}\longrightarrow\Omega_C(D)\longrightarrow 0.$$

Let $h\in\mathrm{Hom}(\Omega_C(D),\mathcal{B}^{\mathcal{A}})$ be a lifting of $df^\vee\in\mathrm{Hom}(f^*\Omega_X,\Omega_C(D))$. Then $\bar\partial h$ lifts to a unique section $(\bar\partial h)^{\mathrm{lift}}\in\mathrm{Hom}(f^*\Omega_X,\Omega^{0,1}_{\mathrm{cpt},C})$. $\psi(\alpha)$ is the cohomology class of $(\bar\partial h)^{\mathrm{lift}}$ in $H^{0,1}_{\bar\partial}(f^*T_X)\equiv H^1(f^*T_X)$. Now let $\beta=(v_1,v_2)\in\mathrm{Ext}^1(\mathcal{D}^\bullet,\mathcal{O}_C)^\dagger$ be an element defined before Lemma 3.3. Then $\varphi_1(\beta)=v_1$ and $d\phi_{\mathbf{V}}(\xi)(\beta)$ is the lift of $\bar\partial v_2$. The homomorphism φ_2 is the obvious map sending a $(0,1)$-form to its cohomology class in $H^1(f^*T_X)$. In case $\alpha=v_1$ we can simply choose h to be v_2. Hence $\varphi_2\circ d\phi_{\mathbf{V}}(\xi)(\beta)=\psi\circ\varphi_1(\alpha)$. This proves the Proposition. □

4. The proof of the comparison theorem. In this section, we will prove that the algebraic and the symplectic construction of GW-invariants yield identical invariants.

We will work with the category of algebraic schemes as well as the category of analytic schemes. Specifically, we will use the words schemes, morphisms and ètale neighborhoods to mean the corresponding objects in algebraic category and use the word analytic maps and open subsets to mean the corresponding objects in analytic category. As before, the words analytic and holomorphic are interchangeable. Also, we will use $\mathcal{O}_S$ to mean the sheaf of algebraic sections or the sheaf of analytic sections when S is an algebraic scheme or an analytic scheme respectively. We will continue to use the complex dimension through out this section.

We now clarify our usage of the notions of cycles and currents. Let W be a scheme. We denote by $Z_k^{\mathrm{alg}}W$ the group of formal sums of finitely many k-dimensional irreducible subvarieties of W with rational coefficients. We call elements of $Z_k^{\mathrm{alg}}W$ k-cycles of W. Now let W be a stratified topological space with stratification $\mathcal{S}$. We say a (complex) k-dimensional current C stratifiable if there is a refinement of $\mathcal{S}$, say $\mathcal{S}'$, such that there are finitely many k-dimensional strata S_i in $\mathcal{S}'$ and rationals $a_i\in\mathbb{Q}$ such that $C=\sum a_i[S_i]$ (All currents in this paper are oriented). Here we assume that each stratum of $\mathcal{S}'$ is oriented a priori and $[S_i]$ is the oriented current defined by S_i. We identify two currents if they define identical measures in the sense of rectifiable currents. We denote the set of all stratifiable k-dimensional currents modulo the equivalence relation by Z_kW. Clearly, if W is a scheme then any k-cycle has an associated current in Z_kW, which defines a map $Z_k^{\mathrm{alg}}W\to Z_kW$. In the following, we will not distinguish a cycle from its associated current. Hence for $C\in Z_k^{\mathrm{alg}}W$ we will view it as an element of Z_kW. Note that if $C\in Z_kW$ has zero boundary in the sense of current and C has compact support, then C defines canonically an element in $H_{2k}(W,\mathbb{Q})$. Finally, if $C=\sum a_i[S_i]\in Z_kW$ and $F\subset W$ is a stratifiable subset, we say that C intersects F transversally if F intersects each S_i transversally as stratified sets (See [GM] for topics on stratifications). In such cases, we can define the intersection current $C \cap F$ if the orientation of the intersection can be defined according to the geometry of W and F.

We begin with a quick review of the algebraic construction of GW-invariants. Let X be any smooth projective variety and let $A \in H_2(X,\mathbb{Z})$ and $g,n \in \mathbb{Z}$ as before be fixed once and for all. We let $\mathfrak{M}_{g,n}(X,A)$ be the moduli scheme of stable morphisms defined before. $\mathfrak{M}_{g,n}(X,A)$ is projective. The GW-invariants of X is defined using the virtual moduli cycle

$$[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}} \in A_*\mathfrak{M}_{g,n}(X,A).$$

To review such a construction, a few words on the obstruction theory of deformations of morphisms are in order. Let $w \in \mathfrak{M}_{g,n}(X,A)$ be any point associated to the stable morphism $\mathcal{X}$. Let $(B,I,\mathcal{X}_{B/I})$ be a collection where B is an Artin ring, $I \subset B$ is an ideal annihilated by the maximal ideal $\mathfrak{m}_B$ of B and $\mathcal{X}_{B/I}$ is a flat family of stable morphisms over $\operatorname{Spec} B/I$ whose restriction to the closed fiber of $\mathcal{X}_{B/I}$ is isomorphic to $\mathcal{X}$. An obstruction theory to deformation of $\mathcal{X}$ consists of a $\mathbb{C}$-vector space V, called the obstruction space, and an assignment that assigns any data $(B,I,\mathcal{X}_{B/I})$ as before to an obstruction class

$$\mathrm{Ob}(B,B/I,\mathcal{X}_{B/I}) \in I \otimes_{\mathbb{C}} V$$

to extending $\mathcal{X}_{B/I}$ to $\operatorname{Spec} B$. Here by an obstruction class we mean that its vanishing is the necessary and sufficient condition for $\mathcal{X}_{B/I}$ to be extendable to a family over $\operatorname{Spec} B$. We also require that such an assignment satisfies the obvious base change property (For reference on obstruction theory please consult [Ob]). In case $\mathcal{X}$ is the map $f: C \to X$ with $D \subset C$ the divisor of its n marked points, the space of the first order deformations of $\mathcal{X}$ is parameterized by $\operatorname{Ext}^1(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C)$, where $\mathcal{D}^\bullet_{\mathcal{X}} = [f^*\Omega_X \to \Omega_C(D)]$ is the complex as before, and the standard obstruction theory to deformation of $\mathcal{X}$ takes values in $\operatorname{Ext}^2(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C)$.

An example of obstruction theories is the following. Let R be the ring of formal power series in m variables and let $\mathfrak{m}_R \subset R$ be its maximal ideal. Let F be a vector space and let $f \in \mathfrak{m}_R \otimes_{\mathbb{C}} F$. We let $(f) \subset R$ be the ideal generated by components of f and let V be the cokernel of $df: (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee \to F$. Then there is a standard obstruction theory to deformations of 0 in $\operatorname{Spec} R/(f)$ taking values in V defined as follows. Let $I \subset B$ be an ideal of an Artin ring as before and let $\varphi_0: \operatorname{Spec} B/I \to \operatorname{Spec} R/(f)$ be any morphism. To extend φ_0 to $\operatorname{Spec} B$, we first pick a homomorphism $\sigma: R \to B$ extending the induced $R \to B/I$, and hence a morphism $\varphi_{\mathrm{pre}}: \operatorname{Spec} B \to \operatorname{Spec} R$. The image $\sigma(f) \in B \otimes F$ is in $I \otimes F$, and is the obstruction to φ_{pre} factor through $\operatorname{Spec} R/(f) \subset \operatorname{Spec} R$. Let $\mathrm{Ob}(B,B/I,\varphi_0)$ be the image of $\sigma(f)$ in $I \otimes V$ via $F \to V$. It is direct to check that $\mathrm{Ob}(B,B/I,\varphi_0) = 0$ if and only if there is an extension $\varphi: \operatorname{Spec} B \to \operatorname{Spec} R/(f)$ of φ_0. This assignment

$$(B,B/I,\varphi_0) \mapsto \mathrm{Ob}(B,B/I,\varphi_0) \in I \otimes V \tag{4.4.1}$$

is the induced obstruction theory of $\operatorname{Spec} R/(f)$.

Let $\mathcal{X}$ be a stable morphism.

DEFINITION 4.1. *A Kuranishi family of the standard obstruction theory of $\mathcal{X}$ consists of a vector space F, a ring of formal power series R with $\mathfrak{m}_R$ its maximal ideal, an $f \in \mathfrak{m}_R \otimes F$, a family $\mathcal{X}_{R/(f)}$ of stable morphisms over* $\operatorname{Spec} R/(f)$ *whose closed fiber over* $0 \in \operatorname{Spec} R/(f)$ *is isomorphic to $\mathcal{X}$ and an exact sequence*

$$0 \longrightarrow \operatorname{Ext}^1(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C) \xrightarrow{\alpha} (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee \xrightarrow{df} F \longrightarrow \operatorname{Ext}^2(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C) \longrightarrow 0 \tag{4.4.2}$$

of which the following holds: First, the composite

$$\mathrm{Ext}^1(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C) \xrightarrow{\alpha} \ker(df) \equiv T_0 \operatorname{Spec} R/(f) \longrightarrow \mathrm{Ext}^1(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C),$$

where the second arrow is the Kodaira-Spencer map of the family $\mathcal{X}_{R/(f)}$, is the identity homomorphism; Secondly, let $I \subset B$ and $\varphi_0 : \operatorname{Spec} B/I \to \operatorname{Spec} R/(f)$ be as before and let

$$\mathrm{Ob}(B, B/I, \varphi_0^* \mathcal{X}_{R/(f)}) \in I \otimes \mathrm{Ext}^2(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C)$$

be the obstruction to extending $\varphi_0^* \mathcal{X}_{B/I}$ to $\operatorname{Spec} B$. *Then it is identical to* $\mathrm{Ob}(B, B/I, \varphi_0)$ *in* (4.4.1) *under the isomorphism*

$$\mathrm{Coker}(df) \cong \mathrm{Ext}^2(\mathcal{D}^\bullet_{\mathcal{X}}, \mathcal{O}_C).$$

We now sketch how the virtual moduli cycle $[\mathfrak{M}_{g,n}(X, A)]^{\mathrm{vir}}$ was constructed. Similar to the situation of the moduli of stable smooth maps, we need to treat $\mathfrak{M}_{g,n}(X, A)$ either as a $\mathbb{Q}$-scheme or as a Deligne-Mumford stack. The key ingredient here is the notion of atlas, which is a collection of charts of $\mathfrak{M}_{g,n}(X, A)$. A chart of $\mathfrak{M}_{g,n}(X, A)$ is a tuple $(S, G, \mathcal{X}_S)$, where G is a finite group, S is a G-scheme (with effective G-action) and $\mathcal{X}_S$ is a G-equivariant family of stable morphisms so that the tautological morphism $\iota : S/G \to \mathfrak{M}_{g,n}(X, A)$ induced by the family $\mathcal{X}_S$ is an étale neighborhood. For details of such notion, please consult [DM, Vi, LT1]. We now let $f : C \to X$ be the representative of $\mathcal{X}_S$ with $D \subset C$ the divisor of the n-marked sections of $\mathcal{X}_S$. Let $\pi : C \to S$ be the projection. We consider the relative extension sheaves $\mathcal{E}xt^i_\pi(\mathcal{D}^\bullet_{\mathcal{X}_S}, \mathcal{O}_C)$, where $\mathcal{D}^\bullet_{\mathcal{X}_S} = [f^*\Omega_X \to \Omega_{C/S}(D)]$ as before. For short, we denote the sheaves $\mathcal{E}xt^i_\pi(\mathcal{D}^\bullet_{\mathcal{X}_S}, \mathcal{O}_C)$ by $\mathcal{T}^i_S$. Because they vanish for $i = 0$ and $i > 2$, for any $w \in S$, the Zariski-tangent space $T_w S$ is $\mathcal{T}^1_S \otimes_{\mathcal{O}_S} k(w)$ and the obstruction space to deformations of w in S is $V_w = \mathcal{T}^2_S \otimes_{\mathcal{O}_S} k(w)$. Now we choose a complex of locally free sheaves of $\mathcal{O}_S$-modules $\mathcal{E}^\bullet = [\mathcal{E}_1 \to \mathcal{E}_2]$ so that it fits into the exact sequence

$$0 \longrightarrow \mathcal{T}^1_S \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{T}^2_S \longrightarrow 0. \tag{4.4.3}$$

We let $F_i(w) = \mathcal{E}_i \otimes_{\mathcal{O}_S} k(w)$. Then we have the exact sequence of vector spaces

$$0 \longrightarrow T_w S \longrightarrow F_1(w) \longrightarrow F_2(w) \longrightarrow V_w \longrightarrow 0. \tag{4.4.4}$$

We let $K_w \in R(w) \otimes_{\mathbb{C}} V_w$ be a Kuranishi map of the obstruction theory to deformations of w, where $R(w) = \varprojlim \oplus_{k=0}^N \mathrm{Sym}^k(F_1(w)^\vee)$, so that (4.4.4) is part of the data of the Kuranishi family specified in Definition 4.1. Let $(K_w) \subset R(w)$ be the ideal generated by the components of K_w and let $\operatorname{Spec} R_w/(K_w) \subset \operatorname{Spec} R_w$ be the corresponding subscheme. It follows that $\operatorname{Spec} R_w/(K_w)$ is isomorphic to the formal completion of S along w, denoted $\hat{w}$. We let N_w be the normal cone to $\operatorname{Spec} R_w/(K_w)$ in $\operatorname{Spec} R_w$. Then N_w is canonically a subcone of $F_2(w) \times \hat{w}$. Here, by abuse of notation we will use $F_2(w)$ to denote the total space of the vector space $F_2(w)$. Note that N_w is the infinitesimal normal cone to S in its obstruction theory at w. To obtain a global cone over S, we need the following existence and uniqueness theorem, which is the main result of [LT1].

In this paper, we will call a vector bundle E the associated vector bundle of a locally free sheaf $\mathcal{E}$ if $\mathcal{O}(E) \cong \mathcal{E}$. For notational simplicity, we will not distinguish a vector bundle from the total space (scheme) of this vector bundle.

THEOREM 4.2 ([LT1]). *Let E be the associated vector bundle of $\mathcal{E}_2$. Then there is a cone scheme $N_S \subset E$ such that for each $w \in S$ there is an isomorphism*

$$F_2(w) \times \hat{w} \cong E \times_S \hat{w} \tag{4.4.5}$$

of cones over $\hat{w}$ extending the canonical isomorphism $F_2(w) \cong E \times_S w$ such that under the above isomorphism N_w is isomorphic to $N_S \times_S \hat{w}$. In particular, the cycle defined by the scheme N_S is uniquely characterized by this condition.

In the previous discussion, if we replace $F_1(w)$ and $F_2(w)$ by $T_w S$ and V_w respectively, we obtain a Kuranishi map and correspondingly a cone scheme in $V_w \times \hat{w}$, denoted by N_w^0.

THEOREM 4.3 ([LT1]). *Let the notation be as before. Then there is a vector bundle homomorphism $r: E \times_S \hat{w} \to V_w \times \hat{w}$ extending the canonical homomorphism $E|_w \equiv F_2(w) \to V_w$ induced by (4.4.4) such that*

$$N_w^0 \times_{V_w \times \hat{w}} E \times_S \hat{w} = N_S \times_S \hat{w}.$$

To construct the virtual cycle $[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}}$, we need to find a global complex over $\mathfrak{M}_{g,n}(X,A)$ analogous to $\mathcal{E}^\bullet$. We will use atlas of analytic charts for the purpose of comparing with the analytic construction of the virtual cycles. We let $\{(U_i, V_i, \phi_i)\}_{i\in\Lambda}$ be the good atlas of the smooth approximation of $[\Phi: \mathbf{B} \to \mathbf{E}]$ chosen in section 2. Then the collection $Z_i = \phi_i^{-1}(0)$ with the tautological family of stable analytic maps (with the last k-marked points discarded) form an atlas of the underlying analytic scheme of $\mathfrak{M}_{g,n}(X,A) \cong \Phi^{-1}(0)$. Since we are only interested in constructing and working with cone cycles in $\mathbb{Q}$-bundles (known as V-bundles) over $\mathfrak{M}_{g,n}(X,A)$, there is no loss of generality that we work with $\mathfrak{M}_{g,n}(X,A)$ with the reduced scheme structure. Hence, for simplicity we will endow $Z_i = \phi_i^{-1}(0)$ with the reduced analytic scheme structure. We let $\mathcal{X}_i$ be the tautological family of the n-pointed stable analytic maps over Z_i that is derived by discarding the last k marked points of the restriction to Z_i of the tautological family over $\tilde{\mathbf{U}}_i$. We let G_i be the finite group associated to the chart (U_i, V_i, ϕ_i), and let $\mathcal{X}_i$ be represented by $f_i: C_i \to X$ with $D_i \subset C_i$ the divisor of the n-marked sections of C_i and $\pi_i: C_i \to Z_i$ the projection. In [LT1], to each i, we have constructed a G_i-equivariant locally free sheaves of $\mathcal{O}_{Z_i}$-modules $\mathcal{E}_i$ such that $\mathcal{E}xt^2_{\pi_i}(\mathcal{D}^\bullet_{\mathcal{X}_i}, \mathcal{O}_{C_i})$ is the quotient sheaf of $\mathcal{E}_i$. It follows from the algebraic and the analytic constructions of charts that each $(Z_i, \mathcal{X}_i)$ can be realized as an analytic open subset of an algebraic chart, say $(S, G, \mathcal{X}_S)$, and $\mathcal{E}_i$ is the restriction to this open subset of an algebraic sheaf $\mathcal{E}$, as in (4.4). Therefore we can apply Theorem 4.2 to obtain a unique analytic cone cycle $M_i^{\mathrm{alg}} \in Z_* E_i$, where E_i is the associated vector bundle of $\mathcal{E}_i$. Let $\iota_i: Z_i/G_i \to \mathfrak{M}_{g,n}(X,A)$ be the tautological map induced by the family $\mathcal{X}_i$. One property that follows from the construction of the $\mathcal{E}_i$ which we did not mention is that to each i, the cone bundle E_i/G_i over Z_i/G_i descends to a cone bundle over $\iota_i(Z_i/G_i)$, denoted by $\tilde{E}_i$, and $\{\tilde{E}_i\}_{i\in\Lambda}$ patch together to form a global cone bundle over $\mathfrak{M}_{g,n}(X,A)$, denoted by $\tilde{E}$. Further, by the uniqueness of the cone cycles $M_i^{\mathrm{alg}} \in Z_* E_i$ in Theorem 3.2 and 3.3, to each i the cone cycle M_i^{alg}/G_i in E_i/G_i descends to a cone cycle $\mathcal{M}_i^{\mathrm{alg}} \in Z_* \tilde{E}_i$, and $\{\mathcal{M}_i^{\mathrm{alg}}\}_{i\in\Lambda}$ patch together to form a cone cycle in $Z_* \tilde{E}$, denoted by $\mathcal{M}^{\mathrm{alg}}$. It follows from [LT1] that $\tilde{E}$ is an algebraic cone over $\mathfrak{M}_{g,n}(X,A)$ and $\mathcal{M}^{\mathrm{alg}}$ is an algebraic cone cycle in $\tilde{E}$. In the end, we let $\eta_E: \mathfrak{M}_{g,n}(X,A) \to \tilde{E}$ be the zero section and let

$$\eta_E^*: \{\text{algebraic cycles in } Z_* \tilde{E}\} \longrightarrow A_* \mathfrak{M}_{g,n}(X,A)$$

be the Gysin homomorphism [Fu, Vi]. Here all cycle groups are of rational coefficient. Then the virtual moduli cycle is

$$[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}} = \eta_E^*[\mathcal{M}^{\mathrm{alg}}] \in A_*\mathfrak{M}_{g,n}(X,A).$$

There is an analogous way to construct the GW-invariants of algebraic varieties using analytic method. We continue to use the notion developed in section 2. Let (U, V, ϕ) be the finite dimensional model of a smooth approximation $(\tilde{\mathbf{U}}, \mathbf{V}, \tilde{\Phi})$ of $[\Phi : \mathbf{B} \to \mathbf{E}]$ constructed in Lemma 2.7. Then we can construct a cone current in the total space of V as follows. Let $\Gamma_{t\phi}$ be the graph of $t\phi$ in V and let $N_{0/\phi}$ be the limit current $\lim_{t\to\infty}\Gamma_{t\phi}$, when it exists. Clearly, if such a limit does exist, then it is contained in $V|_{\phi^{-1}(0)}$. In general, for smooth ϕ there is no guarantee that such a limit will exist. However, if the approximation is analytic, then we will show that such limit does exist as an stratifiable current. Indeed, assume (U, V, ϕ) is an analytic smooth approximation. Since the existence of $\lim \Gamma_{t\phi}$ is a local problem, we can assume that there is a holomorphic frame of V, say $e_1, \dots, e_r$. Then ϕ can be expressed in terms of r holomorphic functions $\phi_1, \dots, \phi_r$. Now let $\mathbf{C}$ be the complex line with complex variable t, let w_i be the dual of e_i and let $\Theta \subset V \times \mathbf{C}$ be the analytic subscheme defined by the vanishing of $tw_i - \phi_i$, $i = 1, \dots, r$. We let Θ_0 be the smallest closed analytic subscheme of Θ that contains $\Theta \cap (V \times \mathbf{C}^*)$, where $\mathbf{C}^* = \mathbf{C} - \{0\}$. By the Weierstrass preparation theorem, such Θ_0 does exist. Then we define $N_{0/\phi}$ to be the associated cycle of the intersection of the scheme Θ_0 with $V \times \{0\}$. By [Fu], $N_{0/\phi}$ is the limit of $\Gamma_{t\phi}$. Obviously, $N_{0/\phi}$ is stratifiable. This shows that for any analytic approximation (U, V, ϕ) the limit $\lim \Gamma_{t\phi}$ does exist.

We now state a simple lemma which implies that if (U, V, ϕ) is a smooth approximation that is finer than the analytic approximation (U', V', ϕ'), then $\lim \Gamma_{t\phi}$ exists as well. We begin with the following situation. Let E be a smooth oriented vector bundle over a smooth oriented manifold M and let $\psi : M \to E$ be a smooth section. Let $E' \subset E$ be a smooth subbundle such that at any $x \in \psi^{-1}(0)$ we have $\mathrm{Im}(d\psi(x)) + E'|_x = E|_x$. Then $M_0 = \psi^{-1}(E')$ is a smooth submanifold of M near $\psi^{-1}(0)$. Let E_0 be the restriction of E' to M_0 and let $\psi_0 : M_0 \to E_0$ be the induced section. We next let $N \subset TM|_{\psi^{-1}(0)}$ be a subbundle complement to $TM_0|_{\psi^{-1}(0)}$ in $TM|_{\psi^{-1}(0)}$. Then the union of $d\psi(x)(N_x)$ for all $x \in \psi^{-1}(0)$ forms a subbundle of $E|_{\psi^{-1}(0)}$. We denote this bundle by $d\psi(N)$. Clearly, $E|_{\psi^{-1}(0)} \equiv E_0|_{\psi^{-1}(0)} \oplus d\psi(N)$. Hence there is a unique projection

$$P : E|_{\psi^{-1}(0)} \to E_0|_{\psi^{-1}(0)} \tag{4.4.6}$$

according to this direct sum decomposition. Note that P depends on the choice of the complement N.

LEMMA 4.4. *Let the notation be as before and let* $l = \dim M$ *and* $l_0 = \dim M_0$. *Then* $\lim \Gamma_{t\psi}$ *exists as an* l*-dimensional current in* $E|_{\psi^{-1}(0)}$ *if and only if* $\lim \Gamma_{t\psi_0}$ *exists as an* l_0*-dimensional oriented current in* $E_0|_{\psi^{-1}(0)}$. *Further, if they do exist then*

$$\lim \Gamma_{t\psi} = P^*(\lim \Gamma_{t\psi_0}).$$

Hence $\lim \Gamma_{t\phi}$ *is stratifiable if* $\lim \Gamma_{t\phi_0}$ *is stratifiable.*

Proof. This is obvious and will be left to the readers. □

Now let $\{(U_\alpha, V_\alpha, \phi_\alpha)\}_{\alpha\in\Xi}$ be a collection of analytic smooth approximations of $[\Phi:\mathbf{B}\to\mathbf{E}]$ such that the images of $Z_\alpha=\phi_\alpha^{-1}(0)$ in $\Phi^{-1}(0)$ covers $\Phi^{-1}(0)$. We choose a good atlas $\{(U_i, V_i, \phi_i)\}_{i\in\Lambda}$ constructed in Lemma 2.7 so that all approximations in Λ are finer than approximations in Ξ. Now let $i\in\Lambda$ and let $x\in Z_i=\phi_i^{-1}(0)\subset U_i$ be any point. Because charts in Ξ cover $\Phi^{-1}(0)$, there is an $\alpha\in\Xi$ such that the image of U_α in $\mathbf{B}$ contains the image of x in $\mathbf{B}$. Then because (U_i, V_i, ϕ_i) is finer than $(U_\alpha, V_\alpha, \phi_\alpha)$, by definition, there is a locally closed submanifold $U_{i,\alpha}\subset U_i$, a local diffeomorphism $f_i^\alpha: U_{i,\alpha}\to U_\alpha$ and a vector bundle inclusion $(f_i^\alpha)^*V_\alpha\subset V_i|_{U_{i,\alpha}}$ such that $(f_i^\alpha)^*(\phi_\alpha)=\phi_i$, as in (2.2.5). This is exactly the situation studied in Lemma 4.4. Let

$$P_{i,\alpha}: V_i|_{U_{i,\alpha}\cap Z_i}\longrightarrow (f_i^\alpha)^*V_\alpha|_{U_{i,\alpha}\cap Z_i} \tag{4.4.7}$$

be the projection as in (4.4.6) associated to the choice of a complement N of $TU_{i,\alpha}\subset TU_i|_{U_{i,\alpha}}$. Then $\lim\Gamma_{t\phi_i}$ exists near fibers of V over x since it is $P_{i,\alpha}^*(\lim\Gamma_{t\phi_\alpha})$, which exists. Because $\{Z_\alpha\}$ covers $\Phi^{-1}(0)$, $\lim\Gamma_{t\phi_i}$ exists everywhere and is a pure dimensional stratifiable current of dimension $\dim U_i$. We denote this current by N_{0/ϕ_i}.

Now it is clear how to construct the GW-invariants of algebraic varieties using these analytically constructed cones. By the property of good coverings, for $j\le i\in\Lambda$ the approximation (U_i, V_i, ϕ_i) is finer than (U_j, V_j, ϕ_j). We let $Z_i=\phi_i^{-1}(0)$ be as before and let $Z_{i,j}=Z_i\cap U_{i,j}\subset Z_i$, where $U_{i,j}$ is defined before (2.2.5). Let $\rho_i^j: Z_{i,j}\to Z_j$ be the restriction of f_i^j to $Z_{i,j}$. Note that $Z_{i,j}$ is an open subset of Z_i and $\rho_i^j: Z_{i,j}\to\rho_i^j(Z_{i,j})$ is a covering. Let F_i be the restriction of V_i to Z_i and let $p_i: F_i\to Z_i$ be the projection. Hence, $(\rho_i^j)^*(F_j)$ is canonically a subvector bundle of $F_i|_{Z_{i,j}}$. By Lemma 4.4, $(\rho_i^j)^*(F_j)$ intersects transversally with $N_{0/\phi_i}\cap p_i^{-1}(Z_{i,j})$ and as currents,

$$N_{0/\phi_i}\cap(\rho_i^j)^*(F_j)=(p_i^j)^*(N_{0/\phi_j}).$$

For convenience, in the following we will call the collection $\{F_i,\rho_i^j\}$ a semi-$\mathbb{Q}$-bundle. We denote it by $\mathfrak{F}$ and denote $\{N_{0/\phi_i}\}$ by $\mathcal{N}^{\mathrm{an}}$. As in section two, we call a collection $\mathfrak{s}=\{s_i\}_{i\in\Lambda}$, where $s_i\in\Gamma_{Z_i}(F_i)$, a global section of $\mathfrak{F}$ if for $j\le i\in\Lambda$ the restriction $s_i|_{Z_{i,j}}\in\Gamma_{Z_{i,j}}(F_i)$ coincides with the pull back section $(\rho_i^j)^*s_j\in\Gamma_{Z_{i,j}}((\rho_i^j)^*F_j)$ under the canonical inclusion $(\rho_i^j)^*F_j\subset F_i|_{Z_{i,j}}$. We say that the section $\mathfrak{s}$ is transversal to $\mathcal{N}^{\mathrm{an}}$ if for each $i\in\Lambda$ the graph of s_i is transversal to N_{0/ϕ_i} in F_i.

Now let $\mathfrak{s}$ be a global section of $\mathfrak{F}$ transversal to $\mathcal{N}^{\mathrm{an}}$. Let $\iota_i': Z_i\to\mathbf{B}$ be the map induced by $\iota_i: U_i\to\mathbf{B}$ and m_i be the number of sheets of the branched covering $\iota_i': Z_i\to\iota_i'(Z_i)$. Then following the argument after Lemma 2.11, currents

$$\frac{1}{m_i}\iota_{i*}'\pi_{i*}(N_{0/\phi_i}\cap\Gamma_{s_i}),\quad i\in\Lambda,$$

patch together to form an oriented current in $\mathbf{B}$ with zero boundary. We denote this current by $\mathfrak{s}^*(\mathcal{N}^{\mathrm{an}})$. It has pure dimension $r_{\exp}$ since the current N_{0/ϕ_i} has dimension $\dim U_i=\operatorname{rank}F_i+r_{\exp}$. Hence it defines a homology class $[\mathfrak{s}^*(\mathcal{N}^{\mathrm{an}})]$ in $H_{2r_{\exp}}(\mathbf{B};\mathbb{Q})$.

PROPOSITION 4.5. *$[\mathfrak{s}^*(\mathcal{N}^{an})]$ is the Euler class $e[\Phi:\mathbf{B}\to\mathbf{E}]$ constructed in section one.*

Proof. Recall that the class $e[\Phi:\mathbf{B}\to\mathbf{E}]$ was constructed by first selecting a collection of perturbations $h_i(s): U_i\to V_i$ of ϕ_i parameterized by $s\in[0,1]$ satisfying certain property and then form the current that is the patch together of the currents $\frac{1}{m_i}\iota_{i*}(\Gamma_{h_i(1)}\cap\Gamma_0)$, where $\Gamma_{h_i(1)}$ and Γ_0 are the graphs of $h_i(1)$ and the zero section $0: U_i \to V_i$ respectively. Without loss of generality, we can assume that the perturbations $h_i(s)$ are such that the graph $\Gamma_{h_i(1)}$ is transversal to both N_{0/ϕ_i} and the graph $\Gamma_{t\phi_i}$ for sufficiently large t. Of course such perturbations do exist following the proof of Proposition 2.13. Let C_t be the current in $\mathbf{B}$ that is the result of patching together the currents $\frac{1}{m_i}\iota_{i*}p_{i*}(\Gamma_{h_i(1)}\cap\Gamma_{t\phi_i})$, where p_i is the projection $V_i \to U_i$. Clearly, for $t \gg 0$, we have $\partial C_t = 0$ and $\operatorname{supp}(C_t)$ is compact. Hence C_t defines a homology class in $H_{2r_{\exp}}(\mathbf{B};\mathbb{Q})$, denoted by $[C_t]$. It follows from the uniqueness argument in the end of section two that for sufficiently large t, the homology class $[C_t]$ in $H_{2r_{\exp}}(\mathbf{B};\mathbb{Q})$ is exactly the Euler class. On the other hand, we let C_∞ be the current in $\mathbf{B}$ that is the patch together of the currents $\frac{1}{m_i}\iota_{i*}p_{i*}(\Gamma_{h_i(1)}\cap N_{0/\phi_0})$. Because N_{0/ϕ_i} is the limit of $\Gamma_{t\phi_i}$, and because $\Gamma_{h_i(1)}$ intersects transversally with $\Gamma_{t\phi_i}$ for $t \gg 0$ and with N_{0/ϕ_i}, the union

$$\bigcup_{t\in[0,\epsilon]} \{t\}\times C_{1/t} \subset [0,\epsilon]\times\mathbf{B},$$

where $1 \gg \epsilon > 0$, is a current whose boundary is $C_{1/\epsilon} - C_\infty$. This implies that

$$[C_\infty] = [C_t] \in H_{2r_{\exp}}(\mathbf{B};\mathbb{Q}) \qquad \text{for } t \gg 0.$$

Further, because N_{0/ϕ_i} is contained in $F_i = V_i|_{Z_i}$, $p_{i*}(N_{0/\phi_i}\cap\Gamma_{h_i(1)})$ as current is identical to $\pi_{i*}(N_{0/\phi_i}\cap\Gamma_{r_i})$, where $r_i \in \Gamma_{Z_i}(F_i)$ is the restriction of $h_i(1)$ to Z_i. Hence $C_\infty = \mathbf{r}^*(\mathcal{N}^{\mathrm{an}})$ with $\mathbf{r} = \{r_i\}$. Finally, it is direct to check that the homology class $[\mathfrak{s}^*(\mathcal{N}^{\mathrm{an}})]$ do not depend on the choice of the section $\mathfrak{s}$ of $\mathfrak{F} = \{F_i\}$ so long as they satisfy the obvious transversality conditions. Therefore,

$$[\mathfrak{s}^*(\mathcal{N}^{\mathrm{an}})] = [\mathbf{r}^*(\mathcal{N}^{\mathrm{an}})] = [C_{1/\epsilon}] = e[\Phi:\mathbf{B}\to\mathbf{E}].$$

This proves the Proposition. □

Let $e:\mathbf{B}\to X^n$ be the evaluation map, $\pi:\mathbf{B}\to\overline{\mathfrak{M}}_{g,n}$ be the forgetful stable contraction. Then the algebraic GW-invariants

$$\gamma^X_{A,g,n}: H^*(X)^{\times n}\times H^*(\overline{\mathfrak{M}}_{g,n}) \longrightarrow \mathbb{Q}$$

is defined by the formula

$$\gamma^X_{A,g,n}(\alpha,\beta) = \int_{[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}}} e^*(\alpha)\cup\pi^*(\beta),$$

where this time $[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}}$ is considered as the corresponding homology class in $H_*(\mathbf{B};\mathbb{Q})$. The symplectic GW-invariants is defined similarly with $[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}}$ replaced by the Euler class $e[\Phi:\mathbf{B}\to\mathbf{E}]$. Hence the main theorem 1.1 will follow from the following theorem:

THEOREM 4.6. *Let X be any smooth projective variety and A, g and n be as before. Then the homology class $[\mathfrak{M}_{g,n}(X,A)]^{\mathrm{vir}}$ in $H_*(\mathbf{B};\mathbb{Q})$ is exactly the Euler class $e[\Phi:\mathbf{B}\to\mathbf{E}]$ constructed in section two.*

In the end, we will compare the algebraic normal cones with the analytic normal cones to demonstrate that the algebraic and the analytic construction of the GW-invariants give rise to the identical invariants.

Here is our strategy. Taking the good atlas $\{(Z_i, \mathcal{X}_i)\}_{i\in\Lambda}$ of $\mathfrak{M}_{g,n}(X,A)$ as before, we have two semi-$\mathbb{Q}$-vector bundles $\mathfrak{E} = \{E_i\}$ and $\mathfrak{F} = \{F_i\}$, two cone currents $\mathcal{M}^{\mathrm{alg}} = \{M_i^{\mathrm{alg}}\}$ and $\mathcal{N}^{\mathrm{an}} = \{N_{0/\phi_i}\}$ of $\mathfrak{E}$ and $\mathfrak{F}$ respectively such that $[\eta_{\mathfrak{E}}^*(\mathcal{M}^{\mathrm{alg}})]$ and $[\eta_{\mathfrak{F}}^*(\mathcal{N}^{\mathrm{an}})]$ are the algebraic and the symplectic virtual moduli cycles of $\mathfrak{M}_{g,n}(X,A)$ respectively. Here $\eta_{\mathfrak{E}}$ and $\eta_{\mathfrak{F}}$ are generic sections of $\mathfrak{E}$ and $\mathfrak{F}$ respectively. To compare these two classes, we will form a new semi-$\mathbb{Q}$-vector bundle $\mathfrak{W} = \{W_i\}$, where $W_i = E_i \oplus F_i$, and construct a stratifiable cone current $\mathcal{P}$ in $\mathfrak{W}$ such that the current $\mathcal{P}$ intersects $\mathfrak{E} \subset \mathfrak{W}$ and $\mathfrak{F} \subset \mathfrak{W}$ transversally and the intersections $\mathcal{P} \cap \mathfrak{E}$ and $\mathcal{P} \cap \mathfrak{F}$ are $\mathcal{M}^{\mathrm{alg}}$ and $\mathcal{N}^{\mathrm{an}}$ respectively. Therefore, if we let $\eta_{\mathfrak{W}}$ be a generic section of $\mathfrak{W}$, then

$$[\eta_{\mathfrak{E}}^*(\mathcal{M}^{\mathrm{alg}})] = [\eta_{\mathfrak{W}}^*(\mathcal{P})] = [\eta_{\mathfrak{F}}^*(\mathcal{N}^{\mathrm{an}})] \in H_*(\mathfrak{M}_{g,n}(X,A);\mathbb{Q}).$$

This will prove the Comparison Theorem.

We now provide the details of this construction. Let $i \in \Lambda$ be any index. We first recall the construction of the vector bundles E_i. Let $f_i : C_i \to X$ be the tautological family of stable maps over Z_i with $D_i \subset C_i$ the divisor of its n-marked sections and $\pi_i : C_i \to Z_i$ the projection. Note that f_i is the restriction of a family of stable morphisms over a scheme to an analytic open subset of the base scheme. Following the construction in [LT1, section 3], after fixing a sufficiently ample line bundle over X, we canonically construct a locally free sheaf of $\mathcal{O}_C$-modules $\mathcal{K}_i$ so that $f_i^*\Omega_X$ is canonically a quotient sheaf of $\mathcal{K}_i$. Let $\mathcal{L}_i$ be the kernel of $\mathcal{K}_i \to f_i^*\Omega_X$. Then the sheaf $\mathcal{E}_i$ mentioned before is the direct image sheaf $\pi_{i*}(\mathcal{L}_i^\vee)$. We let E_i be its associated vector bundle. We define $W_i = E_i \oplus F_i$.

We next construct the cycle (current) $\mathcal{P}_i \subset W_i$. We construct $\mathcal{P}_i$ locally. Let $x \in Z_i$ be any point. By our construction of the collection Λ there is an analytic approximation $\alpha \in \Xi$ so that $\iota_i(x) \subset \iota_\alpha(Z_\alpha)$ and (U_i, V_i, ϕ_i) is finer than $(U_\alpha, V_\alpha, \phi_\alpha)$. Let $U_{i,\alpha} \subset U_i$ and $f_i^\alpha : U_{i,\alpha} \to U_\alpha$ be the comparison map in (2.2.5), let $Z_{i,\alpha} = U_{i,\alpha} \cap Z_i$ and let $\rho_i^\alpha : Z_{i,\alpha} \to Z_\alpha$ be the restriction of f_i^α to $Z_{i,\alpha} \subset U_{i,\alpha}$. Note that $(\rho_i^\alpha)^*(F_\alpha)$ is a smooth subvector bundle of $F_i|_{Z_{i,\alpha}}$. We let

$$P_{i,\alpha} : F_i|_{Z_{i,\alpha}} \longrightarrow (\rho_i^\alpha)^*(F_\alpha) \tag{4.4.8}$$

be the projection given in (4.4.7) We next construct a homomorphism

$$Q_\alpha : F_\alpha \longrightarrow E_\alpha. \tag{4.4.9}$$

By the construction, F_α is generated by l independent sections $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_l$ (see their construction before lemma 3.1). Recall that for each $w \in Z_\alpha$, the restriction $\tilde{\sigma}_j(w)$ of $\tilde{\sigma}_j$ to fiber $C_\alpha|_w$ of C_α over w is a (0,1)-form with values in $f_\alpha^*T_X|_{C_\alpha|_w}$. Namely, if we let $\Omega^{0,1}_{C_\alpha/Z_\alpha}$ be the quotient sheaf of the pull-back homomorphism $\pi_\alpha^*\Omega^{0,1}_{Z_\alpha} \to \Omega^{0,1}_{C_\alpha}$, then $\tilde{\sigma}_j$ are sections of $\Omega^{0,1}_{C_\alpha/Z_\alpha}(f_\alpha^*T_X)$ of compact support (i.e. sections that vanish near nodal points of fibers). Now we look at the exact sequence

$$0 \longrightarrow \mathcal{O}_{C_\alpha}(f_\alpha^*T_X) \xrightarrow{e_1} \mathcal{K}_\alpha^\vee \xrightarrow{e_2} \mathcal{L}_\alpha^\vee \longrightarrow 0.$$

Since $\mathcal{K}_\alpha^\vee$ is assumed to be sufficiently ample, to each j there is a smooth section $h_j \in \Gamma_{C_\alpha}(\mathcal{K}_\alpha^\vee)$ so that the image of $\bar{\partial} h_j$ in $\Omega^{0,1}_{C_\alpha/Z_\alpha}(\mathcal{K}_\alpha^\vee)$ is equal to $\tilde{\sigma}_j$. Then $e_2(h_j)$ is a fiberwise holomorphic section of $\mathcal{L}_\alpha^\vee$. In particular, $e_2(h_j)$ is a smooth section of E_α. Since the choice of $\mathcal{K}_\alpha^\vee$ depends on a choice of ample line bundle on X, without loss of generality we can assume that the rank $\pi_{\alpha*}(\mathcal{L}_\alpha^\vee)$ is sufficiently large. Hence we can assume that sections $e_2(h_1), \cdots, e_2(h_l)$ so constructed span an l-dimensional subvector bundles of E_α. We define Q_α to be the vector bundle homomorphism that sends $\tilde{\sigma}_j$ to $e_2(h_j)$.

We now construct a cone in F_i over $Z_{i,\alpha} \subset Z_i$. Over $Z_{i,\alpha} \subset Z_i$, we let $k_{i,\alpha}: F_i|_{Z_{i,\alpha}} \to (\rho_i^\alpha)^*(E_\alpha)$ be the composition of $P_{i,\alpha}$ in (4.4.8) and Q_α in (4.4.9), let $h_{i,\alpha}: E_i|_{Z_{i,\alpha}} \to E_i|_{Z_{i,\alpha}}$ be the identity map and

$$\gamma_{i,\alpha} = h_{i,\alpha} + k_{i,\alpha} : W_i|_{Z_{i,\alpha}} \equiv F_i|_{Z_{i,\alpha}} \oplus E_i|_{Z_{i,\alpha}} \longrightarrow (\rho_i^\alpha)^*(E_\alpha) \equiv E_i|_{Z_{i,\alpha}} \tag{4.4.10}$$

be the induced homomorphism. This way we obtain a cone $\mathcal{P}_{i,\alpha}$ that is the pull-back cone $\gamma_{i,\alpha}^*(\mathcal{M}_\alpha^{\mathrm{alg}})$. Note that this is well-defined since $\gamma_{i,\alpha}$ is a surjective vector bundle homomorphism.

To construct the cone over Z_i, we first choose $J \subset \Xi$ so that $\{\iota_\alpha(Z_\alpha)\}_{\alpha \in J}$ covers $\iota_i(Z_i)$. Note that by our construction Λ_i is fine than all Λ_α. Applying the previous construction to this covering, to each $\alpha \in J$ we obtain a cone current $\mathcal{P}_{i,\alpha} \subset F_i|_{Z_{i,\alpha}}$.

LEMMA 4.7. *For any $\alpha, \alpha' \in J$, the cones $\mathcal{P}_{i,\alpha}|_{Z_{i,\alpha} \cap Z_{i,\alpha'}} \equiv \mathcal{P}_{i,\alpha'}|_{Z_{i,\alpha} \cap Z_{i,\alpha'}}$.*

Following this Lemma, the collection $\{\mathcal{P}_{i,\alpha}\}_{\alpha \in J}$ patch together to form a single cone $\mathcal{P}_i \subset F_i$ over Z_i. Now let $i > j \in \Xi$ be any two indices and let ρ_i^j and $Z_{i,j}$ be defined before Proposition 4.5.

LEMMA 4.8. *The collection $\{\mathcal{P}_i\}_{i \in \Lambda}$ is a cone in the semi-$\mathbb{Q}$-bundle $\mathfrak{W}$. Namely, for every $i > j \in \Xi$, the pull-back $(\rho_i^j)^*(\mathcal{P}_j)$ is identical to the restriction of $\mathcal{P}_i$ to $Z_{i,j}$. We denote this cone by $\mathcal{P}$.*

LEMMA 4.9. *The cone $\mathcal{P}$ intersects transversally to the sub-bundle $\beta_{\mathfrak{E}}: \mathfrak{E} \to \mathfrak{W}$ and $\beta_{\mathfrak{F}}: \mathfrak{F} \to \mathfrak{W}$. Further, $\mathcal{P} \cap \mathfrak{E} = \mathcal{M}^{alg}$ and $\mathcal{P} \cap \mathfrak{F} = \mathcal{N}^{an}$.*

Clearly, the comparison Theorem follows immediately from these Lemmas. Indeed, let $\eta_{\mathfrak{E}}$ (resp. $\eta_{\mathfrak{F}}$) be section of the semi-$\mathbb{Q}$-vector bundle $\mathfrak{E}$ (resp. $\mathfrak{F}$) so that its graph is transversal to the come $\mathcal{M}^{\mathrm{alg}}$ (resp. $\mathcal{N}^{\mathrm{an}}$). Then by Lemma 4.9, the sections $\tilde{\eta}_{\mathfrak{E}}$ and $\tilde{\eta}_{\mathfrak{F}}$ of $\mathfrak{W}$, which are induced by $\eta_{\mathfrak{E}}$ and $\eta_{\mathfrak{F}}$ via $\mathfrak{E} \subset \mathfrak{W}$ and $\mathfrak{F} \subset \mathfrak{W}$, are transversal to $\mathcal{P}$. Therefore, as homology classes,

$$[\eta_{\mathfrak{E}}^*(\mathcal{M}^{\mathrm{alg}})] = [\tilde{\eta}_{\mathfrak{E}}^*(\mathcal{P})] = [\tilde{\eta}_{\mathfrak{F}}^*(\mathcal{P})] = [\eta_{\mathfrak{F}}^*(\mathcal{N}^{\mathrm{an}})] \in H_*(\mathbf{B}; \mathbb{Q}).$$

This will prove Theorem 4.6, and hence the main theorem of this paper.

We first establish some technical results before we prove Lemma 4.7, 4.8 and 4.9. Let $\{(Z_i, \mathcal{X}_i)\}_{i \in \Lambda}$ be the good atlas of $\mathfrak{M}_{g,n}(X, A)$ choosen before, F_i be the corresponding vector bundle over Z_i. Our first step is to construct, to each $w \in Z_i$ associated to the map $f: C \to X$ with $D \subset C$ the marked points, a canonical homomorphism

$$\eta_i(w): F_i|_w \longrightarrow \mathrm{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C), \tag{4.4.11}$$

where as usual $\mathcal{D}^\bullet = [f^*\Omega_X \to \Omega_C(D)]$. Let $\alpha \in \Xi$ be an index so that $\iota_i(w)$ is contained in $\iota_\alpha(Z_\alpha)$, let $w_\alpha \doteq \rho_i^\alpha(w)$. In Lemma 3.9, we have constructed a canonical homomorphism

$$\eta_\alpha(w_\alpha): V_\alpha|_{w_\alpha} \longrightarrow \mathrm{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C). \tag{4.4.12}$$

Let $P_{i,\alpha}(w): F_i|_w = V_i|_w \to V_\alpha|_{w_\alpha}$ be the restriction to w of (4.4.7). The homomorphism $\eta_i(w)$ in (4.4.11) is the composite $\eta_\alpha(w_\alpha) \circ P_{i,\alpha}(w)$.

LEMMA 4.10. *The homomorphism $\eta_i(w)$ is canonical. Namely, let β be an index either in Λ or Ξ so that $\iota_i(w) \in \iota_\beta(Z_\beta)$ and that $(\Lambda_i, \mathbf{V}_i)$ is finer than $(\Lambda_\beta, \mathbf{V}_\beta)$. Then*

$$\eta_\beta(w_\beta) \circ P_{i,\beta}(w) = \eta_i(w).$$

Proof. When $\beta \in \Lambda$, then by our construction $(\Lambda_\beta, \mathbf{V}_\beta)$ is finer than $(\Lambda_\alpha, \mathbf{V}_\alpha)$. In this case, the conclusion of the Lemma is obvious. Now we consider the case where $\beta \in \Xi$. Note that $(\Lambda_i, \mathbf{V}_i)$ is finer than both $(\lambda_\alpha, \mathbf{V}_\alpha)$ and $(\lambda_\beta, \mathbf{V}_\beta)$. Thus both $V_\alpha|_{w_\alpha}$ and $V_\beta|_{w_\beta}$ are subvector spaces of $V_i|_w$ via pull-back homomorphisms $(f_i^\alpha)^*$ and $(f_i^\beta)^*$, respectively. let $V_{\alpha\beta}$ be the sum of $V_\alpha|_{w_\alpha}$ and $V_\beta|_{w_\beta} \subset V_i|_w$. Following the description of $\eta_\alpha(w_\alpha)$ given in Lemma 3.3.6, we can define canonically a homomorphism

$$\eta_{\alpha\beta} : V_{\alpha\beta} \longrightarrow \operatorname{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C)$$

so that its restriction to $V_\alpha|_{w_\alpha}$ and $V_\beta|_{w_\beta}$ are $\eta_\alpha(w_\alpha)$ and $\eta_\beta(w_\beta)$ respectively.

We now construct a projection $V_i|_w \to V_{\alpha\beta}$. Let $T \subset T_wU_i$ be the sum of the tangent spaces $T_wU_{i,\alpha}$ and $T_wU_{i,\beta}$. Let $N \subset T_wU_i$ be a complement of $T \subset T_wU_i$. Then $d\phi_i(w)(N) \subset V_i|_w$ is a complement of $V_{\alpha\beta}$, where recall ϕ is the section in the finite dimensional model $[\phi_i : U_i \to V_i]$. We let

$$P_{i,\alpha\beta} : V_i|_w \longrightarrow V_{\alpha\beta}$$

be the unique projection associated to the direct sum decomposition $V_i|_w = V_{\alpha\beta} \oplus d\phi_i(w)(N)$. Similarly, we pick a complement $N' \subset T$ of $T_wU_{i,\alpha} \subset T$ and form a complement $d\phi_i(w)(N') \subset V_{\alpha\beta}$ of $V_\alpha|_{w_\alpha} \subset V_{\alpha\beta}$. This way, we obtain a unique projection $P_{\alpha\beta,\alpha} : V_{\alpha\beta} \to V_\alpha|_{w_\alpha}$. By our construction,

$$P_{\alpha\beta,\alpha} \circ P_{i,\alpha\beta} = P_{i,\alpha}(w). \tag{4.4.13}$$

Also, by picking a complement N'' of $T_wU_{i,\beta} \subset T$, we obtain a projection $P_{\alpha\beta,\beta}(w)$: $V_{\alpha\beta} \to V_\beta|_w$ that satisfies

$$P_{\alpha\beta,\beta} \circ P_{i,\alpha\beta} = P_{i,\beta}(w). \tag{4.4.14}$$

Hence to prove the Lemma it suffices to show that

$$\eta_\alpha(w_\alpha) \circ P_{i,\alpha}(w) = \eta_\beta(w_\beta) \circ P_{i,\beta}(w),$$

which by (4.4.13) and (4.4.14) follows from

$$\eta_\alpha(w_\alpha) \circ P_{\alpha\beta,\alpha} = \eta_\beta(w_\beta) \circ P_{\alpha\beta,\beta}.$$

But this identity follows from

$$d\phi_i(w)(N') \text{ and } d\phi_i(w)(N'') \subset \ker\{\eta_{\alpha\beta} : V_{\alpha\beta} \to \operatorname{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C)\}, \tag{4.4.15}$$

since then

$$\eta_\alpha(w_\alpha) \circ P_{\alpha\beta,\alpha} = \eta_{\alpha\beta} = \eta_\beta(w_\beta) \circ P_{\alpha\beta,\beta}.$$

Finally, (4.4.15) is a direct consequence of the exact sequence in Lemma 3.9. This proves the Lemma. □

A similar statement concerns the homomorphism

$$\zeta_i(w) : W_i|_w \longrightarrow \mathrm{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C) \tag{4.4.16}$$

that is the composition of $\gamma_{i,\alpha}$ in (4.4.10) with the canonical

$$\delta_\alpha(w_\alpha) : E_\alpha|_{w_\alpha} \longrightarrow \mathrm{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C) \tag{4.4.17}$$

that is part of the the exact sequence (4.4.4). (We remark that with S in (4.4.4) be Z_α, then the $F_2(w)$ in (4.4.4) is $E_\alpha|_{w_\alpha}$ and the V_w in (4.4.4) is $\mathrm{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C)$.)

LEMMA 4.11. *The homomorphism $\zeta_i(w)$ satisfies the property that its restriction to $F_i|_w \subset W_i|_w$ (resp. $E_i|_w \subset W_i|_w$) is $\eta_i(w)$ (resp. $\delta_i(w)$). In particular, it is independent of the choice of $\alpha \in \Xi$ and $\gamma_{i,\alpha}$.*

Proof. The proof is similar to that of Lemma 4.10, and will be left to the readers. □

The next technical result we need is that the obstruction theory induced by the defining equation ϕ_α, where $\alpha \in \Xi$, coincides with the standard obstruction theory of stable morphisms. Let $w \in Z_\alpha$ be any point as before. Then using the explicit description of $T_w U_\alpha$, $\mathrm{Ext}^1(\mathcal{D}^\bullet, \mathcal{O}_C)$ is the Zariski tangent space of $\phi_\alpha^{-1}(0)$ at w, and hence is canonically the kernel of $d\phi_\alpha(w) : T_w U_\alpha \to V_\alpha|_w$.

LEMMA 4.12. *Let the notation be as before. Then the germ of $\phi_\alpha : U_\alpha \to V_\alpha$ at w is a Kuranishi map of the standard obstruction theory of the deformation of stable morphisms associated to the exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{D}^\bullet, \mathcal{O}_C) \longrightarrow T_w U_\alpha \longrightarrow V_\alpha|_w \longrightarrow \mathrm{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C) \longrightarrow 0.$$

Proof. Let $I \subset B$ be an ideal of an Artin ring annihilated by the maximal ideal $\mathfrak{m}_B$ and let $\varphi : \mathrm{Spec}\, B/I \to R_\alpha$ be a morphism that sends the closed point of $\mathrm{Spec}\, B/I$ to w and such that $\varphi^*(\phi_\alpha) = 0$. By the description of the tautological family $\mathcal{X}_\alpha$ over R_α, the pull back $\varphi^*(\mathcal{X}_\alpha)$ forms an algebraic family of stable morphisms over $\mathrm{Spec}\, B/I$. We continue to use the open covering of the domain $\mathcal{X}_\alpha$ used before. Since R_α is smooth, we can extend φ to $\tilde{\varphi} : \mathrm{Spec}\, B \to R_\alpha$. Let C_B over $\mathrm{Spec}\, B$ be the domain of the pull back of the domain of $\mathcal{X}_\alpha$ via $\tilde{\varphi}$ and let $C_{B/I}$ be the domain of C_B over $\mathrm{Spec}\, B/I$. We let $\{U_i\}$ (resp. $\{\tilde{U}_i\}$) be the induced open covering of $C_{B/I}$ (resp. C_B) and let $f_i : U_i \to X$ be the restriction to U_i of the pull back of the stable maps in $\mathcal{X}_\alpha$. Because $\varphi^*(\phi_\alpha) = 0$, f_i are holomorphic. Hence they define a morphism $f : C_{B/I} \to X$. Now we describe the obstruction to extending f to $\mathrm{Spec}\, B$. Let C_0 be the closed fiber of C_B and let $f_0 : C_0 \to X$ be the restriction of f. For each i, we pick a holomorphic extension $\tilde{f}_i : \tilde{U}_i \to X$ of f_i. Then over $\tilde{U}_{ij} = \tilde{U}_i \cap \tilde{U}_j$, $\tilde{f}_j - \tilde{f}_i$ is canonically an element in $\Gamma(f_0^* \mathcal{T}_X|_{U_i \cap U_j}) \otimes I$, denoted by f_{ij}. Further, the collection $\{f_{ij}\}$ is a cocycle and hence defines an element $[f_{ij}] \in H^1(f_0^* \mathcal{T}_X) \otimes I$. The obstruction to extending f to $\mathrm{Spec}\, B$ is the image of $[f_{ij}]$ in $\mathrm{Ext}^2(\mathcal{D}_w^\bullet, \mathcal{O}_{C_0}) \otimes I$ under the homomorphism in the statement in Lemma 3.9 with z replaced by w. We denote the image by $\mathrm{ob}^{\mathrm{alg}}$.

The obstruction to extending φ to $\tilde{\varphi} : \mathrm{Spec}\, B \to R_\alpha$ so that $\tilde{\varphi}^*(\phi_\alpha) = 0$ can be constructed as follows. Let $g_i : \tilde{U}_i \to X$ be the pull back of the maps in $\mathcal{X}_\alpha$. Note that g_i are well defined since maps in $\mathcal{X}_\alpha$ depend analytically on the base manifold R_α. By the construction of R_α, for each $i > l$ the map g_i is holomorphic. For $i < l$, we have canonical biholomorphism $\tilde{U}_i \cong \operatorname{Spec} B \times (U_i \cap C_0)$. Because $\varphi^*(\phi_\alpha) \equiv 0$, if we let ξ_i be a holomorphic variable of $U_i \cap C_0$, then $\frac{\partial}{\partial \bar{\xi}_i} g_i \cdot d\bar{\xi}_i$, denoted in short $\bar{\partial} g_i$, vanishes over $U_i \subset \tilde{U}_i$. Hence $\bar{\partial} h$ is a section of $\Gamma(\Omega^{0,1}_{\mathrm{cpt}}(f_0^* T_X)|_{U_i \cap C_0}) \otimes I$. Clearly they patch together to form a global section γ of $\Omega^{0,1}_{\mathrm{cpt}}(f_0^* T_X) \otimes I$. The element γ can be also defined as follows. Let $\tilde{\varphi}^* : \mathcal{O}_{R_\alpha} \to B$ be the induced homomorphism on rings. Then since the image of $\tilde{\varphi}^*_\alpha(\phi_\alpha) \in B \otimes_{\mathcal{O}_{R_\alpha}} \mathcal{O}_{R_\alpha}(W_\alpha)$ in $B/I \otimes_{\mathcal{O}_{R_\alpha}} \mathcal{O}_{R_\alpha}(W_\alpha)$ vanishes, it induces an element $\gamma' \in I \otimes W_\alpha|_w$. By our construction of R_α and ϕ_α, γ coincides with γ' under the inclusion $W_\alpha|_w \subset \Gamma_{C_0}(\Omega^{0,1}_{\mathrm{cpt}}(f_0^* T_X))$. Let $\mathrm{ob}^{\mathrm{an}}$ be the image of γ in the cokernel of $d\phi_\alpha(w) : T_w R_\alpha \to W_\alpha|_w$. By definition, $\mathrm{ob}^{\mathrm{an}}$ is the obstruction to extending φ to $\tilde{\varphi} : \operatorname{Spec} B \to \{\phi_\alpha = 0\}$.

To finish the proof of the lemma, we need to show that $\mathrm{ab}^{\mathrm{alg}} = \mathrm{ob}^{\mathrm{an}}$ under the isomorphism

$$\operatorname{Coker}\{d\phi_\alpha(w)\} \cong \operatorname{Ext}^1(\mathcal{D}^\bullet_w, \mathcal{O}_{C_0})$$

given in Lemma 3.9. For this, it suffices to show that the Dolbeault cohomology class of γ, denoted $[\gamma] \in H^{0,1}_{\bar{\partial}}(f_0^* T_X) \otimes I$, coincides with the Cěch cohomology class $[f_{ij}] \in H^1(f_0^* \mathcal{T}_X) \otimes I$ under the canonical isomorphism $H^{0,1}_{\bar{\partial}}(f_0^* T_X) \cong H^1(f_0^* \mathcal{T}_X)$. But this is obvious since $\varphi_i = \tilde{f}_i - g_i$ is in $\Gamma_{U_i \cap C_0}(\Omega^0_{\mathrm{cpt}}(f_0^* T_X)) \otimes I$ such that $\varphi_j - \varphi_i = f_{ij}$ and $\bar{\partial} \varphi_i = -\bar{\partial} g_i$. Hence, $[f_{ij}] = [\gamma]$ under the given isomorphism. This proves the lemma. □

We now prove Lemma 4.7. Let $w \in Z_i$ be any point, $w_\alpha \in Z_\alpha$ and $w_{\alpha'} \in Z_{\alpha'}$ be images of w under ρ_i^α and $\rho_i^{\alpha'}$ respectively. As before, we assume that w corresponds to the map $f : C \to X$ with $D \subset C$ the marked points. We denote by V_w the vector space $\operatorname{Ext}^2(\mathcal{D}^\bullet, \mathcal{O}_C)$.

Let $\hat{w}_\alpha$ be the formal completion of Z_α along w_α. Following the discussion before Theorem 4.2, by using a Kuranishi map of the deformation theory of the stable map f we obtain a cone $N^0_w \subset V_w \times \hat{w}_\alpha$, where $V_w \times \hat{w}_\alpha$ is a vector bundle over $\hat{w}_\alpha$. Following Theorem 4.3, the cone cycle N^0_w has the property that there is a vector bundle homomorphism

$$r_\alpha : E_\alpha \times_{Z_\alpha} \hat{w}_\alpha \longrightarrow V_w \times \hat{w}_\alpha \tag{4.4.18}$$

extending $\delta_\alpha(w_\alpha) : E_\alpha|_{w_\alpha} \to V_w$ such that

$$\mathcal{M}^{\mathrm{alg}}_\alpha \times_{Z_\alpha} \hat{w}_\alpha \equiv N^0_w \times_{V_w \times \hat{w}_\alpha} E_\alpha \times_{Z_\alpha} \hat{w}_\alpha. \tag{4.4.19}$$

Similarly, for α' we have $r_{\alpha'}$ as in (4.4.18) with α replaced by α' extending $\eta_{\alpha'}(w_{\alpha'})$ such that (4.4.19) holds with α replaced by α'. Our first step in proving Lemma 4.7 is to show

$$\operatorname{supp}(\mathcal{P}_{i,\alpha}) \cap W_i|_w \equiv \operatorname{supp}(\mathcal{P}_{i,\alpha'}) \cap W_i|_w. \tag{4.4.20}$$

By our construction, $\operatorname{supp}(\mathcal{P}_{i,\alpha}) \cap W_i|_w$ is $\gamma^*_{i,\alpha}(\mathcal{M}^{\mathrm{alg}}_\alpha) \cap W_i|_w)$. Let $\gamma_{i,\alpha}(w)$ be the restriction of $\gamma_{i,\alpha}$ to fiber over w. Then combined with (4.4.19) and Lemma 4.11, it is

$$\begin{aligned}
&\gamma_{i,\alpha}(w)^* \big(\mathcal{M}^{\mathrm{alg}}_\alpha \cap E_\alpha|_{w_\alpha}\big) = \gamma_{i,\alpha}(w)^* \big(r_\alpha(w_\alpha)^* (N^0_w|_{V_w \times w_\alpha})\big) \\
=&\gamma_{i,\alpha}(w)^* \big(\delta_\alpha(w)^* (N^0_w|_{V_w \times w_\alpha})\big) = \zeta_i(w)^* \big(N^0_w|_{V_w \times w_\alpha}\big).
\end{aligned}$$

For the same reason, the last term is also the $\mathrm{supp}(\mathcal{P}_{i,\alpha'}) \cap W_i|_w$. This proves (4.4.20). Further, since w can be any point in $Z_{i,\alpha} \cap Z_{i,\alpha'}$, we have

$$\mathrm{supp}(\mathcal{P}_{i,\alpha}|_{Z_{i,\alpha}\cap Z_{i,\alpha'}}) = \mathrm{supp}(\mathcal{P}_{1,\alpha'}|_{Z_{i,\alpha}\cap Z_{i,\alpha'}}).$$

It remains to check that at smooth point z of $\mathrm{supp}(\mathcal{P}_{i,\alpha})$ the multiplicities of $\mathcal{P}_{i,\alpha}$ and $\mathcal{P}_{i,\alpha'}$ at z are identical. Indeed, let $z_\alpha = \gamma_{i,\alpha}(z)$ and $z_{\alpha'} = \gamma_{i,\alpha'}(z)$. Then $z_\alpha \in \mathrm{supp}(\mathcal{M}_\alpha^{\mathrm{alg}}) \cap E_\alpha|_{w_\alpha}$ and $z_{\alpha'} \in \mathrm{supp}(\mathcal{M}_{\alpha'}^{\mathrm{alg}}) \cap E_{\alpha'}|_{w_{\alpha'}}$. By Lemma 4.11, $r_\alpha(z_\alpha) = r_{\alpha'}(z_{\alpha'})$. By (4.4.17), $r_\alpha(z_\alpha)$ is a smooth point of $\mathrm{supp}(N_w^0) \subset V_w \times \hat{w}_\alpha$. Hence the multiplicity of $\mathcal{M}_\alpha^{\mathrm{alg}}$ at z_α and the multiplicity of $\mathcal{M}_{\alpha'}^{\mathrm{alg}}$ at $z_{\alpha'}$ coincide since they are the multiplicity of N_w^0 at $r_\alpha(z_\alpha)$. Because $\mathcal{P}_{i,\alpha}$ and $\mathcal{P}_{i,\alpha'}$ are pullbacks of $\mathcal{M}_\alpha^{\mathrm{alg}}$ and $\mathcal{M}_{\alpha'}^{\mathrm{alg}}$ respectively, their multiplicities at z coincide as well. This proves Lemma 4.7.

The proof of Lemma 4.8 is similar, and will be left to the readers.

Now we prove Lemma 4.9. The statement $\mathcal{P} \cap \mathfrak{E} = \mathcal{M}^{\mathrm{alg}}$ follows from the construction. The proof of the other statements are a combination of the proof of Lemma 4.7 with the following comparison result: For any $w_\alpha \in Z_\alpha$, there is a vector bundle homomorphism

$$s_\alpha : F_\alpha \times_{Z_\alpha} \hat{w}_\alpha \longrightarrow V_w \times \hat{w}_\alpha$$

extending $\eta_\alpha(w_\alpha)\colon F_\alpha|_{w_\alpha} \to V_w$ such that

$$\mathcal{N}_\alpha^{\mathrm{an}} \times_{Z_\alpha} \hat{w}_\alpha \equiv N_w^0 \times_{V_w \times \hat{w}_\alpha} F_\alpha \times_{Z_\alpha} \hat{w}_\alpha.$$

This is true because $\mathcal{N}_\alpha^{\mathrm{an}}$ is the come from dilating the graph of ϕ_α, which is a Kuranishi map of the standard obstruction theory of the stable map f, by Lemma 4.12. This completes the proof of Lemma 4.7, 4.8 and 4.9, and hence the main theorem.

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