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REGULARITY OF $\bar{\partial}$ ON PSEUDOCONCAVE COMPACTS AND APPLICATIONS*

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Abstract. We prove the regularity of the $\bar{\partial}$-equation for $\mathcal{O}(m)$-valued (p,q)-forms on pseudoconcave compacts of the complex projective space $\mathbb{CP}_n$. This leads to the vanishing of the cohomology groups of the $(0,q)$-forms with coefficients in the Sobolev space $W^k(\Omega)$, $q \neq n-1$, where Ω is a pseudoconcave domain with Lipschitz (respectively C^2) boundary of $\mathbb{CP}_n$ for $k \geq 2$ (respectively $k \geq 1$). As an application, we show that the holomorphic functions of $W^1(\Omega)$ vanish on pseudoconcave domains Ω with C^2 boundary of $\mathbb{CP}_n$. This implies the Hartogs-Bochner phenomenon for CR functions in the space $W^{1/2}(\partial\Omega)$, where Ω is a domain with Lipschitz boundary of $\mathbb{CP}_n$ such that the complement is connected and contains a pseudoconcave domain with C^2 boundary. We also obtain the dual of the k-weighted Bergman space of a pseudoconvex domain in $\mathbb{CP}_n$ as the cohomology group of the $(n,n-1)$-forms with W^k-coefficients on the complement.

1. Introduction. The classical Hartogs-Bochner theorem [4] states that a CR function f on the boundary of a bounded domain $\Omega \subset \mathbb{C}^n$ $(n \geq 2)$ with smooth connected boundary has a holomorphic extension to Ω.

A proof of this theorem may be obtained by using the jump formula for the Bochner-Martinelli transform: $f = f_+|_{\partial\Omega} - f_-|_{\partial\Omega}$, where f_+ is holomorphic on Ω and f_- is holomorphic on $\mathbb{CP}_n \backslash \overline{\Omega}$, where $\mathbb{CP}_n$ is the n dimensional complex projective space. Since $\mathbb{CP}_n\backslash\Omega$ is connected and contains complex lines, it follows that $f_- \equiv$ *constant*. An essential fact for this proof of the Hartogs-Bochner phenomenon is that $\mathbb{CP}_n\backslash\Omega$ contains a pseudoconcave domain.

It is well-known that the Hartogs-Bochner theorem is true for relatively compact domains in a Stein manifold X ($\dim_{\mathbb{C}} X \geq 2$). Important versions of this theorem were obtained by Fichera [11], Kohn and Rossi [25], Grauert and Riemenschneider [13], Harvey and Lawson [18]. Recently, interesting new facts about this phenomenon were obtained by Napier and Ramachandran [29], Dingoyan [8] and Sarkis [33].

In this paper we obtain the following new version of the Hartogs-Bochner theorem in $\mathbb{CP}_n$, $(n \geq 2)$:

(1) Let Ω be a domain with Lipschitz boundary in $\mathbb{CP}_n$ such that $\mathbb{CP}_n\backslash\Omega$ is connected and contains a pseudoconcave domain with C^2 boundary. Then every CR Sobolev $W^{1/2}$-function on the boundary has a holomorphic extension to Ω.

One of the difficulties to prove this result is that there exist pseudoconvex domains Ω in $\mathbb{CP}_2$, such that $\overline{\Omega}$ has no any Stein neighborhood. So we can not use the above mentioned results. Another difficulty is that there exists pseudoconcave domains in $\mathbb{CP}_n$ which do not contain any algebraic hypersurface of $\mathbb{CP}_n$ [10].

It is interesting to mention that the Hartogs-Bochner phenomenon is also valid in distribution categories for bounded domains in $\mathbb{C}^n$ with rectifiable boundary. However, we give an example of pseudoconvex domains in $\mathbb{CP}_2$ such that the Hartogs-Bochner phenomenon does not work for $W^{-1/2}$-functions on the boundary.

The jump formula for the Bochner-Martinelli transform in $\mathbb{CP}_n$ [20] shows that (1) is equivalent to the following version of Liouville's theorem:

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(2) Let Ω be a pseudoconcave domain with Lipschitz (respectively C^2) boundary in $\mathbb{CP}_n$. Then every holomorphic function in the Sobolev space $W^2(\Omega)$ (respectively $W^1(\Omega)$) is constant.

One of our main results is a cohomological version of (2):

(3) Let Ω be a pseudoconcave domain with Lipschitz (respectively C^2) boundary in $\mathbb{CP}_n$ and $H^{0,q}_{W^k}(\Omega)$ the group of cohomology of the $(0,q)$-forms with coefficients in the Sobolev space $W^k(\Omega)$. Then $H^{0,q}_{W^k}(\Omega) = 0$ for $k \geq 2$ (respectively $k \geq 1$), $q \neq n-1$.

The statement (3) can also be seen as regularity of $\overline{\partial}$ operator on pseudoconcave compacts in $\mathbb{CP}_n$ and it is a new result even for smoothly bounded domains. Indeed, the known results for the regularity of the $\overline{\partial}$ operator on pseudoconvex and pseudoconcave domains in hermitian manifolds of J. J. Kohn [23] and M.-C. Shaw [35] depend on the existence of a strongly plurisubharmonic function in a neighborhood of the boundary. For pseudoconvex domains in $\mathbb{CP}_n$, such functions may not exist.

A special case of statement (3) concerning the regularity of $\overline{\partial}$ operator on smoothly bounded domains in $\mathbb{CP}_n$ with Levi-flat boundaries and $q=1$ was obtained earlier by Y.-T. Siu [36] as an important step in his proof that such domains do not exist.

We can compare (1), (2), (3) with

(4) Let Ω be a non-dense pseudoconvex in $\mathbb{CP}_n$. Then the Bergman space $W^0(\Omega) \cap \mathcal{O}(\Omega)$ separates the points of Ω. Moreover, this is also true for domains $\widetilde{\Omega}$ spread over Ω.

The statement (4) is proved by using Ohsawa-Takegoshi-Manivel approach [32], [26] for the extension of L^2 holomorphic forms.

Another result of this paper is the following quantitative version of the Serre-Martineau duality [34], [27] in $\mathbb{CP}_n$:

(5) The dual of the k-weighted Bergman space of a pseudoconvex domain Ω with Lipschitz boundary in $\mathbb{CP}_n$ is canonicaly isomorphic to the cohomology group of the $(n, n-1)$-forms on Ω with coefficients in the Sobolev space $W^k(\Omega)$.

If Ω is strongly pseudoconvex the statement (5) can be deduced from [19].

We use the L^2-estimates for $\overline{\partial}$ of Andreotti-Vesentini [1] and Hörmander [22] in the form given by Demailly [6], a recent improvement of these estimates due to Berndtsson and Charpentier [3], Takeuchi's characterization of pseudoconvex domains in $\mathbb{CP}_n$ [39] and a formula of Bochner-Martinelli-Koppelman type in $\mathbb{CP}_n$ [20].

2. Preliminaries and notations. Let Ω be a domain in a Kähler manifold X and E a holomorphic hermitian vector bundle over Ω. For $z, z' \in \Omega$, we denote $d(z,z')$ the geodesic distance from z to z' and $d(z)$ the geodesic distance to the boundary of Ω for the Kähler metric on X (we consider only domains $\Omega \subsetneq X$).

Let δ be a C^∞ positive function on Ω and $\alpha \in \mathbf{R}$. We denote by:

-ω the $(1,1)$-form associated to the Kähler metric on X;

-$ic(E)$ the Chern curvature of E;

-$\mathcal{D}_{(p,q)}(\Omega, E)$ the space of (p,q)-forms with compact suport in Ω and values in the bundle E;

-$L^2(\Omega; \delta^\alpha) = \{f \in L^2(\Omega; loc) | \delta^\alpha f \in L^2(\Omega)\}$ endowed with the topology given by the norm $\left(\int_\Omega |f(x)|^2 \delta^{2\alpha}(x) dV\right)^{1/2}$; this norm is denoted $N_{\alpha,\Omega}$ or N_α;

-$L^2_{(p,q)}(\Omega; \delta^\alpha)$ the set of (p,q)-forms on Ω with coefficients in $L^2(\Omega; \delta^\alpha)$;

-$L^2_{(p,q)}(\Omega; E)$ the set of (p,q)-forms on Ω with L^2 coefficients and values in the bundle E;

-$L^2_{(p,q)}(\Omega;\delta^\alpha;E)$ the set of (p,q)-forms on Ω with coefficients in $L^2_{(p,q)}(\Omega;\delta^\alpha)$ and values in the bundle E;

-$\overline{\partial}$ is considered as an unbounded operator $\overline{\partial} = \overline{\partial}_\alpha : L^2_{(p,q)}(\Omega;\delta^\alpha;E) \longrightarrow L^2_{(p,q+1)}(\Omega;\delta^\alpha;E)$;

-$\mathcal{R}^\alpha_{(p,q)}(\Omega;E) = \{f \in L^2_{(p,q)}(\Omega;\delta^\alpha;E)|, f = \overline{\partial}g, \in L^2_{(p,q-1)}(\Omega;\delta^\alpha;E)\}$;

-$W^s(\Omega)$ the Sobolev space of order $s \in \mathbb{R}$ with the norm denoted $\| \|_s$;

-$W^s_{(p,q)}(\Omega)$ the (p,q)-forms on Ω with coefficients in $W^s(\Omega)$;

-$W^s_{(p,q)}(\Omega;E)$ the (p,q)-forms on Ω with coefficients in $W^s(\Omega)$ and values in the bundle E;

-$A^\infty_{(p,q)}(\Omega;E)$ the set of $\overline{\partial}$-closed (p,q)-forms on Ω with values in E which have a C^∞ extension to $\overline{\Omega}$;

-$AW^k_{(p,q)}(\Omega;E)$ the set of $\overline{\partial}$-closed (p,q)-forms contained in $W^k_{(p,q)}(\Omega;E)$.

-$H^{p,q}_{W^k}(\Omega) = Ker\overline{\partial}^k_{(p,q)}/Range\left(\overline{\partial}^k_{(p,q-1)}\right)$ with $\overline{\partial} = \overline{\partial}^{k+1}_{(p,q)} : Dom\overline{\partial} \subset W^k_{(p,q)}(\Omega) \to W^k_{(p,q+1)}(\Omega)$.

We denote by $\mathcal{O}(-1)$ the universal bundle on $\mathbb{CP}_n$, by $\mathcal{O}(1)$ its inverse and by $\mathcal{O}(m)$ the m-th power of $\mathcal{O}(1)$ if $m > 0$, respectively of $\mathcal{O}(-1)$ if $m < 0$.

In order to prove the estimates we need, we will increase the constants without changing the notation.

DEFINITION 2.1. *Let $f \in Dom\overline{\partial}_{-\alpha} \subset L^2_{(p,q)}(\Omega;\delta^{-\alpha};E)$, $\alpha \geq 0$. We say that f vanishes of order greater than α on $\partial\Omega$ (or simply f vanishes on $\partial\Omega$ if $\alpha = 0$) if $\int_\Omega \overline{\partial}f \wedge \varphi = (-1)^{p+q+1}\int_\Omega f \wedge \overline{\partial}\varphi$ for every $\varphi \in Dom\overline{\partial}_\alpha \subset L^2_{(n-p,n-q-1)}(\Omega;\delta^\alpha;E^*)$.*

Let $f \in W^s_{(p,q)}(\Omega;E)$. We say that $h \in W^r_{(p,q)}(\partial\Omega;E)$ is the boundary value of f and we denote $h = bv(f)$ if $\int_\Omega \overline{\partial}f \wedge \varphi = (-1)^{p+q+1}\int_\Omega f \wedge \overline{\partial}\varphi + \int_{\partial\Omega} h \wedge \varphi$ for every $\varphi \in C^\infty_{(n-p,n-q-1)}(\overline{\Omega};E^*)$.

We consider that two forms $f, g \in L^2_{(p,q)}(\partial\Omega)$ are equal if $\int_{\partial\Omega} f\wedge\varphi = \int_{\partial\Omega} g\wedge\varphi$ for every $\varphi \in C^0_{(n-p,n-q-1)}(\partial\Omega)$. A function $f \in L^2(\partial\Omega)$ satisfies the tangential Cauchy-Riemann equations on $\partial\Omega$ ($f \in CR(\partial\Omega)$) if $\int_{\partial\Omega} f\overline{\partial}\varphi = 0$ for every $(n,n-1)$-form φ of class C^∞ in a neighborhood of $\partial\Omega$.

Let V, W complex vector spaces and θ a hermitian form on $V \otimes W$. Following [6], we write $\theta \geq_s 0$ (respectively $\theta >_s 0$) if $\theta(x,x) \geq 0$ (respectively $\theta(x,x) >_s 0$) for every $x \in V \otimes W$ such that $x = \sum_{j=1}^k v_j \otimes w_j$, $v_j \in V$, $w_j \in W$, $k \leq s$. If θ is semi-positive-definite (respectively positive-definite) we omit the index s.

We denote by L the adjoint of the operator of exterior multiplication by ω: $(L\alpha|\beta) = (\alpha|\omega\wedge\beta)$ for every $\alpha,\beta \in Hom_{\mathbb{R}}(TX;\mathbb{C})$ where $(\ |\ )$ is the inner product induced by $dV = \frac{\omega^n}{n!}$.

Let Θ be a $(1,1)-$form with values in $Herm(E;E)$. Then, for $1 \leq q \leq n$, we define the sesquilinear form Θ_q on $\wedge^{n,q}T^*X \otimes E$ by $\Theta_q(\alpha,\alpha) = (\Theta \wedge (L\alpha)|\beta)$.

Suppose $\Theta \geq_{n-q+1} 0$ and let $\alpha \in \wedge^{n,q}T^*X \otimes E$. We put

$$\|\alpha\|_\Theta = sup\{(\alpha|\beta)\,|\,(\Theta \wedge (L\alpha)|\beta) \leq 1\}.$$

Then $\|\alpha\|_\Theta$ is a decreasing function on Θ, $\|\eta\wedge\alpha\|_\Theta \leq |\eta|\,\|\alpha\|_\Theta$ and if $\Theta \geq_{n-q+1} \lambda\omega \otimes Id_E$ with $\lambda > 0$, we have $\|\alpha\|^2_\Theta \leq \frac{1}{q\lambda}|\alpha|^2$ ([6]).

If Ω is relatively compact in X and E is a hermitian bundle in the neighborhood of $\overline{\Omega}$ we define

$$m_p(\Omega;E) = sup\left\{m \in \mathbb{R}|ic(\wedge^{n-p}T\Omega \otimes E) \geq m\omega \otimes Id_{\wedge^{n-p}T\Omega\otimes E}\right\}.$$

DEFINITION 2.2. *Let X be a complex manifold and Ω a domain in X. We say that Ω is Hartogs-pseudoconvex if there exists a Kähler metric on X and a neighborhood U of $\partial\Omega$ such that the restriction of $-logd$ to $U \cap \Omega$ admits a strongly plurisubharmonic extension to Ω.*

In what follows, if Ω is a relatively compact Hartogs-pseudoconvex domain in X, we consider a Kähler metric ω and denote δ a C^∞ positive function on Ω such that $\delta = d$ on a neighborhood of $\partial\Omega$ and $i\partial\overline{\partial}(-log\delta) \geq C_\Omega\omega$ on Ω with $C_\Omega > 0$.

EXAMPLE 2.3. *Every relatively compact pseudoconvex domain in a Stein manifold is Hartogs-pseudoconvex.*

The same is true if we suppose X a complex manifold such that there exists a continuous strongly plurisubharmonic function on X.

Indeed, let X be a Stein manifold and Ω a relatively compact pseudoconvex domain in X. For every Kähler metric ω on X there exists $C \in \mathbb{R}$ such that $i\partial\overline{\partial}(-log\delta) \geq C\omega$ on Ω [40], [9]. Then, for the Kähler metric $\widetilde{\omega} = e^{-K\psi}\omega$, with ψ a smooth strongly plurisubharmonic function on X and $K > 0$ big enough, we have $i\partial\overline{\partial}(-log\widetilde{\delta}) \geq C_\Omega\widetilde{\omega}$ for the corresponding distance $\widetilde{d}$ and its extension $\widetilde{\delta}$, with $C_\Omega > 0$.

EXAMPLE 2.4. *Every pseudoconvex domain in $\mathbb{CP}_n$ is Hartogs-pseudoconvex [39].*

The same is true for relatively compact pseudoconvex domains in Kähler manifolds with positive holomorphic bisectional curvature [9] and for pseudoconvex domains in complete Kähler manifolds with positive holomorphic bisectional curvature [14] : a Kähler manifold X has positive holomorphic bisectional curvature if $ic(TX) >_1 0$.

By [37], a compact Kähler manifold X with a strictly positive holomorphic bisectional curvature is isomorphic to $\mathbb{CP}_n$.

Since $\mathbb{CP}_n$ has strictly positive holomorphic bisectional curvature for the Fubini -Study metric ω, there exists a strictly positive constant $\mathcal{K}_n$ such that $i\partial\overline{\partial}(-log\delta) \geq \mathcal{K}_n\omega$ for every locally pseudoconvex domain in $\mathbb{CP}_n$.

DEFINITION 2.5. *Let X be a complex manifold. A closed set $L \subset X$ is called pseudoconcave (respectively Hartogs-pseudoconcave) if $X \backslash L$ is pseudoconvex (respectively Hartogs-pseudoconvex).*

We remark that there are pseudoconcave compacts which are not pseudoconcave in the sense of Andreotti [2].

EXAMPLE 2.6. *Two important classes of pseudoconcave sets are: the algebraic (smooth or non-smooth) hypersurfaces and the Levi-flat real hypersurfaces in $\mathbb{CP}_n$ (see Example 12.1). From one of Oka's results it follows that a pseudoconcave subset of an open set of $\mathbb{C}^n$ is a complex hypersurface if and only if its $(2n-2)$-Hausdorff measure is locally finite [30], [21].*

3. Estimates for $\overline{\partial}$ in $L^2_{(p,q)}(\Omega;\delta^\alpha;E)$. We use the L^2-estimates for $\overline{\partial}$ of Andreotti-Vesentini [1] and Hörmander [22] in the following form of Demailly [6]:

Let X be a Kähler manifold of dimension n which admits a complete Kähler metric and E a holomorphic hermitian vector bundle over X such that $ic(E) \geq_{n-q+1} 0$. Let $g \in L^2_{(n,q)}(X;E)$ a $\overline{\partial}$-closed form. Then there exists $f \in L^2_{(n,q-1)}(X;E)$ such that $\overline{\partial}f = g$ and $\int_X |f|^2\, dV \leq \int_X \|g\|^2_{ic(E)}\, dV$.

From this statement it follows the following:

PROPOSITION 3.1. *Let Ω be a relatively compact Hartogs-pseudoconvex domain in a n-dimensional Kähler manifold X and E a holomorphic hermitian vector bundle of class C^2 in a neighborhood of $\overline{\Omega}$. Let $f \in L^2_{(p,q)}(\Omega;\delta^\alpha;E)$, $1 \le q \le n$, a $\overline{\partial}$-closed form. Then, for non-negative α, such that $m_p(\Omega;E)+2\alpha C_\Omega > 0$, there exists $u \in L^2_{(p,q-1)}(\Omega;\delta^\alpha;E)$ such that $\overline{\partial}u = f$ and $(N_\alpha(u))^2 \le \frac{1}{q[m_p(\Omega;E)+2\alpha C_\Omega]}(N_\alpha(f))^2$.*

Proof. For non-negative α we have

$$\begin{aligned} &ic(\wedge^{n-p}T\Omega\otimes E) + i2\alpha\partial\overline{\partial}(-log\delta)\otimes Id_{\wedge^{n-p}T\Omega\otimes E} \\ &\ge [m_p(\Omega;E)+2\alpha C_\Omega]\,\omega\otimes Id_{\wedge^{n-p}T\Omega\otimes E}\end{aligned}$$

where ω is the metric of X.

Since $L^2_{(p,q)}(\Omega;E) = L^2_{(n,q)}(\Omega;\wedge^{n-p}T\Omega\otimes E)$, by using the solution of the $\overline{\partial}$-problem for (n,q)-forms with values in a hermitian fiber bundle with the weight function $\varphi = -2\alpha\log\delta$, it follows that there exists $u \in L^2_{(p,q-1)}(\Omega;\delta^\alpha;E)$ such that $\overline{\partial}u = f$ and $(N_\alpha(u))^2 \le \frac{1}{q[m_p(\Omega;E)+2\alpha C_\Omega]}(N_\alpha(f))^2$. □

REMARK 3.2. *From Proposition 3.1 it follows that for positive α such that $m_p(\Omega;E)$ $+2\alpha C_\Omega > 0$, $\mathcal{R}^\alpha_{(p,q)}(\Omega;E)$ is a closed subspace of $L^2_{(p,q)}(\Omega;\delta^\alpha;E)$ and we can find a bounded operator $T^\alpha_{(p,q)} : \mathcal{R}^\alpha_{(p,q)}(\Omega;E) \longrightarrow L^2_{(p,q-1)}(\Omega;\delta^\alpha;E)$ such that $\overline{\partial}Tu = u$ for every $u \in \mathcal{R}^\alpha_{(p,q)}(\Omega;E)$ and $\left\|T^\alpha_{(p,q)}\right\|^2 \le \frac{1}{q[m_p(\Omega;E)+2\alpha C_\Omega]}$.*

LEMMA 3.3. *For every domain $\Omega \subset \mathbb{CP}_n$ we have $m_p(\Omega;\mathcal{O}(m)) = m$ for $p > 0$ and $m_0(\Omega;\mathcal{O}(m)) = m+n+1$.*

It is well known that $T\mathbb{CP}_n$ has no hermitian metric such that $ic(T\mathbb{CP}_n) \ge_2 0$ and $ic(T\mathbb{CP}_n) >_2 0$ in one point (see for ex. [5]). This gives the case $p = n-1$ of Lemma 3.3.

Proof. From [39] it follows that $i\partial\overline{\partial}(-log\delta) \ge \mathcal{K}_n\omega$, where $\omega = i\partial\overline{\partial}log(\|z\|^2)$ is the $(1,1)$-form associated to the Fubini-Study metric, with $z = (z_0, z_1, ..., z_n)$ homogeneous coordinates for $[z] \in \mathbb{CP}_n$.

We have $ic(T\mathbb{CP}_n) \ge 0$, so $ic(\wedge^{n-p}T\mathbb{CP}_n) \ge 0$. Thus

$$\begin{aligned} ic(\wedge^{n-p}T\mathbb{CP}_n\otimes\mathcal{O}(m)) &= mi\mathcal{O}(1)\otimes Id_{\wedge^{n-p}T\mathbb{CP}_n} + ic(\wedge^{n-p}T\mathbb{CP}_n) \\ &\ge m\omega\otimes Id_{\wedge^{n-p}T\mathbb{CP}_n}.\end{aligned}$$

Let $a = [1,0,...,0]$, $e_i = \left(\frac{\partial}{\partial z_i}\right)_a$, $i = 1,...,n$, $I = \{i_1, i_2, ..., i_{n-p}\}$ $e_I = e_{i_1}\wedge ...\wedge e_{i_{n-p}}$ and $x = \sum_{i,I} x_{iI}e_i\otimes e_I \in T_a(\mathbb{CP}_n)\otimes\wedge^{n-p}T_a\mathbb{CP}_n$. We have

$$ic(T\mathbb{CP}_n)_a = \sum_{i,j,k,l} c_{ijkl}dz_i\wedge d\overline{z}_j\otimes e_k^*\otimes e_l \tag{3.1}$$

where

$$-c_{ijkl} = \frac{\partial^2 g_{ij}}{\partial z_k\partial\overline{z}_l} - \sum_{r,s} g^{rs}\frac{\partial g_{ir}}{\partial z_k}\frac{\partial g_{js}}{\partial\overline{z}_l}$$

with

$$ds^2 = \sum_{i,j} g_{ij}dz_i\wedge d\overline{z}_j = 2\frac{(1+\sum z_i\overline{z}_i)(\sum dz_i\wedge d\overline{z}_i) - (\sum\overline{z}_i dz_i)\wedge(\sum z_i d\overline{z}_i)}{(1+\sum z_i\overline{z}_i)^2}$$

the Fubini-Study metric and (g^{rs}) the inverse of the matrix (g_{ij}).
It follows that

$$c_{iijj} = 1 \; if \; i \neq j; \;\; c_{ijji} = 1 \; if \; i \neq j; \;\; c_{iiii} = 2; \;\; c_{ijkl} = 0 \; otherwise \tag{3.2}$$

From (3.1) and (3.2) we have

$$ic(T\mathbb{CP}_n)_a e_r = \sum_{i,j,l} c_{ijrl} dz_i \wedge d\overline{z}_j \otimes e_l = \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i \otimes e_r + \sum_{i=1}^{n} dz_i \wedge d\overline{z}_r \otimes e_i$$

so

$$\begin{aligned} ic(\wedge^{n-p}T\mathbb{CP}_n)_a e_I &= \sum_{r=1}^{n-p} e_{i_1} \wedge \ldots \wedge e_{i_{r-1}} \wedge ic(T\mathbb{CP}_n)_a e_{i_r} \wedge e_{i_{r+1}} \wedge \ldots \wedge e_{i_{n-p}} \\ &= (n-p) \sum_{i=1}^{n} \sum_{r=1}^{n-p} dz_i \wedge d\overline{z}_i \otimes e_I + \sum_{i \in I} dz_i \wedge d\overline{z}_i \otimes e_I \\ &+ \sum_{j \in I} \sum_{i \notin I} \varepsilon_i^j dz_i \wedge d\overline{z}_i \otimes e_{I_j^i} \end{aligned}$$

where $I_i^j = (I \setminus \{j\}) \cup \{i\}$ and ε_i^j is the sign of the permutation which makes I_i^j an ordered set when we replace j by i.
Therefore

$$\begin{aligned} &ic(\wedge^{n-p}T\mathbb{CP}_n)_a(x,x) \\ &= \sum_I \left(\sum_{i \in I} (n-p+1) |x_{iI}|^2 + \sum_{i \notin I} (n-p) |x_{iI}|^2 + \sum_{j \in I} \sum_{i \notin I} \varepsilon_i^j x_{iI} \overline{x_{jI_i^j}} \right) \end{aligned}$$

and we see that $ic(\wedge^{n-p}T\mathbb{CP}_n)_a(x,x) > 0$ for $x \neq 0$ if and only if $p = 0$. Consequently, $m_p(\Omega; \mathcal{O}(m)) = m$ for $p > 0$.
Since $ic(\wedge^n T\Omega) = (n+1)\omega$, it follows that $m_0(\Omega; \mathcal{O}(m)) = m + n + 1$. □
From Proposition 3.1 and Lemma 3.3 we obtain:

COROLLARY 3.4. *Let Ω be a pseudoconvex domain in $\mathbb{CP}_n$, $m \in \mathbf{Z}$ and $f \in L^2_{(p,q)}(\Omega; \delta^\alpha; \mathcal{O}(m))$ a $\overline{\partial}$-closed form.*
a) For $p > 0$, $\alpha \geq 0$ and $2\alpha\mathcal{K}_n + m > 0$, there exists $u \in L^2_{(p,q-1)}(\Omega; \delta^\alpha; \mathcal{O}(m))$ such that $\overline{\partial} u = f$ and $(N_\alpha(u))^2 \leq \frac{1}{q(2\alpha\mathcal{K}_n+m)} (N_\alpha(f))^2$.
b) For $p = 0$, $\alpha \geq 0$ and $2\alpha\mathcal{K}_n + m + n + 1 > 0$, there exists $u \in L^2_{(0,q-1)}(\Omega; \delta^\alpha; \mathcal{O}(m))$ such that $\overline{\partial} u = f$ and $(N_\alpha(u))^2 \leq \frac{1}{q(2\alpha\mathcal{K}_n+m+n+1)} (N_\alpha(f))^2$.

The following proposition is inspired from [3]:

PROPOSITION 3.5. *Let Ω be a relatively compact Hartogs-pseudoconvex domain with C^2 boundary of a n-dimensional Kähler manifold X and E a holomorphic hermitian vector bundle of class C^2 in a neighborhood of $\overline{\Omega}$. Let $0 \leq p \leq n$ such that $m_p(\Omega; E) \geq 0$. Then there exists $\eta > 0$ such that for every $\alpha > -\eta$ and every $\overline{\partial}$-closed form $f \in L^2_{(p,q)}(\Omega; \delta^\alpha; E)$, $q \geq 1$, there exists $u \in L^2_{(p,q-1)}(\Omega; \delta^\alpha; E)$ such that $\overline{\partial} u = f$ and $N_\alpha(u) \leq C_\alpha N_\alpha(f)$, where C_α is a constant independent on f.*

Proof. By [31] there exists $\eta > 0$ such that the function $\varphi = -\delta^\eta$ is strictly plurisubharmonic on Ω. Let $\psi = -\beta \log \delta$, $0 < \beta < \eta$. Then we have $i\partial\psi \wedge \overline{\partial}\psi \leq ir\partial\overline{\partial}\psi$ with $r = \frac{\beta}{\eta} < 1$.

From now on the proof follows [3] and we give it for the convenience of the reader.

For $\gamma > 0$ we denote $\phi = -\gamma \log \delta$ and we have $i\partial\overline{\partial}(\psi + \phi) \geq (\beta + \gamma)C_\Omega \omega$.

Let $\{\Omega_j\}$ be an exhaustion of Ω by pseudoconvex domains. By Proposition 3.1 there exists $u_j \in L^2_{(p,q-1)}(\Omega; \delta^\gamma; E)$ such that $\overline{\partial} u_j = f$ on Ω_j and $\int_{\Omega_j} |u_j|^2 \delta^\gamma dV \leq \frac{1}{2q\gamma C_\Omega} \int_\Omega |f|^2 \delta^\gamma dV$.

We denote by u_j the minimal solution (i.e. the solution u_j which is orthogonal to all $\overline{\partial}-$closed forms of $L^2_{(p,q-1)}(\Omega_j; \delta^\gamma; E)$) and $v_j = u_j e^\psi = u_j \delta^{-\beta}$.

Then v_j is orthogonal to all $\overline{\partial}$-closed forms of $L^2_{(p,q-1)}(\Omega_j; \delta^{\beta+\gamma}; E)$ and we have

$$\int_{\Omega_j} |v_j|^2 \delta^{\beta+\gamma} dV \leq \int_{\Omega_j} \left\|\overline{\partial} v_j\right\|^2_{i\partial\overline{\partial}(\psi+\phi)} \delta^{\beta+\gamma} dV.$$

It follows that

$$\int_{\Omega_j} |u_j|^2 \delta^{-\beta+\gamma} dV \leq \int_{\Omega_j} \left\|\overline{\partial} u_j + \overline{\partial}\psi \wedge u_j\right\|^2_{i\partial\overline{\partial}(\psi+\phi)} \delta^{-\beta+\gamma} dV$$

$$\leq \left(1 + \frac{1}{a}\right) \int_{\Omega_j} \|f\|^2_{i\partial\overline{\partial}(\psi+\phi)} \delta^{-\beta+\gamma} dV + (1+a) \int_{\Omega_j} \left\|\overline{\partial}\psi \wedge u_j\right\|^2_{i\partial\overline{\partial}(\psi+\phi)} \delta^{-\beta+\gamma} dV$$

for every $a > 0$.

Since

$$\left\|\overline{\partial}\psi \wedge u_j\right\|^2_{i\partial\overline{\partial}(\psi+\phi)} \leq |u_j|^2 \left\|\overline{\partial}\psi\right\|^2_{i\partial\overline{\partial}(\psi+\phi)} \leq |u_j|^2 \left\|\overline{\partial}\psi\right\|^2_{i\partial\overline{\partial}\psi} \leq r^2 |u_j|^2,$$

by choosing a such that $(1+a)r^2 < 1$, (i.e. $0 < a < \left(\frac{\eta}{\beta}\right)^2 - 1$) we obtain

$$\int_{\Omega_j} |u_j|^2 \delta^{-\beta+\gamma} dV \leq \frac{(1+\frac{1}{a})}{[1-(1+a)r^2]} \int_{\Omega_j} \|f\|^2_{i\partial\overline{\partial}(\psi+\phi)} \delta^{-\beta+\gamma} dV.$$

Because $\|f\|^2_{i\partial\overline{\partial}(\psi+\phi)} \leq \|f\|^2_{(\beta+\gamma)C_\Omega\omega} \leq \frac{1}{q(\beta+\gamma)C_\Omega} |f|^2$, we have $\int_{\Omega_j} |u_j|^2 \delta^{-\beta+\gamma} dV \leq C \int_\Omega |f|^2 \delta^{-\beta+\gamma} dV$.

The trivial extensions of u_j to Ω are uniformly bounded in $L^2_{(p,q-1)}(\Omega; \delta^{-\beta+\gamma}; E)$, so there exists a weakly convergent subsequence to $u \in L^2_{(0,q-1)}(\Omega; \delta^{-\beta+\gamma}; E)$. It follows that $\overline{\partial} u = f$ and $\int_\Omega |u|^2 \delta^{-\beta+\gamma} dV \leq C \int_\Omega |f|^2 \delta^{-\beta+\gamma} dV$. □

COROLLARY 3.6. *Let Ω be a pseudoconvex domain with C^2 boundary in $\mathbb{CP}_n$. Then for every non-negative vector bundle E over Ω we have $H^{p,q}_{L^2}(\Omega; E) = 0$ for $q \geq 1$. In particular $H^{p,q}_{L^2}(\Omega) = 0$.*

Proof. Since Ω is Hartogs-pseudoconvex and $m_p(\Omega; E) \geq 0$, the result follows from Proposition 3.5. □

REMARK 3.7. *The previous results have important generalizations for manifolds spread over $\mathbb{CP}_n$:*

Let $\widetilde{\Omega}$ be a connected complex manifold and $\pi : \widetilde{\Omega} \to \Omega$ a local biholomorphism onto a domain $\Omega \subset \mathbb{CP}_n$. We say that $\widetilde{\Omega}$ (or $\pi : \widetilde{\Omega} \to \Omega$) is a domain spread over $\mathbb{CP}_n$. The domain $\widetilde{\Omega}$ is a Kähler manifold with the metric $\widetilde{\omega}$ defined as the pull-back of the Fubini-Study metric on $\mathbb{CP}_n$. For $z \in \widetilde{\Omega}$, we consider the boundary distance $\widetilde{d}(z)$ defined as the lower bound of the length of geodesics $t \to \gamma_z(t)$ from z such that for every compact $K \subset \widetilde{\Omega}$ there exists t_K satisfying $\gamma_z(t) \notin K$ for $t > t_K$. By [40] it follows that $\widetilde{d}$ is strongly plurisubharmonic outside a compact subset of $\widetilde{\Omega}$; so Corollary 3.4 and Corollary 3.6 are valid for pseudoconvex domains $\widetilde{\Omega}$ spread over $\mathbb{CP}_n$ with $\widetilde{\mathcal{O}}(m) = \pi^*\mathcal{O}(m)$ instead $\mathcal{O}(m)$ and a strongly plurisubharmonic extension $\widetilde{\delta}$ of $\widetilde{d}$ instead δ.

4. Separation of points by L^2-holomorphic sections of $\mathcal{O}(m)$ in pseudoconvex domains spread over $\mathbb{CP}_n$. In this paragraph we use the following particular case of the extension theorem of Ohsawa-Takegoshi-Manivel [26]:

Let X be a Stein manifold of dimension n, E, L holomorphic line bundles over X and s a holomorphic section of E generically transverse to the zero section. Let $Y = \{x \in X \mid s(x) = 0, ds(x) \neq 0\}$. Suppose that Ψ is a closed positive $(1,1)$-form on X such that $\Psi \otimes Id_E \geq ic(E)$ and there exists $\alpha > 0$ such that $ic(L) \geq \alpha\Psi - i\partial\overline{\partial}log\,|s|^2$. Then, for every holomorphic form $g \in L^2\left(Y; \Lambda^{n-1}T^*Y \otimes L \otimes E^*\right)$, there exists a holomorphic form $G \in L^2(X; \Lambda^n T^*\Omega \otimes L)$ such that $G|_Y = g \wedge ds$.

For $L = \mathcal{O}(m)$ and s a holomorphic section of $\mathcal{O}(k)$, by the Lelong-Poincaré equation we have

$$ic(L) + i\partial\overline{\partial}log\,|s|^2 = (m-k)\,\omega + 2\pi\,[Y] \tag{4.1}$$

where ω is the $(1,1)$-form associated to the Fubini-Study metric on $\mathbb{CP}_n$ and $[Y]$ the current of integration on $Y = \{z \in \mathbb{CP}_n \mid s(z) = 0\}$.

The following result gives a usefull complement for Theorem 0.7 of [16]:

PROPOSITION 4.1. *Let $\pi : \widetilde{\Omega} \to \Omega$ be a domain spread over a non-dense pseudoconvex domain $\Omega \subset \mathbb{CP}_n$. Then the sections of $H^{0,0}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m)) \cap L^2(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m))$ separate the points of $\widetilde{\Omega}$ for every integer $m \geq 0$.*

Proof. Suppose $n = 2$ and let $\widetilde{a}, \widetilde{b}$ be distinct points of $\widetilde{\Omega}$. We put $a = \pi(\widetilde{a})$, $b = \pi\left(\widetilde{b}\right)$ and let $c \in \mathbb{CP}_2 \backslash \overline{\Omega}$. There exists an homogeneous polynomial P of degree 2 such that the Riemann surface $\Gamma = \{[z] \mid P(z) = 0\}$ contains the points a, b, c, where $z = (z_0, z_1, z_2)$ are homogeneous coordinates in $\mathbb{CP}_2$. By Sard's theorem we can choose $c \in \mathbb{CP}_2 \backslash \overline{\Omega}$ such that $dP \neq 0$ on Γ.

Let $\widetilde{\Gamma} = \pi^{-1}(\Gamma)$ and we show firstly that there exists $\widetilde{h} \in H^{1,0}\left(\widetilde{\Gamma}; \widetilde{\mathcal{O}}(m)\right) \cap L^2_{(1,0)}\left(\widetilde{\Gamma}; \widetilde{\mathcal{O}}(m)\right)$ such that $\widetilde{h}(\widetilde{a}) \neq \widetilde{h}(\widetilde{b})$ for every $m \in \mathbb{Z}$. Let Δ be a projective line such that $a, b \in \Delta$ and s a section of $\mathcal{O}(1)$ such that $\Delta = \{[z] \mid s(z) = 0\}$. Then $\widetilde{\Delta} = \pi^{-1}(\Delta) = \left\{\widetilde{z} \in \widetilde{\Omega} \mid \widetilde{s}(\widetilde{z}) = 0\right\}$ with $\widetilde{s} = \pi^* s \in \widetilde{\mathcal{O}}(1)$ and $\widetilde{A} = \widetilde{\Delta} \cap \widetilde{\Gamma}$ a discrete set. Let $\widetilde{\psi}$ be a compactly supported function which is 1 on a neighborhood of $\widetilde{a}$ and vanishes on $\widetilde{A} \backslash \{\widetilde{a}\}$. By using (4.1), from the extension theorem of Ohsawa-Takegoshi-Manivel [26] it follows that there exists an extension $\widetilde{g} \in H^{1,0}\left(\widetilde{\Gamma}; \widetilde{\mathcal{O}}(2)\right) \cap L^2_{(1,0)}\left(\widetilde{\Gamma}; \widetilde{\mathcal{O}}(2)\right)$ of $\widetilde{\psi}_{|\widetilde{A}} \otimes Id_{\widetilde{\mathcal{O}}(1)} = \widetilde{\psi}_{|\widetilde{A}} \otimes Id_{\widetilde{\mathcal{O}}(2) \otimes \widetilde{\mathcal{O}}(-1)}$.

Since $c \notin \overline{\Omega}$, we can find holomorphic sections η of $\mathcal{O}(m-2)$ which are bounded on Γ (it is obvious for $m \geq 2$; for $m < 2$, since Γ is a hypersurface of degree 2 in $\mathbb{CP}_2$, we can find projective lines $R = \{Q(z) = 0\}$ such that $R \cap \Gamma \subset \mathbb{CP}_2 \backslash \overline{\Omega}$ where Q is a homogeneous polynomial of degree 1 and we can consider η the section defined by $\frac{1}{Q^{m-2}}$). Than we can take $\widetilde{h} = (\pi^* \eta)\, \widetilde{g}$.

Now we use (4.1) for $m \geq 3$ and the extension theorem of Ohsawa-Takegoshi-Manivel to show the existence of $\widetilde{f} \in H^{2,0}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m)) \cap L^2_{(2,0)}(\Omega; \widetilde{\mathcal{O}}(m))$ such that $\widetilde{f}|_{\widetilde{\Gamma}\cap\Omega} = \widetilde{h} \wedge d\pi^* P$, so $\widetilde{f}(a) \neq \widetilde{f}(b)$.

For $n \geq 3$, $m \geq n+1$ we proceed by induction: let $\widetilde{a}, \widetilde{b}$ be distinct points of $\widetilde{\Omega}$, $a = \pi(\widetilde{a})$, $b = \pi\left(\widetilde{b}\right)$ and let $c \in \mathbb{CP}_n \backslash \overline{\Omega}$. There exists a hyperplane H through a, b, c and set $\Omega_H = H \cap \Omega$. Then $\widetilde{\Omega_H} = \pi^{-1}(\Omega_H)$ is a domain spread over H. By the induction hypothesis, there exists

$$\widetilde{g} \in H^{n-1,0}(\widetilde{\Omega_H}; \widetilde{\mathcal{O}}(m-1)) \cap L^2_{(n-1,0)}(\widetilde{\Omega_H}; \widetilde{\mathcal{O}}(m-1)) =$$

$$H^{n-1,0}(\widetilde{\Omega_H}; \widetilde{\mathcal{O}}(m) \otimes \widetilde{\mathcal{O}}(-1)) \cap L^2_{(n-1,0)}(\widetilde{\Omega_H}; \widetilde{\mathcal{O}}(m) \otimes \widetilde{\mathcal{O}}(-1))$$

such that $\widetilde{g}(\widetilde{a}) \neq \widetilde{g}(\widetilde{b})$.

We use again the extension theorem of Ohsawa-Takegoshi-Manivel [26] and it follows that there exists $\widetilde{f} \in H^{n,0}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m)) \cap L^2_{(n,0)}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m))$ such that $\widetilde{f}(\widetilde{a}) \neq \widetilde{f}(\widetilde{b})$. Since

$$H^{n,0}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m)) \cap L^2_{(n,0)}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m)) = H^{0,0}(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m-n-1)) \cap L^2(\widetilde{\Omega}; \widetilde{\mathcal{O}}(m-n-1)),$$

Proposition 4.1 is proved. □

REMARK 4.2. *The proposition 4.1 is related to following theorem from [7]: if Ω is a pseudoconvex domain of $\mathbb{CP}_n$ such that the interior of its complement is not empty and $\zeta \in \overline{\Omega}$, the restriction to the diagonal of the Bergman kernel function K_Ω of Ω has the same order of growth in arbitrary small neighborhoods $V \subset\subset U$ of ζ as the restriction to the diagonal of the Bergman kernel function $K_{\Omega\cap U}$ of $\Omega \cap U$ in the sense $C^{-1}K_\Omega < K_{\Omega\cap U} < CK_\Omega$ on $\Omega \cap V$, $C > 0$.*

In the case of domains with C^2 boundary in $\mathbb{CP}_2$, we can improve the result of Proposition 4.1:

PROPOSITION 4.3. *Let Ω be a pseudoconvex domain with C^2 non-empty boundary in $\mathbb{CP}_2$. Then there exists $\eta > 0$ such that for every $\alpha > -\eta$ the sections of $H^{0,0}(\Omega; \delta^\alpha; \mathcal{O}(m)) \cap L^2(\Omega; \delta^\alpha; \mathcal{O}(m))$ separate the points of Ω for every integer $m \geq -1$.*

Proof. Let a, b be distinct points of Ω and $c \in \mathbb{CP}_2 \backslash \overline{\Omega}$. There exists a Riemann surfaces Γ of degree 2 through a, b, c. Let $\Gamma = \{[z] \mid P(z) = 0\}$ where $z = (z_0, z_1, z_2)$ are homogeneous coordinates in $\mathbb{CP}_2$ and P an homogeneous polynomial of degree 2 and we can choose c such that $dP \neq 0$ on Γ. Let Ω' be an open neighborhood of $\overline{\Omega}$ which does not contain c. By [17] there exists a Stein neighborhood V of $\Gamma \cap \Omega'$ and let $h \in H^{0,0}(V; \mathcal{O}(m))$ such that $h(a) \neq h(b)$.

Let χ be a C^∞ function on $\mathbb{CP}_2$ with support contained in V such that $\chi \equiv 1$ on $V \cap \Omega$. By identifying the sections of $\mathcal{O}(m)$ with the m-homogeneous functions in homogeneous coordinates, $\frac{h\overline{\partial}\chi}{P}$ defines a form $g \in C^\infty_{(0,1)}\left(\overline{\Omega}; \mathcal{O}(m-2)\right)$, so $g \in L^2_{(0,1)}(\Omega; \delta^\beta; \mathcal{O}(m-2))$ for every $\beta > -1$. By Proposition 3.5 and Corollary 3.4, there exists $\eta > 0$ such that for every $\alpha > -\eta$ the equation $\overline{\partial} u = g$ has a solution $u \in L^2(\Omega; \delta^\alpha; \mathcal{O}(m-2))$ for every $m \geq -1$. Then $\chi h - Pu$ defines a section $f \in H^{0,0}(\Omega; \delta^\alpha; \mathcal{O}(m)) \cap L^2(\Omega; \delta^\alpha; \mathcal{O}(m))$ such that $f(a) = h(a) \neq h(b) = f(b)$. □

5. Compactly supported solutions of $\overline{\partial}$.

DEFINITION 5.1. *Let Ω be a pseudoconvex domain in a hermitian n-dimensional manifold X, E a hermitian bundle on Ω and $f \in L^2_{(p,q)}(\Omega;\delta^{-\alpha};E)$. We say that f verifies the moment condition of order α if $\int_\Omega f \wedge h = 0$ for every $\overline{\partial}$-closed form $h \in L^2_{(n-p,n-q)}(\Omega;\delta^{\alpha};E^*)$. If $\alpha = 0$, we say only f verifies the moment condition.*

REMARK 5.2. *If $1 \le q \le n-1$, every $\overline{\partial}$-closed form $f \in L^2_{(p,q)}(\Omega;\delta^{-\alpha-1};E)$ verifies the moment condition of order α (see Corollary 5.6).*

The following proposition is a generalisation of a result from [1]:

PROPOSITION 5.3. *Let Ω be a relatively compact Hartogs-pseudoconvex domain in a Kähler n-dimensional manifold X and E a holomorphic hermitian vector bundle in a neighborhood of $\overline{\Omega}$. Let $\alpha \in \mathbb{R}_+$ such that $m_{n-p}(\Omega;E^*) + 2\alpha C_\Omega > 0$ and $f \in L^2_{(p,q)}(\Omega;\delta^{-\alpha};E)$ a $\overline{\partial}$-closed form verifying the moment condition of order α, $1 \le q \le n$. Then there exists $u \in L^2_{(p,q-1)}(\Omega;\delta^{-\alpha};E)$ such that $\overline{\partial}u = f$, u vanishes of order greater than α on $\partial\Omega$ and*

$$(N_{-\alpha}(u))^2 \le \frac{1}{(n-q+1)[m_{n-p}(\Omega;E^*) + \alpha C_\Omega]}(N_{-\alpha}(f))^2.$$

In particular, since $\alpha \ge 0$ and $C^\infty_{(n-p,n-q)}(\overline{\Omega};E) \subset Dom\overline{\partial}_\alpha$, we have $bv(u) = 0$.

Proof. Since $f \in L^2_{(p,q)}(\Omega;\delta^{-\alpha};E)$ and $m_{n-p}(\Omega;E^*) + 2\alpha C_\Omega > 0$ we can define

$$\Phi_f(\varphi) = \int_\Omega f \wedge T^\alpha_{(n-p,n-q+1)}(\varphi)$$

for every $\varphi \in \mathcal{R}^\alpha_{(n-p,n-q+1)}(\Omega;E^*)$, where $T^\alpha_{(n-p,n-q+1)}$ was defined in Remark 3.2.

We have $\int_\Omega f \wedge h = 0$ for every $\overline{\partial}$-closed form $h \in L^2_{(n-p,n-q)}(\Omega;\delta^\alpha;E^*)$, so $\Phi_f(\varphi) = \int_\Omega f \wedge \psi$ for every $\psi \in L^2_{(n-p,n-q)}(\Omega;\delta^\alpha;E^*)$ such that $\overline{\partial}\psi = \varphi$. Since T is continuous it follows that $\Phi_f \in (\mathcal{R}^\alpha_{(n-p,n-q+1)}(\Omega;E^*))'$ and

$$\|\Phi_f\|^2 \le \frac{1}{(n-q+1)[m_{n-p}(\Omega;E^*) + 2\alpha C_\Omega]}(N_{-\alpha}(f))^2$$

where $\|\Phi_f\|$ is the norm of the linear form Φ_f. By the theorem of Hahn-Banach, we may extend Φ_f to a form $\widetilde{\Phi_f} \in (L^2_{(n-p,n-q+1)}(\Omega;\delta^\alpha;E^*))'$ such that $\left\|\widetilde{\Phi_f}\right\| = \|\Phi_f\|$. Since $(L^2_{(n-p,n-q+1)}(\Omega;\delta^\alpha;E^*))' = L^2_{(p,q-1)}(\Omega;\delta^{-\alpha};E)$ by the pairing $(\alpha,\beta) = \int_\Omega \alpha \wedge \beta$ for every $\alpha \in L^2_{(p,q-1)}(\Omega;\delta^{-\alpha};E)$, $\beta \in L^2_{(n-p,n-q+1)}(\Omega;\delta^\alpha;E^*)$, there exists $u \in L^2_{(p,q-1)}(\Omega;\delta^{-\alpha};E)$, such that $N_{-\alpha}(u) = \left\|\widetilde{\Phi_f}\right\| = \|\Phi_f\|$ and $\widetilde{\Phi_f}(\varphi) = \int_\Omega u \wedge \varphi$ for every $\varphi \in L^2_{(n-p,n-q+1)}(\Omega;\delta^\alpha;E^*)$.

We obtain

$$\Phi_f(\varphi) = \int_\Omega f \wedge \psi = \int_\Omega u \wedge \varphi = \int_\Omega u \wedge \overline{\partial}\psi$$

for every $\psi \in L^2_{(n-p,n-q)}(\Omega;\delta^\alpha;E^*)$ such that $\overline{\partial}\psi = \varphi$. In particular $\int_\Omega f \wedge \psi = \int_\Omega u \wedge \overline{\partial}\psi$ for every $\psi \in Dom\overline{\partial}_\alpha \subset L^2_{(n-p,n-q)}(\Omega;\delta^\alpha;E^*)$, so $(-1)^{p+q+1}\overline{\partial}u = f$, u

vanishes of order greater than α on $\partial\Omega$ and

$$(N_{-\alpha}(u))^2 = \|\Phi_f\|^2 \leq \frac{1}{(n-q+1)[m_{n-p}(\Omega;E^*)+2\alpha C_\Omega]}(N_{-\alpha}(f))^2. \quad \square$$

By Corollary 3.4 and Proposition 5.3 we have

COROLLARY 5.4. *Let Ω be a pseudoconvex domain in $\mathbb{CP}_n$ and $f \in L^2_{(p,q)}(\Omega;\delta^{-\alpha};\mathcal{O}(m))$ a $\overline{\partial}$-closed form verifying the moment condition of order α, $q \geq 1$. Then:*

a) For

$$p < n, \alpha \geq 0, m \in \mathbf{Z}, 2\alpha\mathcal{K}_n - m > 0$$

there exists $u \in L^2_{(p,q-1)}(\Omega;\delta^{-\alpha};\mathcal{O}(m))$ such that $\overline{\partial}u = f$, u vanishes of order greater than α on $\partial\Omega$ and $(N_{-\alpha}(u))^2 \leq \frac{1}{(n-q+1)(2\alpha\mathcal{K}_n-m)}(N_{-\alpha}(f))^2$.

b) For

$$p = n, \alpha \geq 0, m \in \mathbf{Z}, 2\alpha\mathcal{K}_n + n + 1 - m > 0$$

there exists $u \in L^2_{(p,q-1)}(\Omega;\delta^{-\alpha};\mathcal{O}(m))$ such that $\overline{\partial}u = f$, u vanishes of order greater than α on $\partial\Omega$ and $(N_{-\alpha}(u))^2 \leq \frac{1}{(n-q+1)(2\alpha\mathcal{K}_n+n+1-m)}(N_{-\alpha}(f))^2$.

Since $m_0(\Omega;\mathcal{O}(-n-1)) = 0$ and $L^2_{(n,q)}(\Omega;\mathcal{O}(n+1)) = L^2_{(0,q)}(\Omega)$, by using Proposition 3.5, we obtain in the same way:

PROPOSITION 5.5. *Let Ω be a pseudoconvex domain with C^2 boundary in $\mathbb{CP}_n$ and $\alpha \geq 0$. Let $q \geq 1$ and $f \in L^2_{(0,q)}(\Omega;\delta^{-\alpha})$ be a $\overline{\partial}$-closed form verifying the moment condition of order α. Then there exists $u \in L^2_{(0,q-1)}(\Omega;\delta^{-\alpha})$ such that $\overline{\partial}u = f$, u vanishes of order greater than α on $\partial\Omega$ and $N_{-\alpha}(u) \leq C_\alpha N_{-\alpha}(f)$ where C_α is a constant independent on f.*

COROLLARY 5.6. *Let Ω be a relatively compact Hartogs-pseudoconvex domain in a Kähler n-dimensional manifold X and E a holomorphic hermitian vector bundle in a neighborhood of $\overline{\Omega}$. Let $\alpha \in \mathbb{R}_+$ such that $m_{n-p}(\Omega;E^*) + 2\alpha C_\Omega > 0$ and $f \in L^2_{(p,q)}(\Omega;\delta^{-\alpha-1};E)$ a $\overline{\partial}$-closed form, $1 \leq q \leq n-1$. Then f verifies the moment condition of order α.*

Proof. Let $h \in L^2_{(n-p,n-q)}(\Omega;\delta^{\alpha};E^*)$ be a $\overline{\partial}$-closed form.

By Proposition 3.1 it follows that for $m_{n-p}(\Omega;E^*) + 2\alpha C_\Omega > 0$ there exists $g \in L^2_{(n-p,n-q-1)}(\Omega;\delta^{\alpha};E^*)$ such that $\overline{\partial}g = h$ and

$$(N_\alpha(g))^2 \leq \frac{1}{(n-q)[m_{n-p}(\Omega;E^*)+\alpha C_\Omega]}(N_\alpha(h))^2.$$

For $\varepsilon > 0$ let $\Omega_\varepsilon = \{z \in \Omega | \delta(z) \geq \varepsilon\}$ and $f_\varepsilon = \chi_\varepsilon f$, where χ_ε is a C^∞ function on Ω such that $\chi_\varepsilon \equiv 1$ on $\Omega_{2\varepsilon}$, $\chi_\varepsilon \equiv 0$ on $\Omega\backslash\Omega_\varepsilon$, $0 \leq \chi_\varepsilon \leq 1$, $|D\chi_\varepsilon| \leq \frac{C}{\varepsilon}$.

We have

$$\begin{aligned} N_{-\alpha}(\overline{\partial}f_\varepsilon) &= N_{-\alpha}(\overline{\partial}\chi_\varepsilon \wedge f) = \left(\int_\Omega \left|\overline{\partial}\chi_\varepsilon \wedge f\right|^2 \delta^{-2\alpha} dV\right)^{\frac{1}{2}} \\ &\leq \frac{C}{\varepsilon} \cdot \left(\int_{\Omega_\varepsilon\backslash\Omega_{2\varepsilon}} |f|^2 \delta^{-2\alpha} dV\right)^{\frac{1}{2}} \leq 2C \left(\int_{\Omega_\varepsilon\backslash\Omega_{2\varepsilon}} |f|^2 \delta^{-2\alpha-2} dV\right)^{\frac{1}{2}} \to 0 \end{aligned}$$

for $\varepsilon \to 0$.

Let $\psi \in L^2_{(n-p,n-q-1)}(\Omega; \delta^\alpha; E^*)$ such that $\overline{\partial}\psi = 0$. Since $\overline{\partial}(f \wedge \psi) = 0$, we have $\int_\Omega \overline{\partial} f_\varepsilon \wedge \psi = \int_\Omega \overline{\partial}\chi_\varepsilon \wedge f \wedge \psi = 0$. Because $m_{n-p}(\Omega; E^*) + 2\alpha C_\Omega > 0$, by Proposition 5.3 there exists $u_\varepsilon \in L^2_{(n-p,n-q)}(\Omega; \delta^{-\alpha}; E)$ such that $\overline{\partial} u_\varepsilon = \overline{\partial} f_\varepsilon$, u_ε vanishes of order greater than α on $\partial\Omega$ and $(N_{-\alpha}(u_\varepsilon))^2 \leq \frac{1}{(n-q)[m_{n-p}(\Omega;E^*)+\alpha C_\Omega]} \left(N_{-\alpha}(\overline{\partial} f_\varepsilon)\right)^2$. Thus $N_{-\alpha}(u_\varepsilon) \to 0$ for $\varepsilon \to 0$ and

$$\left|\int_\Omega u_\varepsilon \wedge h\right| = \left|\int_\Omega u_\varepsilon \delta^{-2\alpha} \wedge h\delta^{2\alpha}\right| \leq N_{-\alpha}(u_\varepsilon) N_\alpha(h) \to 0$$

for $\varepsilon \to 0$.

We have

$$\int_\Omega f \wedge h = lim_{\varepsilon\to 0} \int_\Omega f_\varepsilon \wedge h = lim_{\varepsilon\to 0} \int_\Omega (f_\varepsilon - u_\varepsilon) \wedge h = lim_{\varepsilon\to 0} \int_\Omega (f_\varepsilon - u_\varepsilon) \wedge \overline{\partial} g.$$

Since u_ε vanishes of order greater than α on $\partial\Omega$ we obtain

$$\begin{aligned}\int_\Omega u_\varepsilon \wedge \overline{\partial} g &= (-1)^{p+q+1} \int_\Omega \overline{\partial} u_\varepsilon \wedge g = (-1)^{p+q+1} \int_\Omega \overline{\partial} f_\varepsilon \wedge g \\ &= (-1)^{p+q+1} \int_\Omega \overline{\partial}\chi_\varepsilon \wedge f \wedge g = (-1)^{p+q+1} \int_\Omega \chi_\varepsilon \wedge \overline{\partial}(f \wedge g) = \int_\Omega f_\varepsilon \wedge g\end{aligned}$$

and it follows that $\int_\Omega f \wedge h = 0$. □

6. $\overline{\partial}$-equation on pseudoconcave compacts.

DEFINITION 6.1. *Let X be a complex manifold, E a holomorphic vector bundle on X and $L \subset X$ closed. Let $k \in \mathbb{N}$ and $f, g \in C^k_{(p,q)}(X; E)$. We put $f \overset{(k)}{=} g$ if $f - g$ vanishes to order k on L. We use the notation $C^k_{(p,q)}(L; E)$ for the quotient set of $C^k_{(p,q)}(X; E)$ by the equivalence relation defined by " $\overset{(k)}{=}$ ". If $f \in C^k_{(p,q)}(L; E)$ and $\widetilde{f} \in C^k_{(p,q)}(X; E)$ belongs to the equivalence class of f, we say that $\widetilde{f}$ is a C^k extension of f to X. We define $\overline{\partial}_k : C^{k+1}_{(p,q)}(L; E) \to C^k_{(p,q+1)}(L; E)$ to be the operator induced by the operator $\overline{\partial}$ on X. A form $f \in C^k_{(p,q)}(L; E)$ is $k - \overline{\partial}$-closed if $\overline{\partial}_k f = 0$ and is $k - \overline{\partial}$-exact if there exists $u \in C^{k+1}_{(p,q)}(L; E)$ such that $\overline{\partial}_k u = f$.*

We define $C^\infty_{(p,q)}(L; E) = \bigcap_k C^k_{(p,q)}(L; E)$. For $f \in C^\infty_{(p,q)}(L; E)$ and $u \in C^\infty_{(p,q-1)}(L; E)$ we say that $\overline{\partial}_\infty u = f$ if $\overline{\partial}_k u = f$ for every $k \in \mathbb{N}$. A form $f \in C^\infty_{(p,q)}(L; E)$ is $\infty - \overline{\partial}$-closed if $\overline{\partial}_\infty f = 0$. We denote by $A^\infty_{(p,q)}(L; E)$ the set of $\infty - \overline{\partial}$-closed (p, q)-forms on L.

DEFINITION 6.2. *Let L be a pseudoconcave compact on a complex manifold X. A $k - \overline{\partial}$-closed form $f \in C^k_{(p,q)}(L; E)$, $k \geq 1$, verifies the moment condition of order k on L if there exists an extension $\widetilde{f} \in C^k_{(p,q)}(X; E)$ such that $\overline{\partial}\widetilde{f}$ verifies the moment condition of order $k - 1$ on the pseudoconvex domain $\Omega = X \backslash L$ (see Definition 5.1).*

REMARK 6.3. *Definition 6.2 does not depend on the extension $\widetilde{f}$.*

Indeed, let $\widetilde{f}, \widetilde{f_1} \in C^k_{(p,q)}(X;E)$ be C^k extensions of f which vanish outside a compact subset of X and $h \in L^2_{(n-p,n-q-1)}(\Omega;\delta^{k-1};E^*)$ a $\overline{\partial}$-closed form. We have

$$\begin{aligned}\int_\Omega \overline{\partial}\left(\widetilde{f}-\widetilde{f_1}\right)\wedge h &= lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\overline{\partial}\left(\widetilde{f}-\widetilde{f_1}\right)\wedge h\\ &= lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\left(\widetilde{f}-\widetilde{f_1}\right)\wedge h\\ &= -lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_\varepsilon}\overline{\partial}\left(\widetilde{f}-\widetilde{f_1}\right)\wedge h\end{aligned}$$

with $(\Omega_\varepsilon)_{\varepsilon>0}$ an exhaustion of Ω by smoothly bounded domains such that $\Omega_\varepsilon \supset \{z \in \Omega \mid d(z,\partial\Omega) > \varepsilon\}$ and

$$\begin{aligned}&\left|\int_{\Omega\setminus\Omega_\varepsilon}\overline{\partial}\left(\widetilde{f}-\widetilde{f_1}\right)\wedge h\right|\\ &\le\left(\int_{\Omega\setminus\Omega_\varepsilon}\left|\overline{\partial}\left(\widetilde{f}-\widetilde{f_1}\right)\right|^2\delta^{-2k+2}\right)^{1/2}\left(\int_{\Omega\setminus\Omega_\varepsilon}|h|^2\delta^{2k-2}\right)^{1/2}\underset{\varepsilon\to 0}{\to}0.\end{aligned}$$

In order to prove the $\overline{\partial}$-regularity on compacts we use the following two lemmas:

LEMMA 6.4. *Let Ω be a relatively compact domain in a complex n-dimensional manifold X, E a holomorphic hermitian vector bundle on X and $f \in C^1_{(p,q)}(X;E)$. We suppose that:*

1) f is $\overline{\partial}$-closed on $X\setminus\Omega$;

2) There exists $v \in L^2_{(p,q)}(\Omega;E)$ which vanishes on $\partial\Omega$ such that $f-v$ is $\overline{\partial}$-closed on Ω.

Then the form $F \in L^2_{(p,q)}(X;E)$ defined by $F = f$ on $X\setminus\Omega$ and $F = f-v$ on Ω is $\overline{\partial}$-closed on X.

Proof. Let $\varphi \in \mathcal{D}_{(n-p,n-q)}(X;E)$. We have

$$\begin{aligned}\int_X \overline{\partial}F\wedge\varphi &= (-1)^{p+q+1}\int_X F\wedge\overline{\partial}\varphi = (-1)^{p+q+1}\left(\int_X f\wedge\overline{\partial}\varphi - \int_\Omega v\wedge\overline{\partial}\varphi\right)\\ &= \int_X \overline{\partial}f\wedge\varphi - (-1)^{p+q+1}\int_\Omega v\wedge\overline{\partial}\varphi\\ &= \int_\Omega \overline{\partial}f\wedge\varphi - (-1)^{p+q+1}\int_\Omega v\wedge\overline{\partial}\varphi.\end{aligned}$$

Since v which vanishes on $\partial\Omega$, we obtain

$$\int_\Omega v\wedge\overline{\partial}\varphi = (-1)^{p+q+1}\int_\Omega \overline{\partial}v\wedge\varphi$$

so

$$\int_X \overline{\partial}F\wedge\varphi = \int_\Omega \overline{\partial}f\wedge\varphi - \int_\Omega \overline{\partial}v\wedge\varphi = 0. \quad \square$$

LEMMA 6.5. *Let Ω be a domain of a compact Kähler n-dimensional manifold X, E a holomorphic hermitian vector bundle on X and $v \in L^2_{(p,q)}(X;E)$, $q \ge 1$, such* *that $v = 0$ on $X \backslash \Omega$ and $v \in L^2_{(p,q)}(\Omega; d^{-k+1}; E)$. We denote by G the Green operator on X and $\overline{\partial}^*$ the L^2-adjoint of $\overline{\partial}$. Then $\overline{\partial}^* Gv \in W^k_{(p,q-1)}(X; E)$ and $\left\|\overline{\partial}^* Gv\right\|_{j,\Omega} \leq CN_{-j+1,\Omega}(v)$ for every $0 \leq j \leq k$, where C is a constant independent of v.*

Proof. G is an integral operator and we have

$$Gv(x) = \int_X G(x,y) \wedge v(y)dy = \int_\Omega G(x,y) \wedge v(y)dy$$

with $|G(x,y)| \leq Cd(x,y)^{-2n+2}$.

For $0 \leq i \leq j \leq k$ we have

$$\left|D^i_x \left[\overline{\partial}^* G(x,y)\right]\right| d^{j-1}(y) \leq Cd(x,y)^{-2n+1-i} d^{j-1}(y) \leq Cd(x,y)^{-2n}. \tag{6.1}$$

It follows that

$$D^i_x \overline{\partial}^* Gv(x) = \int_\Omega D^i_x \left[\overline{\partial}^* G(x,y)\right] d^{j-1}(y) \wedge v(y) d^{-j+1}(y)dy$$

is a classical singular integral with $vd^{-j+1} \in L^2_{(p,q)}(\Omega; E)$ and $D^i_x \overline{\partial}^* Gv \in L^2_{(p,q)}(\Omega; E)$ for $0 \leq i \leq j \leq k$ [38]; so $\overline{\partial}^* Gv \in W^k_{(p,q)}(X; E)$ and

$$\left\|\overline{\partial}^* Gv\right\|_{j,\Omega} \leq CN_{-j+1,\Omega}(v). \quad \square$$

DEFINITION 6.6. *Let X be a hermitian manifold and L a compact subset of X. We set $\Omega = X \backslash L$. We say that L is L^2-regular if there exists $N \in \mathbb{N}$ such that $d^{-N} \notin L^2(V_x \cap \Omega)$ for every $x \in \partial L$ and every neighborhood V_x of x. The least natural number N verifying this property is called the L^2-rank of L.*

EXAMPLE 6.7. *The closure of non-dense domains with Lipschitz boundary and the real analytic sets in $\mathbb{CP}_n$ are examples of L^2-regular compacts.*

THEOREM 6.8. *Let X be a compact Kähler n-dimensional manifold, E a holomorphic hermitian vector bundle on X and L a Hartogs-pseudoconcave L^2-regular compact subset of X. Let $0 \leq p \leq n$, $1 \leq q \leq n-2$ such that $H^{p,q}(X; E) = 0$ and $f \in C^\infty_{(p,q)}(L; E)$ a $\infty - \overline{\partial}$-closed form. Then, for every integer $k \geq 1$ such that $m_{n-p}(X \backslash L; E^*) + 2(k-1)C_\Omega > 0$, there exists $u_k \in C^k_{(p,q-1)}(L; E)$ such that $\overline{\partial}_k u_k = f$. For $q = n-1$, the same is true if $f \in C^\infty_{(p,n-1)}(L; E)$ verifies the moment condition of order $k + r$, where $r = \max(k+n+2, k+N+1)$ with N the L^2-rank of L.*

Proof. Suppose $1 \leq q \leq n-2$. Since $\overline{\partial}_\infty f = 0$, for every $k \in \mathbb{N}$ there exists an extension $\widetilde{f}_k \in C^{k+1}_{(p,q)}(X; E)$ of f such that $\overline{\partial}\widetilde{f}_k$ vanishes to order k on L. It follows that $\overline{\partial}\widetilde{f}_k \in L^2_{(p,q+1)}(\Omega; \delta^{-k}; E)$, where Ω is the Hartogs pseudoconvex domain $X \backslash L$. By Proposition 5.3 and Corollary 5.6 there exists $v_k \in L^2_{(p,q)}(\Omega; \delta^{-k+1}; E)$ such that $\overline{\partial} v_k = \overline{\partial}\widetilde{f}_k$.

We define the (p,q)-form F_k on X by $F_k = \widetilde{f}_k$ on L, $F_k = \widetilde{f}_k - v_k$ on Ω; hence $F_k \in L^2_{(p,q)}(X; E)$. Since $bv(v_k) = 0$, from Lemma 6.4 we conclude that $\overline{\partial} F_k = 0$. By using Lemma 6.5, we obtain $F_k = \overline{\partial}\widetilde{u}_k$ with $\widetilde{u}_k = \overline{\partial}^* GF_k \in W^k_{(p,q-1)}(X; E)$.

Since L is L^2-regular there exists $N \in \mathbb{N}$ such that $\delta^{-N} \notin L^2(V_{x_0} \cap \Omega)$ for every $x_0 \in \partial L$.

Let $m \in \mathbb{N}^*$ fixed and $k_m = \max(m+n+2, m+N+1)$. Than $\widetilde{u}_{k_m}$ defines a form $u_m \in C^{m+1}_{(p,q-1)}(L;E)$ such that $\overline{\partial}_m u_m = f$.

Indeed if $x_0 \in \partial L$ is such that $D^j\left(\overline{\partial}\widetilde{u}_{k_m} - \widetilde{f}_{k_m}\right)(x_0) \neq 0$ with $j \leq m \leq k_m - N - 1$, there exists a neighborhood $V_{x_0}$of x_0 such that $|v_{k_m}| \geq C\delta^j$ on $V_{x_0} \cap \Omega$. Since $\delta^{-N} \notin L^2(V_{x_0} \cap \Omega)$, we have $|v_{k_m}|\,\delta^{-k_m+1} \geq |v_{k_m}|\,\delta^{-N-j} \notin L^2(V_{x_0} \cap \Omega)$ and we obtain a contradiction.

If $q = n-1$, the same proof works for forms satisfying the moment condition of order $k+r$. $\square$

7. Boundary regularity of $\overline{\partial}$ in pseudoconcave domains. We prove now boundary regularity of $\overline{\partial}$ for (p,q)-forms with values in a bundle E on Hartogs-pseudoconcave domains Ω_- with Lipschitz boundary of a compact Kähler manifold. Firstly we consider that $H^{p,q}(X;E) = 0$ and we study the situations $1 \leq q \leq n-2$ and $q = n-1$ with the moment condition. If $H^{p,q}(X;E) \neq 0$, we obtain extension operators of $\overline{\partial}$-closed forms on Ω_- to $\overline{\partial}$-closed forms on X, such that a form is $\overline{\partial}$-exact on Ω_- if and only if its extension is $\overline{\partial}$-exact on X.

We use the following result of P. Grisvard (see for ex. [15], theorem 1.4.4.4):

Let Ω be an bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and $k \in \mathbb{N}^*$. Let u a function belonging to the closure of $\mathcal{D}(\Omega)$ in $W^k(\Omega)$. Then $u \in L^2(\Omega;\delta^{-k})$ and

$$N_{-k}(u) \leq C_k \|u\|_k \tag{7.1}$$

where C_k is independent on u.

THEOREM 7.1. *Let Ω_- be a domain with Lipschitz boundary of a compact Kähler n-dimensional manifold X such that $\overline{\Omega_-}$ is Hartogs-pseudoconcave and E a holomorphic hermitian vector bundle on X. Let $0 \leq p \leq n$, $1 \leq q \leq n-1$ such that $H^{p,q}(X;E) = 0$ and $k_0 \geq 1$ an integer such that $m_{n-p}(\Omega;E^*) + 2(k_0-1)C_\Omega > 0$, where $\Omega = X \backslash \overline{\Omega_-}$.*

a) Suppose that $q < n-1$. Then for every $\overline{\partial}$-closed form $f \in C^\infty_{(p,q)}(\overline{\Omega_-};E)$ and every $k \geq k_0$, there exists $u \in W^k_{(p,q-1)}(\Omega_-;E) \cap C^\infty_{(p,q-1)}(\Omega_-;E)$ such that $\overline{\partial}u = f$ and $\|u\|_j \leq C_j \|f\|_j$ for every $k_0 \leq j \leq k$, where C_j is a constant independent of f.

b) If $q = n-1$ the same statement is true for $\overline{\partial}$-closed forms $f \in C^\infty_{(p,q)}(\overline{\Omega_-};E)$ verifying the moment condition of order k.

Proof. Let $\{U_i\}_{1\leq i\leq N}$ be a finite covering of $\partial\Omega_-$ with coordinate charts of X and $U_0 = \Omega_- \cup U_i$. We consider an orthonormal basis for the (p,q)-forms on U_i, $1 \leq i \leq N$ and by using the extension theorem for the Sobolev spaces on domains with Lipschitz boundaries (see for ex. [15]) we may consider extensions $\widetilde{f}_i \in C^\infty_{(p,q)}(U_i;E)$ of $f_{|U_i\cap\overline{\Omega_-}}$ such that $\left\|\widetilde{f}_i\right\|_j \leq C_j \|f\|_j$. for every j. A partition of unity subordinate to $\{U_i\}_{0\leq i\leq N}$, gives an extension $\widetilde{f}$ of f in a neighborhood Ω'_- of $\overline{\Omega_-}$ such that $\left\|\widetilde{f}\right\|_j \leq C_j \|f\|_j$ and a multiplication with a C^∞ function φ with support contained in Ω'_- such that $\varphi \equiv 1$ on $\overline{\Omega_-}$, allows us to assume that $\widetilde{f} \in C^\infty_{(p,q)}(X;E)$.

Suppose that $1 \leq q \leq n-2$ and let $k_0 \leq j \leq k$. Since f is $\overline{\partial}$-closed, we have $\overline{\partial}\widetilde{f} \in L^2_{(p,q+1)}(\Omega;\delta^{-j};E)$.

By Proposition 5.3 and Corollary 5.6 it follows that there exists $v \in L^2_{(p,q)}(\Omega;\delta^{-j+1};E)$ such that $\overline{\partial}v = \overline{\partial}\widetilde{f}$ and $N_{-j+1,\Omega}(v) \leq C_j N_{-j+1,\Omega}(\overline{\partial}\widetilde{f})$.

We define the (p,q)-form F on X by $F = f$ on $\overline{\Omega_-}$, $F = \widetilde{f} - v$ on Ω; so $F \in L^2_{(p,q)}(X;E)$. Since $bv(v) = 0$, by Lemma 6.4 we obtain that $\overline{\partial} F = 0$.

Since $H^{p,q}(X;E) = 0$, by Hodge theory on compact complex manifolds it follows that $F = \overline{\partial} u$ with $u = \overline{\partial}^* GF \in L^2_{(p,q-1)}(X;E)$ and G the Green operator on X.

As in Lemma 6.5, we have $GF(x) = \int_X G(x,y) \wedge F(y)dy$ with $|G(x,y)| \leq Cd(x,y)^{-2n-2}$.

Therefore,

$$GF(x) = \Phi_1(x) - \Phi_2(x) \tag{7.2}$$

where

$$\Phi_1(x) = \int_X G(x,y) \wedge \widetilde{f}(y)dy, \quad \Phi_2(x) = \int_\Omega G(x,y) \wedge v(y)dy. \tag{7.3}$$

As $\widetilde{f} \in C^\infty_{(p,q)}(X;E)$, $\overline{\partial}^* \Phi_1 \in C^\infty_{(p,q-1)}(X;E)$ and

$$\left\| \overline{\partial}^* \Phi_1 \right\|_{j+1} \leq C_j \left\| \widetilde{f} \right\|_j \leq C_j \left\| f \right\|_j . \tag{7.4}$$

Because $\Phi_2 \in C^\infty_{(p,q)}(\Omega_-;E)$ it follows that $u = \overline{\partial}^* \Phi_1 - \overline{\partial}^* \Phi_2 \in C^\infty_{(p,q-1)}(\Omega_-;E)$.

By Lemma 6.5 it follows that $\overline{\partial}^* \Phi_2 \in W^k_{(p,q-1)}(X;E)$ and

$$\left\| \overline{\partial}^* \Phi_2 \right\|_{j,\Omega_-} \leq CN_{-j+1,\Omega}(v) \leq C_j N_{-j+1,\Omega}(\overline{\partial}\widetilde{f}). \tag{7.5}$$

Since $\overline{\partial}\widetilde{f}$ vanishes to infinite order on $\partial\Omega$, by (7.1) we have

$$N_{-j+1,\Omega}(\overline{\partial}\widetilde{f}) \leq C_j \left\| \overline{\partial}\widetilde{f} \right\|_{j-1} \tag{7.6}$$

and by using (7.5) and (7.6) we obtain

$$\left\| \overline{\partial}^* \Phi_2 \right\|_{j,\Omega_-} \leq C_j \left\| \overline{\partial}\widetilde{f} \right\|_{j-1} \leq C_j \left\| \widetilde{f} \right\|_j \leq C_j \left\| f \right\|_j . \tag{7.7}$$

Finally (7.4) and (7.7) give

$$\left\| u \right\|_j = \left\| \overline{\partial}^* \Phi_1 - \overline{\partial}^* \Phi_2 \right\|_j \leq C_j \left\| f \right\|_j . \quad \square$$

COROLLARY 7.2. *Under the hypothesis of Theorem 7.1 , suppose that Ω_- is with arbitrary boundary. Then there exists $u \in W^k_{(p,q-1)}(\Omega_-;E)$ such that $\overline{\partial} u = f$ and $\left\| u \right\|_k \leq C_k \left\| f \right\|_{k+n+2}$, where C_k is a constant independent of f.*

Proof. We can use the extension theorem of Whitney for smooth functions to obtain an extension $\widetilde{f}$ of f to X such that $\left\| \widetilde{f} \right\|_{C^{k+1}(X)} \leq C_k \left\| f \right\|_{C^{k+1}(\overline{\Omega_-})}$. Since

$$\left\| \widetilde{f} \right\|_{k+1} \leq C_k \left\| \widetilde{f} \right\|_{C^{k+1}(X)} \leq C_k \left\| f \right\|_{C^{k+1}(\overline{\Omega_-})} \leq C_k \left\| f \right\|_{k+n+2}$$

the only change in the proof of Theorem 7.1 is that $\left\| \widetilde{f} \right\|'_k \leq C_k \left\| f \right\|_{k+n+2}$. $\square$

COROLLARY 7.3. *Let Ω_- be a domain with Lipschitz boundary of a compact Kähler n-dimensional manifold X such that $\overline{\Omega_-}$ is Hartogs-pseudoconcave and E a holomorphic hermitian vector bundle on X. Let $0 \le p \le n$, $1 \le q \le n-2$, such that $H^{p,q}(X;E) = 0$ and $f \in C^\infty_{(p,q)}(\overline{\Omega_-};E)$ a $\overline{\partial}$-closed form. Then there exists $u \in C^\infty_{(p,q-1)}(\overline{\Omega_-};E)$ such that $\overline{\partial}u = f$.*

Proof. The proof follows [24], pag.230 and we give it below for the convenience of the reader.

It is sufficient to prove that for every $k \ge k_0$, there exists $u_k \in W^k_{(p,q-1)}(\Omega_-;E) \cap C^\infty_{(p,q-1)}(\Omega_-;E)$ such that $\overline{\partial}u_k = f$ and $\|u_{k+1} - u_k\|_k \le 2^{-k}$ (then we can take $u = u_{k_0} + \sum_{k_0}^\infty (u_{k+1} - u_k) \in W^k_{(p,q-1)}(\Omega_-;E)$).

Suppose we found u_k for $k_0 \le k \le m-1$. By Theorem 7.1 there exists $v_m \in W^m_{(p,q-1)}(\Omega_-;E) \cap C^\infty_{(p,q-1)}(\Omega_-;E)$ such that $\overline{\partial}v_m = f$. Let $\{U_j\}_{0\le j\le N}$ be a covering of $\overline{\Omega_-}$, where $U_0 \subset\subset \Omega$ and $\{U_j\}_{1\le j\le N}$ is a covering of $\partial\Omega_-$ with balls centered at $\zeta_j \in \partial\Omega_-$ such that $\overline{\Omega_-} \cap (U_j - \varepsilon\nu_j) \subset \Omega_-$ for $0 < \varepsilon < \varepsilon_0$ and ν_j suitable directions. Let $\{\chi_j\}_{0\le j\le N}$ be a C^∞ partition of unity subordinate to the covering $\{U_j\}_{0\le j\le N}$. We put $h = v_m - u_{m-1} \in W^{m-1}_{(p,q-1)}(\Omega_-;E)$, $h_\varepsilon(z) = \sum_{j=0}^N \chi_j(z)\, h(z - \varepsilon\nu_j)$ and we remark that $h_\varepsilon \in C^\infty_{(p,q-1)}(\overline{\Omega_-};E)$ for $0 < \varepsilon < \varepsilon_0$. Since $\|h - h_\varepsilon\|_{m-1} \to 0$ for $\varepsilon \to 0$ and $\overline{\partial}h_\varepsilon(z) = \sum_{j=0}^N \overline{\partial}\chi_j(z) \wedge [h(z-\varepsilon\nu_j) - h(z)]$ we conclude that $\left\|\overline{\partial}h_\varepsilon\right\|_{m-1} \to 0$ for $\varepsilon \to 0$.

By Theorem 7.1 there exists $\varphi_\varepsilon \in W^m_{(p,q-1)}(\Omega_-;E)$ such that $\overline{\partial}\varphi_\varepsilon = \overline{\partial}h_\varepsilon$ and $\|\varphi_\varepsilon\|_{m-1} \to 0$ for $\varepsilon \to 0$. For ε small enogh we have

$$\|h - h_\varepsilon - \varphi_\varepsilon\|_{m-1} = \|v_m - h_\varepsilon - \varphi_\varepsilon - u_{m-1}\|_{m-1} \le 2^{-m}$$

and we may choose $u_m = v_m - h_\varepsilon - \varphi_\varepsilon$. □

REMARK 7.4. *The same methods work to obtain C^k regular solutions of $\overline{\partial}$ up to the boundary on a domain $\Omega = \Omega_1 \backslash \Omega_2 \subset \mathbb{C}^n$, where Ω_1 is a pseudoconvex domain with piece-wise smooth boundary and Ω_2 is a pseudoconvex domain with Lipschitz boundary. This statement generalizes for Ω_2 with Lipschitz boundary a result from [28], where Ω_2 is supposed with C^2 piece-wise smooth boundary.*

COROLLARY 7.5. *Let Ω_- be a pseudoconcave domain with Lipschitz boundary in $\mathbb{CP}_n$. Let $f \in C^\infty_{(0,q)}(\overline{\Omega_-};\mathcal{O}(m))$, $\overline{\partial}f = 0$, $1 \le q \le n-2$, $m \in \mathbf{Z}$. Then for every integer $k \ge 1$ such that $2(k-1)\mathcal{K}_n - m > 0$, there exists $u \in W^k_{(0,q-1)}(\Omega_-;\mathcal{O}(m))$ such that $\overline{\partial}u = f$ and $\|u\|_k \le C_k \|f\|_k$, where C_k is a constant independent of f. For $q = n-1$, the same is true if f verifies the moment condition of order k. If Ω_- has C^2 boundary, the statement is also valid for $m = 0$.*

Proof. Since $H^{0,q}(\mathbb{CP}_n;\mathcal{O}(m)) = 0$ Corollary 7.5 follows from Theorem 7.1, Corollary 5.6 and Corollary 5.4. If Ω_- has C^2 boundary and $m = 0$, we can apply Proposition 5.5. □

In general, we can state the following:

THEOREM 7.6. *Let Ω_- be a domain with Lipschitz boundary of a compact Kähler n-dimensional manifold X such that $\overline{\Omega_-}$ is Hartogs-pseudoconcave and E a holomorphic hermitian vector bundle on X. For every integer $k \ge 1$ such that*

$m_{n-p}(\Omega; E^*) + 2(k-1)C_\Omega > 0$, where $\Omega = X \backslash \overline{\Omega_-}$, there exists continuous extension operators

$$E^k_{p,q} : A^\infty_{(p,q)}(\Omega_-; E) \to AW^k_{(p,q)}(X; E) \qquad 0 \le q \le n-2$$
$$E^k_{p,n-1} : A^{\infty,(M)}_{(p,n-1)}(\Omega_-; E) \to AW^k_{(p,n-1)}(X; E)$$

where $\Omega_- = X \backslash \Omega$ and $A^{\infty,(M)}_{(p,n-1)}(\Omega_-; E)$ are the $\overline{\partial}$-closed $(p, n-1)$- forms on Ω_- of class C^∞ up to the boundary which verify the moment condition of order k.

Moreover, f is $\overline{\partial}$-exact on $\overline{\Omega_-}$ if and only if $E^k_{p,q} f$ is $\overline{\partial}$-exact on X.

Proof. As in the proof of Theorem 7.1 we consider an extension $\widetilde{f} \in C^\infty_{(p,q)}(X; E)$ and a solution $v_k \in L^2_{(p,q)}(\Omega; \delta^{-k+2}; E)$ of the equation $\overline{\partial} v_k = \overline{\partial}\widetilde{f}$. Then we define the (p,q)-form F_k on X by $F_k = f$ on $\overline{\Omega_-}$, $F_k = \widetilde{f} - v_k$ on Ω and it follows that $F_k \in L^2_{(p,q)}(X; E)$ is a $\overline{\partial}$-closed form. By Hodge theory on compact complex manifolds we have $F_k = \overline{\partial}\overline{\partial}^* G F_k + H F_k$, where G is the Green operator and H is the projection on the finite dimensional space $\mathcal{H}_{(p,q)}(X; E) = Ker\, \overline{\partial} \cap Ker\, \overline{\partial}^*$ of harmonic forms on X. By Lemma 6.5, $\overline{\partial}^* G F_k \in W^{k+1}_{(p,q)}(X; E)$ and $\left\| \overline{\partial}^* G F \right\|_{k+1} \le C \left\| f \right\|_{k+1}$; so $F_k \in W^k_{(p,q)}(X; E)$ and we can define $E^k_{p,q} f = F_k$.

Suppose that f is $\overline{\partial}$-exact and let $u \in C^\infty_{(p,q)}(\overline{\Omega_-}; E)$ such that $\overline{\partial} u = f$. We denote by $\widetilde{u}$ a C^∞ extension of u to X.

It is enough to prove that $(F_k, h)_{L^2(X;E)} = 0$ for every $h \in \mathcal{H}_{(p,q)}(X; E)$, where $*$ is the Hodge star operator. Since

$$\overline{\partial}(\overline{*h}) = \overline{\partial(*h)} = \overline{*(-*\partial * h)} = \overline{*\overline{\partial}^* h} = 0$$

and Ω is Stein it follows that there exists $\psi \in C^\infty_{(n-p,n-q-1)}(\Omega; E)$ such that $\overline{\partial}\psi = \overline{*h}$. Let $(\Omega_\varepsilon)_{\varepsilon > 0}$ be an exhaustion of Ω by relatively compact domains in Ω.

We have

$$(7.8) \qquad \begin{aligned} (F_k, h)_{L^2(X;E)} &= \int_X F_k \wedge \overline{*h} \\ &= \int_{\Omega_-} f \wedge \overline{*h} + \int_\Omega \left(\widetilde{f} - v_k\right) \wedge \overline{\partial}\psi; \end{aligned}$$

$$(7.9) \qquad \begin{aligned} \int_{\Omega_-} f \wedge \overline{*h} &= \int_{\Omega_-} \overline{\partial} u \wedge \overline{*h} = \int_{\partial\Omega_-} u \wedge \overline{*h} = -\lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} \widetilde{u} \wedge \overline{*h} \\ &= -\lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} \widetilde{u} \wedge \overline{\partial}\psi = (-1)^{p+q} \lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} \overline{\partial}\widetilde{u} \wedge \psi. \end{aligned}$$

Since v_k vanishes on $\partial\Omega$ we have also

$$(7.10) \qquad \int_\Omega \left(\widetilde{f} - v_k\right) \wedge \overline{\partial}\psi = (-1)^{p+q+1} \lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} \widetilde{f} \wedge \psi$$

and by (7.8), (7.9) and (7.10) it follows that

$$\begin{aligned} (F_k, h)_{L^2(X;E)} &= (-1)^{p+q} \lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} \left(\overline{\partial}\widetilde{u} - \widetilde{f}\right) \wedge \psi \\ &= \lim_{\varepsilon \to 0} \int_{\Omega \backslash \Omega_\varepsilon} \left(\overline{\partial}\widetilde{u} - \widetilde{f}\right) \wedge \overline{*h} = 0. \quad \square \end{aligned}$$

8. Vanishing of the W^k-cohomology for $(0,q)$-forms in pseudoconcave domains of $\mathbb{CP}_n$. Let Ω be a domain with Lipschitz boundary in $\mathbb{CP}_n$. We denote by $S(1)$ the unit shere in $\mathbb{C}^{n+1}$ and $\widetilde{\Omega} = \{z \in S(1) \mid \pi(z) \in \Omega\}$ where $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}_n$ is the universal line bundle. If g is a $\mathcal{O}(m)$-valued $(0,q)$-form on $\mathbb{CP}_n$, we put $\widetilde{g} = \pi^* g$. For $1 \le q \le n-1$ let

$$K_q(t,z) = \frac{(-1)^q}{(2\pi i)^{n+1}} C^q_{n-1} \frac{det(\overline{t}, \overline{z}, \overbrace{d\overline{z}}^{q-1}, \overbrace{d\overline{t}}^{n-q})}{\Phi(t,z)^{n-q+1} \Phi^*(t,z)^q} \wedge \omega(t)$$

where $\overline{t}, \overline{z}, d\overline{t}$ and $d\overline{z}$ are columns of an $(n+1, n+1)$-matrix and the determinant is computed by the usual rule, where the place of each factor in the exterior product is determined by the index of the column to which the factor belongs;

$$\Phi(t,z) = \sum_{j=0}^{n} \overline{t_j}(t_j - z_j) = < \overline{t}, t - z > = 1 - < \overline{t}, z >;$$

$$\Phi^*(t,z) = \sum_{j=0}^{n} \overline{z_j}(t_j - z_j) = < \overline{z}, t - z > = < \overline{z}, t > -1;$$

$$\omega(t) = dt_0 \wedge \cdots \wedge dt_n.$$

For $f \in L^2_{(0,q)}(\Omega; \mathcal{O}(m)) \cap L^2_{(0,q)}(\partial\Omega; \mathcal{O}(m))$, we remind a formula of Bochner-Martinelli-Koppelman type in $\mathbb{CP}_n$ from [20]:

$$f = T_{q+1}(\overline{\partial} f) + \overline{\partial} T_q f + Q_q f \quad if\, 1 \le q \le n-1, \tag{8.1}$$

$$f = T_0 f + T_1(\overline{\partial} f) + Q_1 f \quad if\, q = 0, \tag{8.2}$$

where

$$\widetilde{T_q f}(z) = \int_{t \in \widetilde{\Omega}} \widetilde{f}(t) \wedge K_{q-1}(t,z) \quad if\, 1 \le q \le n, \tag{8.3}$$

$$\widetilde{Q_q f}(z) = \int_{t \in \partial\widetilde{\Omega}} \widetilde{f}(t) \wedge K_q(t,z) \quad if\, 1 \le q \le n-1, \tag{8.4}$$

$$\widetilde{T_0 f}(z) = \frac{1}{(2\pi i)^{n+1}} \int_{t \in \widetilde{\Omega}} \widetilde{f}(t) \wedge \frac{det(\overline{t}, \overbrace{d\overline{t}}^{n})}{\Phi(t,z)^{n+1}} \wedge \omega(t). \tag{8.5}$$

LEMMA 8.1. *Let Ω be a domain with Lipschitz boundary in $\mathbb{CP}_n$ and $f \in L^2_{(0,q)}(\partial\Omega; \mathcal{O}(m))$ such that the support of f is contained in a coordinate subset. Then, in local coordinates $z = (z_1, ..., z_n)$, we have*

$$Q_q f(z) = \int_{\partial\Omega} f(\xi) \wedge BMK_q(\xi,z) + \int_{\partial\Omega} f(\xi) \wedge R_q(\xi,z)$$

and

$$T_q f(z) = \int_{\Omega} f(\xi) \wedge BMK_{q-1}(\xi,z) + \int_{\Omega} f(\xi) \wedge R_{q-1}(\xi,z)$$

where $BMK_q(\xi,z)$ is the Bochner-Martinelli-Koppelman kernel in $\mathbb{C}^n$ and $|R_q(\xi,z)| \le C \frac{1}{|\xi - z|^{2n-2}}$.

Proof. In (8.4) we put

$$t=\lambda\xi,\xi=(\xi_0,...,\xi_n),|\lambda|=1,|\xi|^2=|\xi_0|^2+\cdots+|\xi_n|^2=1,\xi_0\in\mathbb{R}$$

and we obtain

$$\widetilde{Q_q f}(z)=\int_{\partial\Omega\times\{|\lambda|=1\}}\widetilde{f}(\lambda\xi)\wedge K_q(\lambda\xi,z).$$

Since $\widetilde{f}(\lambda\xi)=\lambda^m\overline{\lambda}^{-q}\widetilde{f}(\xi)$ and $\omega(\lambda\xi)=\lambda^n d\lambda\wedge\omega'(\xi)$, $\omega'(\xi)=\sum_{j=0}^n(-1)^j\xi_j d\xi_0\wedge\cdots\wedge\widehat{d\xi_j}\wedge\cdots\wedge d\xi_n$, it follows that

$$\widetilde{Q_q f}(z)=\frac{1}{2\pi i}\int_{\partial\Omega}\widetilde{f}(\xi)\wedge H_q(\xi,z)\int_{|\lambda|=1}\frac{\lambda^{n+q+m}d\lambda}{(\lambda-<\overline{\xi},z>)(\lambda<\overline{z},\xi>-1)^{q+1}}d\lambda$$

where

$$H_q(\xi,z)=\frac{(-1)^q}{(2\pi i)^n}C^q_{n-1}det(\overline{\xi},\overline{z},\overbrace{d\overline{z}}^{q},\overbrace{d\overline{\xi}}^{n-q-1})\wedge\omega'(\xi).$$

A simple computation of the second integral gives

$$\widetilde{Q_q f}(z)=\sum_{j=0}^{n-q-1}\int_{\partial\Omega}(-1)^j C^{n-q-1-j}_{n+q+m}C^j_{-q-1}\frac{|<\overline{\xi},z>|^{2j}}{\left(1-|<\overline{\xi},z>|^2\right)^{2(q+j+1)}}\widetilde{f}(\xi)\wedge H_q(\xi,z)$$
$$+P^q_{m+n+q-1}f(z)$$

with

$$P^q_r f(z)=\begin{cases}0 \quad if\, r\geq 0\\ (-1)^{n+r-1}\sum_{j=0}^r\int_{\partial\Omega}C^j_{-n+q}C^{r-j}_{-q-1}<\overline{\xi},z>^{r-2j}\widetilde{f}(\xi)\wedge H_q(\xi,z)\ otherwise\end{cases}$$

(for $s\in\mathbb{N}$ and $t\in\mathbb{Z}$ we denote $C^s_t=\frac{t(t-1)...(t-s+1)}{s!}$).

Suppose $z=(1,...,0)$. Then $<\overline{\xi},z>=\xi_0=\pm\left(1-|\xi'|^2\right)^{1/2}$, with $\xi'=(\xi_1,...,\xi_n)$ and we obtain

$$\widetilde{Q_q f}(z)=\int_{\partial\Omega}\frac{\overline{\xi_0}^{n+1}|\xi_0|^{2(n-q-1)}}{|\xi'|^{2n}}\widetilde{f}(\xi)\wedge H_q(\xi',z)+\int_{\partial\Omega}\frac{0\left(|\xi'|^2\right)}{|\xi'|^{2n}}\widetilde{f}(\xi)\wedge H_q(\xi',z).$$

This means that in local coordinates we have

(8.6)
$$Q_q f(z)=\int_{\partial\Omega}f(\xi')\wedge BMK_q(\xi',z')+\int_{\partial\Omega}f(\xi')\wedge R_q(\xi',z')=I_q(z')+J_q(z')$$

where $z'=(z_1,...,z_n)$, $BMK_q(\xi',z')$ is the Bochner-Martinelli-Koppelman kernel in $\mathbb{C}^n$ and $|R_q(\xi',z')|\leq C\frac{1}{|\xi'-z'|^{2n-2}}$. □

A similar computation gives the analoguous formula for T_q.

REMARK 8.2. *In fact, Lemma 8.1 shows that, in local coordinates, we have* $K_q=\Phi(BMK_q)$, *with* Φ *a smooth function which does not vanish on the diagonal and* BMK_q *the Bochner-Martinelli-Koppelman kernel in* $\mathbb{C}^n$ *(see 8.6).*

The Lemma 8.3 below is based on an approximation method inspired from the solution of Cousin's problems:

LEMMA 8.3. *Let Ω be a relatively compact domain with Lipschitz boundary in a complex manifold, E a holomorphic bundle on X. Suppose that there exists a fundamental system of neighborhoods $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of Ω with the following property: for every $\overline{\partial}$-exact form $\Phi = \overline{\partial}\psi$ with $\psi \in A^\infty_{(p,q)}(\Omega_\varepsilon; E)$, there exists $0 < \varepsilon' < \varepsilon$ and $\varphi \in W^s_{(p,q)}(\Omega_{\varepsilon'}; E) \cap C^\infty_{(p,q)}(\Omega_{\varepsilon'}; E)$ such that $\overline{\partial}\varphi = \Phi$ and $\|\varphi\|_{s,\Omega_{\varepsilon'}} \leq C\|\Phi\|_{s,\Omega_{\varepsilon'}}$ with C independent of Φ and ε. Then, every $f \in AW^s_{(p,q)}(\Omega; E) \cap C^\infty_{(p,q)}(\Omega; E)$ belongs to the closure of $A^\infty_{(p,q)}(\Omega; E)$ in $W^s_{(p,q)}(\Omega; E)$.*

Proof. Let $\{U_j\}_{0\leq j\leq N}$ be a covering of $\overline{\Omega}$, where $U_0 \subset\subset \Omega$ and $\{U_j\}_{1\leq j\leq N}$ is a covering of $\partial\Omega$ with balls centered at $\zeta_j \in \partial\Omega$ such that $\overline{\Omega} \cap (U_j - \varepsilon\nu_j) \subset \Omega$ for $0 < \varepsilon < \varepsilon_0$ and ν_j suitable directions.

Let $f \in AW^s_{(p,q)}(\Omega; E) \cap C^\infty_{(p,q)}(\Omega; E)$. We denote $f_0^\varepsilon(z) = f(z)$ on U_0 and $f_j^\varepsilon(z) = f(z - \varepsilon\nu_j)$ on $\overline{\Omega} \cap U_j$ and we have $f_j^\varepsilon \in A^\infty_{(p,q)}(\Omega_{\varepsilon'} \cap U_j; E) \cap AW^s_{(p,q)}(\Omega_\varepsilon \cap U_j; E)$ for $0 < \varepsilon' < \varepsilon < \varepsilon_0$.

Let $\{\chi_j\}_{0\leq j\leq N}$ be a C^∞ partition of unity subordinate to the covering $\{U_j\}_{0\leq j\leq N}$.

We put $f_{jk}^\varepsilon = f_j^\varepsilon - f_k^\varepsilon \in AW^s_{(p,q)}(\Omega_\varepsilon \cap U_j \cap U_k; E)$, $F_\varepsilon = \sum_{j=0}^N \chi_j f_j^\varepsilon \in W^s_{(p,q)}(\Omega_\varepsilon \cap U_j; E) \cap A^\infty_{(p,q)}(\Omega_{\varepsilon'}; E)$ and $g_k^\varepsilon = \sum_{j=0}^N \chi_j f_{jk}^\varepsilon \in W^s_{(p,q)}(\Omega_\varepsilon \cap U_k; E)$.

We have $g_k^\varepsilon = \sum_{j=0}^N \chi_j(f_j^\varepsilon - f_k^\varepsilon) = F_\varepsilon - f_k^\varepsilon$ and $\overline{\partial}g_k^\varepsilon = \overline{\partial}F_\varepsilon = \sum_{j=0}^N \overline{\partial}\chi_j \wedge f_{jk}^\varepsilon$ on U_k, so $\overline{\partial}F_\varepsilon \in A^\infty_{(p,q+1)}(\Omega_{\varepsilon'}; E) \cap AW^s_{(p,q+1)}(\Omega_\varepsilon; E)$. As $\left\|f_{jk}^\varepsilon\right\|_{s,U_j\cap U_k} \to 0$ for $\varepsilon \to 0$ and $\overline{\partial}F_\varepsilon = \overline{\partial}g_k^\varepsilon = \sum_{j=0}^N \overline{\partial}\chi_j \wedge f_{jk}^\varepsilon$ on U_k we obtain $\left\|\overline{\partial}F_\varepsilon\right\|_{s,\Omega_\varepsilon} \to 0$ for $\varepsilon \to 0$.

By hypothesis, there exists $u_\varepsilon \in W^s_{(p,q)}(\Omega_{\varepsilon''}; E) \cap C^\infty_{(p,q)}(\Omega_{\varepsilon''}; E)$, $0 < \varepsilon'' < \varepsilon' < \varepsilon_0$, such that $\overline{\partial}u_\varepsilon = \overline{\partial}F_\varepsilon$ and $\|u_\varepsilon\|_{s,\Omega_{\varepsilon''}} \leq C\left\|\overline{\partial}F_\varepsilon\right\|_{s,\Omega_{\varepsilon''}}$, so $\|u_\varepsilon\|_{s,\Omega} \to 0$ for $\varepsilon \to 0$.

We have

$$f - (F_\varepsilon - u_\varepsilon) = \sum_{j=0}^N \chi_j f - \sum_{j=0}^N \chi_j f_j^\varepsilon + u_\varepsilon = \left[\sum_{j=1}^N \chi_j(f - f_j^\varepsilon)\right] + u_\varepsilon.$$

Thus $\|f - h_\varepsilon\|_{s,\Omega} \to 0$ for $\varepsilon \to 0$ with $h_\varepsilon = F_\varepsilon - u_\varepsilon \in A^\infty_{(p,q)}(\Omega; E)$. □

LEMMA 8.4. *Let $\Omega_- \subset \mathbb{CP}_n$ be a pseudoconcave domain with Lipschitz (respectively C^2) boundary in $\mathbb{CP}_n$ and $f \in AW^k_{(0,q)}(\Omega_-; \mathcal{O}(m))$, $1 \leq q \leq n-1$, $m \in \mathbb{Z}$, $k \in \mathbb{N}^*$, $2(k-1)\mathcal{K}_n - m > 0$, (respectively $k \geq 1$ and $m = 0$). Then there exist h belonging to the closure of $A^\infty_{(0,q)}(\Omega_-; \mathcal{O}(m))$ in $AW^k_{(0,q)}(\Omega_-; \mathcal{O}(m))$ and $g \in AW^{k+1}_{(0,q-1)}(\Omega_-; \mathcal{O}(m))$, $\|g\|_{k+1} \leq C_k\|f\|_k$ such that $f = \overline{\partial}g + h$.*

Proof. Since $\Omega = \mathbb{CP}_n \backslash \overline{\Omega_-}$ is pseudoconvex, by [39] it follows that the domains $\Omega_-^\varepsilon = \mathbb{CP}_n \backslash \{z \in \Omega | \delta(z) \geq \varepsilon\}$, $0 < \varepsilon < \varepsilon_0$, form a strongly pseudoconcave neighborhood system of Ω_-. We consider an extension $F \in AW^k_{(0,q)}(\Omega_-^{\varepsilon_0}; \mathcal{O}(m))$ of f such that $supp\, F \subset \Omega_-^{\varepsilon_0}$ and $\|F\|_{k,\Omega_-^{\varepsilon_0}} \leq C_k\|f\|_{k,\Omega_-}$. From the Bochner-Martinelli-Koppelman formula 8.1 we obtain

$$F = T_{q+1}\left(\overline{\partial}F\right) + \overline{\partial}\left(T_qF\right)$$

with $\widetilde{T_qF}(z) = \int_{t\in\widetilde{\Omega_-^{\varepsilon_0}}} \widetilde{F}(t) \wedge K_{q-1}(t,z)$, (see (8.3)).

Let $g = T_q F$ and $h = T_{q+1}\left(\overline{\partial} F\right)$. By Remark 8.2 it follows that $g \in W^{k+1}_{(0,q)}(\Omega^{\varepsilon}_{-};\mathcal{O}(m))$, $\|g\|_{k+1,\Omega^{\varepsilon}_{-}} \leq \|F\|_{k,\Omega^{\varepsilon}_{-}} \leq C_k \|f\|_{k,\Omega_-}$ and $h \in AW^{k}_{(0,q)}(\Omega_-;\mathcal{O}(m))$ for $0 < \varepsilon < \varepsilon_0$.

To finish the proof, it is enough to show that $h = \lim\limits_{\varepsilon \to 0} h_\varepsilon$, with $h_\varepsilon \in A^{\infty}_{(0,q)}(\Omega_-;\mathcal{O}(m))$.

Since $\overline{\partial} F = 0$ on Ω_-, $\widetilde{T_{q+1}\,\overline{\partial} F}\,(z) = \int_{t \in \widetilde{\Omega^{\varepsilon_0}_{-} \setminus \Omega_-}} \widetilde{\overline{\partial} F}(t) \wedge K_{n-1}(t,z)$ so $h \in C^{\infty}_{(0,q)}(\Omega_-;\mathcal{O}(m))$. According to Lemma 8.3, it is enough to verify that for every $0 < \varepsilon < \varepsilon_0$ and every $\overline{\partial}$-exact form $\Phi = \overline{\partial}\psi$ with $\psi \in A^{\infty}_{(0,q)}\left(\Omega^{\varepsilon}_{-};\mathcal{O}(m)\right)$, there exists $\varphi \in W^{k}_{(0,q)}\left(\Omega^{\varepsilon}_{-};\mathcal{O}(m)\right) \cap C^{\infty}_{(0,q)}\left(\Omega^{\varepsilon}_{-};\mathcal{O}(m)\right)$ such that $\overline{\partial}\varphi = \Phi$ and $\|\varphi\|_{k,\Omega^{\varepsilon}_{-}} \leq C\,\|\Phi\|_{k,\Omega^{\varepsilon}_{-}}$ with C independent of f and ε. Since every $\overline{\partial}$-exact form $\Phi \in A^{\infty}_{(0,q+1)}\left(\Omega^{\varepsilon}_{-};\mathcal{O}(m)\right)$ verifies the moment condition of any order, this follows from Corollary 7.5. □

LEMMA 8.5. *Let $\Omega_- \subset \mathbb{CP}_n$ be a pseudoconcave domain with Lipschitz boundary in $\mathbb{CP}_n$ and h a form belonging to the closure of $A^{\infty}_{(0,q)}(\Omega_-;\mathcal{O}(m))$ in $AW^{k}_{(0,q)}(\Omega_-;\mathcal{O}(m))$, $k \in \mathbb{N}^*$. Then there exists an extension $\widetilde{h} \in W^{k}_{(0,q)}(\mathbb{CP}_n;\mathcal{O}(m))$ of h such that $\overline{\partial}\widetilde{h}_{|\Omega} \in L^2_{(0,q+1)}\left(\Omega;\delta^{-k+1};\mathcal{O}(m)\right)$ and $N_{-k+1,\Omega}\left(\overline{\partial}\widetilde{h}\right) \leq C_k \left\|\overline{\partial}\widetilde{h}\right\|_{k-1,\Omega}$, where $\Omega = \mathbb{CP}_n \setminus \overline{\Omega_-}$.*

Proof. Let $h_j \in A^{\infty}_{(0,q)}(\Omega_-)$ such that $\|h - h_j\|_{k,\Omega_-} \to 0$. We denote by $\widetilde{h_j}$ a C^∞ extension of h_j to $\mathbb{CP}_n$ such that $\left\|\widetilde{h_j}\right\|_{i,\mathbb{CP}_n} \leq C_i \|h_j\|_{i,\Omega_-}$ for every non-negative integer $i \leq k$ and every j. In particular, $\left(\widetilde{h_j}\right)_j$ is bounded in $W^{k}_{(0,q)}(\mathbb{CP}_n)$, so there exists a weakly convergent subsequence to $\widetilde{h} \in W^{k}_{(0,q)}(\mathbb{CP}_n)$ and $\widetilde{h}_{|\Omega_-} = h$. Since $\overline{\partial}\widetilde{h_j}$ vanishes to infinite order on $\partial\Omega$, by (7.1) we have $N_{-k+1,\Omega}(\overline{\partial}\widetilde{h_j}) \leq C_k \left\|\overline{\partial}\widetilde{h_j}\right\|_{k-1,\Omega}$ and $\left(\overline{\partial}\widetilde{h_j}\right)_j$ has also a weakly convergent subsequence in $L^2_{(0,q+1)}\left(\Omega;\delta^{-k+1}\right)$. So $\overline{\partial}\widetilde{h}_{|\Omega} \in L^2_{(0,q+1)}\left(\Omega;\delta^{-k+1}\right)$ and $N_{-k+1,\Omega}\left(\overline{\partial}\widetilde{h}\right) \leq C_k \left\|\overline{\partial}\widetilde{h}\right\|_{k-1,\Omega}$. □

DEFINITION 8.6. *Let $\Omega_- \subset \mathbb{CP}_n$ be a pseudoconcave domain in $\mathbb{CP}_n$ and , $k \in \mathbb{N}^*$. A form $f \in AW^{k}_{(0,n-1)}(\Omega_-;\mathcal{O}(m))$ verifies the moment condition of order k if there exists an extension $\widetilde{f} \in W^{k}_{(0,n-1)}(\mathbb{CP}_n;\mathcal{O}(m))$ of f such that $\overline{\partial}\widetilde{f}_{|\Omega} \in L^2\left(\Omega;\delta^{-k+1};\mathcal{O}(m)\right)$ and verifies the moment condition of order $k-1$ on Ω, where $\Omega = \mathbb{CP}_n \setminus \overline{\Omega_-}$.*

THEOREM 8.7. *Let $\Omega_- \subset \mathbb{CP}_n$ be a pseudoconcave domain with Lipschitz (respectively C^2) boundary in $\mathbb{CP}_n$. Let $k \in \mathbb{N}^*, m \in \mathbb{Z}$, $2(k-1)\mathcal{K}_n - m > 0$, (respectively $k \geq 1$ and $m = 0$).*

a) Suppose that $q < n-1$. Then for every $f \in AW^{k}_{(0,q)}(\Omega_-;\mathcal{O}(m))$ there exists $u \in W^{k}_{(0,q-1)}(\Omega_-;\mathcal{O}(m))$ such that $\overline{\partial}u = f$ and $\|u\|_k \leq C_k \|f\|_k$, where C_k is a constant independent of f.

b) If $q = n-1$ the same statement is true for forms $f \in W^{k}_{(0,n-1)}(\Omega_-;\mathcal{O}(m))$ verifying the moment condition of order k.

Proof. By Lemma 8.4, there exist h belonging to the closure of $A^{\infty}_{(0,q)}(\Omega_-;\mathcal{O}(m))$ in $AW^{k}_{(0,q)}(\Omega_-;\mathcal{O}(m))$ and $g \in AW^{k+1}_{(0,q-1)}(\Omega_-;\mathcal{O}(m))$, $\|g\|_{k+1} \leq C_k \|f\|_k$ such that $f = \overline{\partial}g + h$. So it is enough to solve the $\overline{\partial}$-equation for h.

According to Lemma 8.5 there exists an extension $\widetilde{h} \in W^k_{(0,q)}(\mathbb{CP}_n; \mathcal{O}(m))$ of h such that $\overline{\partial}\widetilde{h}_{|\Omega} \in L^2_{(0,q+1)}\left(\Omega; \delta^{-k+1}; \mathcal{O}(m)\right)$ and $N_{-k+1,\Omega}\left(\overline{\partial}\widetilde{h}\right) \leq C_k \left\|\overline{\partial}\widetilde{h}\right\|_{k-1,\Omega}$, where $\Omega = \mathbb{CP}_n \backslash \overline{\Omega_-}$. From now on the proof of the Theorem 7.1 works without any change to find $u_1 \in W^k_{(0,q-1)}(\Omega_-; \mathcal{O}(m))$ such that $\overline{\partial}u_1 = h$ and $\|u_1\|_k \leq C_k \|h\|_k$. □

A consequence of Theorem 8.7 is the following:

COROLLARY 8.8. *Let $\Omega_- \subset \mathbb{CP}_n$ be a pseudoconcave domain with Lipschitz (respectively C^2) boundary in $\mathbb{CP}_n$. Then $H^{0,q}_{W^k}(\Omega_-) = 0$ for every integer $k \geq 2$ (respectively $k \geq 1$) and $0 < q < n-1$.*

9. The dual of the weighted Bergman space on pseudoconvex domain of $\mathbb{CP}_n$. Let Ω be a domain with Lipschitz boundary in a complex n-dimensional manifold X, $\Omega_- = X \backslash \overline{\Omega}$ and $k \in \mathbb{N}$. In this paragraph we use the following notations:

$$\mathcal{B}_k(\Omega) = \mathcal{O}(\Omega) \cap L^2\left(\Omega; \delta^k\right)$$

endowed with the norm N_k induced by $L^2\left(\Omega; \delta^k\right)$;

$$\left(\mathcal{B}_k(\Omega)\right)'_{\#} = \left\{\Phi \in \left(\mathcal{B}_k(\Omega)\right)' \mid \langle \Phi, 1\rangle = 0\right\}$$

where $\left(\mathcal{B}_k(\Omega)\right)'$ is the dual of $\mathcal{B}_k(\Omega)$.

Let $[\varphi] \in H^{n,n-1}_{k+1}(\Omega_-)$ be the cohomology class of the $\overline{\partial}$-closed form $\varphi \in W^{k+1}_{(n,n-1)}(\Omega_-)$. Since every $h \in \mathcal{B}_k(\Omega)$ is harmonic, it admits a boundary value $bv(h) \in W^{-k-1/2}(\partial\Omega)$. So we can define $T : H^{n,n-1}_{W^{k+1}}(\Omega_-) \to \left(\mathcal{B}_k(\Omega)\right)'_{\#}$ by

$$\langle T[\varphi], h\rangle = \int_{\partial\Omega} \langle bv(h), bv(\varphi)\rangle$$

where $\langle\ ,\ \rangle$ is the duality between $W^{-k-1/2}(\partial\Omega)$ and $W^{k+1/2}(\partial\Omega)$.

Indeed, if $[\varphi] = 0$, ie. $\varphi = \overline{\partial}\psi$ with $\psi \in W^{k+1}_{(n,n-2)}(\Omega_-)$, by Stokes' formula we obtain $T[\varphi] = 0$.

Since

$$|\langle T[\varphi], h\rangle| \leq \|bv(h)\|_{-k-1/2,\partial\Omega} \|bv(\varphi)\|_{k+1/2,\partial\Omega} \leq C_\varphi \|bv(h)\|_{-k,\Omega} \leq C_{k,\varphi} N_{k,\Omega}(h)$$

it follows that $T[\varphi]$ is a continuous functional on $\mathcal{O}_k(\Omega)$.

Because φ is $\overline{\partial}$-closed, by Stokes' formula we have

$$\langle T[\varphi], 1\rangle = \int_{\partial\Omega} bv(\varphi) = \int_{\Omega} \overline{\partial}\varphi = 0$$

and we conclude that $T[\varphi] \in \left(\mathcal{B}_k(\Omega)\right)'_{\#}$ and T is well defined.

THEOREM 9.1. *Let $\Omega_- \subset \mathbb{CP}_n$ be a pseudoconcave domain with Lipschitz (respectively C^2) boundary, $\Omega = \mathbb{CP}_n \backslash \overline{\Omega_-}$ and $k \in \mathbb{N}^*$ (respectively $k \in \mathbb{N}$). Then $T : H^{n,n-1}_{W^{k+1}}(\Omega_-) \to \left(\mathcal{B}_k(\Omega)\right)'_{\#}$ is an isomorphism. In particular $\dim_{\mathbb{C}} H^{n,n-1}_{W^{k+1}}(\Omega_-) = \infty$.*

Proof. Suppose that $T[\varphi] = 0$. Lemma 8.4 implies that $[\varphi] = [\psi]$ with ψ belonging to the closure of $A^\infty_{(n,n-1)}(\Omega_-)$ in $W^{k+1}_{(n,n-1)}(\Omega_-)$. According to Lemma 8.5 there exists an extension $\widetilde{\psi} \in W^{k+1}_{(n,n-1)}(\mathbb{CP}_n)$ such that $\overline{\partial}\widetilde{\psi} \in L^2_{(n,n)}\left(\Omega;\delta^{-k}\right)$. Therefore, for every $h \in \mathcal{B}_k(\Omega)$ we have

$$\int_\Omega h\overline{\partial}\widetilde{\psi} = \int_{\partial\Omega} \langle bv(h), bv(\varphi)\rangle = T[\varphi](h) = 0$$

and it follows that ψ verifies the moment condition of order k. By Theorem 8.7 there exists $u \in W^{k+1}_{(n,n-1)}(\Omega_-)$ such that $\overline{\partial}u = \psi$, so $[\varphi] = 0$ and T is injective.

Let $\Phi \in (\mathcal{B}_k(\Omega))'_{\#}$. By the theorem of Hahn-Banach there exists an extension $\widetilde{\Phi} \in \left(L^2\left(\Omega;\delta^k\right)\right)' = L^2_{(n,n)}\left(\Omega;\delta^{-k}\right)$ of Φ, so there exists $\Psi \in L^2_{(n,n)}\left(\Omega;\delta^{-k}\right)$ such that $\langle\Phi, h\rangle = \int_\Omega h\Psi$ for every $h \in \mathcal{B}_k(\Omega)$. Let $\widetilde{\Psi}$ be the trivial extension of Ψ to $\mathbb{CP}_n$. Since

$$\int_{\mathbb{CP}_n} \left\langle \widetilde{\Psi}, *1\right\rangle dV = \int_{\mathbb{CP}_n} \widetilde{\Psi} = \int_\Omega \Psi = \langle\Phi, 1\rangle = 0$$

it follows that $\widetilde{\Psi}$ is orthogonal to the harmonic space $\mathcal{H}_{(n,n)}(\mathbb{CP}_n)$. Therefore, by Hodge's theorem $\widetilde{\Psi} = \overline{\partial}\widetilde{\varphi}$ with $\widetilde{\varphi} = \overline{\partial}^* G\widetilde{\Psi}$ and G the Green operator. From Lemma 6.5 we conclude that $\widetilde{\varphi} \in W^{k+1}_{(n,n-1)}(\mathbb{CP}_n)$ and let $\varphi = \widetilde{\varphi}\left|_{\Omega_-}\right.$. Since $\overline{\partial}\widetilde{\varphi} = \widetilde{\Psi}$ and $\widetilde{\Psi}$ vanishes on Ω_-, φ defines a cohomology class $[\varphi] \in H^{n,n-1}_{W^{k+1}}(\Omega_-)$ such that

$$\langle T[\varphi], h\rangle = \int_{\partial\Omega} \langle bv(h), bv(\varphi)\rangle = \int_\Omega h\overline{\partial}\widetilde{\varphi} = \int_\Omega h\Psi = \langle\Phi, h\rangle$$

for every $h \in \mathcal{B}_k(\Omega)$. This shows the surjectivity of T and completes the proof. □

10. Liouville's theorem on pseudoconcave compacts of $\mathbb{CP}_n$. The following Proposition is a generalisation of Liouville's theorem for pseudoconcave compacts in $\mathbb{CP}_n$:

PROPOSITION 10.1. *Let L be a pseudoconcave compact in $\mathbb{CP}_n$, $n \geq 2$. Then:*
i) $A^\infty_{(n,0)}(L;\mathcal{O}(m)) = 0$ for $m \leq n$;
ii) $A^\infty_{(n,0)}(L;\mathcal{O}(m))$ is isomorphic to the space of $m-n-1$-homogeneous complex polynomials in $n+1$ variables for $m \geq n+1$.

Proof. Let $k \in \mathbb{N}$ such that $2k\mathcal{K}_n + n + 1 - m > 0$ (we can choose $k = 0$ if $m \leq n$) and $f \in A^\infty_{(n,0)}(L;\mathcal{O}(m))$. We consider an extension $\widetilde{f} \in C^{k+3}_{(n,0)}(\mathbb{CP}_n;\mathcal{O}(m))$ of f and we have $\overline{\partial}\widetilde{f} \in L^2_{(n,1)}(\Omega;\delta^{-k-1};\mathcal{O}(m))$, where $\Omega = \mathbb{CP}_n \backslash L$. By Corollary 5.6 and Corollary 5.4 b), there exists $v \in L^2_{(n,0)}(\Omega;\mathcal{O}(m))$ such that $\overline{\partial}v = \overline{\partial}\widetilde{f}$ and $bv(v) = 0$. Let $F \in L^2_{(n,0)}(\mathbb{CP}_n;\mathcal{O}(m))$ defined by $F = f$ on Ω_- and $F = \widetilde{f} - v$ on Ω. By Lemma 6.4 it follows that $\overline{\partial}F = 0$ on $\mathbb{CP}_n$, so F is a holomorphic $(n,0)$-form on $\mathbb{CP}_n$ with values in $\mathcal{O}(m)$. Since the holomorphic $(n,0)$-forms are holomorphic sections of $\mathcal{O}(m-n-1)$, Proposition 10.1 follows. □

An immediate consequence of the case $m = n+1$ of Proposition 10.1 is the following:

COROLLARY 10.2. *A pseudoconcave compact in $\mathbb{CP}_n$ is connected.*

PROPOSITION 10.3. *Let Ω_- be a domain with Lipschitz boundary of a compact Kähler n-dimensional manifold X such that $\overline{\Omega_-}$ is Hartogs-pseudoconcave and E a* *holomorphic hermitian vector bundle on X. Let $0 \leq p \leq n$, $0 \leq q \leq n-1$ such that $H^{p,q+1}(X;E) = 0$ and $k \geq 1$ an integer such that $m_{n-p}(\Omega;E^*) + 2(k-1)C_\Omega > 0$, where $\Omega = X\backslash\overline{\Omega_-}$. Then $A^\infty_{(p,q)}(\Omega_-;E)$ is dense in $AW^k_{(p,q)}(\Omega_-;E)$.*

Proof. Since $\Omega = \mathbb{CP}_n\backslash\overline{\Omega_-}$ is pseudoconvex, by [39] it follows that the domains $\Omega^\varepsilon_- = \mathbb{CP}_n\backslash\{z \in \Omega | d(z,\Omega_-) \geq \varepsilon\}$ form a strongly pseudoconcave neighborhood system of Ω_-. According to Theorem 7.1, for every $\overline{\partial}$-exact form $\Phi = \overline{\partial}\psi$ with $\psi \in A^\infty_{(p,q)}(\Omega^\varepsilon_-;E)$, there exists $0 < \varepsilon' < \varepsilon$ and $\varphi \in W^s_{(p,q)}\left(\Omega^{\varepsilon'}_-;E\right) \cap C^\infty_{(p,q)}\left(\Omega^{\varepsilon'}_-;E\right)$ such that $\overline{\partial}\varphi = \Phi$ and $\|\varphi\|_{s,\Omega_{\varepsilon'}} \leq C\|\Phi\|_{s,\Omega_{\varepsilon'}}$ with C independent of Φ and ε (because Φ verifies the moment condition of any order). So we obtain the result by applying Lemma 8.3. □

COROLLARY 10.4. *Let Ω_- be a pseudoconcave domain with Lipschitz boundary in $\mathbb{CP}_n$ and $m \in \mathbb{Z}$, $m \leq n$. Then $A^\infty_{(n,0)}(\Omega_-;\mathcal{O}(m))$ is dense in $AW^k_{(n,0)}(\Omega_-;\mathcal{O}(m))$, $k \geq 1$. If Ω_- has C^2 boundary we have the same result for $m = n+1$.*

Proof. If Ω_- has Lipschitz boundary, Corollary 10.4 follows directly by Proposition 10.3 by identifying the holomorphic $\mathcal{O}(m)$-valued $(n,0)$-forms with holomorphic sections of $\mathcal{O}(m-n-1)$. Under the hypothesis of C^2 boundary, we conclude similarly by using Theorem 8.7. □

From Proposition 10.1 and Corollary 10.4 we obtain the following form of Liouville's theorem:

THEOREM 10.5. *Let Ω_- be a pseudoconcave domain with Lipschitz boundary in $\mathbb{CP}_n$. Then $H^{0,0}(\Omega_-;\mathcal{O}(m)) \cap W^1(\Omega_-;\mathcal{O}(m)) = 0$ for $m \leq -1$ and $H^{0,0}(\Omega_-) \cap W^2(\Omega_-) = \mathbb{C}$. If Ω_- has C^2 boundary we have also $H^{0,0}(\Omega_-) \cap W^1(\Omega_-) = \mathbb{C}$.*

11. Hartogs-Bochner theorem in $\mathbb{CP}_n$.

THEOREM 11.1. *Let Ω be a domain with Lipschitz boundary in $\mathbb{CP}_n$ such that $\Omega_- = \mathbb{CP}_n\backslash\overline{\Omega}$ is connected and contains a pseudoconcave domain with C^2 boundary and $f \in CR(\partial\Omega) \cap W^{\frac{1}{2}}(\partial\Omega)$. Then there exists $\widetilde{f} \in \mathcal{O}(\Omega) \cap W^1(\Omega)$ such that $bv(\widetilde{f}) = f$.*

Proof. We consider the current T_f of bidegree $(0,1)$ on $\mathbb{CP}_n$ given by $\langle T_f, \varphi\rangle = \int_{\partial\Omega} f\varphi$ for every $\varphi \in C^\infty_{(n,n-1)}(\mathbb{CP}_n)$. Since $f \in CR(\partial\Omega)$ we have $\overline{\partial}T_f = 0$, so $T_f = \overline{\partial}KT_f$, where $K \in \mathcal{D}'_{(n,n-1)}(\mathbb{CP}_n \times \mathbb{CP}_n)$ is the fundamental solution for $\overline{\partial}$ given by the formula of Bochner-Martinelli-Koppelmann type in $\mathbb{CP}_n$ [20].

We denote $F(z) = KT_f(z)$ for $z \in \mathbb{CP}_n\backslash\partial\Omega$ and we have

$$F(z) = -\int_{\partial\Omega} K(\xi,z)f(\xi)d\xi.$$

Since $suppT_f \subset \partial\Omega$, F is holomorphic on $\mathbb{CP}_n\backslash\partial\Omega$. We denote F_+ the restriction of F to Ω and F_- the restriction of F to Ω_- and since $f \in W^{\frac{1}{2}}(\partial\Omega)$, by Lemma 8.1 it follows that $F_+ \in H^{0,0}(\Omega) \cap W^1(\Omega)$, $F_- \in H^{0,0}(\Omega_-) \cap W^1(\Omega_-)$, so from Theorem 10.5 it follows that $F_- \equiv C \equiv$ *constant* on Ω_-.

For every $\varphi \in C^\infty_{(n,n-1)}(\mathbb{CP}_n)$ we have

$$\begin{aligned}\langle T_f, \varphi\rangle &= \int_{\partial\Omega} f\varphi = \langle\overline{\partial}KT_f, \varphi\rangle = -\int_{\mathbb{CP}_n} KT_f(z)\overline{\partial}\varphi(z) \\ &= -\int_\Omega F_+\overline{\partial}\varphi - \int_{\Omega_-} F_-\overline{\partial}\varphi = \int_{\partial\Omega} bv(-F_+ + C)\varphi\end{aligned}$$

thus $f = bv(-F_+ + C)$ with $\widetilde{f} = -F_+ + C \in \mathcal{O}(\Omega) \cap W^1(\Omega)$. □

REMARK 11.2. *Both hypothesis in Theorem 11.1 are essential. Indeed, if* $\Omega = \mathbb{CP}_n \backslash \overline{D}$, *where* D *is a bounded domain in* $\mathbb{C}^n$, *than the restriction to* $\partial\Omega$ *of a non-constant entire function cannot extend holomorphically to* Ω; *if* $\mathbb{CP}_n \backslash \Omega$ *has two connected components, a* CR *function which takes different constants as values on the boundaries of these components cannot extend holomorphically to* Ω.

If Ω is a bounded domain in $\mathbb{C}^n$ such that $\partial\Omega$ is connected, Theorem 11.1 is equivalent to the generalization of the classical Hartogs-Bochner theorem proved by Fichera [11]. For smooth CR functions (ie. for functions f on $\partial\Omega$ which admit a C^∞ extension $\widetilde{f}$ in a neighborhood of $\partial\Omega$ such that $\overline{\partial}\widetilde{f}$ vanishes to infinite order on $\partial\Omega$), we can use directly Proposition 10.1 to obtain in a similar way the following:

THEOREM 11.3. *Let* Ω *be a domain in* $\mathbb{CP}_n$ *such that* $vol_{2n}(\partial\Omega) = 0$, $\partial\Omega = \partial\left(\mathbb{CP}_n \backslash \overline{\Omega}\right)$, $\mathbb{CP}_n \backslash \Omega$ *is connected and contains a pseudoconcave compact with non-empty interior. Then for every* $f \in CR(\partial\Omega) \cap C^\infty(\partial\Omega)$ *there exists* $\widetilde{f} \in \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$ *such that* $\widetilde{f}\,|_{\partial\Omega} = f$.

Proof. As in the proof of Theorem 11.1 (by transforming the integrals on the boundaryin integrals over the domains) we have $f = F_{+|\partial\Omega} - F_{-|\partial\Omega}$, where $F_+ \in \mathcal{O}(\Omega) \cap C^\infty\left(\overline{\Omega}\right)$ and $F_- \in \mathcal{O}(\Omega_-) \cap C^\infty\left(\overline{\Omega_-}\right)$ with $\Omega_- = \mathbb{CP}_n \backslash \overline{\Omega}$.

Let $L \subset \overline{\Omega_-}$ be a pseudoconcave compact such that $\overset{\circ}{L} \neq \emptyset$. Then $F_- \in A^\infty(L)$ and by Proposition 10.1, $F_- \equiv$ *constant* on L. Since $\overset{\circ}{L} \subset \Omega_-$, it follows that $F_- \equiv$ *constant* on Ω_- and Theorem 11.3 is proved. □

REMARK 11.4. *If* Ω *is a bounded domain in* $\mathbb{C}^n$ *such that* $\mathbb{C}^n \backslash \Omega$ *is connected, the classical theorem of Hartogs-Bochner is a particular case of Theorem 11.3 because* $\mathbb{CP}_n \backslash \Omega$ *contains the pseudoconcave compact* $\mathbb{CP}_n \backslash B$, *where* B *is a suitable ball of* $\mathbb{C}^n$.

REMARK 11.5. *All the results are valid for domains* Ω *in algebraic manifolds* $X \subset \mathbb{CP}_n$ *such that* $\Omega = \widetilde{\Omega} \cap X$, *with* $\widetilde{\Omega}$ *pseudoconvex domain in* $\mathbb{CP}_n$ *and* $\partial\widetilde{\Omega}$ *transverse to* X *(the condition* $\Omega = \widetilde{\Omega} \cap X$ *is not valid in general, see for ex. [12]).*

12. A (counter)example. In this paragraph we consider the domain

$$\Omega = \{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid |z_1| < |z_0|\} \tag{12.1}$$

where (z_0, z_1, z_2) are homogeneous coordinates in $\mathbb{CP}_2$. One can compare this domain with the well-known Hartogs triangle $\left\{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\right\}$. We study below some of the properties of the domain Ω in connection with our results:

EXAMPLE 12.1. *The domain* Ω *defined in (12.1) has the following properties:*

1. Ω *is simultaneously pseudoconvex and pseudoconcave;*
2. *There do not exist any Stein neighborhood of* $\overline{\Omega}$;
3. *There exist* $\overline{\partial}$*-closed forms* $g \in C^\infty_{(0,1)}\left(\overline{\Omega}; \mathcal{O}(m)\right)$ *such that the equation* $\overline{\partial}u = f$ *has no solution* $u \in C\left(\overline{\Omega}; \mathcal{O}(m)\right)$ *for any* $m \in \mathbb{Z}$;
4. *The Bergman space* $L^2(\Omega) \cap \mathcal{O}(\Omega)$ *separates the points of* Ω;
5. *There exist no non-constant holomorphic functions in* $W^1(\Omega)$;
6. *The Hartogs-Bochner theorem for* $CR(\partial\Omega) \cap W^{-1/2}(\partial\Omega)$ *is not valid.*

Proof.

1. We denote $U_j = \{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid z_j \neq 0\}$. Since $\Omega \subset U_0$, in local coordinates Ω is the product of the unit disc and the complex plane, so it is pseudoconvex. But $\mathbb{CP}_2 \backslash \Omega = \{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid |z_1| > |z_0|\}$ and by the same arguments as before in U_1, it follows that $\mathbb{CP}_2 \backslash \Omega$ is also pseudoconvex, so Ω is pseudoconcave.
2. The boundary $\partial\Omega = \{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid |z_1| = |z_0|\}$ is smooth exceptly at $[0,0,1]$. It contains the compact Riemann surfaces
$$S_\theta = \left\{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid z_1 = e^{i\theta} z_0\right\},\ \theta \in \mathbb{R}.$$
This means that $\overline{\Omega}$ has no any Stein neighborhood.
3. Let a, b distinct points of Ω such that the complex projective line $\mathcal{L}$ through a and b has a nonempty intersection with $\mathbb{CP}_2 \backslash \overline{\Omega}$. Let Ω' be an open neighborhood of $\overline{\Omega}$ such that $\mathcal{L} \cap \left(\mathbb{CP}_2 \backslash \overline{\Omega'}\right) \neq \emptyset$. By [17] there exists a Stein neighborhood V of $\mathcal{L} \cap \Omega'$.
Suppose $m \leq -1$. Since V is Stein, there exists $h \in H^{0,0}(V; \mathcal{O}(m+1))$ such that $h(a) \neq h(b)$.
Let χ be a C^∞ function on $\mathbb{CP}_2$ with support contained in Ω' such that $\chi \equiv 1$ on V and let L a linear form such that $\mathcal{L} = \{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid L(z) = 0\}$. Then $\frac{h\overline{\partial}\chi}{L}$ defines a form $g \in C^\infty_{(0,1)}\left(\overline{\Omega}; \mathcal{O}(m)\right)$. Suppose that there exists $u \in C\left(\overline{\Omega}; \mathcal{O}(m)\right)$ such that $\overline{\partial} u = g$. Then the function $\chi h - Lu$ defines a section $f \in \mathcal{O}(\Omega; \mathcal{O}(m+1)) \cap C\left(\overline{\Omega}; \mathcal{O}(m+1)\right)$ such that $f = h$ on $\mathcal{L}$. But $\partial\Omega$ is foliated by compact Riemann surfaces which have a common point, so every holomorphic section of $\mathcal{O}(m+1)$ is constant on $\partial\Omega$ and this gives a contradiction.
Suppose $m \geq 0$. We choose homogeneous coordinates $z = (z_0, z_1, z_2)$ for a point $[z] \in \mathbb{CP}_2$ such that $\mathcal{L} = \{[z] \,|\, z_0 = 0\}$. Let $h \in H^{0,0}(V; \mathcal{O}(m+1))$ such that $\frac{\partial h}{\partial z_1^{m+1}}$ is non-constant on $\mathcal{L}$.
Let χ be a C^∞ function on $\mathbb{CP}_2$ with support contained in Ω' such that $\chi \equiv 1$ on V. Then $\frac{h\overline{\partial}\chi}{z_0}$ defines a form $g \in C^\infty_{(0,1)}\left(\overline{\Omega}; \mathcal{O}(m)\right)$. Suppose that there exists $u \in C\left(\overline{\Omega}; \mathcal{O}(m)\right)$ such that $\overline{\partial} u = g$. Then the $(m+1)$-homogeneous function $\varphi = h\chi - z_0 u$ defines a section $f \in \mathcal{O}(\Omega; \mathcal{O}(m+1)) \cap C\left(\overline{\Omega}; \mathcal{O}(m+1)\right)$ such that $\frac{\partial\varphi}{\partial z_1^{m+1}} = \frac{\partial h}{\partial z_1^{m+1}}$ on $\mathcal{L}$. Since $\partial\Omega$ is foliated by compact Riemann surfaces which have a common point, $\frac{\partial\varphi}{\partial z_1^{m+1}}$ is constant fonction on each leaf, so constant on Ω, and we obtain a contradiction.
4. This property follows from Proposition 4.1.
5. Let $f \in \mathcal{O}(\Omega) \cap W^1(\Omega)$. We consider the complex projective lines
$$\Delta_\lambda = \{[z_0, z_1, z_2] \in \mathbb{CP}_2 \mid z_1 = \lambda z_0\},\ \lambda \in \mathbb{C},\ |\lambda| < 1$$
and we have $\Delta_\lambda \backslash \{[0,0,1]\} \subset \Omega$.
Let $\zeta = (\zeta_0, \zeta_1)$ be local coordinates in U_2 and $B_r = \{\zeta \mid |\zeta| \leq r\}$ a ball centered at the origin. For $|\lambda| < 1$ we denote $\varphi_\lambda : \mathbb{C}^* \to \mathbb{C}$ the function defined by $\varphi_\lambda(z) = f(\zeta_0, \zeta_1)$ with $\zeta_0 = z$ and $\zeta_1 = \lambda z$.
Since
$$\int_{B_r \cap \Omega} |f|^2 \, d\zeta_0 d\zeta_1 d\overline{\zeta}_0 d\overline{\zeta}_1 = \int_{|\lambda|<1} d\lambda d\overline{\lambda} \int_{D_r} |z|^2 \, |f(z, \lambda z)|^2 \, dz d\overline{z}$$

with $D_r = \{z \in \mathbb{C} | \, |z| \leq r\}$, the function $z\varphi_\lambda \in L^2(D_r) \cap \mathcal{O}(D_r \setminus \{0\})$ for almost all $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and it follows that $\varphi_\lambda(z) = O\left(\frac{1}{|z|}\right)$ and $\varphi'_\lambda(z) = O\left(\frac{1}{|z|^2}\right)$. Because $f \in W^1(\Omega)$, we obtain as before that $z\varphi'_\lambda \in L^2(D_r)$. If φ_λ has a first order pole at the origin, then $z\varphi'_\lambda(z) = O\left(\frac{1}{|z|}\right)$ and $z\varphi'_\lambda \notin L^2(D_r)$; so the origin is a regular point for φ_λ. It follows that f is constant on Δ_λ for almost all $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and therefore constant on Ω.

6. By Proposition 4.1 there exist non-constant holomorphic L^2 functions on Ω; their boundary values belong to $CR(\partial\Omega) \cap W^{-1/2}(\partial\Omega)$ and cannot extend to holomorphic functions of $L^2(\mathbb{CP}_2 \setminus \Omega)$. □

REFERENCES

[1] A.Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Publ. Math. IHES, 24-25 (1965), pp. 81–150.

[2] A. Andreotti, *Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves*, Bull. Soc. Math. France, 91 (1963), pp. 1–38.

[3] B. Berndtsson and P. Charpentier, *A Sobolev mapping property of the Bergman kernel*, to appear in Math. Z.

[4] S. Bochner, *Analytic and meromorphic continuation by means of Green's formula*, Ann. of Math., 44 (1943), pp. 652–673.

[5] J.-P. Demailly, *Analytic Geometry*, Springer-Verlag, to appear.

[6] ———, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe sémi-positif*, Ann. Scient. Ec. Norm. Sup., 15 (1982), pp. 457–511.

[7] K. Diederich and T. Ohsawa, *On pseudoconvex domains in P_n*, Tokyo J. of Math., 21 (1998), pp. 353–358.

[8] P. Dingoyan, *Un phénommène de Hartogs dans les variétés projectives*, Math. Z., 232 (1999), pp. 217–240.

[9] G. Elencwajc, *Pseudo-convexité locale dans les variétés kähleriennes*, Ann. Inst. Fourier, 25 (1975), pp. 295–314.

[10] B. Fabre, *Sur l'intersection d'une surface de Riemann avec des hypersurfaces algébriques*, C. R. Acad. Sc. Paris, 322 (1996), pp. 371–376.

[11] G. Fichera, *Caratterizzazione della tracia, sulla frontiera di un campo, di una funzione analitica di piu variabili complesse*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Nat. Natur., 22 (1957), pp. 706–715.

[12] H. Grauert, *Bemerkenswerte pseudoconvexe Mannigfaltigkeiten*, Math. Z., 81 (1963), pp. 377–391.

[13] H. Grauert and O. Riemenschneider, *Kählersche Mannigfältigkeiten mit hyper-q-konvexen Rand*, in Problems in Analysis (A Symposium in Honor of S. Bochner), A Smposium in Honor of S. Bochner, Princeton, 1969, Princeton University Press, Princeton, 1970, pp. 61–79.

[14] R. E. Greene and H. Wu, *On Kähler manifolds of positive bisectional curvature and a theorem of Hartogs*, Abh. Math. Sem. Univ. Hamburg, 47 (1978), pp. 171–185.

[15] P. Grisvard, *Elliptic Problems in Non-Smooth Domains*, Monographs and Studies in Mathematics 24, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1985.

[16] M. Gromov, G. Henkin, and M. Shubin, *Holomorphic L^2 functions on coverings of pseudo-convex manifolds*, GAFA, 8 (1998), pp. 552–585.

[17] R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[18] R. Harvey and B. Lawson, *On boundaries of complex analytic varieties I*, Ann. of Math., 102 (1975), pp. 233–290.

[19] G. M. Henkin, *The Lewy equation and analysis on pseudoconvex manifolds*, Uspekhi Math. Nauk, 32 (1977), pp. 57–118. English Transl: Russ. Math. Surv., 32, 59-130 (1977).

[20] G. M. Henkin and P. L. Polyakov, *Homotopy formulas for the $\overline{\partial}$ -operator on and the Radon-Penrose transform transform on CP_n*, Izv. Akad. Nauk. SSSR, 50 (1986). Engl. transla. in Math. USSR Izvestiya, 28 (1987), n.3.

[21] A. Hirschowitz, *Entre les hypersurfaces et les ensembles pseudoconcaves*, Ann. Scuola Norm. Sup. Pisa, 28 (1974), pp. 873–887.

[22] L. HÖRMANDER, *L^2 estimates and existence theorems for the $\overline{\partial}$ operator*, Acta Math., 113 (1965), pp. 89–152.

[23] J. J. KOHN, *Global regularity for $\overline{\partial}$ on weakly pseudoconvex manifolds*, J.A.M.S., 181 (1973), pp. 272–292.

[24] ———, *Methods of partial differential equations in complex analysis*, Proc. Symp. Pure Math., 30 (1977), pp. 215–237.

[25] J. J. KOHN AND H. ROSSI, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math., 81 (1965), pp. 451–472.

[26] L. MANIVEL, *Un théorème de prolongement L^2 de sections holomorphes d'un fibré hermitien*, Math. Z., 212 (1993), pp. 107–122.

[27] A. MARTINEAU, *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann., 163 (1966), pp. 62–88.

[28] J. MICHEL AND M.-C.SHAW, *The $\overline{\partial}$ problem on domains with piecewise smooth boundaries with applications*, Trans. Am. Math. Soc., 351 (1999), pp. 147–162.

[29] T. NAPIER AND M. RAMACHANDRAN, *The Bochner-Hartogs dichotomy for weakly 1-complete Kähler manifolds*, Ann. Inst. Fourier, 47 (1997), pp. 1345–1365.

[30] T. NISHINO, *Sur les ensembles pseudoconcaves*, J. Math. Kyoto Univ., 1 (1961/62), pp. 225–245.

[31] T. OHSAWA AND N. SIBONY, *Bounded P.S.H. functions and pseudoconvexity in a Kähler manifold*, Nagoya Math. J., 149 (1998), pp. 1–8.

[32] T. OHSAWA AND K. TAKEGOSHI, *On the extension of L^2 holomorphic functions*, Math. Z., 195 (1987), pp. 197–204.

[33] F. SARKIS, *CR-meromorphic extension and the non-embedding of the Andreotti-Rossi structure in the projective space*, Int. J. Math., 10 (1999), pp. 897–905.

[34] J.-P. SERRE, *Un théorème de dualité*, Comment. Math. Helv., 29 (1955), pp. 9–26.

[35] M.-C. SHAW, *Global solvability and regularity for $\overline{\partial}$ on an annulus between two weakly pseudoconvex domains*, Trans. Amer. Math. Soc., 291 (1985), pp. 255–267.

[36] Y.-T. SIU, *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces àf dimension* $\geqslant 3$, Ann. of Math., 151 (2000), pp. 1217–1243.

[37] Y.-T. SIU AND S.-T. YAU, *Compact Kähler manifolds of positive bisectional curvature*, Inventiones Math., 59 (1980), pp. 189–204.

[38] E. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[39] A. TAKEUCHI, *Domaines pseudoconvexes infinis et la métrique riemanienne dans un espace projectif*, J. Math. Soc. Japan, 16 (1964), pp. 159–181.

[40] ———, *Domaines pseudoconvexes sur les variétés kählériennes*, J. Math. Kyoto Univ., 6 (1967), pp. 323–357.