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A NUMERICAL BOUND FOR SMALL PRIME SOLUTIONS OF SOME TERNARY LINEAR EQUATIONS, II*

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1. Introduction. Since Vinogradov's remarkable result [V] that every sufficiently large odd integer can be represented as a sum of three primes in 1937, there are many results on the solvability to the Diophantine linear equation

$$p_1 + p_2 + p_3 = b \tag{1.1}$$

with prime variables. Hua's result[Hu1] is more general than the others; he proved in 1938 that every sufficiently large odd integer b can be written as the form

$$p_1 + p_2 + p_3^k = b \tag{1.2}$$

where $k \geq 1$ is a fixed integer. Then Hua established in [Hu2] the additive theory of prime numbers based on a series of generalizations of the works of Vinogradov on prime number theory. Alan Baker[B] took a step further and considered in 1967 the problem on bounds for small prime solutions of the linear equation

$$a_1p_1 + a_2p_2 + a_3p_3 = b \tag{1.3}$$

where a_1, a_2, a_3 are integers such that

$$a_1a_2a_3 \neq 0 \quad \text{and} \quad (a_1,\, a_2,\, a_3) = 1, \tag{1.4}$$

and b is any integer satisfying

$$b \equiv a_1 + a_2 + a_3 (\bmod 2) \quad \text{and} \quad (b,\, a_i,\, a_j) = 1 \quad \text{for} \quad 1 \leq i < j \leq 3. \tag{1.5}$$

Put

$$A := \max\{|a_1|,\, |a_2|,\, |a_3|\}. \tag{1.6}$$

As the culmination of a series of earlier discoveries[L1], [L2] on the bound of the small prime solutions to equation (1.3), it was proved [LT1]:

THEOREM 0. *Assume the conditions of (1.4) and (1.5) on a_j's and b.*

If all the a_j's are positive, then there is a computable absolute constant $V > 0$ such that equation (1.3) is solvable in prime variables p_1, p_2 and p_3 when $b \gg A^V$;

If all the a_j's are not of the same sign, then there is a computable absolute constant $B > 0$ such that equation (1.3) has a prime solution p_1, p_2, p_3 with

$$\max\{|a_1|p_1,\, |a_2|p_2,\, |a_3|p_3\} \ll \max\{|b|,\, A^B\}.$$

Here $\gg$ and $\ll$ are the Vinogradov symbol and the implied constants in the symbols are computable absolute constants.

As indicated in [LT1, §1] and [LT2, Remark 1.1 and Remark 2.1], conditions (1.4) and (1.5) are necessary and Theorem 0 implies Linnik's theorem that the least prime in any arithmetic progression $\{nq + \ell\}_{n=1,2,\ldots}$ with $(q,\, \ell) = 1$, is $\ll q^L$, where

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L is an absolute constant. Recall that the best known numerical upper bound for the infimum $\mathfrak{L}$ (called the Linnik Constant) of all possible values of L is ≤ 5.5 where the implied constants in $\ll$ is computable. This result is due to Heath-Brown [H-B]. Accordingly, it is worthwhile to estimate the bound for the infimum $\mathcal{B}$ of all the possible values of the constant B in Theorem 0. We now call $\mathcal{B}$ the Baker Constant. Note that $\mathfrak{L} \leq \mathcal{B}$. In [CLT], it was proved that $\mathcal{B} \leq 4$ under the Generalized Riemann Hypothesis. Unconditionally, Choi [C] proved that $\mathcal{B} \leq 4191$ in 1997 for the first time. Most recently, based on the results of the zero-free regions for Dirichlet L-functions in [H-B] and an explicit zero-density estimate near the line $\sigma = 1$, we are able to improve upon this bound for $\mathcal{B}$ considerably and obtained [LW] that $\mathcal{B} \leq 45$ (and also infimum $V \leq 45$) where the corresponding constants to the Vinogradov symbols $\gg$ and $\ll$ in Theorem 0 above are non-computable in [LW]. Here we would like to remark that it is not surprising if one can reduce the 45 further to a two-digit bound by modifying the arguments in [LW]. However, it seems not easy to obtain an one-digit bound for $\mathcal{B}$ which should be the next worthy target in the investigation of Baker's Constant.

In the present paper we shall continue the investigation in [LW] to obtain the following Theorem 1 where there are no non-computable constants at all.

THEOREM 1. *Under the conditions of (1.4) and (1.5) there is a computable absolute constant $C > 0$. If all the a_j's are positive, then the equation (1.3) is solvable in prime variables p_1, p_2 and p_3 when $b \geq CA^{144}$;*

If all the a_j's are not of the same sign, then the equation (1.3) has a prime solution p_1, p_2, p_3 satisfying

$$\max\{|a_1|p_1, |a_2|p_2, |a_3|p_3\} \leq \max\{3|b|, CA^{144}\}.$$

In order to obtain the computable absolute constant C in our Theorem 1 above we have to use the effective upper bound for $\tilde{\beta}$ in (2.8) below which supersedes the Siegel theorem in [LW, (2.13)]. Note that the Siegel theorem was used only in those proofs of [LW, Lemmas 6.1 and 7.3] where $\omega \leq \varepsilon_2$ was assumed. So we shall use (2.8) to establish the corresponding results in our Lemmas 3.1 and 3.2 below when $\omega \leq \varepsilon_2$. These two lemmas will give bounds for Gallagher's type triple sums $\sum_1$ and $\sum_2$ (see Lemmas 3.1 and 3.2) to estimate the M_2 in Lemma 5.4 below. On the other hand, when $\omega > \varepsilon_2$, we shall apply all the results in [LW] directly except that the second inequality in [LW, (2.13)] now is redundant, and so all the arguments there become effective.

REMARK. Concerning Vinogradov's result in [V], under the Generalized Riemann Hypothesis (GRH) it was obtained in [DERZ] that for any odd integer $b \geq 9$, the equation (1.1) holds. Therefore, the Three Primes Goldbach Conjecture is true under the GHR. Unconditionally, very recently it was proved by the first two authors that for any odd integer $b \geq \exp(3100)$, the equation (1.1) holds.

2. Auxiliary lemmas. From now on, we shall use the following notations. Let N be a sufficiently large positive number and let

$$\theta := 1/(12 - \varepsilon_1), \quad Q := N^{\theta}, \quad T := Q^3, \quad \mathcal{L} := \log Q, \quad \tau := N^{-1}Q^{1+\varepsilon_1}, \tag{2.1}$$

where and throughout this paper, ε_1 is a fixed sufficiently small positive number. We assume that

$$Q \geq A^{12+\varepsilon_1}. \tag{2.2}$$

For $1 \le j \le 3$, let

$$N_j := N\,|a_j|^{-1}, \quad N_j' := N\,(4\,|a_j|)^{-1}. \tag{2.3}$$

Let $L(s,\chi)$ denote the Dirichlet L-function of $s = \sigma + it$ where χ is a Dirichlet character modulo q. As a starting point, we first give the following well known result.

LEMMA 2.1. *Under the notations of (2.1), there exists a small computable absolute constant c_0 such that the function*

$$\prod_{q \le Q} \prod_{\chi \bmod q}^{*} L(s,\chi) \tag{2.4}$$

has at most one zero in the region

$$\sigma \ge 1 - c_0/\mathcal{L}, \quad |t| \le T. \tag{2.5}$$

Such a zero $\tilde{\beta}$ (called the exceptional zero or the Siegel zero), if it exists, is real and simple, and corresponds to a non-principal, real, primitive character $\tilde{\chi}$ to a modulus $\tilde{r} \le Q$. Here the $*$ in (2.4) indicates that the product $\prod^{*}_{\chi \bmod q}$ is over all primitive characters $\chi(\bmod\, q)$.

Proof. One can see, for example, [D, §14].

In what follows, except for the last part of the Proof of Theorem 1 in §6, we shall always assume that the exceptional zero $\tilde{\beta}$ in Lemma 2.1 indeed exists and satisfies

$$\omega := \left(1 - \tilde{\beta}\right)\mathcal{L} \le \varepsilon_2, \tag{2.6}$$

where and throughout this paper, ε_2 is a fixed sufficiently small positive number.

LEMMA 2.2. *If the exceptional zero $\tilde{\beta}$ in Lemma 2.1 indeed exists, then for any constant c with $0 < c < 1$ and for any small $\varepsilon > 0$ there is $K(c,\varepsilon) > 0$ depending on c and ε only such that for any zero $\rho = \beta + i\gamma \ne \tilde{\beta}$ (corresponding to $\chi(\bmod\, q)$) of the function (2.4) we have*

$$\beta \le 1 - \min\left\{\frac{c}{6},\ \frac{(1-c)(2/3-\varepsilon)}{\log[\tilde{r},\, q]\,|\gamma|}\log\left(\frac{(1-c)(2/3-\varepsilon)}{(1-\tilde{\beta})\log[\tilde{r},\, q]\,|\gamma|}\right)\right\}, \tag{2.7}$$

if $[\tilde{r}, q]|\gamma| > K(c,\varepsilon)$. Moreover there exists a computable absolute constant $c_1 > 0$ such that

$$1 - c_0/\mathcal{L} \le \tilde{\beta} \le 1 - c_1 \tilde{r}^{-1/2} \log^{-2} \tilde{r}. \tag{2.8}$$

Proof. (2.7) is a direct consequence of [G, Theorem 10.1]. For the second inequality in (2.8), one can see, for example, [D, §14].

LEMMA 2.3. *For any $x \ge 1$ and $y \ge 2$, let*

$$N_q(\alpha,y) := \sum_{\chi \bmod q}^{*} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma| \le y \\ \beta \ge \alpha}} 1 \quad \text{and} \quad N(\alpha,\, x,\, y) := \sum_{q \le x} N_q(\alpha, y), \tag{2.9}$$

where the $*$ in (2.9) indicates that the summation $\sum^{*}_{\chi \bmod q}$ is over all primitive characters χ modulo q, and $\rho = \beta + i\gamma$ is any non-trivial zero of $L(s,\chi)$. Then

$$N(\alpha,\, x,\, y) \ll \left(x^2 y\right)^{3(1-\alpha)/(2-\alpha)} \log^9\left(x^2 y\right), \quad \text{for } 1/2 \le \alpha \le 4/5, \tag{2.10}$$

and

$$N(\alpha, x, y) \ll (x^2 y)^{(2+\varepsilon)(1-\alpha)}, \quad \text{for } 4/5 \le \alpha \le 1. \tag{2.11}$$

Proof. (2.10) is [PP, Theorem 4.4], and (2.11) is [J, Theorem 1].

LEMMA 2.4. *Let $\rho = \beta + i\gamma$ be any complex number satisfying $1/2 \le \beta \le 1$. Then for any real η we have*

$$\int_{N/4}^{N} x^{\rho-1} e(x\eta)\,dx \ll \begin{cases} \min\{N^\beta, |\eta|^{-\beta}\}, & \text{if } |\gamma| \ll 1, \quad (2.12) \\ N^\beta |\gamma|^{-1}, & \text{if } |\eta| \le |\gamma|/(4\pi N), \quad (2.13) \\ N^\beta |\gamma|^{-1/2}, & \text{if } |\gamma|/(4\pi N) < |\eta| \le 4|\gamma|/(\pi N), \quad (2.14) \\ N^{\beta-1}|\eta|^{-1}, & \text{if } |\eta| > 4|\gamma|/(\pi N). \quad (2.15) \end{cases}$$

Proof. This is [LW, Lemma 4.3] or [LT, Lemma 3.2].

3. Gallagher's type triple sums. Now we come to estimate Gallagher's type triple sums in Lemmas 3.1 and 3.2 below, under the assumption $\omega \le \varepsilon_2$ in (2.6).

LEMMA 3.1. *Under the notations of (2.1) and (2.3) and the assumption of (2.6), there is $K(\varepsilon_2) > 0$ depending on ε_2 only such that if $Q \ge K(\varepsilon_2)$ we have for $1 \le j \le 3$,*

$$\sum\nolimits_1 := \sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \sideset{}{'}\sum_{|\gamma| \le Q^{1+\varepsilon_1} q^{-1}} N_j^{\beta-1} \ll \begin{cases} \varepsilon_2, & \text{for any } \tilde{r} \le Q, \\ \varepsilon_2^{1/64}\omega^3, & \text{for } \tilde{r} \le Q^{1/4}, \end{cases}$$

where the $'$ indicates that the last summation runs over all the zeros $\rho = \beta + i\gamma \ne \tilde{\beta}$ of $L(s,\chi)$ with $\beta \ge 1/2$.

Proof. For any zero $\rho = \beta + i\gamma \ne \tilde{\beta}$ of $L(s,\chi)$ with $|\gamma| \le Q^{1+\varepsilon_1} q^{-1}$, by (2.6), $\tilde{r} \le Q$ and Lemma 2.2

$$\beta \le 1 - \min\left\{\frac{c}{6}, \frac{(1-c)(2/3-\varepsilon)}{\log(\tilde{r}Q^{1+\varepsilon_1})} \log\left(\frac{(1-c)(2/3-\varepsilon)}{(2+\varepsilon_1)\omega}\right)\right\}. \tag{3.1}$$

By (2.6) and (2.8), we have

$$\omega \gg \mathcal{L}\tilde{r}^{-1/2}\log^{-2}\tilde{r} \gg Q^{-1/2}\mathcal{L}^{-1}, \tag{3.2}$$

and

$$\omega \gg Q^{-1/8}\mathcal{L}^{-1}, \quad \text{for} \quad \tilde{r} \le Q^{1/4}. \tag{3.3}$$

Let $c = (1/2) + \varepsilon$. Then by (3.2), the second term inside the curly brackets in (3.1) is

$$\le \frac{(1/2-\varepsilon)(2/3-\varepsilon)}{\log(\tilde{r}Q^{1+\varepsilon_1})} \log\left(\mathcal{L}^{1+\varepsilon}\tilde{r}^{1/2}\right).$$

The above is increasing with respect to $\tilde{r}$. Hence in view of $\tilde{r} \le Q$, the above can be estimated further as

$$\le \frac{(1/2-\varepsilon)(2/3-\varepsilon)}{(2+\varepsilon_1)\mathcal{L}} \log\left(\mathcal{L}^{1+\varepsilon}Q^{1/2}\right) \le \frac{(1/2-\varepsilon)(2/3-\varepsilon)}{(2+\varepsilon_1)}(1/2+\varepsilon) \le c/6.$$

This shows that the second term inside the curly brakets in (3.1) is always smaller than the first one if one sets $c = (1/2) + \varepsilon$. With this choice of c, by (3.1) and $\tilde{r} \le Q$ we have

$$\beta \le 1 - (1/6 - \varepsilon(\varepsilon_1))\,\mathcal{L}^{-1}\log(1/8\omega) =: 1 - \eta_1(Q), \tag{3.4}$$

where $\varepsilon(\varepsilon_1)$ is a sufficiently small positive constant depending only on ε_1. Similarly, by (3.3), it is easy to verify that the second term inside the last curly brakets in (3.1) is always smaller than $c/6$ if one assumes $\tilde{r} \le Q^{1/4}$ and sets $c = (2/7) + \varepsilon$. With this choice of c, by (3.1) we have

$$\beta \le 1 - (8/21 - \varepsilon(\varepsilon_1))\, \mathcal{L}^{-1} \log(1/6\omega) =: 1 - \eta_2(Q). \tag{3.5}$$

Now we let $\eta(Q) = \eta_1(Q)$ in general and $\eta(Q) = \eta_2(Q)$ if $\tilde{r} \le Q^{1/4}$. Setting $y = Q^{1+\varepsilon_1} q^{-1}$ in the first equality of (2.9) and then letting $N^*(\alpha, Q) := \sum_{q \le Q} N_q(\alpha, y)$ instead of $N(\alpha, x, y)$ in (2.9), one has

(3.6)

$$\begin{aligned} \sum\nolimits_1 &\le -\int_{1/2}^{1-\eta(Q)} N_j^{\alpha-1} \mathrm{d}N^*(\alpha, Q) \\ &= N_j^{-1/2} N^*(1/2, Q) + \left\{ \int_{1/2}^{4/5} + \int_{4/5}^{1-\eta(Q)} \right\} N^*(\alpha, Q)\, N_j^{\alpha-1} \log N_j \mathrm{d}\alpha. \end{aligned}$$

From (2.9), we have for $1/2 \le \alpha \le 1$,

$$N^*(\alpha, Q) \ll \mathcal{L} \max_{1 \le M \le Q} \sum_{M \le q \le 2M} N_q\left(\alpha, Q^{1+\varepsilon_1} M^{-1}\right). \tag{3.7}$$

Note that, in view of (1.6), (2.1), (2.3) and $Q \ge A^{12}$ (by (2.2)), we have for $1 \le j \le 3$,

$$N_j \ge Q^{12-(1/12)-\varepsilon_1} =: Q^{c_2}. \tag{3.8}$$

Hence by (3.7), (2.10), (3.8) and (3.2), the first term on the right hand side of (3.6) can be estimated as

$$\ll N_j^{-1/2} \mathcal{L} \max_{1 \le M \le Q} \{M^2 Q^{1+\varepsilon_1} M^{-1} \mathcal{L}^9\} \ll Q^{-1} \omega^3. \tag{3.9}$$

When $1/2 \le \alpha \le 4/5$, by (2.10) we see that (3.7) is

$$\ll \mathcal{L} \max_{1 \le M \le Q} \left\{ \left(M^2 Q^{1+\varepsilon_1} M^{-1}\right)^{3(1-\alpha)/(2-\alpha)} \mathcal{L}^9 \right\} \ll Q^{6(1+\varepsilon_1)(1-\alpha)/(2-\alpha)} \mathcal{L}^{10},$$

whence by (3.8), (3.3), and $c_2 = 12 - (1/12) - \varepsilon_1$, the first integral on the right hand side of (3.6) is

(3.10)

$$\begin{aligned} &\ll \mathcal{L}^{11} \int_{1/2}^{4/5} Q^{c_2(\alpha-1)} Q^{6(1+\varepsilon_1)(1-\alpha)/(2-\alpha)} \mathrm{d}\alpha \ll \mathcal{L}^{11} Q^{\{c_2 - 6(1+\varepsilon_1)/(2-4/5)\}(4/5-1)} \\ &\ll Q^{-1.38} \ll \begin{cases} Q^{-1.38}, & \text{for any } \tilde{r} \le Q, \\ Q^{-1}\omega^3, & \text{for } \tilde{r} \le Q^{1/4}. \end{cases} \end{aligned}$$

To estimate the last integral on the right hand side of (3.6), instead of (3.7), we first use (2.11) to bound $N^*(\alpha, Q)$ as follows. Let $d_k = 1 - 2^{-k}$ for $k \ge 0$. So $2d_k = 1 + d_{k-1}$ for $k \ge 1$. Let u be a fixed sufficiently large constant, then by (2.9) and (2.11) we have for $\alpha \in [4/5,\ 1]$,

$$
\begin{aligned}
N^*(\alpha, Q) &\le \sum_{k=1}^{u} \sum_{Q^{d_{k-1}} \le q \le Q^{d_k}} N_q\left(\alpha, Q^{1+\varepsilon_1 - d_{k-1}}\right) + \sum_{Q^{d_u} \le q \le Q} N_q\left(\alpha, Q^{1+\varepsilon_1-d_u}\right) \\
&\ll \sum_{k=1}^{u} Q^{(1+2d_k+\varepsilon_1-d_{k-1})(2+\varepsilon)(1-\alpha)} + Q^{(2+1+\varepsilon_1-d_u)(2+\varepsilon)(1-\alpha)} \\
&\ll Q^{(2+2\varepsilon_1)(2+\varepsilon)(1-\alpha)},
\end{aligned}
$$

providing that $2^{-u} \le \varepsilon_1$ where the implied constants in the above $\ll$ depend on u. Hence by (3.8), the last integral in (3.6) can be estimated as

$$
(3.11) \qquad \ll \int_{4/5}^{1-\eta(Q)} Q^{\{c_2-(2+2\varepsilon_1)(2+\varepsilon)\}(\alpha-1)} \mathcal{L} d\alpha \ll Q^{-\{c_2-(2+2\varepsilon_1)(2+\varepsilon)\}\eta(Q)}.
$$

Thus, in general, by (2.6), (3.4), and $c_2 = 12 - (1/12) - \varepsilon_1$, the above is

$$
\begin{aligned}
(3.12) \qquad &\ll \exp\{-(c_2 - (2+\varepsilon)(2+2\varepsilon_1))(1/6 - \varepsilon(\varepsilon_1))\log(1/8\omega)\} \\
&\ll \omega^{(c_2-(2+\varepsilon)(2+2\varepsilon_1))(1/6-\varepsilon(\varepsilon_1))} \ll \omega \ll \varepsilon_2.
\end{aligned}
$$

When $\tilde{r} \le Q^{1/4}$, by (2.6) and (3.5), we can estimate (3.11) further as

$$
\begin{aligned}
(3.13) \qquad &\ll \exp\{-(c_2 - (2+\varepsilon)(2+2\varepsilon_1))(8/21 - \varepsilon(\varepsilon_1))\log(1/6\omega)\} \\
&\ll \omega^{3+(1/64)} \ll \varepsilon_2^{1/64}\omega^3.
\end{aligned}
$$

Combining (3.6), (3.9), (3.10), (3.12) and (3.13), the proof of Lemma 3.1 is complete.

LEMMA 3.2. *Under the conditions of Lemma 3.1, we have*

$$
\sum\nolimits_2 := \sum_{q \le Q} q^{-1} \sum_{\chi \bmod q}^{*} \sum_{Q^{1+\varepsilon_1}q^{-1} \le |\gamma| \le T}' N_j^{\beta-1}|\gamma|^{-1} \ll \begin{cases} Q^{-1-0.9\varepsilon_1}, & \textit{for any } \tilde{r} \le Q, \\ Q^{-1-0.9\varepsilon_1}\omega^3, & \textit{for } \tilde{r} \le Q^{1/4}. \end{cases}
$$

Proof. Note that

(3.14)

$$
\begin{aligned}
\sum\nolimits_2 &\ll \mathcal{L} \max_{1 \le U \le T} \left\{ \sum_{q \le Q} q^{-1} \sum_{\chi \bmod q}^{*} \sum_{Q^{1+\varepsilon_1}q^{-1}U \le |\gamma| \le 2Q^{1+\varepsilon_1}q^{-1}U}' N_j^{\beta-1}|\gamma|^{-1} \right\} \\
&\ll Q^{-1-\varepsilon_1}\mathcal{L}^2 \max_{1 \le U \le T} \max_{1 \le V \le Q} \left\{ U^{-1} \sum_{V \le q \le 2V} \sum_{\chi \bmod q}^{*} \sum_{|\gamma| \le 2Q^{1+\varepsilon_1}q^{-1}U}' N_j^{\beta-1} \right\}.
\end{aligned}
$$

Noting $U \le Q^3$ (see (2.1)) and $|\gamma| \le 2Q^{1+\varepsilon_1}q^{-1}U$, by Lemma 2.2 and (2.6), the β in (3.14) is

$$
\le 1 - \min\left\{ \frac{c}{6},\ \frac{(1-c)(2/3-\varepsilon)}{\log(\tilde{r}Q^{1+\varepsilon_1}U)} \log\left(\frac{(1-c)(2/3-\varepsilon)}{6\omega}\right) \right\}.
$$

In view of (3.3), if $\tilde{r} \le Q^{1/4}$, then by similar arguments as between (3.3) and (3.4) it can be verified that the second term inside the last curly brackets is always smaller than $c/6$ if one sets $c = (2/7) + \varepsilon$. With this choice of c, the above is

$$
(3.15) \qquad \le 1 - \frac{10/21 - \varepsilon}{\log(\tilde{r}Q^{1+\varepsilon_1}U)} \log\left(\frac{1}{13\omega}\right) =: 1 - \eta_3(Q, U), \quad \text{say.}
$$

If $\tilde{r} > Q^{1/4}$, we shall use the trivial bound $\beta \le 1$, that is, in this case, we let $\eta_3(Q,U) = 0$. Thus in view of (2.9) the last triple summation over q in (3.14) is

$$\begin{aligned}(3.16)\qquad &\le -\int_{1/2}^{1-\eta_3(Q,U)} N_j^{\alpha-1} \mathrm{d}N\left(\alpha,\, 2V,\, 2Q^{1+\varepsilon_1}UV^{-1}\right) \\ &\le N_j^{-1/2} N\left(1/2,\, 2V,\, 2Q^{1+\varepsilon_1}UV^{-1}\right) + \left\{\int_{1/2}^{4/5} + \int_{4/5}^{1-\eta_3(Q,U)}\right\} \\ &\quad \times N\left(\alpha,\, 2V,\, 2Q^{1+\varepsilon_1}UV^{-1}\right) N_j^{\alpha-1} \log N_j \mathrm{d}\alpha.\end{aligned}$$

By (2.10), the first term on the right of (3.16) is $\ll N_j^{-1/2} Q^{1+\varepsilon_1} UV\mathcal{L}^9$. Hence by (3.14), (3.8), $c_2 = 12 - (1/12) - \varepsilon_1$, and (3.2) its contribution to $\sum_2$ is

$$(3.17)\qquad \ll Q^{-1-\varepsilon_1}\mathcal{L}^{11} \max_{1\le U\le T}\max_{1\le V\le Q}\left\{U^{-1}N_j^{-1/2}Q^{1+\varepsilon_1}UV\right\} \ll Q^{-3}\omega^3.$$

Again by (2.10), the first integral on the right of (3.16) is

$$\ll \mathcal{L}^{10}\int_{1/2}^{4/5} N_j^{\alpha-1}\left(Q^{1+\varepsilon_1}UV\right)^{3(1-\alpha)/(2-\alpha)} \mathrm{d}\alpha,$$

hence by (3.14), its contribution to $\sum_2$ is

$$\ll Q^{-1-\varepsilon_1}\mathcal{L}^{12}\max_{1\le U\le T}\left\{U^{-1}\int_{1/2}^{4/5} N_j^{\alpha-1}\left(Q^{2+\varepsilon_1}U\right)^{3(1-\alpha)/(2-\alpha)} \mathrm{d}\alpha\right\}.$$

Note that $3(1-\alpha)/(2-\alpha) \le 1$ for $1/2 \le \alpha \le 4/5$. Then by (3.8) and (3.3) and $c_2 = 12 - (1/12) - \varepsilon_1$, the above is

$$\begin{aligned}(3.18)\qquad &\ll Q^{-1-\varepsilon_1}\mathcal{L}^{12}\int_{1/2}^{4/5} Q^{\{c_2 - 6(1+\varepsilon_1)/(2-\alpha)\}(\alpha-1)}\mathrm{d}\alpha \\ &\ll Q^{-1-\varepsilon_1}Q^{-1.38} \ll \begin{cases} Q^{-2}, & \text{for any } \ \tilde{r} \le Q, \\ Q^{-2}\omega^3, & \text{for } \ \tilde{r} \le Q^{1/4}. \end{cases}\end{aligned}$$

Next, by (2.11), the last integral on the right of (3.16) is

$$\ll \mathcal{L}\int_{4/5}^{1-\eta_3(Q,U)} N_j^{\alpha-1}\left(Q^{1+\varepsilon_1}UV\right)^{(2+\varepsilon)(1-\alpha)} \mathrm{d}\alpha,$$

hence by (3.14), its contribution to $\sum_2$ is

(3.19)

$$\begin{aligned}&\ll Q^{-1-\varepsilon_1}\mathcal{L}^3\max_{1\le U\le T}\left\{U^{-1}\int_{4/5}^{1-\eta_3(Q,U)} N_j^{\alpha-1}\left(Q^{2+\varepsilon_1}U\right)^{(2+\varepsilon)(1-\alpha)} \mathrm{d}\alpha\right\} \\ &=: Q^{-1-\varepsilon_1}\mathcal{L}^3\max_{1\le U\le T}\{f(U)\}, \quad \text{say.}\end{aligned}$$

We now come to prove that for any $1 \le U \le T$,

$$(3.20)\qquad \text{(i)}\ \ f(U) \ll 1, \quad \text{for any } \ \tilde{r} \le Q,$$

and

$$(3.21)\qquad \text{(ii)}\ \ f(U) \ll \omega^3, \quad \text{for } \ \tilde{r} \le Q^{1/4}.$$

For (3.20), noting $U^{(2+\varepsilon)(1-\alpha)} < U$ for $\alpha \in [4/5,\ 1]$, we have by (3.8),

$$f(U) \le \int_{4/5}^{1} Q^{\{c_2-(2+\varepsilon_1)(2+\varepsilon)\}(\alpha-1)} \mathrm{d}\alpha \ll 1,$$

as desired. To prove (3.21), we consider the upper bounds for U at $Q^{0.006}$, $Q^{0.025}$, $Q^{0.087}$, $Q^{0.35}$ and T $(= Q^3)$. Note that the $U^{-1+(2+\varepsilon)(1-\alpha)}$ in (3.19) is decreasing with respect to $U \ge 1$. When $1 \le U \le Q^{0.006}$ and $\tilde{r} \le Q^{1/4}$, by (3.8), (3.15) and $c_2 = 12 - (1/12) - \varepsilon_1$, we have

$$\begin{aligned} f(U) &\le Q^{-\{c_2-(2+\varepsilon_1)(2+\varepsilon)\}\eta_3(Q,U)} \\ &\ll \omega^{\{c_2-(2+\varepsilon_1)(2+\varepsilon)\}(10/21-\varepsilon)\mathcal{L}/\log\left(Q^{1/4+1+\varepsilon_1+0.006}\right)} \le \omega^{3.001}. \end{aligned}$$

When $Q^{0.006} \le U \le Q^{0.025}$ and $\tilde{r} \le Q^{1/4}$, by (3.8), (3.3) and (3.15) we get

$$\begin{aligned} f(U) &\le Q^{-0.006} \int_{4/5}^{1-\eta_3(Q,U)} N_j^{\alpha-1} \left(Q^{2.006+\varepsilon_1}\right)^{(2+\varepsilon)(1-\alpha)} \mathrm{d}\alpha \\ &\ll (\mathcal{L}\omega)^{8(0.006)} \int_{4/5}^{1-\eta_3(Q,U)} Q^{\{c_2-(2.006+\varepsilon_1)(2+\varepsilon)\}(\alpha-1)} \mathrm{d}\alpha \\ &\ll (\mathcal{L}\omega)^{8(0.006)} \mathcal{L}^{-1} Q^{-\{c_2-(2.006+\varepsilon_1)(2+\varepsilon)\}\eta_3(Q,U)} \\ &\le \omega^{8(0.006)} \omega^{\{c_2-(2.006+\varepsilon_1)(2+\varepsilon)\}(10/21-\varepsilon)\mathcal{L}/\log\left(Q^{(1/4)+1+\varepsilon_1+0.025}\right)} \le \omega^{3.00025}. \end{aligned}$$

Similarly, when $\tilde{r} \le Q^{1/4}$ we get

$$f(U) \ll \begin{cases} \omega^{3.001} & \text{if } Q^{0.025} \le U \le Q^{0.087}\ , \\ \omega^{3.003} & \text{if } Q^{0.087} \le U \le Q^{0.35}\ , \\ \omega^{3.6} & \text{if } Q^{0.35} \le U \le Q^{3} = T\ . \end{cases}$$

This proves (3.21). By (3.20) and (3.21), we see that (3.19) is

$$\ll \begin{cases} Q^{-1-\varepsilon_1}\mathcal{L}^3, & \text{for any } \ \tilde{r} \le Q, \\ Q^{-1-\varepsilon_1}\mathcal{L}^3\omega^3, & \text{for } \ \tilde{r} \le Q^{1/4}. \end{cases} \tag{3.22}$$

From (3.17), (3.18) and (3.22), the proof of Lemma 3.2 is complete.

4. Triple sums on integrals.

LEMMA 4.1. *Under the conditions of Lemma 3.1, we have for* $1 \leq j \leq 3$,

$$S_{1,j} := \sum_{q\leq Q} \sum_{\chi \bmod q}^{*} \sum_{|\gamma|\leq T}' \left\{ \int_{-\tau/q}^{\tau/q} \left| \int_{N_j'}^{N_j} x^{\rho-1} e(a_j x \eta)\, dx \right|^2 d\eta \right\}^{1/2} \ll |a_j|^{-1} N^{1/2}.$$

where $e(\alpha) := \exp(i2\pi\alpha)$ *for any real* α.

Proof. We write $Q(q) := 15Q^{1+\varepsilon_1}q^{-1}$ and $S_{1,j}$ as

$$S_{1,j} = \left\{ \sum_{q\leq Q} \sum_{\chi \bmod q}^{*} \sum_{|\gamma|\leq Q(q)}' + \sum_{q\leq Q} \sum_{\chi \bmod q}^{*} \sum_{Q(q)<|\gamma|\leq T}' \right\} \left\{ \int_{-\tau/q}^{\tau/q} \left| \int_{N_j'}^{N_j} x^{\rho-1} e(a_j x \eta)\, \mathrm{d}x \right|^2 \mathrm{d}\eta \right\}^{1/2} \tag{4.1}$$

By (2.3) the above integral with respect to η is

$$= |a_j|^{-2\beta} \int_{-\tau/q}^{\tau/q} \left| \int_{N/4}^{N} x^{\rho-1} e(x\eta) \mathrm{d}x \right|^2 \mathrm{d}\eta.$$

For simplicity, we denote the intervals of η in $[-\tau/q, \tau/q]$ satisfying the conditions in (2.13), (2.14) and (2.15) by J_1, J_2, J_3 respectively. Thus by Lemma 2.4 and (2.3), the innermost sum $\sum'_{|\gamma|\leq Q(q)}$ in the first triple sum in (4.1) is

$$\begin{aligned}
&\ll \sum_{|\gamma|\leq 1}' |a_j|^{-\beta} \left\{ \int_{-\tau/q}^{\tau/q} \min\left\{N^{2\beta},\ |\eta|^{-2\beta}\right\} \mathrm{d}\eta \right\}^{\frac{1}{2}} \\
&\quad + \sum_{1\leq|\gamma|\leq Q(q)}' |a_j|^{-\beta} \left\{ \int_{J_3} \left(N^{\beta-1}|\eta|^{-1}\right)^2 \mathrm{d}\eta + \int_{J_2} \left(N^{\beta}|\gamma|^{-1/2}\right)^2 \mathrm{d}\eta + \int_{J_1} \left(N^{\beta}|\gamma|^{-1}\right)^2 \mathrm{d}\eta \right\}^{\frac{1}{2}} \\
&\ll \sum_{|\gamma|\leq 1}' |a_j|^{-\beta} \left\{ \int_{0}^{N^{-1}} N^{2\beta} \mathrm{d}\eta + \int_{N^{-1}}^{\tau/q} \eta^{-2\beta} \mathrm{d}\eta \right\}^{1/2} \\
&\quad + \sum_{1\leq|\gamma|\leq Q(q)}' |a_j|^{-\beta} \left\{ N^{2(\beta-1)}|\gamma|^{-1}N + N^{2\beta}|\gamma|^{-1}|\gamma|N^{-1} + N^{2\beta}|\gamma|^{-2}|\gamma|N^{-1} \right\}^{\frac{1}{2}} \\
&\ll |a_j|^{-1} N^{1/2} \sum_{|\gamma|\leq Q(q)}' N_j^{\beta-1}.
\end{aligned}$$

Hence by Lemma 3.1, the first triple sum in (4.1) is $\ll |a_j|^{-1} N^{1/2}$. For the second triple sum in (4.1), in view of $\tau = N^{-1}Q^{1+\varepsilon_1}$ in (2.1), we have $|\eta| \leq \tau q^{-1} \leq |\gamma|/(4\pi N)$ if $|\gamma| \geq Q(q)$. Thus by (2.13), (2.1) and (2.3), the innermost sum $\sum'_{Q(q)<|\gamma|\leq T}$ in the second triple sum in (4.1) is

$$\ll \sum_{Q(q)\le|\gamma|\le T}{}' |a_j|^{-\beta}\left\{\int_0^{\tau/q}\left(N^\beta|\gamma|^{-1}\right)^2 \mathrm{d}\eta\right\}^{1/2} \ll \left(\tau q^{-1}\right)^{1/2}\sum_{Q(q)\le|\gamma|\le T}{}' |a_j|^{-\beta} N^\beta|\gamma|^{-1}$$
$$\ll |a_j|^{-1} N^{1/2}Q^{1+\varepsilon_1/2}q^{-1}\sum_{Q(q)\le|\gamma|\le T}{}' N_j^{\beta-1}|\gamma|^{-1}.$$

Thus by Lemma 3.2, the second triple sum in (4.1) is $\ll Q^{-0.4\varepsilon_1}|a_j|^{-1}N^{1/2}$. The proof of Lemma 4.1 is complete.

LEMMA 4.2. *Under the conditions of Lemma 3.1, we have for* $1\le j\le 3$,

$$S_{2,j} := \sum_{q\le Q}\sum_{\chi \bmod q}{}^{*}\sum_{|\gamma|\le T}{}'\left\{\int_{-\tau/q}^{\tau/q}\left|\int_{N_j'}^{N_j} x^{\rho-1}e(a_jx\eta)\,dx\right|^3 d\eta\right\}^{1/3}$$
$$\ll \begin{cases} \varepsilon_2|a_j|^{-1}N^{2/3}, & \text{for any } \tilde r\le Q,\\ \varepsilon_2^{1/64}\omega^3|a_j|^{-1}N^{2/3}, & \text{for } \tilde r\le Q^{1/4}.\end{cases}$$

Proof. Using $Q(q)$ again we write $S_{2,j}$ as

$$S_{2,j} = \left\{\sum_{q\le Q}\sum_{\chi \bmod q}{}^{*}\sum_{|\gamma|\le Q(q)}{}' + \sum_{q\le Q}\sum_{\chi \bmod q}{}^{*}\sum_{Q(q)<|\gamma|\le T}{}'\right\}\left\{\int_{-\tau/q}^{\tau/q}\left|\int_{N_j'}^{N_j} x^{\rho-1}e(a_jx\eta)\,\mathrm{d}x\right|^3 \mathrm{d}\eta\right\}^{1/3}. \tag{4.2}$$

By a similar argument as in the proof of Lemma 4.1, the innermost sum $\sum_{|\gamma|\le Q(q)}{}'$ in the first triple sum in (4.2) is

$$\ll |a_j|^{-1}N^{2/3}\sum_{|\gamma|\le Q(q)}{}' N_j^{\beta-1}.$$

Thus by Lemma 3.1, the first triple sum is

$$\ll \begin{cases} \varepsilon_2|a_j|^{-1}N^{2/3}, & \text{for any } \tilde r\le Q,\\ \varepsilon_2^{1/64}\omega^3|a_j|^{-1}N^{2/3}, & \text{for } \tilde r\le Q^{1/4}.\end{cases} \tag{4.3}$$

For the second triple sum in (4.2), in view of $|\eta|\le\tau q^{-1}$ and $\tau = N^{-1}Q^{1+\varepsilon_1}$, there always exists $|\eta| < |\gamma|/(4\pi N)$ if $|\gamma| > Q(q)$. Thus similar to the above lemma, by (2.13) the innermost sum $\sum_{Q(q)<|\gamma|\le T}{}'$ in the second triple sum in (4.2) can be estimated as

$$\ll |a_j|^{-1}N^{2/3}Q^{1+\varepsilon_1/3}q^{-1}\sum_{Q(q)<|\gamma|\le T}{}' N_j^{\beta-1}|\gamma|^{-1}.$$

Thus by Lemma 3.2, the second triple sum can be bounded by (4.3). The proof of Lemma 4.2 is complete.

5. The circle method. In this section we apply the circle method, and our Lemmas 4.1 and 4.2 to obtain some useful results for our proof of Theorem 1. Since our arguments are very similar to [LW, Sections 4 and 5] we shall make use of some lemmas in [LW] directly. For convenience in mentioning these lemmas we shall use the same notations as in [LW] except now our $\theta = 1/(12-\varepsilon_1)$ (see (2.1)). For $j = 1,2,3$ define $I(b)$, $I_1(b)$, $I_2(b)$, $I_j(y)$, $\tilde{I}_j(y)$, $G(a_jh,\chi)$, $G(a_jh,q)$, $\mathcal{G}_j(h,q,\eta)$, $H_j(h,q,\eta)$, $\delta(q)$, M_j, $s(p)$, $Z(q;\chi_1,\chi_2,\chi_3)$ and M_0 as in [LW, (4.7), (4.9), (4.10), (4.11), (5.1), (5.4), (5.5) and (5.10)] respectively. We need the following lemmas.

LEMMA 5.1. *Let $I_1(b)$ and $H_j(h,q,\eta)$ be defined as in [LW, (4.7) and (4.11)]. Under the assumptions of Lemma 3.1, we have*

$$I_1(b) = \sum_{q\le Q}\varphi(q)^{-3}\sum_{h=1}^{q}{}' e\left(-\frac{b}{q}h\right)\int_{-\tau/q}^{\tau/q} e(-b\eta)\prod_{j=1}^{3} H_j(h,q,\eta)d\eta + O\left(\Omega_1\right),$$

where $\sum_{h=1}^{q}{}'$ is the summation over all $1\le h\le q$, $(h,q)=1$, and

$$\Omega_1 := N^2 Q^{2.5+\varepsilon_1}\left|a_1a_2a_3\right|^{-1}T^{-1}\mathcal{L}^2. \tag{5.1}$$

Proof. The lemma can be proved by precisely the same way as that of [LW, Lemma 4.7] with the use of Lemma 4.1 instead of [LW, Lemma 4.6].

LEMMA 5.2. *Let M_0, M_1 and M_3 be defined as in [LW, (5.10) and (5.1)]. Then we have*

$$M_1 + M_3 \ge \omega^3 M_0 + O\left(\tilde{r}\Omega_2 + N^2Q^{-1-\varepsilon_1}\right).$$

where

$$\Omega_2 := N^2Q^{-1+\varepsilon_1}|a_1\,a_2\,a_3|^{-1}. \tag{5.2}$$

Proof. It can be proved by precisely the same way as that of [LW, Lemma 5.5] with [LW, (5.15)] replaced by the following (5.3). In view of (2.6) we have $\omega\le\varepsilon_2$. Thus noting $N_j'\ge 4^{-1}Q^{12-\varepsilon_1-(1/12)}$ (similar to (3.8)) we have for $1\le j\le 3$,

$$\begin{aligned}1 - N_j'^{\tilde{\beta}-1} &\ge 1-\left(4^{-1}Q^{12-\varepsilon_1-(1/12)}\right)^{\tilde{\beta}-1}\\ &= 1-\exp\left\{\left(\tilde{\beta}-1\right)\log\left(4^{-1}Q^{12-\varepsilon_1-(1/12)}\right)\right\}\\ &\ge 1-\exp\left\{(12-2\varepsilon_1-(1/12))\left(\tilde{\beta}-1\right)\mathcal{L}\right\}\\ &= 1-\exp\left\{-\left(12-2\varepsilon_1-(1/12)\right)\omega\right\}\\ &\ge \omega,\end{aligned} \tag{5.3}$$

providing that ε_2 is sufficiently small.

LEMMA 5.3. *Let M_0 and $s(p)$ be defined as in [LW, (5.10) and (5.4)]. If*
(i) all the a_j's are positive and $b=N$, or
(ii) not all the a_j's are of the same sign and $N\ge 3|b|$, then

$$M_0 \gg N^2\left|a_1a_2a_3\right|^{-1}\prod_p s(p).$$

Proof. This is [LW, Lemma 7.4].

LEMMA 5.4. *Under the conditions of Lemma 3.1 and Lemma 5.3, we have*

$$M_2 \ll \begin{cases}\varepsilon_2 M_0, & \text{for any } \tilde{r}\le Q,\\ \varepsilon_2^{1/64}\omega^3 M_0, & \text{for } \tilde{r}\le Q^{1/4}.\end{cases}$$

Proof. Recall from [LW, (5.1)] that there are 19 terms in the integrand for M_2 and they are of the 6 types as in [LW, (i)-(vi) below Lemma 7.4]. The treatments for these six types are quite similar. We illustrate the details with a term belonging to the fifth type, namely,

$$M_{25} := \sum_{q\le Q} \varphi(q)^{-3} \sum_{h=1}^{q}{}' e\left(-\frac{b}{q}h\right) \int_{-\tau/q}^{\tau/q} \delta(q) G(a_1 h, \tilde{\chi}\chi_0)\, \tilde{I}_1(\eta) e(-b\eta) \prod_{j=2}^{3} \mathcal{G}_j(h,q,\eta) \mathrm{d}\eta.$$

In view of [LW, (4.9), (4.10) and (5.5)], the above is

(5.4)

$$\begin{aligned} M_{25} &= \sum_{\substack{q\le Q\\ \tilde{r}|q}} \varphi(q)^{-3} \sum_{h=1}^{q}{}' e\left(-\frac{b}{q}h\right) G(a_1 h, \tilde{\chi}\chi_0) \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} \left(\prod_{j=2}^{3} G(a_j h, \bar{\chi}_j) \right) \\ &\quad \times \sum_{|\gamma_2|\le T}{}' \sum_{|\gamma_3|\le T}{}' \int_{-\tau/q}^{\tau/q} e(-b\eta) \tilde{I}_1(\eta) \left(\prod_{j=2}^{3} \int_{N_j'}^{N_j} x^{\rho_j - 1} e(a_j x\eta)\, \mathrm{d}x \right) \mathrm{d}\eta \\ &= \sum_{r_2\le Q} \sum_{\chi_2 \bmod r_2}^{*} \sum_{|\gamma_2|\le T}{}' \sum_{r_3\le Q} \sum_{\chi_3 \bmod r_3}^{*} \sum_{|\gamma_3|\le T}{}' \sum_{\substack{q\le Q\\ [\tilde{r},\, r_2,\, r_3]|q}} \varphi(q)^{-3} Z(q; \tilde{\chi}, \bar{\chi}_2, \bar{\chi}_3) \\ &\quad \times \int_{-\tau/q}^{\tau/q} e(-b\eta) \tilde{I}_1(\eta) \left(\prod_{j=2}^{3} \int_{N_j'}^{N_j} x^{\rho_j - 1} e(a_j x\eta)\, \mathrm{d}x \right) \mathrm{d}\eta. \end{aligned}$$

Noting that $[\tilde{r}, r_2, r_3]\,|q$ implies $r_2, r_3 \le q$ (so $\tau/q \le \tau/r_j$, $j = 2, 3$), applying Hölder's inequality, the absolute value of the last integral with respect to η in (5.4) is

(5.5)

$$\le \left\{ \int_{-\tau/q}^{\tau/q} \left| \int_{N_1'}^{N_1} x^{\tilde{\beta}-1} e(a_1 x\eta)\, \mathrm{d}x \right|^3 \mathrm{d}\eta \right\}^{1/3} \prod_{j=2}^{3} \left\{ \int_{-\tau/r_j}^{\tau/r_j} \left| \int_{N_j'}^{N_j} x^{\rho_j-1} e(a_j x\eta)\, \mathrm{d}x \right|^3 \mathrm{d}\eta \right\}^{1/3}.$$

By (2.12), the first term in the above product can be estimated easily as

$$\ll |a_1|^{-1} N^{2/3}. \tag{5.6}$$

Now we substitute (5.6) into (5.5), and then into (5.4). Then by Lemma 4.2 and using [LW, (5.8)] to estimate the sum over q, (5.4) can be estimated further as

$$\begin{aligned} &\ll \begin{cases} \varepsilon_2^2 N^2 |a_1 a_2 a_3|^{-1} \prod_p s(p), & \text{for any } \tilde{r} \le Q, \\ \varepsilon_2^{1/32} \omega^6 N^2 |a_1 a_2 a_3|^{-1} \prod_p s(p), & \text{for } \tilde{r} \le Q^{1/4}, \end{cases} \\ &\ll \begin{cases} \varepsilon_2^2 M_0, & \text{for any } \tilde{r} \le Q, \\ \varepsilon_2^{1/32} \omega^6 M_0, & \text{for } \tilde{r} \le Q^{1/4}. \end{cases} \end{aligned} \tag{5.7}$$

Similarly, by the use of [LW, (5.8)] and Lemma 4.2 we have

$$M_{21},\ M_{22},\ M_{23} \ll \begin{cases} \varepsilon_2 M_0, & \text{for any } \ \tilde{r} \le Q, \\ \varepsilon_2^{1/64}\omega^3 M_0, & \text{for } \ \tilde{r} \le Q^{1/4}; \end{cases}$$

$$M_{24} \ll \begin{cases} \varepsilon_2^2 M_0, & \text{for any } \ \tilde{r} \le Q, \\ \varepsilon_2^{1/32}\omega^6 M_0, & \text{for } \ \tilde{r} \le Q^{1/4}; \end{cases} \quad M_{26} \ll \begin{cases} \varepsilon_2^3 M_0, & \text{for any } \ \tilde{r} \le Q, \\ \varepsilon_2^{3/64}\omega^9 M_0, & \text{for } \ \tilde{r} \le Q^{1/4}. \end{cases}$$

Combining all the above estimates, the proof of Lemma 5.4 is complete.

6. Proof of Theorem 1.

LEMMA 6.1. *Let $I_2(b)$ be defined as in [LW, (4.7)], then we have*

$$I_2(b) \ll N^2 Q^{-1/2} \left|a_1 a_2 a_3\right|^{-1/2} \mathcal{L}^5.$$

Proof. This is [LW, Lemma 8.1].

LEMMA 6.2. *Let $I_1(b)$ be defined as in [LW, (4.7)] and θ be given as in (2.1). Under the assumptions of Lemma 5.3, and (2.2) we have*

$$I_1(b) \gg N^2 \left|a_1 a_2 a_3\right|^{-1} Q^{-3/8} \mathcal{L}^{-3}.$$

Proof. We consider two cases according to $\tilde{r} \le Q^{1/4}$ or not. If $\tilde{r} \le Q^{1/4}$, then by [LW, (5.2)] and Lemmas 5.2, 5.4, and then by (5.1) and (5.2), we get

$$\begin{aligned} &I_1(b) \\ &\ge \omega^3 M_0 + O\left(\varepsilon_2^{1/64}\omega^3 M_0 + \Omega_1 + \tilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}\right) \\ &\gg \omega^3 M_0 + O\left(N^2 Q^{2.5+\varepsilon_1} \left|a_1 a_2 a_3\right|^{-1} T^{-1} \mathcal{L}^2 + \tilde{r} N^2 Q^{-1+\varepsilon_1} \left|a_1 a_2 a_3\right|^{-1} + N^2 Q^{-1-\varepsilon_1}\right). \end{aligned}$$

In view of (3.3), Lemma 5.3, [LW, (5.7)] and $T = Q^3$ (in (2.1)), the above O-term can be absorbed and hence

$$I_1(b) \gg N^2 \left|a_1 a_2 a_3\right|^{-1} Q^{-3/8} \mathcal{L}^{-3},$$

providing that $Q \ge \max\left\{A^5,\ K(\varepsilon_2)\right\}$, where $K(\varepsilon_2)$ is a sufficiently large positive number depending only on the given small $\varepsilon_2 > 0$.

If $\tilde{r} > Q^{1/4}$, then by the same arguments as above except that now we use [LW, Lemma 5.6] instead of our Lemma 5.2 and that we do not use (3.3), we have

$$\begin{aligned} I_1(b) &\ge M_0 + O(\varepsilon_2 M_0) + O\left(\Omega_1 + \Omega_2 + N^2 \tilde{r}^{-1} \log^3 \mathcal{L} + N^2 Q^{-1-\varepsilon_1}\right) \\ &\gg M_0 + O\Big\{N^2 Q^{2.5+\varepsilon_1} \left|a_1 a_2 a_3\right|^{-1} T^{-1} \mathcal{L}^2 + N^2 Q^{-1+\varepsilon_1} \left|a_1 a_2 a_3\right|^{-1} \\ &\quad + N^2 \tilde{r}^{-1} \log^3 \mathcal{L} + N^2 Q^{-1-\varepsilon_1}\Big\} \gg N^2 \left|a_1 a_2 a_3\right|^{-1}, \end{aligned}$$

providing that $Q \ge \max\left\{A^{12+\varepsilon_1}, K(\varepsilon_2)\right\}$. The proof of Lemma 6.2 is complete.

Proof of Theorem 1. We separate our arguments into two cases, according to (2.6) holds or not, to prove that $I(b)$ (see [LW, (4.6)]), has the lower bound $N^{1.9}$. Then Theorem 1 follows immediately. When the exceptional zero $\tilde{\beta}$ in Lemma 2.1 exists and satisfies (2.6), then we write $I(b)$ in the form [LW, (4.7)]. Thus by [LW, (4.7)], Lemmas 6.1 and 6.2 and (2.1), we have, under the assumptions (i) or (ii) in Lemma 5.3,

$$\begin{aligned} I(b) = I_1(b) + I_2(b) &\gg N^2 \left|a_1 a_2 a_3\right|^{-1} Q^{-3/8} \mathcal{L}^{-3} + O\left(N^2 Q^{-1/2} \left|a_1 a_2 a_3\right|^{-1/2} \mathcal{L}^5\right) \\ &\gg N^2 \left|a_1 a_2 a_3\right|^{-1} Q^{-3/8} \mathcal{L}^{-3} \gg N^2 Q^{-5/8} \mathcal{L}^{-3} \gg N^{1.9}, \end{aligned}$$

providing that $Q \geq \max\{A^{12+\varepsilon_1}, K(\varepsilon_2)\}$, where $K(\varepsilon_2)$ is a sufficiently large positive number depending only on ε_2. That is to say $N \gg A^{144}$.

When the exceptional zero $\tilde{\beta}$ in Lemma 2.1 does not exist or it exists and satisfies $\omega > \varepsilon_2$, then, in view of (2.5) and $(1-\tilde{\beta})\mathcal{L} = \omega$ (in (2.6)), all zeros ρ with $|\mathrm{Im}\rho| \leq T$ of the function defined by (2.4) lie in the region

$$\sigma \leq 1 - \min\{c_0,\ \varepsilon_2\}/\mathcal{L}. \tag{6.1}$$

Note that the parameter Q in [LW, (4.1)] equals to $N^{1/(15-11\varepsilon_1)}$, while $Q = N^{1/(12-\varepsilon_1)}$ (see (2.1)) in this paper. Thus by (6.1), we see that the possible exceptional $\tilde{\beta}$ in [LW, Proposition 2.3] satisfies

$$\tilde{\beta} \leq 1 - 4\min\{c_0,\ \varepsilon_2\}/(5\log N^{1/(15-11\varepsilon_1)}).$$

Therefore, the second inequality in [LW, (2.13)] now is redundant. Consequently, all the arguments in [LW] become effective. Since the ω there now is always $\gg 1$, we can use the final result there on the lower bound for $I(b)$ (i.e. $I(b) \gg \omega^3 N^2 |a_1 a_2 a_3|^{-1}$) to give

$$I(b) \gg N^{1.9},$$

provided that $N \geq CA^{45}$ or $N \geq \max\{3|b|,\ CA^{45}\}$ according to the validity of conditions (i) or (ii) in Lemma 5.3. As the non-effective second inequality in [LW, (2.13)] has not been used, the above positive C are sufficiently large computable absolute constants. The proof of Theorem 1 is complete.

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