ASIAN J. MATH.
Vol. 4, No. 4, pp. 977–986, December 2000

014

ON THE DISTRIBUTION OF CERTAIN HUA SUMS*

S. J. PATTERSON†

1. Introduction. Let $f(x)$ be a polynomial with integral coefficients. Let, for $c \geq 1$

$$S\big(f(x), c\big) = \sum_{j=0}^{c-1} e^{2\pi i f(j)/c}$$

be the corresponding Hua sum. One has the estimate, due to Hua,

$$S\big(f(x), c\big) = O(c^{1-\frac{1}{n}})$$

where $n = \deg(f)$. In this paper we shall study the asymptotic behaviour of

$$\sum_{c \leq X} S\big(f(x), c\big)$$

in the special case of $f(x) = a\ x^3$. In general there is some numerical evidence to suggest that

$$\sum_{c \leq X} S\big(f(x), c\big) \sim K(f) \cdot X^{1+\frac{1}{n}}$$

unless f is symmetric in the sense that there exists $r \in \mathbb{Z}$ so that $f(r - x) = f(x)$. If this is so, and it can only happen if n is even, there is a little evidence to suggest that

$$\sum_{c \leq X} S\big(f(x), c\big) \sim K'(f) \cdot X^{1+\frac{2}{n}}$$

when $n > 2$. I shall discuss these problems in a paper which is, at present, in preparation.

In the case of $\deg(f) = 3$ then it is possible to prove results of this type for a certain class of f over $\mathbb{Z}[\omega](\omega^2 + \omega + 1 = 0)$. This is the subject of a forthcoming paper by R. Livné and the author ([LP]). The method used there is based on the theory of cubic metaplectic forms and an identity due to W. Duke and H. Iwanieć, [DI]. Unexpectedly for certain very special f, namely those of the form $a_3 \cdot x^3$ one has, at least for some a_3

$$\sum_{\substack{c: N(c) \leq X \\ c \equiv 1 \pmod 3}} \widetilde{S}(a_3 x^3, c) \sim k(a_3) \cdot X^{\frac{4}{3}} \log X$$

where, here, $\widetilde{S}$ ist the analogue of S for $\mathbb{Z}[\omega]$ and $c \in \mathbb{Z}[\omega]$. The purpose of this paper is to use the theory of metaplectic forms again to prove the following:

*Received June 29, 2000; accepted for publication October 4, 2000.

†Department of Mathematics, Georg-August-University, Bunsenstrasse 3-5, D-37073 Goettingen, Germany (sjp@uni-math.gwdg.de).

THEOREM. *Let $A \in \mathbb{Z}, A \neq 0$. Then there is a constant $k(A)$ so that, for any $\varepsilon > 0$*

$$\sum_{c \leq X} S(A \cdot x^3, c) = k(A) X^{\frac{4}{3}} + O(X^{\frac{5}{4}+\varepsilon}).$$

One can give an explicit formula for $k(A)$ but it is not illuminating (see §3).

It is worth noting that D.R. Heath-Brown ([HB]) has shown that, for any $\varepsilon > 0$,

$$\sum_{\substack{p \leq X \\ p \text{ prime}}} S(Ax^3, p) = O(X^{\frac{4}{3}+\varepsilon}).$$

This is a considerable sharpening of a result due to Heath-Brown and Patterson [HBP].

This paper has its origins in a joint project with R. Livné (Jerusalem).

2. The sums $S(Ax^3, c)$. In this section we shall gather together the information on the sums $S(Ax^3, c)$ that we need. Most of this is well-known but for our purposes we have to be a little more precise than is necessary in other applications.

LEMMA 2.1. *If c_1, c_2 are coprime then*

$$S(Ax^3, c_1\, c_2) = S(A\, c_2^2\, x^3, c_1) \cdot S(A\, c_1^2\, x^3, c_2).$$

Proof. See [V], Lemma 2.10.

LEMMA 2.2. *One has*

$$S(dA\, x^3, dc) = dS(Ax^3, c)$$

Proof. This is clear.

LEMMA 2.3. *Suppose that p is a prime, $p \nmid A$. Then*

$$p^{-\frac{2}{3}k} S(Ax^3, p^k)$$

depends only on $k \pmod 3$. In fact one has:

$$\begin{aligned} p^{-\frac{2}{3}k} S(Ax^3, p^k) &= 1 && \text{if } k \equiv 0 \pmod 3 \\ &= p^{-\frac{2}{3}} S(Ax^3, p) && \text{if } k \equiv 1 \pmod 3 \\ &= p^{-\frac{1}{3}} && \text{if } k \equiv 2 \pmod 3, p \neq 3 \\ &= p^{-\frac{1}{3}} \left(1 - 2\cos\left(\tfrac{2\pi A}{9}\right)\right) && \text{if } k \equiv 2 \pmod 3, p = 3 \end{aligned}$$

Also if $p \equiv 2 \pmod 3$, or if $p = 3$ then

$$S(Ax^3, p) = 0.$$

This lemma is the basis of the proof of Hua's lemma [H], §13 specialized to this case.

Proof. Let $S^*(Ax^3, p^k) = \sum_{\substack{j \pmod{p^k} \\ (j,p)=1}} e(Aj^3, p^k)$. Then, by the usual linearization argument (see [H], *loc. cit.*)

$$S^*(Ax^3, p^k) = 0 \quad \text{if } p \neq 3 \text{ and } k \geq 2 \text{ or } p = 3 \text{ and } k \geq 3.$$

On the other hand

$$S(Ax^3, p^k) = \sum_{j=0}^{k} S^*(p^{2j}Ax^3, p^{k-j})$$

if we organize the summands by the power of p dividing them. If $p \neq 3$ and $k-3j \geq 2$ or if $p = 3$ and $k - 3j \geq 3$ then the corresponding summand will vanish. If $k - 3j \leq 0$ we obtain $S^*(p^{2j}Ax^3, p^{k-j}) = \varphi(p^{k-j})$. Thus we have, if $p \neq 3$

$$S(Ax^3, p^k) = \sum_{\substack{3j \geq k \\ j \leq k}} \varphi(p^{k-j}) + p^{2j_1} \cdot S^*(Ax^3, p)$$

where $j_1 : k - 3j_1 = 1$; if j_1 does not exist then the last term is taken to be zero. If $p = 3$ then we have:

$$\begin{aligned} S(Ax^3, p^k) = \sum_{\substack{3j \geq k \\ i \leq k}} \varphi(p^{k-j}) \quad &+ p^{2j_1} S^*(Ax^3, p) \\ &+ p^{2j_2} S^*(Ax^3, p^2) \end{aligned}$$

where $j_1 : k - 3j_1 = 1$ and $j_2 : k - 3j_2 = 2$ with the same convention as above. These simplify to, if $p \neq 3$,

$$S(Ax^3, p^{3K+a}) = \begin{cases} p^{2K}, & \text{if } a = 0 \\ p^{2K} S(Ax^3, p), & \text{if } a = 1 \\ p^{2K+1}, & \text{if } a = 2. \end{cases}$$

If $p \equiv 2 \pmod 3$ then it follows easily that $S(Ax^3, p) = 0$. If $p = 3$ then we deduce that

$$S(Ax^3, p^{3K+a}) = \begin{cases} p^{2K}, & \text{if } a = 0 \\ p^{2K} S(Ax^3, p), & \text{if } a = 1 \\ p^{2K} S(Ax^3, p^2), & \text{if } a = 2. \end{cases}$$

One verifies easily in this case that $S(Ax^3, p) = 0$ and $S(Ax^3, p^2) = 6 \cdot \cos\left(\frac{2\pi A}{9}\right) + 3$. This completes the proof of the lemma.

LEMMA 2.4. *Let $p \equiv 1 \pmod 3$ and let $p = \pi\bar{\pi}$ be a decomposition of p in prime factors in $\mathbb{Z}[\omega](\omega^2 + \omega + 1 = 0)$, with $\pi \equiv 1 \pmod 3$. Then*

$$S(2Ax^3, p) = g(A, \varepsilon, \pi) + \overline{g(A, \varepsilon, \pi)}$$

where

$$g(A, \varepsilon, \pi) = \sum_{y \pmod \pi} \varepsilon\left(\left(\frac{y}{\pi}\right)_3\right) e\left(\mathrm{Tr}\left(\frac{Ay}{\pi}\right)\right);$$

here ε is an injection of the group $\{1, \omega, \omega^2\}$ into $\mathbb{C}^\times$ and $(\div)_3$ denotes the cubic residue symbol in $\mathbb{Z}[\omega]$. The function Tr *is the trace $\mathbb{Q}(\omega) \to \mathbb{Q}$.*

Proof. This is well-known - see [HB-P].

We now deduce from Lemmas 2.1, 2.2. and 2.3 the following:

LEMMA 2.5. *If there exists a prime p so that*

$$\mathrm{ord}_p(c) \geq 3 + \mathrm{ord}_p(A)$$

then

$$\sum_{d:d^3|c} S(Ax^3, c/d^3)\cdot \mu(d)d^2 = 0.$$

In view of Hua's lemma the series

$$F_1(A,s) = \sum_{c=1}^{\infty} S(Ax^3, c)c^{-s}$$

converges if $\mathrm{Re}(s) > \frac{5}{3}$. We define

$$F(A,s) = F_1(A,s)\zeta(3s-2)^{-1},$$

which is also defined for $\mathrm{Re}(s) > \frac{5}{3}$. We also note that by Lemma 2.5 if

$$F(A,s) = \sum_{c=1}^{\infty} t(A,c)c^{-s}$$

where $t(A,c) = 0$ if there is a prime p so that $\mathrm{ord}_p(c) \geq \mathrm{ord}_p(A) + 3$. Next we have from Lemma 2.1 together with the observation that

$$S(A\delta^3x^3, c) = S(Ax^3, c)$$

if δ is a coprime to c, the following fact.

LEMMA 2.6. *One has*

$$t(A, c_1c_2) = t(c_1^2A, c_2)t(c_2^2A, c_1)$$

if c_1 and c_2 are coprime.

We shall now transform $\sum t(A,c)\cdot c^{-3}$ by writing

$$c = c_1c_2$$

where c_2 is coprime to $2\cdot 3A$ and the only primes dividing c_1 are those dividing $2\cdot 3\cdot A$. We then have

$$F(A,S) = \sum_{c_1} c_1^{-s} \sum_{c_2} t(c_2^2A, c_1)\cdot t(c_1^2A, c_2)c_2^3.$$

We observe that the sum over c_1 is *finite* and that $t(c_2^2A, c_1)$ depends only on the residue class of $c_2 \pmod{c_1}$. Thus we can write $F(A,s)$ as

$$\sum_{c_1} c_1^{-s} \sum_{\substack{d \pmod{c_1} \\ (c_1,d)=1}} t(d^2A, c_1)\cdot \sum_{\substack{c_2\equiv d \pmod{c_1} \\ (c_2,6A)=1}} t(c_1^2A, c_2)c_2^{-s}.$$

In order to bring this into a usable form we need the following lemma:

LEMMA 2.7. *If c, A are such that $(c, 6A) = 1$ then $t(A, c) = 0$ unless c is a norm from $\mathbb{Z}[\omega]$; in this case*

$$t(A,c) = \sum_{\substack{\gamma : N(\gamma) = c \\ \gamma \equiv 1 \pmod 3}} g(4A, \varepsilon, \gamma)$$

where $g(A, \varepsilon, \gamma)$ is defined as

$$g(A,\varepsilon,\gamma) = \sum_{y \pmod \gamma} \varepsilon\left(\left(\frac{y}{\gamma}\right)_3\right) e\left(Tr\left(\frac{Ay}{\gamma}\right)\right)$$

are before.

Proof. We verify first that it suffices to prove this formula in the case of a prime power. This is the case because when we have $c = c_1 c_2$ with $(c_1, c_2) = 1$ then the $\gamma : N(\gamma) = c$ are of the form $\gamma_1\gamma_2$, uniquely as $\gamma \equiv 1 \pmod 3, \gamma_1 \equiv 1 \pmod 3, \gamma_2 \equiv 1 \pmod 3$. Now

$$g(A,\varepsilon,\gamma_1\gamma_2) = \varepsilon\left(\left(\frac{\gamma_1}{\gamma_2}\right)_3\right) \cdot \varepsilon\left(\left(\frac{\gamma_2}{\gamma_1}\right)_3\right) \cdot g(A,\varepsilon,\gamma_1) \cdot g(A,\varepsilon,\gamma_2).$$

Now we note that

$$\begin{aligned} \left(\frac{\overline{\gamma_1}}{\gamma_2}\right)_3 &= \overline{\left(\frac{\gamma_1}{\overline{\gamma_2}}\right)_3} \\ &= \overline{\left(\frac{\overline{\gamma_2}}{\gamma_1}\right)_3} \\ &= \left(\frac{\overline{\gamma_2}}{\gamma_1}\right)_3^{-1}. \end{aligned}$$

The $\bar{\ }$ denotes the non-trivial involution of $\mathbb{Q}(\omega)/\mathbb{Q}$; the first equality follows by functorality, the second by the law of cubic reciprocity; the last one is trivial. Consequently

$$\begin{aligned} g(A,\varepsilon,\gamma_1\gamma_2) &= \varepsilon\left(\left(\frac{\gamma_1\overline{\gamma_1}}{\gamma_2}\right)_3\right) \cdot \varepsilon\left(\left(\frac{\gamma_2\overline{\gamma_2}}{\gamma_1}\right)_3\right) g(A,\varepsilon,\gamma_1) g(A,\varepsilon,\gamma_2) \\ &= \varepsilon\left(\left(\frac{c_1}{\gamma_2}\right)_3\right) \varepsilon\left(\left(\frac{c_2}{\gamma_1}\right)_3\right) g(A,\varepsilon,\gamma_1) g(A,\varepsilon,\gamma_2) \\ &= g(c_2^2 A,\varepsilon,\gamma_1) \cdot g(c_1^2 A,\varepsilon,\gamma_2). \end{aligned}$$

That $\sum_{\substack{\gamma \pmod c \\ \gamma \equiv 1 \pmod 3}} g(A,\varepsilon,\gamma)$ has the multiplicativity of Lemma 2.6 follows immediately, and therefore that we need only prove the formula in the case of a prime power. Suppose $p \neq 3$; then

$$t(A,p) = S(Ax^3, p)$$

and the result follows from Lemma 2.3. If $p \equiv 1 \pmod 3$ then $t(A,p^2) = p = g(A,\varepsilon,\pi) \cdot \overline{g(A,\varepsilon,\pi)} = g(A,\varepsilon,\pi) \cdot g(A,\varepsilon,\bar{\pi})$. If $p \equiv 2 \pmod 3$ then $t(A,p^2) = g(A,\varepsilon,p) = p$ as is well known, so that the identity also holds here. If $k \geq 3$ the $t(A,p^k) = 0$ so that the result also holds in this case as well. We have therefore verified it in all cases.

We can now summarize the conclusions of this section in the following theorem:

THEOREM 2.8. *The series* $\sum_{c=1}^{\infty} S(Ax^3, c)c^{-s}$ *converges absolutely if* $\mathrm{Re}(8) > \frac{5}{3}$ *and it is equal to*

$$\zeta(3s-2) \cdot \sum_{c_1} c_1^{-s} \cdot \sum_{\substack{d \pmod{c_1} \\ (d,c_1)=1}} t(d^2A, c_1) \sum_{\substack{\gamma : N(\gamma) \equiv d \pmod{c_1} \\ (\gamma, 6A)=1 \\ \gamma \equiv 1 \pmod 3}} g(4A, \varepsilon, \gamma) N(\gamma)^{-s}$$

where c_1 *contains only primes dividing* $6A$ *and* t *is defined by*

$$t(A,c) = \sum_{\delta^3 | c} S(Ax^3, c/\delta^3)\mu(\delta) \cdot \delta^2.$$

The sum over c_1 *ist finite; indeed for each* p *dividing* $6A$ *we have* $\mathrm{ord}_p(c_1) \le \mathrm{ord}_p(A) + 3$.

COROLLARY 2.9. *The series* $\sum_{c=1}^{\infty} S(Ax^3, c) \cdot c^{-s}$ *converges for* $\mathrm{Re}(s) > \frac{3}{2}$.

Proof. This follows by using the majorant

$$\sum_{\substack{\gamma : N(\gamma) \equiv d \pmod{c_1} \\ (\gamma, 6A)=1 \\ \gamma \equiv 1 \pmod 3}} | g(4A, \varepsilon, \gamma) | N(\gamma)^{-s},$$

which clearly converges in $\mathrm{Re}(s) > \frac{3}{2}$.

3. The analytic theory. In this section we shall recall some aspects of the theory of cubic metaplectic forms over $\mathbb{Q}(\omega)$ which we shall need. From now on we shall fix ε by $\varepsilon(\omega) = e^{2\pi i/3}$ and drop it from our notations. Let, for $\mathrm{Re}(s) > \frac{3}{2}$

$$\psi(r,s) = \sum_{\gamma \equiv 1 \pmod 3} g(r,\gamma) \cdot N(\gamma)^{-s}$$

where the sum is taken over $\mathbb{Z}[\omega]$. Let $k = \mathbb{Q}(\omega)$ and let $\zeta_k(s)$ be the Dedekind zeta function of k. Then (see[P1], Theorem 6.1) it is known that $\psi(r,s)\zeta_k(3s-2)$ has a analytic continuation as a meromorphic function to the entire plane. It is of finite order and has only one pole in $\mathrm{Re}(s) \ge 1$, at $s = \frac{4}{3}$ and it is simple. The residue at $s = \frac{4}{3}$ is (by [P1], Theorem 9.1)

$$K\tau(r)/N(r)^{\frac{1}{6}}$$
$$\text{where} \quad K = \frac{(2\pi)^{\frac{5}{3}}}{3^2 \cdot 2^4} \frac{1}{\Gamma(\frac{2}{3})} \frac{1}{\zeta_k(2)}$$

and, if $\lambda = \sqrt{-3}$

$$\tau(r) = \begin{cases} 2\overline{g(\lambda^2, c)} \mid d/c \mid 3^{n/2} & \text{if } r = \pm\lambda^{3n-1}cd^3 \\ 2e^{-2\pi i/9}\overline{g(\omega\lambda^2, c)} \mid d/c \mid 3^{n/2} & \text{if } r = \pm\omega\lambda^{3n-1}cd^3 \\ 2e^{2\pi i/9}\overline{g(\omega^2\lambda^2, c)} \mid d/c \mid \cdot 3^{n/2} & \text{if } r\,\omega^2\lambda^{3n-1}cd^3 \\ 2\overline{g(1, c)} \mid d/c \mid 3^{n/2} & \text{if } r = \pm\lambda^{3n-3}cd^3 \\ 0 & \text{otherwise} \end{cases}$$

where $c, d \equiv 1 \pmod 3$, c is square-free and $n \geq 0$. Note that this is a multiple of the function of [P1] by the factor $2/3^{5/2}$.

Let now T be a finite set of finite places of k and, for each $v \in T$ let q_0 be the order of the residue class field at v. Then let

$$\psi_T(r,s) = \sum_{\substack{\gamma \equiv 1 \pmod 3 \\ \gamma \text{ coprime to } T}} g(r,\gamma)N(\gamma)^{-s}$$

where γ coprime to T means that for each $v \in T$ we have $|\gamma|_v = 1$. Then we have that $\psi_T(r,s)$ also has an analytic continuation of the same type as above and that

$$\operatorname*{Res}_{s=4/3} \psi_T(r,s) = \prod_{v \in T}(1+q_v^{-1})^{-1} \cdot \frac{4}{3} \cdot K \cdot \tau(r)/N(r)^{1/6};$$

see [P2], p. 180 and [KP], §II.3. We next need twisted versions of this which, fortunately, follow from it. Let λ denote the place of k dividing 3 and let $U_\lambda = \{x \in k_\lambda \mid \mathrm{ord}_\lambda(x-1) \geq 2\}$. Then $[U_\lambda : U_\lambda \cap k_\lambda^{x3}] = 9$. For $v \neq \lambda$ let $U_v : \{x \in k_v \mid \mathrm{ord}_v(x) = 0\}$. Then $[U_v : U_v \cap k_v^{x3}] = 3 (v \neq \lambda)$. The results quoted above show in fact further that if V is an open subgroup of $\prod_{v \in T}(U_v \cap k_v^{x3}) \subset \prod_{v \in} k_v^x$ and if $\alpha \in \prod_{v \in T} U_v$ then

$$\psi_T(r,\alpha V,s) = \sum_{\substack{\gamma \in \mathbb{Z}[\omega] \\ \gamma \in x V}} g(r,\gamma)N(\gamma)^{-s}$$

has analogous properties to $\psi_T(r,s)$ and $\operatorname*{Res}_{s=4/3} \psi_T(r,\alpha V,s) = [V_0 : V]^{-1} \operatorname*{Res}_{s=4/3} \psi_T(r,\alpha V_0,s)$ where $V_0 = \prod_{v \in T}(U_v \subset k_v^{x3})$. We note here also that $U_v \cap k_v^{x3} = U_v^3$. For each $v \in T, v \neq \lambda$ let π_v be the prime associated with v satisfying $\pi_v \equiv 1 \pmod 3$. Let $(\cdot,\cdot)_v$ denote the cubic Hilbert symbol at v. Then every character of order 3 on $\prod_{v \in T} U_v$ is of the form

$$\prod_{v \neq \lambda} \varepsilon(\pi_v^{f_v},\cdot)_v \cdot \varepsilon(\lambda^{f_\lambda}\omega^{g_\lambda},\cdot)_\lambda$$

where $f_v \in \mathbb{Z}/3\mathbb{Z} (v \neq \lambda)$, $f_\lambda, g_\lambda \in \mathbb{Z}/3\mathbb{Z}$. Let

$$M(T) = \left\{ \prod_{\substack{v \in T \\ v \neq \lambda}} \pi_v^{f_v} \cdot \lambda^{f_\lambda} \cdot \omega^{g_\lambda} \mid 0 \leq f_v < 3 (v \in T), 0 \geq g\lambda < 3 \right\}.$$

Then $\mathrm{Card}(M(T)) = 3^{1+\mathrm{Card}(T)}$. Moreover, for $\mu \in M(T)$ we have

$$\sum_{\gamma \equiv 1 \pmod 3} g(r,c)\varepsilon\Big(\big(\frac{\mu}{c}\big)_3\Big)^{-1} \cdot N(c)^{-s} = \sum_{\gamma \equiv 1 \pmod 3} g(r\mu,c)N(c)^{-s}.$$

Then, from the results of [P2], p. 180, [KP], §II.3 we have:

THEOREM 3.1. *Let $T, U_v(\nu \in T), V_0$ be as above and let $\alpha \in \mathbb{Z}[\omega] \cap \prod_{v \in T} U_v$. Let V be an open subgroup of V_0. Then $\psi_T(r,\alpha V,s)$ has an analytic continuation as a* *meromorphic function to* $\mathbb{C}$. *If* $\mathrm{Re}(s) \geq 1, s \neq \frac{4}{3}$ *then* $\psi_T(r, \alpha V, s)$ *is holomorphic at* $s; \psi_T(r, \alpha V, s)$ *has at most a simple pole at* $s = 4/3$. *Let* $\varepsilon > 0$; *then we have*

$$\psi_T(r, \alpha V, \sigma + i\,t) = O(|\,t\,|^{3-2\sigma+\varepsilon})$$

in $1+\varepsilon \leq \sigma \leq \frac{3}{2}+\varepsilon, |\,t\,| \geq 1$. *The residue of* $\psi_T(r, \alpha V, s)$ *at* $s = \frac{4}{3}$ *is*

$$\frac{4}{9} \cdot \frac{K}{[V_0 : V]} \prod_{v \in T} \frac{q_v}{3(1+q_v)} \sum_{\mu \in M(T)} \tau(\mu r) N(\mu r)^{-\frac{1}{6}} \cdot \varepsilon\big((\frac{\mu}{\alpha})_3\big).$$

Note that the very formulation of this theorem assumes the law of cubic reciprocity. We can now combine this theorem with Theorem 2.8. Let, for any $x \in \mathbb{Z}[\omega]$, Supp (χ) denote the set of places v of k for which $|\,\chi\,|_\nu < 1$. Then we have:

THEOREM 3.2. *The function* $F_1(A, s)$, *defined in* $\mathrm{Re}(s) > \frac{3}{2}$ *by* $\sum_{c=1}^{\infty} S(Ax^3, c)c^{-s}$ *has an analytic continuation as a meromorphic function to* $\mathbb{C}$. *In* $\mathrm{Re}(s) \geq 1$ *it has at most a simple pole at* $s = 4/3$, *and the residue is*

$$\frac{2\pi^2 K}{3} \sum_{c_1} c_1^{-4/3} N([3, c_1])^{-1} \sum_{\substack{\gamma_1 (\mathrm{mod}\ [3, c_1]) \\ (\gamma_1, c_1) = 1 \\ \gamma_1 \equiv 1 (\mathrm{mod}\ 3)}} \sum_{\mu \in M(\mathrm{Supp}(\lambda c_1))} \prod_{v/3c_1} \frac{q_v}{3(1+q_0)}$$
$$\cdot t(N(\gamma_1)^2 A c_1) \tau(4A\mu) N(4A\mu)^{-\frac{1}{6}} \varepsilon\big((\frac{\mu}{\gamma_1})_3\big).$$

For any $\varepsilon > 0$ *we have in* $\sigma : 1+\varepsilon < \sigma < \frac{3}{2}+\varepsilon, t :|\,t\,| > 1$

$$F_1(A, \sigma + i\,t) = O(|\,t\,|^{3-2\sigma+\varepsilon}).$$

REMARK. The sum over γ_1 in Theorem 3.2. is taken in $\mathbb{Z}[\omega]$. Unfortunately I have not been able to simplifiy this rather opaque expression.

Proof. This is mainly a matter of organizing the formulae. First of all we can rewrite the representation of $F_1(A, s)$ of Theorem 2.8 as

$$\zeta(3s-2) \sum_{c_1} c_1^{-s} \sum_{\substack{\gamma_1 \ (\mathrm{mod}\ c_1) \\ (\gamma_1, c_1) = 1}} t(N(\gamma_1)^2 A, c_1) \cdot \sum_{\substack{\gamma \equiv \gamma_1 \pmod{c_1} \\ (\gamma, 6A) = 1 \\ \gamma \equiv 1 \pmod 3}} g(4A, \gamma) N(\gamma)^{-s}$$

where $\gamma_1 \in \mathbb{Z}[\omega]$ and ((mod c_1) is to be understood in the context of $\mathbb{Z}[\omega]$. Now let, with $T = \mathrm{Supp}(6A)$,

$$V(c_1) = \prod_{v \in T} \{U \in U_v \mid U \equiv 1 \pmod{c_1}\}$$

where the (mod c_1) is to be understood in the ring of integers of k_ν. Then $F_1(A, s)$ is

$$\zeta(3s-2) \sum_{c_1} c_1^{-s} \sum_{\substack{\gamma_1 \pmod{[3, c_1]} \\ (\gamma_1, c_1) = 1 \\ \gamma_1 \equiv 1 \pmod 3}} t(N(\gamma_1)^2 A, c_1) \cdot \psi_T(4A, \gamma_1 V(c_1), s).$$

where [,] denotes the least common multiple. Since the sum over c_1 is finite the theorem follows from this representation.

The Theorem of §1 follows immediately from this in the usual way.

REFERENCES

[DI] W. Duke and H. Iwanieć, *A relation between cubic exponential sums and Kloosterman sums*, Contemp. Math., 143 (1993), pp. 255–258.

[HB] D. R. Heath-Brown, *Kummer's conjecture for cubic Gauss sums*, Israel J. Math., To appear.

[HBP] D. R. Heath-Brown and S. J. Patterson, *The distribution of Kummer sums at prime arguments*, J. reine angew. Math., 310 (1979), pp. 111–130.

[KP] D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*, Publ. Math. IHES, 59 (1984), pp. 35–142.

[LP] R. Livné and S. J. Patterson, *The first moment of cubic exponential sums*, In preparation.

[P1] S. J. Patterson, *A cubic analogue of the theta series*, J. reine angew. Math., 296 (1977), pp. 125–161 & pp. 217–220.

[P2] S. J. Patterson, *The distribution of general Gauss sums at prime arguments*, in Recent Progress in Analytic Number Theory, vol. 2, H. Halberstam and C. Hooley, eds., Academic Press, 1981, pp. 171–182.

[V] R. C. Vaughan, *The Hardy-Littlewood Method*, Cambridge Tracts in Mathematics 80, CUP, 1981.