THE CHOW GROUP MODULO l FOR THE TRIPLE PRODUCT OF A GENERAL ELLIPTIC CURVE*

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0. Introduction. The purpose of this note is to prove

(0.1) **THEOREM.** *Let $\mathbb{Q} \subset k$ be an algebraically closed extension of positive transcendence degree. Let E be an elliptic curve over k whose J -invariant is transcendental over $\mathbb{Q}$. If $l \in \{5, 7, 11, 13, 17\}$, then $CH^2(E^3)/l$ is infinite.*

The proof is based on a fundamental result of Bloch and Esnault [Bl-Es] which provides a criterion for a nullhomologous cycle to give a non-zero element in the second Chow group modulo l . In order to apply the criterion, one must have information concerning the image of the cycle under the l -adic Abel-Jacobi map. We will work with the Ceresa cycle [Ce] constructed from a particular genus 3 curve whose Jacobian is isogenous to the triple product of a general elliptic curve. In this case computations of l -adic Abel-Jacobi maps were done in [Bu-Sch-Top] using techniques adapted from a paper of Bloch, [Bl]. One may conclude that the class of the Ceresa cycle is not divisible by l in an appropriate Chow group. To show that the Chow group modulo l is indeed infinite, we study the orbit of the Ceresa cycle under the action of modular correspondences as in Nori's paper, [No].

Examples of varieties, W , over algebraically closed fields of characteristic 0 and primes l for which $CH^2(W)/l$ is infinite were given in [Sch]. The results of that paper apply only to varieties which have complex multiplication cycles or to closely related varieties. Since one may speculate that the Chow group modulo l is infinite for at least a moderately large class of varieties, it seems desirable to have examples which don't depend on the presence of complex multiplication cycles. This paper provides one such example.

Notations. $Z^r(W)$ = the group of cycles of codimension r on a separated scheme, W , of finite type over a field.

$CH^r(W)$ = the group of cycles of codimension r modulo cycles which are rationally equivalent to zero [Fu, §1].

$A[m] := \text{Ker} : A \xrightarrow{\cdot m} A$, where A is an abelian group and $\cdot m$ is multiplication by an integer, m .

$A[l^\infty] := \bigcup_{n \in \mathbb{N}} A[l^n]$.

1. Overview of the proof. Let F denote the field of rational functions in one variable, $\mathbb{Q}(a)$, and let $\bar{F}$ denote its algebraic closure. Define an elliptic curve, E/F , by

$$(1.1) \quad f_E(x, y, z) := y^2 z - [(a^2 - 4)x^3 + (2a^2 - 4a)x^2 z + (a^2 - 4)xz^2] = 0.$$

Theorem (0.1) will follow easily from

(1.2) **THEOREM.** *If $l \in \{5, 7, 11, 13, 17\}$, then $CH^2(E_{\bar{F}}^3)/l$ is not finite.*

Proof. To begin, we use the genus 3 curve,

$$(1.3) \quad C : x^4 + y^4 + z^4 + a(x^2 y^2 + y^2 z^2 + z^2 x^2) = 0,$$

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and the map [Bu-Sch-Top, 4.2],

$$\pi : C \rightarrow E,$$

corresponding to the field extension, $F(E) \rightarrow F(C)$,

$$(1.4) \quad x/z \mapsto (2(x/z)^2 + (y/z)^2 + a)^2, \quad y/z \mapsto (y/z)(2(x/z)^2 + (y/z)^2 + a),$$

to construct a cycle on E^3 . The element, $\sigma \in \text{Aut}(\mathbb{P}^2)$, given by

$$x \circ \sigma = y, \quad y \circ \sigma = z, \quad z \circ \sigma = x,$$

stabilizes C and gives rise to an embedding,

$$\varrho : C \rightarrow E^3, \quad \varrho = (\pi, \pi \circ \sigma, \pi \circ \sigma^2).$$

Define a cycle,

$$(1.5) \quad \Xi := \varrho(C) - (-1)_* \varrho(C) \in Z^2(E^3).$$

It is homologous to zero because $-1 \in \text{Aut}(E^3)$ operates trivially on $H^4(E_F^3, \mathbb{Z}_l(2))$.

The proof of (1.2) now proceeds in three steps:

1. We recall the theorem of Bloch and Esnault [Bl-Es] which provides a criterion for a nullhomologous cycle to give a non-zero class in the Chow group modulo l .

2. We show that the image of a specialization of Ξ under the l -adic Abel-Jacobi map is not divisible by l . This allows us to use the theorem of Bloch and Esnault to conclude that the class of Ξ in $CH^2(E_F^3)/l$ is not zero.

3. We apply Nori's argument [No] using modular correspondences to create a subgroup, $G \subset \text{Aut}(CH^2(E_F^3)/l)$, such that the orbit of the class of Ξ under G is not finite.

2. The theorem of Bloch and Esnault. Let A be a smooth projective variety of dimension d defined over $\mathbb{Q}$. Define

$$CH_{hom}^r(A) := \text{Ker} [CH^r(A) \rightarrow \prod_{l \text{ prime}} H^{2r}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_l(r))].$$

Set $G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. There is an l -adic Abel-Jacobi map [Bu-Sch-Top, §1],

$$\mathbf{a}_l^r : CH_{hom}^r(A) \rightarrow J_l^r(A, \mathbb{Q}) := H^1(G_{\mathbb{Q}}, H^{2r-1}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_l(r))).$$

Write $\bar{\mathbf{a}}_l^r$ for the composition of $\mathbf{a}_l^r$ with the tautological map, $J_l^r(A, \mathbb{Q}) \rightarrow J_l^r(A, \mathbb{Q})/l$. Recall the definition of the first level in the coniveau filtration:

$$N^1 H^j(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)) := \text{Ker} [H^j(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)) \rightarrow H^j(\bar{\mathbb{Q}}(A), \mathbb{Z}/l(2))].$$

(2.1) THEOREM. *Let A be an Abelian variety defined over $\mathbb{Q}$ of dimension d . Let l be a prime number.*

(i) *The cycle class map of Bloch [Bl2] gives an isomorphism $\lambda_l^2 : CH^2(A_{\bar{\mathbb{Q}}})[l] \rightarrow N^1 H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$.*

(ii) *If A has good ordinary reduction at l and $d \geq 3$, then $N^1 H^3(A, \mathbb{Z}/l(2)) \neq H^3(A, \mathbb{Z}/l(2))$.*

Proof. (i) This is a consequence of [Bl2] and [Me-Su, §18] (cf. [Ra, 3.6(2)]). Details may be found in [Sch, 6.1].

(ii) This is [Bl-Es, 1.2] plus [Bl-Ka, 7.4]. $\square$

Recall that a correspondence, $P \in CH^d(A \times A) \otimes \mathbb{Z}_l$, acts on Chow groups and cohomology by,

$$P_*(z) := pr_{2*}(P \cdot pr_1^*(z)).$$

This action is compatible with the cycle class map, the coniveau filtration, and the l -adic Abel-Jacobi map [Bu-Sch-Top, 1.9(6)].

(2.2) THEOREM. *Let A be an Abelian variety defined over $\mathbb{Q}$ of dimension d which has good ordinary reduction at a prime, l . Suppose $P_*N^1H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)) = 0$. If $z \in CH_{hom}^2(A)$ is such that $\bar{\alpha}_l^2(P_*z) \neq 0$, then the class of z in $CH_{hom}^2(A_{\bar{\mathbb{Q}}})/l$ is not zero.*

Proof. This is easily deduced from [Bl-Es, §4]. For details see [Sch, 6.2]. $\square$

(2.3) COROLLARY. *Let A be an Abelian variety defined over $\mathbb{Q}$ of dimension $d \geq 3$ which has good ordinary reduction at a prime, l . Suppose that $P \in CH^d(A \times A) \otimes \mathbb{Z}_l$ is an idempotent in the ring of correspondences and*

- (i) $P_*H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$ is an irreducible $G_{\mathbb{Q}}$ -module,
 - (ii) $(Id - P)_*H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$ is contained in the image of $\lambda_l^2 : CH^2(A_{\bar{\mathbb{Q}}})[l] \rightarrow H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$,
 - (iii) there is an element, $z \in CH_{hom}^2(A)$, such that $\bar{\alpha}_l^2(P_*z) \neq 0$,
- then the class of z in $CH_{hom}^2(A_{\bar{\mathbb{Q}}})/l$ is not zero.

Proof. By (ii) and (2.1)(i) $(Id - P)_*H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)) = (Id - P)_*N^1H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$. By (2.1)(ii) $P_*N^1H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)) \neq P_*H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$. Now (i) and the compatibility of the coniveau filtration with the action of the Galois group implies that $P_*N^1H^3(A_{\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)) = 0$. The corollary follows from (2.2). $\square$

3. The nonvanishing of Ξ in $CH_{hom}^2(E_{\bar{F}}^3)/l$. This step in the proof of (1.2) relies heavily on the paper, [Bu-Sch-Top], which is an extension of a fundamental paper of Bloch [Bl].

For fixed $a \in \mathbb{Q} - \{-1, \pm 2\}$ the equation (1.1) defines an elliptic curve, E_a over the rational numbers. Similarly (1.3) defines a genus 3 curve over $\mathbb{Q}$, C_a , which admits a map, $\pi_a : C_a \rightarrow E_a$, given by (1.4). Formula (1.5) defines a cycle $\Xi_a \in Z_{hom}^2(E_a^3)$. In [Bu-Sch-Top, 2.3] a correspondence $P \in Z^3(E_a^3 \times E_a^3)$ is defined. We make this into an idempotent by multiplying by $\frac{1}{3}$. By a mild abuse of notation this idempotent will henceforth be denoted P .

- (3.1) PROPOSITION. *Suppose that $(a, l) \in \{(5, 5), (1, 7), (\frac{2}{7}, 11), (\frac{5}{2}, 13), (47, 17)\}$.*
- (i) $P_*H^3(E_{a\bar{\mathbb{Q}}}^3, \mathbb{Z}/l(2))$ is an irreducible $G_{\mathbb{Q}}$ -module.
 - (ii) $(Id - P)_*H^3(E_{a\bar{\mathbb{Q}}}^3, \mathbb{Z}/l(2))$ is contained in the image of $\lambda_l^2 : CH^2(E_{a\bar{\mathbb{Q}}}^3)[l] \rightarrow H^3(E_{a\bar{\mathbb{Q}}}^3, \mathbb{Z}/l(2))$.
 - (iii) $\bar{\alpha}_l^2(P_*\Xi_a) \neq 0$.
 - (iv) E_a^3 has good ordinary reduction at l .
 - (v) The class of Ξ_a in $CH_{hom}^2(E_{a\bar{\mathbb{Q}}}^3)/l$ is non-zero.

Proof. (i) Using the Künneth formula and the definition of P one checks that

$$P_*H^3(E_{a\bar{\mathbb{Q}}}^3, \mathbb{Z}/l(2)) \simeq \text{Sym}^3 H^1(E_{a\bar{\mathbb{Q}}}, \mathbb{Z}/l(2)).$$

The action of $G_{\mathbb{Q}}$ on $H^1(E_{a\bar{\mathbb{Q}}}, \mathbb{Z}/l)$ gives rise to a representation, $\kappa : G_{\mathbb{Q}} \rightarrow GL(2, \mathbb{Z}/l)$. The third symmetric power of the tautological representation of $SL(2, \mathbb{Z}/l)$ is irreducible for primes $l \geq 5$. Thus one need only show that $\kappa(G_{\mathbb{Q}})$ contains $SL(2, \mathbb{Z}/l)$. Let $\bar{\kappa}(G_{\mathbb{Q}})$ denote the image of $\kappa(G_{\mathbb{Q}})$ in $PGL(2, \mathbb{Z}/l)$. According to [La, p.183 and p.185] if $\kappa(G_{\mathbb{Q}})$ does not contain $SL(2, \mathbb{Z}/l)$, then either $\kappa(G_{\mathbb{Q}})$ is contained in a Borel subgroup of $GL(2, \mathbb{Z}/l)$ or the order of $\kappa(G_{\mathbb{Q}})$ is prime to l and one of the following holds:

- $\bar{\kappa}(G_{\mathbb{Q}})$ is cyclic and $\kappa(G_{\mathbb{Q}})$ is contained in a Cartan subgroup.
- $\bar{\kappa}(G_{\mathbb{Q}})$ is dihedral and $\kappa(G_{\mathbb{Q}})$ is contained in the normalizer of a Cartan sub-

group.

- $\kappa(G_{\mathbb{Q}})$ is isomorphic to A_4 , S_4 , or S_5 .

In the second row of the following table we give for each pair, (a, l) , a prime p of good reduction for E_a with the property that the geometric Frobenius, $\kappa(\text{Frob}_p^{-1})$, lies in a split Cartan subgroup and the order of $\kappa(\text{Frob}_p^{-1})$, $\mathfrak{o}_p$, is the same as the order of its image in $PGL(2, \mathbb{Z}/l)$. In the third row a prime p is given such that $\kappa(\text{Frob}_p^{-1})$ lies in a non-split Cartan subgroup of $GL(2, \mathbb{Z}/l)$ and has the indicated order, $\mathfrak{o}_p$. This information is easily deduced from the characteristic polynomial of Frob_p^{-1} , $t^2 - a_p t + p$, where $a_p = 1 + p - |E_a(\mathbb{F}_p)|$. The software, GP-PARI, was used to compute a_p .

(a, l)	(5, 5)	(1, 7)	(2/7, 11)	(5/2, 13)	(47, 17)
$(p, \mathfrak{o}_p)$	(13, 4)	(13, 3)	(17, 10)	(5, 12)	(19, 8)
$(p, \mathfrak{o}_p)$	(43, 24)	(5, 48)	(5, 60)	(19, 168)	(11, 96)

One deduces easily from the table that $SL(2, \mathbb{Z}/l) \subset \kappa(G_{\mathbb{Q}})$ for each choice of (a, l) .

(ii) The symmetric group, S_3 , acts on E_a^3 by permuting the factors. Using the Künneth formula one may decompose $H^3(E_{a\bar{\mathbb{Q}}}^3, \mathbb{Z}/l(2))$ as the sum of $\text{Sym}^3 H^1(E_{a\bar{\mathbb{Q}}}, \mathbb{Z}/l(2))$ and the $\mathbb{Z}/l[S_3]$ -module, H , generated by

$$H^0(E_{a\bar{\mathbb{Q}}}, \mathbb{Z}/l) \otimes H^2(E_{a\bar{\mathbb{Q}}}, \mu_l) \otimes H^1(E_{a\bar{\mathbb{Q}}}, \mu_l) \quad \text{and} \quad \Delta \cdot H^1(E_{a\bar{\mathbb{Q}}}, \mu_l),$$

where Δ generates $\Lambda^2 H^1(E_{a\bar{\mathbb{Q}}}^2, \mathbb{Z}/l(1)) \subset H^2(E_{a\bar{\mathbb{Q}}}^2, \mu_l)$. Now using the isomorphism from the Kummer sequence, $CH^1(E_{a\bar{\mathbb{Q}}})[l] \simeq H^1(E_{a\bar{\mathbb{Q}}}, \mu_l)$, and the fact that $H^0(E_{a\bar{\mathbb{Q}}}, \mathbb{Z}/l) \otimes H^2(E_{a\bar{\mathbb{Q}}}, \mu_l) \oplus \Delta \mathbb{Z}/l$ is in the image of the cycle class map, $N.S.(E_a^2) \rightarrow H^2(E_{a\bar{\mathbb{Q}}}, \mu_l)$, one sees that H is in the image of λ_l^2 . To complete the proof, note that $(Id - P)_* H = (Id - P)_* H^3(E_{a\bar{\mathbb{Q}}}^3, \mathbb{Z}/l(2))$.

(iii) The procedure used to calculate $\bar{\alpha}_l^2(P_* \Xi_a)$ is described in [Bu-Sch-Top, §3-5]. To summarize briefly, one reduces $E_{a\bar{\mathbb{Q}}}^3$ modulo an appropriate prime p of good reduction. One then computes

$$p_{3*}(D \cdot P_* \Xi_{a\mathbb{F}_p})_l \in CH^1(E_{a\mathbb{F}_p})[l^\infty]$$

for D a suitable divisor in $E_{a\mathbb{F}_p}^3$, where $p_3 : E_{a\mathbb{F}_p}^3 \rightarrow E_{a\mathbb{F}_p}$ is projection on the third factor, the subscript, l , means take the l -primary component of a torsion element in the Chow group, and $\Xi_{a\mathbb{F}_p}$ is the nullhomologous cycle obtained by reducing Ξ_a modulo p . The association, $P_* \Xi_a \mapsto p_{3*}(D \cdot P_* \Xi_{a\mathbb{F}_p})_l$, factors through the l -adic Abel-Jacobi map [Bu-Sch-Top, 4.8(1)], so if

$$(3.2) \quad CH^1(E_{a\mathbb{F}_p})[l^\infty] = CH^1(E_{a\mathbb{F}_p})[l],$$

then $p_{3*}(D \cdot P_* \Xi_{a\mathbb{F}_p})_l \neq 0$ implies $\bar{\alpha}_l^2(P_* \Xi_a) \neq 0$. The actual calculation of $p_{3*}(D \cdot P_* \Xi_{a\mathbb{F}_p})$ is sufficiently involved to require a computer. The results of these calculations for the triples, $(a, l, p) \in \{(5, 5, 13), (1, 7, 29), (2/7, 11, 41), (5/2, 13, 43), (47, 17, 73)\}$ (and for numerous others) are summarized in [Bu-Sch-Top, Table 9.2]. In each of the five cases $p_{3*}(D \cdot P_* \Xi_{a\mathbb{F}_p})_l \neq 0$. Also (3.2) holds because in each case the Riemann hypothesis implies that no elliptic curve over the field $\mathbb{F}_p$ can have a point of order l^2 .

(iv) It suffices to show that E_a has good ordinary reduction at l . Since the discriminant of the Weierstrass model for E_a in [Bu-Sch-Top, p. 113] is $256(a+1)(a+2)^4(-a+2)^6$ it is easy to check that E_a has good reduction at l . To show that it has ordinary reduction one need only check that $|E_a(\mathbb{F}_l)| \neq l+1$ [Sil, V.Ex5.10].

(v) This follows from (2.3). $\square$

(3.3) PROPOSITION. *If $l \in \{5, 7, 11, 13, 17\}$, then the class of Ξ in $CH_{hom}^2(E_{\bar{F}}^3)/l$ is non-zero.*

Proof. For any finite extension, $F \subset F'$, there is a specialization map, $CH^2(E_{F'}^3) \rightarrow CH^2(E_{a\mathbb{Q}}^3)$. Passing to the direct limit gives a specialization map, $CH^2(E_{\bar{F}}^3) \rightarrow CH^2(E_{a\mathbb{Q}}^3)$. There is also a specialization map on etale cohomology and the two specialization maps are compatible with the cycle class map [Fu, 20.3.5]. Thus specialization gives a map,

$$CH_{hom}^2(E_{\bar{F}}^3)/l \rightarrow CH_{hom}^2(E_{a\mathbb{Q}}^3)/l,$$

which sends Ξ to Ξ_a . Given $l \in \{5, 7, 11, 13, 17\}$, we may chose $a \in \mathbb{Q}$ such that (3.1)(v) implies that the class of Ξ_a in $CH_{hom}^2(E_{a\mathbb{Q}}^3)/l$ is non-zero. The proposition follows. $\square$

4. The action of modular correspondences on $CH^2(E_{\bar{F}}^3)/l$. In this final step in the proof of (1.2) we use modular correspondences to create a subgroup, $G \subset \text{Aut}(CH^2(E_{\bar{F}}^3)/l)$, such that the orbit of the class of Ξ is infinite. The argument is taken from [No] with modifications added to adapt it to the present context.

The first task is to relate the elliptic curve E of (1.1) to the universal elliptic curve over a modular curve. The quadratic twist, $\mathcal{E}$, of E by the field extension,

$$F \subset F' := F(\sqrt{2-a}),$$

is the elliptic curve over F defined by $f_{\mathcal{E}}(x, y, z) = 0$, where

$$(4.1) \quad f_{\mathcal{E}}(x, y, z) = y^2 z + [(a+2)x^3 + (2a)x^2 z + (a+2)xz^2].$$

Note that $f_{\mathcal{E}}$ is obtained from f_E in (1.1) by dividing all terms except the first by $(2-a)$.

(4.2) LEMMA. *$\mathcal{E}$ is isomorphic to the generic fiber of the universal elliptic curve with a point of order 4.*

Proof. There is a fine moduli space for elliptic curves with a point of order 4 [Ed, 4.3]. We denote it by $\dot{X}_1(4)$ and its compactification by $X_1(4)$. It is a degree 6 cover of the J -line. The point of order 4, $(-1 : 2 : 1) \in \mathcal{E}(F)$, gives rise to a morphism from $\mathcal{E}$ to the universal elliptic curve over $\dot{X}_1(4)$. The J -invariant of $\mathcal{E}$ is

$$J = -16 \frac{(a^2 - 12a - 12)^3}{(a+2)^4(a+1)}.$$

As $[F : \bar{\mathbb{Q}}(J)] = 6$, it follows that $F \simeq \bar{\mathbb{Q}}(X_1(4))$ and $\mathcal{E}$ is isomorphic to the generic fiber of the universal family of elliptic curves with a point of order 4. $\square$

Write $\phi : E_{\bar{F}} \rightarrow \mathcal{E}_{\bar{F}}$ for the isomorphism induced by applying *Proj* to the $\bar{F}$ -algebra homomorphism,

$$(4.3) \quad \bar{F}[x, y, z]/(f_{\mathcal{E}}) \rightarrow \bar{F}[x, y, z]/(f_E), \quad x \mapsto x, \quad y \mapsto (2-a)^{-1/2}y, \quad z \mapsto z.$$

Define $[\Xi] := \phi_*^3(\Xi) \in Z_{hom}^2(\mathcal{E}_{\bar{F}}^3)$.

We now turn to the construction of modular correspondences on $\mathcal{E}_{\bar{F}}$.

Let $N \in \mathbb{N}$ be odd, square free and prime to l . Let $X_0(N)$ denote the compactification of the coarse moduli space for the functor, elliptic curves with a cyclic subgroup of order N . Write $X_0(4N)$ for the normalization of the fiber product over the J -line, $X_1(4) \times_{\mathbb{P}^1} X_0(N)$. (The notation, $X_0(4N)$, agrees with standard notation because the tautological map, $X_1(4) \rightarrow X_0(4)$, is an isomorphism.) The curve, $\check{X}_0(4N)$, obtained by removing the cusps from $X_0(4N)$ represents the functor, elliptic curves with a point of order 4 and a cyclic subgroup of order N . Let $\varsigma_N : \mathcal{E}_N \rightarrow \check{X}_0(4N)$ denote the universal elliptic curve. It is isomorphic to the pullback of the universal elliptic curve over $\check{X}_1(4)$ by the “forget the level N -structure” map, $h_N : \check{X}_0(4N) \rightarrow \check{X}_1(4)$. Let p be a prime dividing N . Write $H_N \subset \mathcal{E}_N$ for the cyclic subgroup of order N and $H_N(p) \subset H_N$ for the subgroup of order p . The elliptic curve, $\mathcal{E}_N/H_N(p)$, inherits a point of order 4 from $\mathcal{E}_N$ and has a cyclic subgroup of order N given by $(\mathcal{E}_N[p]/H_N(p)) \times (H_N/H_N(p))$. Thus it may be realized as a pullback of the universal elliptic curve over $\check{X}_0(4N)$. This gives a commutative diagram in which the square on the right is Cartesian and the map on the upper left is tautological:

$$(4.4) \quad \begin{array}{ccccc} \mathcal{E}_N & \longrightarrow & \mathcal{E}_N/H_N(p) & \xrightarrow{\tilde{w}_p} & \mathcal{E}_N \\ \varsigma_N \downarrow & & \gamma \downarrow & & \varsigma_N \downarrow \\ \check{X}_0(4N) & \xlongequal{\quad} & \check{X}_0(4N) & \xrightarrow{w_p} & \check{X}_0(4N). \end{array}$$

Write $u_p : \mathcal{E}_N \rightarrow \mathcal{E}_N$ for the map in the top row of (4.4). Let $m_n \in \text{End}(\mathcal{E}_N)$ denote multiplication by n and let $\epsilon : \mathbb{Z} \rightarrow \{\pm 1\}$ denote the quadratic Dirichlet character of conductor 4.

(4.5) LEMMA. $w_p^2 = \text{Id}$ and $u_p \circ u_p = m_{\epsilon(p)p}$.

Proof. Taking the quotient in the right hand square of (4.4) by subgroups of order p gives rise to the left hand square in the Cartesian diagram,

$$(4.6) \quad \begin{array}{ccccc} \mathcal{E}_N/\mathcal{E}_N[p] & \xrightarrow{\tilde{w}_p} & \mathcal{E}_N/H_N(p) & \xrightarrow{\tilde{w}_p} & \mathcal{E}_N \\ \downarrow & & \gamma \downarrow & & \varsigma_N \downarrow \\ \check{X}_0(4N) & \xrightarrow{w_p} & \check{X}_0(4N) & \xrightarrow{w_p} & \check{X}_0(4N). \end{array}$$

Observe that $\mathcal{E}_N/\mathcal{E}_N[p]$ has a level $4N$ structure consisting of the point of order 4 which comes from the tautological point of order 4 on $\mathcal{E}_N$ via the canonical quotient map and the cyclic subgroup of order N which is the image of

$$(H_N/H_N(p)) \times (\mathcal{E}_N/H_N(p))[p] \subset \mathcal{E}_N/H_N(p)$$

under the map, $\mathcal{E}_N/H_N(p) \rightarrow \mathcal{E}_N/\mathcal{E}_N[p]$. One checks immediately that $\tilde{w}_p \circ \tilde{w}_p$ maps this level $4N$ structure to the tautological level $4N$ structure on $\mathcal{E}_N$. Since both squares in (4.6) are Cartesian, the map, $w_p \circ w_p$, is the map to the fine moduli space, $\check{X}_0(4N)$, induced by the level $4N$ structure on $\mathcal{E}_N/\mathcal{E}_N[p]$. Since $\mathcal{E}_N/\mathcal{E}_N[p]$ has a unique cyclic subgroup of order N and a unique cyclic subgroup of order 4, one of the two isomorphisms of $\check{X}_0(4N)$ -schemes, $\xi : \mathcal{E}_N/\mathcal{E}_N[p] \rightarrow \mathcal{E}_N$, must preserve the level $4N$ structure. Thus $w_p^2 = \text{Id}$. When we compose the canonical quotient map, $\mathcal{E}_N \rightarrow \mathcal{E}_N/\mathcal{E}_N[p]$, with the isomorphism, ξ , preserving the level $4N$ structure, we get $m_{\pm p}$ with the sign determined by $p \bmod 4$. $\square$

Set $F_N := \mathbb{Q}(X_0(4N))$. For $N|N'$ there is a natural inclusion, $F_N \subset F_{N'}$, corresponding to the map, forget the level N'/N -structure, $X_0(4N') \rightarrow X_0(4N)$. Thus

there is a tower of fields,

$$F \subset F_N \subset F_{N'} \subset \dots,$$

which we embed in $\bar{F}$. Set $F_\infty := \cup_N F_N \subset \bar{F}$, where the union is over square free, odd N prime to l . For each odd prime p distinct from l and multiple, N , of p , there is an involution $w_p \in \text{Aut}(\text{Spec}(F_N))$. If $N|N'$, then $w_p \in \text{Aut}(\text{Spec}(F_{N'}))$ induces $w_p \in \text{Aut}(\text{Spec}(F_N))$. This gives rise to an involution of $\text{Spec}(F_\infty)$ denoted by w_p as well. Fix a lifting, $\bar{w}_p \in \text{Aut}(\text{Spec}(\bar{F}))$, of w_p . For N a multiple of p define a map of schemes,

$$\bar{u}_p : \mathcal{E}_{\bar{F}} \simeq \mathcal{E}_N \times_{\dot{X}_0(4N)} \text{Spec}(\bar{F}) \xrightarrow{u_p \times \bar{w}_p} \mathcal{E}_N \times_{\dot{X}_0(4N)} \text{Spec}(\bar{F}) \simeq \mathcal{E}_{\bar{F}},$$

which makes sense because both u_p and $\bar{w}_p$ are compatible with the map, w_p , on $\dot{X}_0(4N)$. The map is independent of the choice of N . Similarly there is a map,

$$U_p := \bar{u}_p \times \bar{u}_p \times \bar{u}_p : \mathcal{E}_{\bar{F}}^3 \rightarrow \mathcal{E}_{\bar{F}}^3,$$

because $\bar{u}_p$ is compatible with $\bar{w}_p \in \text{Aut}(\text{Spec}(\bar{F}))$. Now U_p induces maps on Chow groups and etale cohomology which are compatible with the cycle class map. Thus it gives rise to an element, $U_p^* \in \text{End}(CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l)$.

(4.7) LEMMA. U_p^* is invertible.

Proof. Recall the exact sequence,

$$0 \rightarrow CH_{\text{alg}}^2(\mathcal{E}_{\bar{F}}^3) \rightarrow CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3) \rightarrow \text{Griff}^2(\mathcal{E}_{\bar{F}}^3) \rightarrow 0,$$

defining the Griffiths group as the quotient of CH_{hom}^2 by the subgroup generated by cycles algebraically equivalent to zero. As $CH_{\text{alg}}^2(\mathcal{E}_{\bar{F}}^3)$ is generated by Jacobians of curves via correspondences, it is a divisible group. Thus

$$(4.8) \quad CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l \simeq \text{Griff}^2(\mathcal{E}_{\bar{F}}^3)/l.$$

The map,

$$\bar{u}'_p := u_p \times \bar{w}_p^{-1} : \mathcal{E}_N \times_{X_0(4N)} \text{Spec}(\bar{F}) \rightarrow \mathcal{E}_N \times_{X_0(4N)} \text{Spec}(\bar{F}),$$

is well defined since both u_p and $\bar{w}_p^{-1}$ are compatible with $w_p \in \text{Aut}(\dot{X}_0(4N))$. Furthermore $\bar{u}_p \circ \bar{u}'_p$ may be identified with multiplication by $\epsilon(p)p$ in $\text{End}(\mathcal{E}_{\bar{F}})$ by (4.5). Write $M_n \in \text{End}(\mathcal{E}_{\bar{F}}^3)$ for multiplication by n . Set $U'_p := (\bar{u}'_p)^3$. Then $U_p \circ U'_p = M_{\epsilon(p)p}$. Now $M_{\epsilon(p)p} \circ M_{\epsilon(p)p}^*$ acts by multiplication by $p^6 = \deg(M_{\epsilon(p)p})$ on $CH(\mathcal{E}_{\bar{F}}^3)$ by the projection formula. On the other hand, $M_{\epsilon(p)p}^* \circ M_{\epsilon(p)p}$ takes a cycle to the sum of its translates by the points of order p on $\mathcal{E}_{\bar{F}}^3$. It thus acts on $\text{Griff}^2(\mathcal{E}_{\bar{F}}^3)$ by multiplication by p^6 . Since $p \neq l$, $M_{\epsilon(p)p}$ acts invertibly on $CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l$. Hence U_p does as well. $\square$

(4.9) Remark. In fact both $M_{\epsilon(p)p}^*$ and $M_{\epsilon(p)p}$ act by multiplication by $\epsilon(p)p^3$ on $\text{Griff}^2(\mathcal{E}_{\bar{F}}^3) \otimes \mathbb{Z}[\frac{1}{2}]$ [Bl-Sri, §4 Proposition].

Let $\mathcal{P} = \{p \text{ prime} : p \equiv 1 \pmod{4l}\}$.

(4.10) THEOREM. The set $\mathcal{U} := \{U_p^*[\Xi] \in CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l : p \in \mathcal{P}\}$ is infinite.

Proof. The set $\mathcal{P}$ is infinite by Dirichlet's theorem on primes in an arithmetic progression. We will show that the classes $U_p^*[\Xi] \in CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l$ for $p \in \mathcal{P}$ are linearly independent over $\mathbb{Z}/l$ by studying the action of $\text{Gal}(\bar{F}/F_\infty)$ on $CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l$.

Recall $F' = F(\sqrt{2-a})$. Define $F'_N := F'F_N$ and $F'_\infty := F'F_\infty \subset \bar{F}$. Write χ for the composition,

$$\text{Gal}(\bar{F}/F_\infty) \rightarrow \text{Gal}(F'_\infty/F_\infty) \simeq \{\pm 1\}.$$

Let χ_p denote the character which results when $\bar{w}_p(F'_\infty)$ is substituted for F'_∞ in the above expression. In other words, $\chi_p(g) = \chi(\bar{w}_p g \bar{w}_p^{-1})$.

Now (4.10) is an immediate consequence of

(4.11) PROPOSITION.

- (i) Each $g \in \text{Gal}(\bar{F}/F_\infty)$ acts on $[\Xi] \in Z^2(\mathcal{E}_{\bar{F}}^3)$ by $g[\Xi] = \chi(g)[\Xi]$.
- (ii) For $p \in \mathcal{P}$ the action of each $g \in \text{Gal}(\bar{F}/F_\infty)$ on $CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l$ satisfies

$$g U_p^*[\Xi] = \chi_p(g) U_p^*[\Xi].$$

(iii) For distinct primes $p \in \mathcal{P}$ the characters, $\chi_p : \text{Gal}(\bar{F}/F_\infty) \rightarrow \{\pm 1\}$, are distinct.

Proof. (i) The Galois group, $\text{Gal}(\bar{F}/F_\infty)$, acts on $E_{\bar{F}}$ and $\mathcal{E}_{\bar{F}}$ by acting on the coefficients in (4.3). For $g \in \text{Gal}(\bar{F}/F_\infty)$ we have,

$$g \circ \phi = \phi \circ \chi(g) \text{Id}_E \circ g,$$

where $\chi(g) \text{Id}_E \in \text{Aut}(E_{\bar{F}})$ denotes inversion in the group law if $\chi(g) = -1$ and the identity if $\chi(g) = 1$. Since inversion in the group law on $E_{\bar{F}}^3$ acts by -1 on $\Xi \in Z^2(E_{\bar{F}}^3)$ and $\text{Gal}(\bar{F}/F)$ fixes this element, (i) follows.

(ii) From

$$\bar{u}_p g \bar{u}'_p = (u_p \times \bar{w}_p) \circ (\text{Id} \times g) \circ (u_p \times \bar{w}_p^{-1}) = (u_p \circ u_p) \times \bar{w}_p g \bar{w}_p^{-1} = m_{\epsilon(p)p} \times \bar{w}_p g \bar{w}_p^{-1}$$

we deduce

$$U_p g U'_p = M_{\epsilon(p)p} \circ (\text{Id} \times \bar{w}_p g \bar{w}_p^{-1}).$$

By (4.8) and (4.9) $M_{\epsilon(p)p}$ acts as the identity on $CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l$ for $p \in \mathcal{P}$. Taking $g = \text{Id}$ in the previous equation gives that U'_p acts as the inverse of U_p on $CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l$. Thus

$$(4.12) \quad (U_p^*)^{-1} g U_p^* = (U_p g U'_p)^* = (\bar{w}_p g \bar{w}_p^{-1})^* \in \text{End}(CH_{\text{hom}}^2(\mathcal{E}_{\bar{F}}^3)/l)^*.$$

Now (ii) follows from (i) by applying (4.12) to $[\Xi]$.

(iii) Write $\dot{X}' \rightarrow \dot{X}_1(4)$ for the proper map of irreducible curves corresponding to the field extension, $F \subset F' = F(\sqrt{2-a})$. The branch locus, $B \subset \dot{X}_1(4)$, corresponds to the parameter value, $a = 2$. Write Q for the elliptic curve over $\mathbb{Q}$ obtained by setting $a = 2$ in (4.1). The J -invariant of Q is $2^{11}3^{-1}$. As it is not an integer, Q does not have complex multiplication [Sil2, II.6].

Let p_1 and p_2 be distinct odd primes. Then $\bar{w}_{p_1}(F'_\infty)$ and $\bar{w}_{p_2}(F'_\infty)$ are quadratic extensions of F_∞ . They coincide exactly when $\bar{w}_{p_1}(\sqrt{2-a}) \in \bar{w}_{p_2}(F'_\infty)$. Since $\bar{w}_{p_2}(F'_\infty) = \lim_N \bar{w}_{p_2}(F'_N)$, this means $\bar{w}_{p_1}(\sqrt{2-a}) \in \bar{w}_{p_2}(F'_N)$ for some N divisible by $p_1 p_2$. Equivalently $\bar{w}_{p_1}(F'_N) = \bar{w}_{p_2}(F'_N)$. The quadratic extension, $F_N \subset \bar{w}_{p_i}(F'_N) = F_N(\bar{w}_{p_i}(\sqrt{2-a}))$, corresponds to a degree two map of algebraic curves, $\tau_{N, p_i} : \dot{X}'_{N, p_i} \rightarrow \dot{X}_0(4N)$, which is branched at $w_{p_i}(h_N^{-1}(B))$. To show $\bar{w}_{p_1}(F'_N) \neq \bar{w}_{p_2}(F'_N)$ it suffices to show $w_{p_1}(h_N^{-1}(B)) \neq w_{p_2}(h_N^{-1}(B))$. In fact we claim that $w_{p_1}(h_N^{-1}(B)) \cap w_{p_2}(h_N^{-1}(B)) = \emptyset$. The fiber of $\varsigma_N : \mathcal{E}_N \rightarrow \dot{X}_0(4N)$ over a point in $w_{p_i}(h_N^{-1}(B))$ is isomorphic to Q/H_i , where $H_i \subset Q$ is a subgroup of order p_i . If subgroups H_i of order p_i were to exist such that $Q/H_1 \simeq Q/H_2$, then there would be an endomorphism,

$$\nu : Q \rightarrow Q/H_1 \simeq Q/H_2 \xrightarrow{\tau} Q,$$

where the kernel of τ is $Q[p_2]/H_2$ and the kernel of the ν has order $p_1 p_2$. This contradicts the observation that Q does not have complex multiplication and proves the claim. This completes the proof of (4.11) and (4.10). $\square$ $\square$

Proof of Theorem (1.2). Since $H^4(\mathcal{E}_F^3, \mathbb{Z}_l(2))$ is torsion free, $\text{Tor}\left(\frac{\mathbb{Z}}{l}, \frac{CH^2(\mathcal{E}_F^3)}{CH_{\text{hom}}^2(\mathcal{E}_F^3)}\right) = 0$. Thus $CH_{\text{hom}}^2(\mathcal{E}_F^3)/l$ infinite implies that $CH^2(\mathcal{E}_F^3)/l$ is infinite. $\square$

Proof of Theorem (0.1). Let k be an algebraically closed field of positive transcendence degree over $\mathbb{Q}$. Let $\mathbf{E}_k$ be an elliptic curve over k whose J -invariant, $J \in k$, is transcendental over $\mathbb{Q}$. Write $\mathbf{E}$ for an elliptic curve over $\mathbb{Q}(J)$ which becomes isomorphic to $\mathbf{E}_k$ after extension of scalars. There is an isomorphism, $\mathbb{Q}(J) \simeq \bar{F}$, which lifts to an isomorphism, $\mathbf{E} \simeq \mathcal{E}_{\bar{F}}$. Now (0.1) follows from (1.2) and the fact that the Chow group modulo l is unchanged under base extension by algebraically closed fields [Le].

(4.13) *Remark.* The methods of this section do not apply directly to elliptic curves over $\mathbb{Q}$ because modular correspondences do not give self maps of elliptic curves over $\mathbb{Q}$. There is one instance known to the author in which $CH^2(\mathbf{E}^3)/l$ has been shown to be infinite for an elliptic curve, $\mathbf{E}$, defined over $\mathbb{Q}$. In this case the J -invariant is 0 [Sch, §10].

(4.14) *Remark.* An application of Theorem (0.1) is given in [Sch2].

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